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LIMIT THEOREMS IN THE AREA OF LARGE DEVIATIONS FOR SOME DEPENDENT RANDOM VARIABLES[1]

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A magnetic body can be considered to consist of n sites, where n is large. The magnetic spins at these n sites, whose sum is the total magnetization present in the body, can be modelled by a triangular array of random variables $(X_1^{(n)}, \ldots, X_n^{(n)})$. Standard theory of physics would dictate that the joint distribution of the spins can be modelled by $dQ_n(\mathbf{x}) = z_n^{-1}\exp[-H_n(\mathbf{x})]\prod dP(x_j)$, where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathscr{R}^n$, where H_n is the Hamiltonian, z_n is a normalizing constant and P is a probability measure on $\mathscr{R}$. For certain forms of the Hamiltonian H_n, Ellis and Newman (1978b) showed that under appropriate conditions on P, there exists an integer $r \geq 1$ such that $S_n/n^{1-1/2r}$ converges in distribution to a random variable. This limiting random variable is Gaussian if $r = 1$ and non-Gaussian if $r \geq 2$. In this article, utilizing the large deviation local limit theorems for arbitrary sequences of random variables of Chaganty and Sethuraman (1985), we obtain similar limit theorems for a wider class of Hamiltonians H_n, which are functions of moment generating functions of suitable random variables. We also present a number of examples to illustrate our theorems.

1. Introduction. In this article we obtain limit theorems for some dependent random variables which are used to describe the distribution of magnetic spins present in a ferromagnetic crystal. A ferromagnetic crystal consists of a large number of sites. The amount of magnetic spin present at site i will be denoted by $X_i^{(n)}$, $i = 1, \ldots, n$, where n is a positive integer. The magnetic spin present at any site interacts with the magnetic spins at its neighboring sites and hence gives rise to some dependency among the random variables $X_i^{(n)}$'s. In the Ising model, the joint distribution, at a fixed temperature $T > 0$, of the spin random variables $(X_1^{(n)}, \ldots, X_n^{(n)})$, is given by

$$dQ_n(\mathbf{x}) = z_n^{-1}\exp\left[-\frac{H_n(\mathbf{x})}{T}\right]\prod dP(x_j), \tag{1.1}$$

where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathscr{R}^n$ and P is a probability measure on $\mathscr{R}$. The function $H_n(\mathbf{x})$ is known as the Hamiltonian and it represents the energy of the crystal at the configuration $\mathbf{x}$, and z_n is a normalizing constant which is also known as the partition function. In many cases, an explicit evaluation of z_n is very difficult and physicists usually try to evaluate the limiting free energy per site $\xi(T)$, at the temperature T, defined as follows:

$$\xi(T) = -\lim_{n\to\infty}\left[\frac{\log(z_n)}{n}\right]. \tag{1.2}$$

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For some particular types of Hamiltonians, it has been shown by physicists that there exists a temperature level T_c such that the function $\xi(T)$ is non-differentiable at $T = T_c$ [see Kac (1966)]. A phase transition is said to occur at the critical temperature T_c. As pointed out by Ellis and Newman (1978b), the existence of the critical temperature can be demonstrated in yet another way. We may be able to show that for $T > T_c$, there is a weak dependence among the random variables $(X_1^{(n)}, \dots, X_n^{(n)})$ and a standard central limit theorem is valid for $S_n/n^{1/2}$ and that for $T = T_c$, there exists a $\delta \in (1, 2)$ such that $S_n/n^{\delta/2}$ converges to a non-Gaussian limit and for $T < T_c$, due to the strong dependence among the $X_i^{(n)}$'s, the random variables tend to cluster in several ergodic components. This shows a marked discontinuity in the asymptotic distribution of S_n as the temperature T is allowed to vary and represents our approach to demonstrating a phase transition.

In Section 2, we consider a special case for the Hamiltonian, H_n, by setting it to be equal to $-(1/2n)\Sigma\Sigma x_i x_j$. This is known as the Curie–Weiss model. The asymptotic distribution of S_n for this model when P (which appears in Theorem 2.1) is symmetric Bernoulli is obtained by Simon and Griffiths (1973). In a two-paper series, Ellis and Newman (1978a, 1978b) extended Theorem 2.1 of Simon and Griffiths to the class of probability measures L, defined in (2.2) [see also Ellis and Rosen (1979)]. We state their extension precisely in Theorem 2.6. Recently Jeon (1979) in his Ph.D. dissertation gave a simpler and statistically motivated proof of Theorem 2.6. The goal of this article is to extend Theorem 2.6 for a larger class of Hamiltonians H_n and probability measures P. Our main result, Theorem 3.7 is stated in Section 3. The proof of Theorem 3.7 relies on a recent large deviation local limit theorem of Chaganty and Sethuraman (1985), which is restated in Section 3 as Theorem 3.4.

Let T_n, $n \geq 1$, be an arbitrary sequence of random variables with analytic moment generating function $\phi_n(z)$. We assume that T_n satisfies the conditions of Theorem 3.4. In our generalized model the Hamiltonian $H_n(\mathbf{x})$ is taken to be equal to $-\log[\phi_n(s_n/n)]$, where $s_n = x_1 + \cdots + x_n$. Thus, the joint distribution of the spin random variables $(X_1^{(n)}, \dots, X_n^{(n)})$ is given by

$$dQ_n(\mathbf{x}) = z_n^{-1}\phi_n\left(\frac{s_n}{n}\right)\prod dP(x_j), \tag{1.3}$$

where P is an arbitrary probability measure. Let $S_n = X_1^{(n)} + \cdots + X_n^{(n)}$. Under appropriate conditions on the probability measure P we show in Theorems 3.7 and 3.18, there exists an integer $r \geq 1$ such that $S_n/n^{1-1/2r}$ converges in distribution to a random variable Y_r^*, which has a nonnormal distribution when $r \geq 2$ and normal distribution when $r = 1$. The technique of our proof is to introduce a new random variable W_n, conditional on which, $X_1^{(n)}, \dots, X_n^{(n)}$ become i.i.d. It is easy to obtain the limiting distribution of W_n and the conditional asymptotic distribution of $S_n/n^{1-1/2r}$. Using the results of Sethuraman (1961) we deduce the asymptotic distribution of $S_n/n^{1-1/2r}$.

