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MARKOV ADDITIVE PROCESSES I. EIGENVALUE PROPERTIES AND LIMIT THEOREMS

BY P. NEY[1] AND E. NUMMELIN

University of Wisconsin, Madison and University of Helsinki

We consider a Markov additive process $\{(X_n, S_n): n = 0, 1, \dots\}$, where $\{X_n\}$ is a M.C. on a general state space and S_n is an $\mathbb{R}^d$-valued additive component. Limit theory for S_n is studied via properties of the eigenvalues and eigenfunctions of the kernel of generating functions associated with the transition function of the process. The emphasis is on large deviation theory, but some other limit theorems are also given.

1. Introduction and summary. Let $\{X_n;\ n = 0, 1, \dots\}$ be a Markov chain on a state space $\mathbb{E}$ with σ-field $\mathscr{E}$, which is irreducible with respect to a maximal irreducibility measure φ on $(\mathbb{E}, \mathscr{E})$ and is aperiodic. Let $\{\xi_n;\ n = 1, 2, \dots\}$ be a sequence of $\mathbb{R}^d$-valued random variables such that $\{(X_n, \xi_n);\ n = 1, 2, \dots\}$ is a Markov chain on $(\mathbb{E} \times \mathbb{R}^d, \mathscr{E} \times \mathscr{R}^d)$ ($\mathscr{R}^d$ = Borel sets) having transition function

$$(1.1)\qquad \begin{aligned} &P\{(X_{n+1}, \xi_{n+1}) \in A \times \Gamma | X_n = x, \mathscr{F}_n\} \\ &\quad = P\{(X_{n+1}, \xi_{n+1}) \in A \times \Gamma | X_n = x\} \\ &\quad \equiv P(x, A \times \Gamma), \qquad x \in \mathbb{E},\ A \in \mathscr{E},\ \Gamma \in \mathscr{R}^d, \end{aligned}$$

where $\mathscr{F}_n$ is the σ-field generated by $(X_0, \dots, X_n, \xi_1, \dots, \xi_n)$. Let $S_n = S_0 + \sum_{i=1}^n \xi_i$. The pair $\{(X_n, S_n);\ n = 0, 1, \dots\}$ is called a *Markov-additive process* or chain (abbreviated MA-process), and $P(x, A \times \Gamma)$ is called an MA-transition kernel. For basic results on construction and properties of MA-processes see Cinlar (1972a, b).

The limit theory of an MA-process can be studied via the "spatial" transform or generating function of its transition function:

$$(1.2)\qquad \hat{P}(\alpha) = \hat{P}(x, A; \alpha) = \int_{\mathbb{R}^d} e^{\langle \alpha, s \rangle} P(x, A \times ds), \qquad \alpha \in \mathbb{R}^d,\ x \in \mathbb{E},\ A \in \mathscr{E},$$

and of its iterates $\hat{P}^n(\alpha)$. ($\langle \cdot, \cdot \rangle$ denotes inner product.)

The main purpose of this paper (part I) is to study, from a general viewpoint, the existence and (especially) the regularity properties of the eigenvalues and eigenfunctions of the above "Feynman–Kac" operator. These properties seem of interest in their own right and are useful in proving limit theorems for MA-processes. Our particular interest is in large deviation (LD) theorems. We will

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prove a logarithmic LD theorem, and under stronger hypotheses, a sharper limit theorem.

The present results are also essential preliminaries to part II of the paper, where a general logarithmic LD theorem is proved under minimal hypotheses.

There is extensive literature on LD theorems for Markov chains. An early work by Miller (1961) considered a real valued MA-process defined on a finite Markov chain, and used properties of the transform matrix to prove a local LD theorem. Kim and David (1979) and the present authors with Iscoe (1985) proved Perron–Frobenius-type theorems for $\hat{P}(\alpha)$ (and thence LD theorems) for general state space M.C.'s subject to strong uniformity restrictions. The most definitive work on additive functionals of M.C.'s is Donsker and Varadhan (1975a, b), (1976) and (1983). Upper bounds have also been obtained somewhat more generally by de Acosta (1985). Recent books by Stroock (1984) and Ellis (1985) also contain many related results. Azencott (1980) is an excellent survey of LD theory.