We now briefly give our reasons for calling these theorems on the asymptotic distribution of S_n under Q_n, defined in (1.3), as limit theorems in the area of large deviations. A standard technique to obtain the asymptotic distribution of

S_n under Q_n is to first obtain the asymptotic distribution of S_n under P_n, where

$$dP_n(\mathbf{x}) = \prod dP(x_j) \tag{1.4}$$

and then to use contiguity arguments, as in Le Cam (1960). This technique breaks down completely if $r \geq 2$. For the various models considered in physics which are described in greater detail in Sections 2 and 3,

$$|L_n(\mathbf{x})| = \left|\log \frac{dQ_n(\mathbf{x})}{dP_n(\mathbf{x})}\right| = \left|\frac{H_n(\mathbf{x})}{T} + \log z_n\right| \tag{1.5}$$

converges to ∞ in probability under P_n and thus contiguity arguments are not applicable here. Under P_n, $S_n/n^{1/2}$ has a limiting normal distribution, and $|L_n(\mathbf{x})|$ is small in the area of ordinary deviations of S_n, that is, when $S_n/n^{1/2}$ is finite, while $|L_n(\mathbf{x})|$ is large otherwise. Thus from the point of view of P_n, we are looking for the asymptotic distribution of S_n, when P_n is modified by $L_n(\mathbf{x})$, which is substantially different from 1 in the area of large deviations of S_n. This view point helps in a statistically motivated proof of the asymptotic distribution of S_n under Q_n and describes the background behind the title of this article. One should also note that the normalizing factor on S_n in its asymptotic distribution under Q_n is different from the corresponding factor under P_n.

2. A brief summary of the Curie–Weiss model and its extensions. In a ferromagnetic system with only pair interactions and with no external magnetic field, the Hamiltonian H_n, may be taken to be $-\frac{1}{2}\sum\sum a_{ij}x_ix_j$, where $a_{ij} \geq 0$. The Curie–Weiss model assumes that $a_{ij} = 1/n$ for all i and j, that is to say that each spin interacts equally with every other spin with strength $1/n$, and takes P to be symmetric Bernoulli, i.e., $P(\{-1\}) = P(\{1\}) = \frac{1}{2}$. Replacing P by $P_T(x) = P(xT^{1/2})$, we get

$$dQ_n(\mathbf{x}) = z_n^{-1}\exp\left(\frac{s_n^2}{2n}\right)\prod dP(x_j), \tag{2.1}$$

where $s_n = x_1 + \cdots + x_n$. This model has the advantage that the limiting free energy per site can be solved exactly. The existence of the critical temperature and phase transition for this model was demonstrated by Kac (1968). The asymptotic distribution for the total magnetism, S_n, for this model was obtained by Simon and Griffiths (1973). This is contained in Theorem 2.1.

THEOREM 2.1 (Simon and Griffiths). *Let $X_j^{(n)}$, $j = 1,\ldots,n$, be a triangular array of random variables whose joint distribution is given by* (2.1) *and P be symmetric Bernoulli. Then $S_n/n^{3/4}$ converges in distribution to a random variable whose density function is proportional to* $\exp(-y^4/12)$.

Theorem 2.1 was extended to the class of probability measures L, which is defined below, by Ellis and Newman (1978b).

DEFINITION 2.2. Let L be the class of probability measures P on $\mathscr{R}$ such that

$$\int_{-\infty}^{\infty} \exp\left(\frac{x^2}{2}\right) dP(x) < \infty. \tag{2.2}$$

Fix $P \in L$. It can be shown that condition (2.2) guarantees the existence of the moment generating function (m.g.f.), $m(u)$, of P. Let $h(u) = \log m(u)$ be the cumulant generating function (c.g.f.) of P. The function $G(u) = u^2/2 - h(u)$ plays an important role in Theorem 2.6 below.

DEFINITION 2.3. A real number m is said to be a global minimum for G if $G(u) \geq G(m)$ for all u.

DEFINITION 2.4. A global minimum m for G is said to be of type r if

$$G(u+m) - G(m) = \frac{c_{2r}u^{2r}}{(2r)!} + O\left(|u|^{2r+1}\right), \quad \text{as } u \to 0, \tag{2.3}$$

where $c_{2r} = G^{(2r)}(m)$ is strictly positive.

DEFINITION 2.5. A probability measure P is said to be pure if G has a unique global minimum.

Let Y_r, $r \geq 1$, be a sequence of random variables with density function $p_r(y)$, where

$$p_r(y) = \begin{cases} d_r \exp\left[-c_{2r}y^{2r}/(2r)!\right], & \text{if } r \geq 2, \\ N(0, (1-c_2)/c_2), & \text{if } r = 1, \end{cases} \tag{2.4}$$

and where d_r is the appropriate normalizing constant. With these definitions and notation we are now in a position to state the generalization of Theorem 2.1, due to Ellis and Newman (1978b).

THEOREM 2.6 (Ellis and Newman). *Let $P \in L$. Let P be pure, that is, let m be the unique global minimum of type r for G. Let $X_j^{(n)}$, $j = 1, \ldots, n$, be a triangular array of random variables with joint distribution given by* (2.1). *Let $S_n = X_1^{(n)} + \cdots + X_n^{(n)}$. Then*

$$\frac{S_n - nm}{n^{1-1/2r}} \to_d Y_r, \tag{2.5}$$

where Y_r is a random variable with density function given by (2.4).

It is easily verified that the symmetric Bernoulli measure is pure and belongs to the class L with the corresponding value of r equal to 2. Thus Theorem 2.6 contains Theorem 2.1.

Note that the moment generating function $M(s)$ of the standard normal is given by $\exp(s^2/2)$. Thus we can write (2.1) as

$$dQ_n(\mathbf{x}) = z_n^{-1}\left[M\left(\frac{s_n}{n}\right)\right]^n \prod dP(x_j). \tag{2.6}$$

One might ask the question whether it is possible to obtain limit theorems of the type (2.5) when $[M(s)]^n$ is replaced by the m.g.f. $\phi_n(s)$ of a random variable T_n, satisfying some conditions. We answer this question in the affirmative in the next section.

3. Further extensions of the Curie–Weiss model. In this section we propose to extend Theorem 2.6 by enlarging the class of Hamiltonians as well as the class of probability measures L. The large deviation local limit theorem for an arbitrary sequence T_n, $n \geq 1$, of random variables of Chaganty and Sethuraman (1985) (stated below) plays a key role in this extension. The Hamiltonians, H_n, in our generalized model (3.13) are taken to be the cumulant generating functions of these random variables T_n.

Let $\{T_n,\ n \geq 1\}$ be a sequence of nonlattice valued random variables with m.g.f.'s $\phi_n(s)$, $n \geq 1$, finite for real values of s such that $|s| < c \leq \infty$. Assume that $\phi_n(z)$, $n \geq 1$, are analytic and nonvanishing for complex z in $\Omega = \{z\colon |z| < c_1\}$, where $0 < c_1 \leq c$. Let $I = (-a, a)$ and $\Omega_a = \{z\colon |z| < a\}$, where $0 < a < c_1$. For values of z such that $\phi_n(z)$ is nonvanishing we let

$$\psi_n(z) = \frac{1}{n}\log \phi_n(z) \tag{3.1}$$

and

$$\gamma_n(u) = \sup_{|s|<c}\left[us - \psi_n(s)\right], \quad \text{for } u \in \mathscr{R}. \tag{3.2}$$

Let $\mathscr{A}_n = \{\psi_n'(s)\colon s \in I\}$. For $u \in \mathscr{A}_n$, we have $\gamma_n(u) = [us_n - \psi_n(s_n)]$, where $s_n \in I$ satisfies $\psi_n'(s_n) = u$. Let P be a probability measure on $(-c, c)$ which satisfies the following condition:

$$\int_{-c}^{c} \exp[\psi_n(x)]\, dP(x) < \infty, \quad \text{for all } n \geq 1. \tag{3.3}$$

Let $h(u)$ denote the c.g.f. of P. It is easy to check that condition (3.3) implies that $h(u)$ is finite for $u \in \mathscr{R}_n$, where

$$\mathscr{R}_n = \{u\colon \gamma_n(u) < \infty\}. \tag{3.4}$$

Let

$$V_n(u) = \begin{cases} \gamma_n(u) - h(u), & \text{for } u \in \mathscr{R}_n, \\ \infty, & \text{for } u \notin \mathscr{R}_n. \end{cases} \tag{3.5}$$

The function V_n plays the same role as the function G of Section 2.

DEFINITION 3.1. Let L^* be the class of all probability measures P on $(-c, c)$ satisfying condition (3.3). We assume that there exist $l,\ p_1 > 0$, such that

$$\int_{\mathscr{R}_n} \exp[-lV_n(u)]\,du = O(n^{p_1}), \tag{3.6}$$

and the V_n's have a unique global minimum at some point m_n. Furthermore, there exists $\eta_1 > 0$ such that

$$\inf_{|u|>\delta}[V_n(m_n + u) - V_n(m_n)] = \min_{s=-1,1}[V_n(m_n + s\delta) - V_n(m_n)], \quad \text{for all } 0 < \delta < \eta_1. \tag{3.7}$$

REMARK 3.2. Condition (3.7) is used mainly in inequality (3.27) of Lemma 3.13. An easily verifiable sufficient conditions for (3.7) is

$$V_n'(u) > 0, \quad \text{for } u > m_n \quad \text{and} \quad V_n'(u) < 0, \quad \text{for } u < m_n. \tag{3.8}$$

In all the examples of Section 4 we will be verifying (3.8) instead of (3.7).

REMARK 3.3. Suppose that $\mathscr{R}_n = (-\infty, \infty)$. If $\gamma_n(u)/|u|$ converges to ∞ as $|u| \to \infty$, then condition (3.3) implies (3.7) as seen below:

$$\begin{aligned} \exp[-V_n(u)] &= \exp[-\gamma_n(u) + h(u)] \\ &= \exp[-\gamma_n(u)]\left[\int_{|x|\le A} \exp[ux]\,dP(x) + \int_{|x|>A} \exp[ux]\,dP(x)\right] \\ &\le \exp[-\gamma_n(u) + |u|A] + \int_{|x|>A} \exp[\psi_n(x)]\,dP(x) \\ &\le \exp\left[-|u|\left(\frac{\gamma_n(u)}{|u|} - A\right)\right] + \int_{|x|>A} \exp[\psi_n(x)]\,dP(x). \end{aligned} \tag{3.9}$$

The right-hand side can be made close to zero first by choosing A and then letting $|u| \to \infty$. This shows that $V_n(u) \to \infty$ as $|u| \to \infty$. Since m_n is the unique global minimum of V_n, this also shows that condition (3.7) holds.

Let $m_n \in \mathscr{A}_n$. Then there is a τ_n in I such that $\psi_n'(\tau_n) = m_n$. For $t \in I$, define

$$G_n(t) = \psi_n(\tau_n) + itm_n - \psi_n(\tau_n + it). \tag{3.10}$$

The following theorem, which provides an asymptotic expansion for the density function k_n of T_n/n at m_n, in terms of the large deviation rate γ_n, is due to Chaganty and Sethuraman (1985). In fact, in that paper, it was shown that (3.11) holds for any $m_n \in \mathscr{A}_n$.

THEOREM 3.4. *Assume the following conditions for* T_n:

(A) *There exists* $\beta > 0$ *such that* $|\psi_n(z)| < \beta$ *for* $z \in \Omega_a$ *and* $n \ge 1$.
(B) *There exists* $\alpha > 0$ *such that* $\psi_n''(\tau) \ge \alpha$ *for* $\tau \in I$ *and* $n \ge 1$.
(C) *There exists* $\eta > 0$ *such that for any* $0 < \delta < \eta$,

$$\inf_{|t|\ge\delta} \text{Real}(G_n(t)) = \min[\text{Real}(G_n(\delta)), \text{Real}(G_n(-\delta))], \quad \textit{for } n \ge 1.$$

(D) *There exists $p > 0$ such that*

$$\sup_{\tau \in I} \int_{-\infty}^{\infty} \left| \frac{\phi_n(\tau + it)}{\phi_n(\tau)} \right|^{1/n} dt = O(n^p).$$

Then

$$k_n(m_n) = \left[\frac{n}{2\pi\psi_n''(\tau_n)} \right]^{1/2} \exp(-n\gamma_n(m_n)) \left[1 + O\left(\frac{1}{n} \right) \right]. \tag{3.11}$$

REMARK 3.5. When T_n is the sum of n i.i.d. random variables, condition (C) is automatically satisfied and conditions (A), (B) and (D) are easy to verify, since they do not depend on n.

REMARK 3.6. Suppose that m is an interior point of $\bigcap \mathscr{A}_n$. Then there exists $\xi_n \in I$ such that $\psi_n'(\xi_n) = m$, for $n \geq 1$. If conditions (A) and (B) of the above theorem are satisfied, then one can verify that $\psi_n''(\xi_n)$ is bounded above uniformly in n and $(\psi_n''(\xi_n))^{1/2}/(\psi_n''(\tau_n))^{1/2} = [1 + O(|m_n - m|)]$ [see (2.6) and (2.25) of Chaganty and Sethuraman (1985)]. Thus we can rewrite (3.11) as

$$k_n(m_n) = \left[\frac{n}{2\pi\psi_n''(\xi_n)} \right]^{1/2} e^{-n\gamma_n(m_n)} \left[1 + O(|m_n - m|) + O\left(\frac{1}{n} \right) \right]. \tag{3.12}$$

For each integer $r \geq 1$, let Y_r^* be a random variable with probability density function given by $d_r \exp[-c_{2r} y^{2r}/[h''(m)]^{2r}(2r)!]$ if $r \geq 2$ and $N(0, h''(m)[h''(m) + c_2]/c_2)$ if $r = 1$, where c_{2r} is the constant that appears in Theorem 3.7 below and d_r is the normalizing factor. With these assumptions and notation, we are in a position to state the main theorem of this section.