There are a number of departures between our present work and the above works. First, our techniques are quite different, being very probabilistic, and relying on the construction of a regeneration structure for the MA-process. This simplifies much of the analysis, and provides an alternative approach to the Donsker–Varadhan theory. Our hypotheses differ in several respects from those of Donsker and Varadhan. We have no topological restrictions on the Markov chain; only irreducibility with respect to the reference measure φ.

Both of our results cover random sequences of the form $\Sigma_1^n F(X_{i-1}, X_i)$. In this paper we restrict ourselves to $\mathbb{R}^d$-valued F's, whereas Donsker and Varadhan (and other references cited above) cover more general spaces, including measure valued functions. On the other hand, we have essentially no further restrictions on F, such as boundedness or growth properties. (See also the remarks below.) Our setting also allows $F(x)$ or $F(x, y)$, for each $x, y \in \mathbb{E}$, to be random variables. This would be the case, for instance, if S_n is the time up to the nth jump in a semi-Markov process on a general state space. The absence of restrictions on F and on $\{X_i\}$ (other than irreducibility) becomes most striking in the lower bound theorem of part II.

Finally, there is a technical distinction between our setup and that of some other approaches (e.g., Donsker and Varadhan, de Acosta or Stroock). We work with the *convergence parameter* of $\hat{P}(\alpha)$ (defined below) as the basis for determining the rate function, whereas they use the spectral radius (or an equivalent function). The convergence parameter leads to LD theorems for

$$\mathbb{P}_x\{S_n/n \in \Gamma,\ X_n \in A\}, \qquad \Gamma \in \mathscr{R}^d,\ A \in \mathscr{E}, \tag{1.3}$$

for A's that are "sufficiently small" in a suitable sense; the spectral radius leads to theorems for

$$\mathbb{P}_x\{S_n/n \in \Gamma\}$$

(i.e., $A = \mathbb{E}$). When the convergence parameter and the spectral radius are the same, then (1.3) will have the same rate function for all φ-positive sets A, but in general this will not be so. (Our results shed some light on this.)

This point is illustrated in Example 6.3 below, for a sequence $S_n = \Sigma_1^n F(X_i)$, where X_i takes values in $\mathbb{E} = \{0, 1, \dots\}$ and F is a (finite) integer valued function. Our theory yields an LD result for

$$\mathbb{P}_i\{S_n \geq an,\ X_n = j\}.$$

But the spectral radius turns out to be infinite, and hence tells us nothing about $\mathbb{P}_i\{S_n \geq an\}$.

The basis of the regeneration technique is the construction of random times $\{T_i;\ i = 0, 1, \dots\}$, in terms of which the process is decomposed into independent blocks, and one can then define the generating function

$$\psi(\alpha, \zeta) = E\exp\{\langle\alpha, S_{T_{i+1}} - S_{T_i}\rangle - \zeta(T_{i+1} - T_i)\}, \qquad \alpha \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^1,$$

which is independent of i. Letting $\exp\{-\Lambda(\alpha)\}$ be the convergence parameter of $\hat{P}(\alpha)$, one can show that if $\hat{P}$ is $e^{-\Lambda}$-recurrent, then

$$\psi(\alpha, \Lambda(\alpha)) = 1. \tag{1.4}$$

More importantly, there is also a converse. Namely, if ψ has open domain of convergence $\mathscr{W}$, then $\psi(\alpha, \zeta) = 1$ has a unique solution $\zeta = \Lambda(\alpha)$, and $e^{\Lambda(\alpha)}$ is an eigenvalue of $\hat{P}(\alpha)$ with associated nonnegative invariant function $\{r(x; \alpha);\ x \in \mathbb{E}\}$ and measure $\{l(A; \alpha);\ A \in \mathscr{E}\}$. Furthermore, the properties of Λ, r and l can be developed directly from (1.4) (which we thus call the characteristic equation of Λ). These properties are used to prove limit theorems about S_n.