THEOREM 3.7. *Let $X_j^{(n)}$, $j = 1, \ldots, n$, be a triangular array of random variables satisfying $|X_j^{(n)}| < c$ and having a joint distribution given by*

$$dQ_n(\mathbf{x}) = z_n^{-1} \phi_n\left(\frac{s_n}{n} \right) \prod dP(x_j), \tag{3.13}$$

where ϕ_n is the m.g.f. of T_n and $P \in L^*$. Assume that V_n, defined in (3.5), *has a unique global minimum of type r at $m_n \in \mathscr{A}_n$. Let $m_n \to m$ and $V_n^{(2r)}(m_n) \to c_{2r}$ as $n \to \infty$, where m is an interior point of $\bigcap \mathscr{A}_n$. Let $S_n = X_1^{(n)} + \cdots + X_n^{(n)}$. If T_n satisfies the conditions of Theorem* 3.4, *then*

$$\frac{S_n - n\tau_n}{n^{1-1/2r}} \to_d Y_r^*, \tag{3.14}$$

where $\psi_n'(\tau_n) = m_n$ and Y_r^* is as defined above.

The proof of the above theorem is postponed until the end of Lemma 3.13.

REMARK 3.8. The distribution function $Q_n(\mathbf{x})$ is well defined because

$$z_n = \int \exp\left(n\psi_n\left(\frac{s_n}{n}\right)\right)\prod dP(x_j) \le \left[\int_{-c}^{c} \exp[\psi_n(x)]\, dP(x)\right]^n < \infty,$$

wherein we have used condition (3.3) and the fact that ψ_n is a convex function.

For $y \in \mathscr{R}$, let

$$g(y) = \exp\left[-\frac{y^{2r}c_{2r}}{(2r)!}\right] \tag{3.15}$$

and

$$g_n(y) = \left[\frac{2\pi\psi_n''(\xi_n)}{n}\right]^{1/2} k_n\left(m_n + n^{-1/2r}y\right) \times \exp\left[n\left(h\left(m_n + n^{-1/2r}y\right) + V_n(m_n)\right)\right], \tag{3.16}$$

where the ξ_n's are defined as in Remark 3.6. The functions g_n, $n \ge 1$, arise in the proof of Theorem 3.7. Lemma 3.9 shows that $g_n(y)$ converges to $g(y)$ as $n \to \infty$ for each y. The next four lemmas, Lemmas 3.10–3.13, show that

$$\int_{-\infty}^{\infty} g_n(y)\, dy \to \int_{-\infty}^{\infty} g(y)\, dy, \quad \text{as } n \to \infty. \tag{3.17}$$

LEMMA 3.9. *Suppose that V_n has a unique global minimum of type r at the point $m_n \in \mathscr{A}_n$. Let m_n converge to m, where m is an interior point of $\bigcap \mathscr{A}_n$. Suppose that $V_n^{(2r)}(m_n) = c_{2r,n}$ converges to c_{2r} as $n \to \infty$. Then*

$$g_n(y) \to g(y), \quad \text{as } n \to \infty. \tag{3.18}$$

PROOF. Fix $y \in \mathscr{R}$. Let $m_{n,r}(y) = m_n + n^{-1/2r}y$. Then $m_{n,r}(y)$ converges to m and $m_{n,r}(y) \in \mathscr{A}_n$ for sufficiently large n. Applying Theorem 3.4 together with Remark 3.6, with m_n replaced by $m_{n,r}(y)$ we get

$$\begin{aligned} g_n(y) &= \exp\left[-n\gamma_n(m_{n,r}(y)) + n\left(h(m_{n,r}(y)) + V_n(m_n)\right)\right] \\ &\quad\times\left[1 + O\left(|m_{n,r}(y) - m|\right) + O\left(\frac{1}{n}\right)\right] \\ &= \exp\left[-n\left(V_n(m_{n,r}(y)) - V_n(m_n)\right)\right] \\ &\quad\times\left[1 + O\left(|m_{n,r}(y) - m|\right) + O\left(\frac{1}{n}\right)\right] \\ &= \exp\left[-\frac{y^{2r}c_{2r,n}}{(2r)!} + n\,o\left(\frac{|y|^{2r}}{n}\right)\right]\left[1 + O\left(|m_{n,r}(y) - m|\right) + O\left(\frac{1}{n}\right)\right] \\ &\to g(y), \quad \text{as } n \to \infty. \end{aligned} \tag{3.19}$$

□

LEMMA 3.10. *Suppose that the V_n's have a unique global minimum of type r at the point $m_n \in \mathscr{A}_n$. Then there exists an N such that*

$$n\left[V_n\left(m_n + n^{-1/2r}y\right) - V_n(m_n)\right] \geq \frac{y^{2r}c_{2r}}{2(2r)!}, \tag{3.20}$$

for all $n \geq N$, and $|y| < n^{1/4r}$.

PROOF. Let $0 < \varepsilon < c_{2r}/2$. Since $c_{2r,n}$ converges to c_{2r} we can find N_1 such that $c_{2r,n} > c_{2r}/2 + \varepsilon$ for all $n \geq N_1$. Recall that $\gamma_n(m_n) = m_n\tau_n - \psi_n(\tau_n)$, where τ_n is such that $\psi_n'(\tau_n) = m_n$. It is easy to verify that $\gamma_n'(m_n) = \tau_n$ and $\gamma_n''(m_n) = [\psi_n''(\tau_n)]^{-1}$. Also, $\gamma_n^{(2r+1)}(m_n)$ is the ratio of a polynomial of $(2r+1)$ derivatives of ψ_n at τ_n to $[\psi_n''(\tau_n)]^{2r}$. Conditions (A) and (B) of Theorem 3.4 imply that all these derivatives are bounded uniformly in n and that $\psi_n''(\tau_n) \geq \alpha > 0$ [see (2.6) of Chaganty and Sethuraman (1985)]. Hence $\gamma_n^{(2r+1)}(m_n)$ is uniformly bounded in n and consequently $V_n^{(2r+1)}(m_n) = \gamma_n^{(2r+1)}(m_n) - h^{(2r+1)}(m_n)$ is also uniformly bounded in n. Therefore