In a principal application of the above results we approximate a general MA-process by a sequence of processes satisfying the above hypothesis on $\mathscr{W}$, and thereby prove a large deviation theorem *under little more hypothesis than the irreducibility of the underlying Markov chain* (e.g., no recurrence on $\{X_n\}$, only a mild nonsingularity hypothesis on the additive component). As this result requires its own constructions and arguments we have given it a fairly self-contained treatment in the sequel "part II."

Here is an outline of the rest of this paper:

Section 2. A quick survey of some known results about nonnegative kernels, plus a few lemmas about such kernels which will be needed later.

Section 3. Definition of $\Lambda(\alpha)$ via the characteristic equation $\psi(\alpha, \Lambda(\alpha)) = 1$. Proof of analyticity and essential smoothness of $\Lambda(\alpha)$ under the condition that $\mathscr{W} = \{(\alpha, \zeta) \in \mathbb{R}^{d+1}\colon \psi(\alpha, \zeta) < \infty\}$ is an open set.

Section 4. Proof that $e^{-\Lambda(\alpha)}$ is the convergence parameter of $\hat{P}(\alpha)$ and $e^{\Lambda(\alpha)}$ is an eigenvalue. Determination of representation formulas for the associated (right) eigenfunction $\{r(x; \alpha);\ x \in \mathbb{E}\}$ and (left) eigenmeasure $\{l(A; \alpha);\ A \in \mathscr{E}\}$. Proof that $r(x; \alpha)$ and $l(A; \alpha)$ are analytic for $x \notin$ a fixed set of φ-measure 0, and A "sufficiently small" in a suitable sense. Proof of geometric ergodicity of $\hat{P}(\alpha)$.

Section 5. Limit theorems for S_n.

Section 6. Examples are given of MA-processes that satisfy the hypotheses of this paper, and some that do not.

2. Nonnegative kernels. In this section we summarize some definitions and known results, plus a few simple additional facts, about nonnegative kernels.

For proofs and details about results not explicitly proved here see Vere-Jones (1967, 1968), Tweedie (1974a, b), Athreya and Ney (1982) and particularly Nummelin (1984).

After looking at the basic definitions, the reader can skip to Section 3 and later refer back to Section 2 when needed.

Let $\{K(x, A);\ x \in \mathbb{E},\ A \in \mathscr{E}\}$ be a σ-finite nonnegative φ-irreducible kernel. For any function h: $\mathbb{E} \to \mathbb{R}$ and measure ν on $(\mathbb{E}, \mathscr{E})$, we write $Kh(x) = \int K(x, dy)h(y)$, $\nu K(A) = \int \nu(dx)K(x, A)$ and

$$(h \otimes \nu)(x, A) = h(x)\nu(A), \qquad \nu h(A) = \int_A \nu(dx)h(x), \qquad \nu h = \nu h(\mathbb{E}).$$

Let

$$G(x, A) = \sum_{n=0}^{\infty} K^n(x, A), \qquad G^{(\rho)} = \sum_{n=0}^{\infty} \rho^n K^n. \tag{2.1}$$

Call R the *convergence parameter* of K if $G^{(\rho)}$ is "finite" for $\rho < R$, "infinite" for $\rho > R$ [see Tweedie (1974a, b) or Nummelin (1984) for a precise definition]. A σ-finite irreducible, nonnegative kernel *always* has a convergence parameter $0 \le R < \infty$. We will assume K is such that $R > 0$. Given any K as above, one can show that there always exists an integer $k_0 < \infty$, a function h and a nonzero measure ν such that

$$h \otimes \nu \le K^{k_0}, \qquad \varphi h > 0. \tag{2.2}$$

This condition is equivalent to the existence of C-sets [see Orey (1971) and Nummelin (1984)]. In the interest of simplicity we will assume that $k_0 = 1$; namely,

$$h \otimes \nu \le K, \tag{2.3}$$

but this is not a necessary condition for our results.