$$V_n(m_n + u) - V_n(m_n) = \frac{u^{2r}c_{2r,n}}{(2r)!} + K_n u^{2r+1},$$

as $u \to 0$, where $|K_n| \leq K < \infty$ for all n. Thus

$$\begin{aligned} n\left[V_n\left(m_n + n^{-1/2r}y\right) - V_n(m_n)\right] &= \frac{y^{2r}c_{2r,n}}{(2r)!} + \frac{K_n y^{2r+1}}{n^{1/2r}} \\ &\geq \frac{y^{2r}c_{2r}}{2(2r)!} + y^{2r}\left[\frac{\varepsilon}{(2r)!} - \frac{Ky}{n^{1/2r}}\right] \\ &\geq \frac{y^{2r}c_{2r}}{2(2r)!}, \end{aligned}$$

if $|y| < n^{1/4r}$ and $n \geq N = \max\{N_1, (K(2r)!/\varepsilon)^{4r}\}$. This completes the proof of Lemma 3.10. □

LEMMA 3.11. *Let g and g_n be as defined in* (3.15) *and* (3.16). *Then under the hypothesis of Lemma* 3.9 *we have*

$$\int_{|y| \leq n^{1/4r}} g_n(y)\,dy \to \int_{-\infty}^{\infty} g(y)\,dy, \quad \text{as } n \to \infty. \tag{3.21}$$

PROOF. Note that $n^{-1/2r}y$ converges to zero uniformly in y for $|y| < n^{1/4r}$. Since m is an interior point of $\bigcap \mathscr{A}_n$ there exists N_3 (independent of y) such that $m_{n,r}(y) = (m_n + n^{-1/2r}y) \in \mathscr{A}_n$ for $n \geq N_3$. Applying Theorem 3.4 for

$n \geq N_3$, we get

$$\int_{|y| \leq n^{1/4r}} g_n(y)\, dy = \left[\frac{2\pi\psi_n''(\xi_n)}{n}\right]^{1/2} \int_{|y| \leq n^{1/4r}} \exp[n(h(m_{n,r}(y)) + V_n(m_n))] \times k_n(m_{n,r}(y))\, dy$$

$$(3.22) \qquad = \int_{|y| \leq n^{1/4r}} \exp[-n(V_n(m_{n,r}(y)) - V_n(m_n))] \times \left[1 + O(|m_{n,r}(y) - m|) + O\left(\frac{1}{n}\right)\right] dy = \int_{-\infty}^{\infty} \lambda_n(y)\, dy,$$

where

$$\lambda_n(y) = I(|y| \leq n^{1/4r}) \exp[-n(V_n(m_{n,r}(y)) - V_n(m_n))] \times \left[1 + O(|m_{n,r}(y) - m|) + O\left(\frac{1}{n}\right)\right],$$

and $I(\cdot)$ is the indicator function. It follows from Lemma 3.10 that $|\lambda_n(y)|$ is bounded by an integrable function. We can now conclude from Lemma 3.9 and the Lebesgue dominated convergence theorem that

$$(3.23) \qquad \int_{-\infty}^{\infty} \lambda_n(y)\, dy \to \int_{-\infty}^{\infty} g(y)\, dy, \quad \text{as } n \to \infty.$$

The proof Lemma 3.11 is now complete. □

The next Lemma 3.12 is needed in the proof of Lemma 3.13.

LEMMA 3.12. *Let T_n, $n \geq 1$, be a sequence of random variables satisfying the conditions of Theorem* 3.4. *Then*

$$(3.24) \qquad \sup_y [\exp(n\gamma_n(m_n + y)) k_n(m_n + y)] = O(n^{p+1}), \quad \text{as } n \to \infty.$$

PROOF. An application of the inversion formula yields [see (2.12) of Chaganty and Sethuraman (1985)],

$$[\exp[n((m_n + y)s - \psi_n(s))] k_n(m_n + y)] = \left| \frac{n}{2\pi} \int_{-\infty}^{\infty} \exp[n(\psi_n(s + it) - \psi_n(s) - it(m_n + y))]\, dt \right| \leq \frac{n}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\phi_n(s + it)}{\phi_n(s)} \right|^{1/n} dt.$$

Taking the supremum with respect to $s \in I$ and using condition (D) of Theorem

3.4 we get

$$\sup_y \left[\exp(n\gamma_n(m_n + y))k_n(m_n + y)\right] = O(n^{p+1}). \qquad \square$$

LEMMA 3.13. *Suppose that V_n has a unique global minimum at the point $m_n \in \mathscr{A}_n$ and let g_n be as defined in* (3.16). *Then*

$$\int_{|y|>n^{1/4r}} g_n(y)\,dy \to 0, \quad \text{as } n \to \infty. \tag{3.25}$$

PROOF. Let $m_{n,r}(y) = m_n + n^{-1/2r}y$. By (3.16) we have

$$\begin{aligned}\int_{|y|>n^{1/4r}} g_n(y)\,dy &= \left[\frac{2\pi\psi_n''(\xi_n)}{n}\right]^{1/2} \int_{|y|>n^{1/4r}} k_n(m_{n,r}(y)) \\ &\qquad \times \exp\left[n(h(m_{n,r}(y)) + V_n(m_n))\right] dy \\ &= \left[\frac{2\pi\psi_n''(\xi_n)}{n}\right]^{1/2} \int_{|y|>n^{1/4r}} k_n(m_{n,r}(y)) \\ &\qquad \times \exp\left[\left[-n(V_n(m_{n,r}(y)) - V_n(m_n))\right] + n\gamma_n(m_{n,r}(y))\right] dy.\end{aligned} \tag{3.26}$$

Substituting $u = n^{-1/2r}y$, we get

$$\begin{aligned}&\int_{|y|>n^{1/4r}} g_n(y)\,dy \\ &\quad = \sqrt{2\pi\psi_n''(\xi_n)}\, n^{-(1-1/r)/2} \int_{|u|>n^{-1/4r}} \left[\exp\left[-n(V_n(m_n + u) - V_n(m_n))\right]\right] \\ &\qquad \times \left[\exp(n\gamma_n(m_n + u))k_n(m_n + u)\right] du \\ &\quad \le O(n^{p+(1+1/r)/2}) \int_{|u|>n^{-1/4r}} \left[\exp\left[-n(V_n(m_n + u) - V_n(m_n))\right]\right] du.\end{aligned}$$

The last inequality follows from Lemma 3.12 and the fact that $\psi_n''(\xi_n)$ is uniformly bounded in n (see Remark 3.6). It is easy to verify that $V_n(m_n) = m_n\tau_n - \psi_n(\tau_n) - h(m_n)$ is uniformly bounded in n under the conditions of Theorem 3.4 and because $m_n \to m$. Thus we get

$$\begin{aligned}&\int_{|y|>n^{1/4r}} g_n(y)\,dy \\ &\quad \le O(n^{p+(1+1/r)/2}) \max_{|u|>n^{-1/4r}} \exp\left[-(n-l)(V_n(m_n + u) - V_n(m_n))\right] \\ &\qquad \times \int_{|u|>n^{-1/4r}} \exp\left[-l(V_n(m_n + u) - V_n(m_n))\right] du \\ &\quad \le O(n^{p+(1+1/r)/2}) \max_{|u|>n^{-1/4r}} \exp\left[-(n-l)(V_n(m_n + u) - V_n(m_n))\right] \\ &\qquad \times \int_{\mathscr{R}_n} \exp\left[-l(V_n(u))\right] du.\end{aligned}$$

This together with condition (3.6) yields

$$\left|\int_{|y|>n^{1/4r}} g_n(y)\,dy\right| \le O(n^q) \max_{|u|>n^{-1/4r}} \exp[-(n-l)(V_n(m_n+u)-V_n(m_n))] = O(n^q)\exp[-(n-l)L_n], \tag{3.27}$$

where

$$q = p_1 + p + \frac{1+1/r}{2},$$

and

$$L_n = \min_{|u|>n^{-1/4r}} [V_n(m_n+u) - V_n(m_n)].$$

This minimum is attained at $z = \pm n^{-1/4r}$ by condition (3.7). Therefore,

$$L_n = \min[(V_n(m_n+n^{-1/4r}) - V_n(m_n)), (V_n(m_n - n^{-1/4r}) - V_n(m_n))] = \frac{c_{2r,n}}{(2r)!}\frac{1}{n^{1/2}} + K_n n^{-(2r+1)/4r}.$$

Hence

$$\left|\int_{|y|>n^{1/4r}} g_n(y)\,dy\right| = O(n^q)\exp\left[-(n-l)\left[\frac{c_{2r,n}}{(2r)!}\frac{1}{n^{1/2}} + K_n n^{-(2r+1)/4r}\right]\right],$$

which goes to zero since $|K_n| \le K$ for all n. The proof of Lemma 3.13 is now complete. □

PROOF OF THEOREM 3.7. We first express dQ_n defined in (3.13) as follows:

$$dQ_n(\mathbf{x}) = z_n^{-1}\phi_n\left(\frac{s_n}{n}\right)\prod dP(x_j) = z_n^{-1}\int \exp(ys_n)k_n(y)\,dy \prod dP(x_j). \tag{3.28}$$

Substituting $m_{n,r}(y) = m_n + n^{1/2r}y$, we get

$$\begin{aligned} dQ_n(\mathbf{x}) &= z_n^{-1}n^{-1/2r}\int \exp(m_{n,r}(y)s_n)k_n(m_{n,r}(y))\,dy \prod dP(x_j) \\ &= z_n^{-1}n^{-1/2r}\int \prod \exp(x_j m_{n,r}(y) - h(m_{n,r}(y)))\,dP(x_j) \\ &\quad \times k_n(m_{n,r}(y))\exp(nh(m_{n,r}(y)))\,dy \\ &= \int \prod dM_{n,y}(x_j)\, f_n(y)\,dy, \end{aligned} \tag{3.29}$$

where

$$dM_{n,y}(x_j) = \exp\big(x_j m_{n,r}(y) - h(m_{n,r}(y))\big)\, dP(x_j) \tag{3.30}$$

and

$$f_n(y) = z_n^{-1} n^{-1/2r} k_n(m_{n,r}(y)) \exp\big(nh(m_{n,r}(y))\big). \tag{3.31}$$

Since $\int dQ_n(\mathbf{x}) = 1$ and $\int dM_{n,y}(x_j) = 1$ for each y and j, we have $\int f_n(y)\,dy = 1$. Thus we can introduce random variables W_n with probability density function $f_n(y)$ and the representation (3.29) of $dQ_n(\mathbf{x})$ shows that given $W_n = y$, $X_j^{(n)}$, $j = 1,\dots,n$, are i.i.d. with common distribution $M_{n,y}(x)$.

We now proceed to obtain the limiting distribution of $(S_n - n\tau_n)/n^{1-1/2r}$ under $dM_{n,y}(x)$.

We first note that

$$\begin{aligned}
\log E_{M_{n,y}} \exp\left[\frac{t(S_n - n\tau_n)}{n^{1-1/2r}}\right]
&= n\left[-\frac{t\tau_n}{n^{1-1/2r}} + h\left(\frac{t}{n^{1-1/2r}} + m_{n,r}(y)\right) - h(m_{n,r}(y))\right] \\
&= n\left[-\frac{t\tau_n}{n^{1-1/2r}} + h'(m_{n,r}(y))\frac{t}{n^{1-1/2r}} \right. \\
&\qquad \left. + h''(m_{n,r}(y))\frac{t^2}{2n^{2-1/r}} + o(n^{-1})\right] \\
&= h''(m_n)ty + \frac{h''(m_n)t^2}{2n^{1-1/r}} + o(1),
\end{aligned} \tag{3.32}$$

since $\tau_n = h'(m_n)$. Thus

$$\log E_{M_{n,y}} \exp\left[\frac{t(S_n - n\tau_n)}{n^{1-1/2r}}\right] \to \begin{cases} h''(m)ty, & \text{if } r \geq 2, \\ h''(m)ty + \dfrac{h''(m)t^2}{2}, & \text{if } r = 1. \end{cases} \tag{3.33}$$

This shows that the limiting distribution of $(S_n - n\tau_n)/n^{1-1/2r}$ given $W_n = y$ is degenerate at $h''(m)y$ if $r > 1$ and $N(h''(m)y,\ h''(m))$ if $r = 1$. Next we note that

$$\begin{aligned}
f_n(y) &= z_n^{-1} n^{-1/2r} k_n(m_{n,r}(y)) \exp\big(nh(m_{n,r}(y))\big) \\
&= \frac{g_n(y)}{\int g_n(y)\,dy},
\end{aligned} \tag{3.34}$$

where $g_n(y)$ is as defined in (3.16). By Lemmas 3.9, 3.11 and 3.13 it follows that

$$f_n(y) \to f(y) = \frac{g(y)}{\int g(y)\,dy}, \quad \text{as } n \to \infty, \tag{3.35}$$

where $g(y) = \exp[-y^{2r}c_{2r}/(2r)!]$. Thus the limiting distribution of W_n is $f(y)$. The unconditional limiting distribution of $(S_n - n\tau_n)/n^{1-1/2r}$ is just the mixture of the limiting conditional distribution and $f(y)$, by Theorem 3.15 of Sethuraman (1961). This completes the proof of Theorem 3.7. □

REMARK 3.14. When T_n is the sum of independent, normally distributed random variables with mean zero and variance one, $\phi_n(s_n/n)$ becomes $\exp[s_n^2/2n]$ and the class of probability measures L^* reduces to the class L. Thus Theorem 3.7 generalizes Theorem 2.6 to a larger class of Hamiltonians and probability measures.

We now state the theorem of Sethuraman (1961) which was crucially used to obtain the limiting marginal distribution of $(S_n - n\tau_n)/n^{1-1/2r}$ in the proof of Theorem 3.7.

THEOREM 3.15 (Sethuraman, 1961). *Let Λ_n be a sequence of probability measures on $V \times W$, where V and W are topological spaces. Let μ_n be the marginal probability measure of Λ_n and V and $\nu_n(v, \cdot)$ be the conditional probability measure on W. Assume that there exists a probability measure μ such that $\mu_n(A)$ converges to $\mu(A)$ for every measurable set $A \subset V$. Suppose that for almost all v with respect to μ, $\nu_n(v, \cdot)$ converges weakly to $\nu(v, \cdot)$. Then Λ_n converges weakly to Λ, where*

$$\Lambda(A \times B) = \int_A \nu(v, B)\, d\mu(v), \tag{3.36}$$

for every measurable rectangular set $A \times B$.

We now turn our attention to the case where T_n, $n \geq 1$, are lattice valued random variables with spans h_n, $n \geq 1$. The following theorem, which is analogous to Theorem 3.4, was proved by Chaganty and Sethuraman (1985).

THEOREM 3.16. *Let T_n, $n \geq 1$, be a sequence of lattice valued random variables with spans h_n, $n \geq 1$. Let m_n belong to the range of T_n/n. Assume that conditions* (A) *and* (B) *of Theorem* 3.4 *hold and replace conditions* (C) *and* (D) *by the following*:

(C′) *There exists $\eta > 0$ such that for any $0 < \delta < \eta$,*

$$\inf_{\delta \leq |t| \leq \pi/|h_n|} \operatorname{Real}(G_n(t)) = \min[\operatorname{Real}(G_n(\delta)), \operatorname{Real}(G_n(-\delta))], \quad \textit{for } n \geq 1,$$

where $G_n(t)$ is defined by (3.10).

(D′) *There exists $p > 0$ such that $|h_n| = O(n^{-p})$.*

Then

$$\frac{n^{1/2}}{|h_n|}\Pr\left(\frac{T_n}{n} = m_n\right) = \left[\frac{1}{2\pi\psi_n''(\tau_n)}\right]^{1/2} \exp(-n\gamma_n(m_n))\left[1 + O\left(\frac{1}{n}\right)\right]. \tag{3.37}$$

As before for a probability measure P on $\mathscr{R}$, define $V_n(u)$ as in (3.5). The class of probability measures that are of interest is defined below.

DEFINITION 3.17. Let L_1^* be the class of probability measures P satisfying conditions (3.3), (3.7) and (3.38) (defined below).

$$\sum_{u \in \mathscr{R}_n} \exp[-lV_n(u)] = O(n^{p_1}), \quad \text{for some } l, p_1 > 0. \tag{3.38}$$

Note that (3.38) is the appropriate replacement of (3.6) for the lattice valued case.

For Hamiltonians which are functions of the moment generating functions of lattice valued random variables we have the following theorem almost identical to Theorem 3.7.

THEOREM 3.18. *Let $P \in L_1^*$. Let $X_j^{(n)}$, $j = 1, \ldots, n$, be a triangular array of random variables satisfying $|X_j^{(n)}| < c$ and having a joint distribution given by*

$$dQ_n(\mathbf{x}) = z_n^{-1}\phi_n\left(\frac{s_n}{n}\right)\prod dP(x_j), \tag{3.39}$$

where ϕ_n is the m.g.f. of the lattice valued random variables T_n. Let $S_n = X_1^{(n)} + \cdots + X_n^{(n)}$. Let V_n have a unique global minimum of type r at the point $m_n \in \mathscr{A}_n$. Let m_n converge to a point m belonging to the interior of $\bigcap \mathscr{A}_n$. If T_n satisfies the conditions of Theorem 3.16, *then*

$$\frac{S_n - n\tau_n}{n^{1-1/2r}} \to_d Y_r^*, \tag{3.40}$$

where Y_r^* and τ_n are as defined in Theorem 3.7.

The proof of the above theorem parallels the proof of Theorem 3.7. We therefore outline briefly the modifications that need to be done. Note that dQ_n can be written as

$$dQ_n(\mathbf{x}) = \sum_y \prod dM_{n,y}(x_j)\, f_n^*(y), \tag{3.41}$$

where $f_n^*(y) = z_n^{-1}k_n(m_n + n^{-1/2r}y)\exp[nh(m_n + n^{-1/2r}y)]$ is a probability mass function of a lattice valued distribution with span $h_n' = h_n/n^{1-1/2r}$, and $dM_{n,y}(x_j)$ is as defined in (3.30). We introduce discrete random variables W_n^* with p.m.f. f_n^*. It suffices to show that W_n^* converges weakly to a continuous random variable W with probability density function f, defined in (3.35). The rest of the proof is identical to the proof of Theorem 3.7. Note that the span, h_n', of W_n^* converges to zero. By a theorem of Okamoto (1959), the sequence of random variables W_n^* will converge in distribution to W, once we prove the following:

LEMMA 3.19. *For $y \in \mathscr{R}$, define $y_n = h'_n[y/h'_n]$. Let the probability mass function f_n^* and the probability density function f be as defined above. Then*

$$\frac{1}{|h'_n|} f_n^*(y_n) \to f(y), \quad \text{as } n \to \infty, \tag{3.42}$$

uniformly on bounded intervals of y.

PROOF (outline). Note that $f(y) = g(y)/\int g(y)\,dy$, where $g(y)$ is as defined in (3.15). We first write

$$f_n^*(y_n) = \frac{g_n^*(y_n)}{\Sigma g_n^*(y)},$$

where

$$g_n^*(y_n) = \frac{\sqrt{2\pi\psi_n''(\xi_n)}}{n^{(1-1/r)/2}} k_n\left(m_n + n^{-1/2r}y_n\right) \times \exp\left[n\left(h\left(m_n + n^{-1/2r}y_n\right) + V_n(m_n)\right)\right]. \tag{3.43}$$

Imitating the proofs of Lemmas 3.9–3.13, one can show the following:

(i) $$\frac{1}{|h'_n|} g_n^*(y_n) \to g(y), \quad \text{as } n \to \infty,$$

uniformly on bounded intervals of y;

(ii) $$\sum_{|y_n| \le n^{1/4r}} g_n^*(y_n) \to \int_{-\infty}^{\infty} g(y)\,dy, \quad \text{as } n \to \infty;$$

(iii) $$\sum_{|y_n| > n^{1/4r}} g_n^*(y_n) \to 0, \quad \text{as } n \to \infty.$$

The above three steps (i), (ii) and (iii) complete the proof of Lemma 3.19. □

4. Applications. In this section we illustrate the main theorems of Section 3 with four applications and demonstrate limit theorems in quite complicated situations of dependent variables. The model (3.13) for the joint distribution of $(X_1^{(n)}, \dots, X_n^{(n)})$ is completely specified if T_n and P that arise in it are specified. To simplify matters, in all the examples of this section we let T_n be the sum of n i.i.d. random variables with common distribution function F. The four examples below contain all occurrences of lattice and nonlattice T_n, and continuous and discrete P. The limit distribution of the normalized sum $S_n/n^{1-1/2r}$ is normal ($r = 1$, in the notation of Theorems 3.7 and 3.18) in Example 4.1 and is nonnormal ($r = 2$) in Examples 4.2, 4.3 and 4.4. The results of Ellis and Newman (1978b) show that limit distributions with every possible value of $r > 2$ can also arise in suitable models.

In all the examples below we will specify F and P and write down the joint distribution Q_n. Since T_n is the sum of independent random variables, the functions $\psi_n \equiv \psi$, $\gamma_n \equiv \gamma$ and $V_n \equiv V$, independent of n and therefore it is straightforward to verify the four conditions of Theorem 3.4 or Theorem 3.16 depending on whether T_n is nonlattice or lattice. We can also show that in all the examples considered, V is symmetric around the origin and $V'(u) > 0$ for $u > 0$. Hence V has a unique global minimum at the origin. The verification of conditions (3.3), (3.6) and (3.8) that insure that $P \in L^*$, in the case of continuous P and conditions (3.3), (3.8) and (3.38) that insure that $P \in L_1^*$, in the case of discrete P, does not pose any difficulties. The details are left to the reader.

EXAMPLE 4.1. Let the distribution function F and the probability measure P be defined by the probability density functions $\frac{1}{2}\exp(-|x|)$, $-\infty < x < \infty$, and $\frac{3}{4}(1 - x^2)$, $|x| < 1$, respectively. Then the joint distribution Q_n given in (3.13) becomes

$$dQ_n(\mathbf{x}) = z_n^{-1}\left(\frac{3}{4}\right)^n\left(1 - \frac{s_n^2}{n^2}\right)^{-n}\prod\left(1 - x_j^2\right)\prod dx_j. \tag{4.1}$$

In this case we can show that for $u \in \mathscr{R}$,

$$\begin{aligned} V(u) = {} & \left[-1 + \sqrt{1 + u^2}\right] + \log|u| + \log\left[-1 + \sqrt{1 + u^2}\right] - |u| \\ & -\log\left[|u|(1 + e^{-2|u|}) - (1 - e^{-2|u|})\right] + \log\left(\tfrac{4}{3}\right). \end{aligned} \tag{4.2}$$

Since $V''(0) = \frac{3}{10} > 0$, the global minimum of V is of type 1. We can therefore conclude in this example that

$$S_n/n^{1/2} \to_d N\left(0, \tfrac{1}{3}\right). \tag{4.3}$$

EXAMPLE 4.2. Let F be the triangular distribution function on the interval $(-2b, 2b)$ with $b = 3^{1/2}/2^{1/2}$. Let P be the standard normal probability measure. The joint distribution Q_n is given by

$$dQ_n(\mathbf{x}) = z_n^{-1}(2\pi)^{-n/2}\left[\frac{n\sinh(bs_n/n)}{(bs_n)}\right]^{2n}\exp\left[-\frac{1}{2}\sum x_j^2\right]\prod dx_j. \tag{4.4}$$

With our choice of $b = 3^{1/2}/2^{1/2}$ one can verify that $V''(0) = 0$ and $V^{(4)}(0) = \frac{3}{5} > 0$. Therefore V has a unique global minimum of order 2 at the origin and hence by the conclusion of Theorem 3.7 we get

$$\frac{S_n}{n^{3/4}} \to_d Y_2^*, \tag{4.5}$$

where the p.d.f. of Y_2^* is given by $d_2\exp(-y^4/40)$, $-\infty < y < \infty$.

EXAMPLE 4.3. Let F be as defined in Example 4.2. Let P be symmetric Bernoulli, i.e., $P(\{-1\}) = P(\{1\}) = \frac{1}{2}$. The joint distribution Q_n is given by

$$dQ_n(\mathbf{x}) = z_n^{-1}\left[\frac{n\sinh(bs_n/n)}{(2^{1/2}bs_n)}\right]^{2n}, \tag{4.6}$$

where $x_j = \pm 1$ for all $1 \le j \le n$, and $b = 3^{1/2}/2^{1/2}$. In this example we can show that V has a unique minimum of order 2 at the origin and $V^{(4)}(0) = \frac{13}{5}$. Thus by the conclusion of Theorem 3.7 we get

$$\frac{S_n}{n^{3/4}} \to_d Y_2^*, \tag{4.7}$$

where the p.d.f. of Y_2^* is given by $d_2 \exp[-13y^4/120]$, $-\infty < y < \infty$.

EXAMPLE 4.4. Let F be symmetric Bernoulli distribution and P be the standard normal probability measure. The joint distribution in this example is given by

$$dQ_n(\mathbf{x}) = z_n^{-1}(2\pi)^{-n/2}\left[\cosh\left(\frac{s_n}{n}\right)\right]^n \exp\left[-\frac{1}{2}\sum x_j^2\right]\prod dx_j. \tag{4.8}$$

It is easy to check that zero is the unique global minimum of order 2 for the function V and $V^{(4)}(0) = 2$. Note that in this example T_n's are lattice valued random variables. Thus by the conclusion of Theorem 3.18 we get

$$\frac{S_n}{n^{3/4}} \to_d Y_2^*, \tag{4.9}$$

where Y_2^* is distributed as $d_2 \exp(-y^4/12)$, $-\infty < y < \infty$.

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