Let

$$\begin{aligned} G_{h,\nu} &= \sum_{n=0}^{\infty} (K - h \otimes \nu)^n, \\ G_{h,\nu}^{(\rho)} &= \sum_{n=0}^{\infty} \rho^n (K - h \otimes \nu)^n. \end{aligned} \tag{2.4}$$

K is called *R-recurrent* if $G^{(R)}(x, A) = \infty$ for $x \in \mathbb{E}$, $\varphi(A) > 0$, otherwise R-transient. R-recurrence is equivalent to $\nu G^{(R)}h = \infty$. If K is R-recurrent for some R, call it recurrent. Let

$$\begin{aligned} u_0 &= 1, \qquad u_n = \nu K^{n-1}h, \qquad n \ge 1, \\ b_0 &= 0, \qquad b_n = \nu(K - h \otimes \nu)^{n-1}h, \qquad n \ge 1. \end{aligned} \tag{2.5}$$

Then $\{u_n\}_0^\infty$ is the renewal sequence generated by $\{b_n\}_0^\infty$ [Nummelin (1984) and Athreya and Ney (1982)]. Define

$$\hat{\mu}(\rho) = \sum_0^\infty \rho^n u_n, \qquad \hat{b}(\rho) = \sum_0^\infty \rho^n b_n.$$

Let $R' = \sup\{\rho\colon \hat{u}(\rho) < \infty\} = \sup\{\rho\colon \hat{b}(\rho) < 1\}$. Then $R' = R$, and $\hat{b}(R) \le 1$. Furthermore,

$$K \text{ is } R\text{-recurrent} \Leftrightarrow \hat{b}(R) = 1 \Leftrightarrow \hat{u}(R) = \infty. \tag{2.6}$$

K is called *positive R-recurrent* if $\Sigma nR^n b_n < \infty$ and just positive recurrent if it is positive R-recurrent for some $R > 0$ (we write p-recurrent). An R-recurrent K is called *geometrically R-recurrent* if

$$\nu G_{h,\nu}^{(\rho)} h < \infty, \quad \text{for some } \rho > R,$$

or, equivalently,

$$\sum \rho^n b_n < \infty, \quad \text{for some } \rho > R.$$

(We again say g-recurrent for short.)

A function $r(x) \ge 0$ (not $\equiv \infty$) with $\varphi r > 0$ is called ρ-subinvariant if $\rho K r \le r$, invariant if $\rho K r = r$. In the latter case call $\lambda = \rho^{-1}$ an eigenvalue. There may be any number of eigenvalues (or none) and R may or may not be an eigenvalue. For $\rho \le R$, ρ-subinvariant functions *always* exist. When $\rho < R$, or when $\rho = R$ and K is R-transient, then a ρ-subinvariant function is given by

$$r(x) = (G^{(\rho)}h)(x) \quad [h \text{ as in } (2.3)].$$

If K is R-recurrent, then there exists an R-invariant function given by

$$r(x) = \left(RG_{h,\nu}^{(R)}h\right)(x) \quad [h, \nu \text{ as in } (2.3)], \tag{2.7}$$

and an R-invariant measure

$$l(A) = \left(R\nu G_{h,\nu}^{(R)}\right)(A), \qquad A \in \mathscr{E}. \tag{2.8}$$

Note that

$$\hat{b}(R) = R\nu G_{h,\nu}^{(R)} = \nu r = lh \tag{2.9}$$

and this $= 1$ if K is R-recurrent. Thus in the R-recurrent case R^{-1} is an eigenvalue with

$$r = RKr, \qquad l = RlK. \tag{2.10}$$

In the R-transient case there *may* still be an R-invariant function. Thus the existence of such a function does *not* imply R-recurrence. Let $K_\varepsilon = K - \varepsilon h \otimes \nu$.

LEMMA 2.1. *If $K \ge h \otimes \nu$ and K is g-recurrent, then $K_\varepsilon = K - \varepsilon h \otimes \nu$ is g-recurrent for all sufficiently small $\varepsilon \ge 0$.*

PROOF. Let R be the convergence parameter of K. Then by (2.6)

$$\hat{b}(R) = \sum_{n=1}^{\infty} R^n \nu (K - h \otimes \nu)^{n-1} h = 1$$

and

$$\infty > \sum_{n=1}^{\infty} \rho_0^n \nu (K - h \otimes \nu)^{n-1} h = c > 1,$$

for some $\rho_0 > R$. Let

$$\hat{b}_\varepsilon(\rho) = \sum \rho^n \nu(K_\varepsilon - (1-\varepsilon)h \otimes \nu)^{n-1}(1-\varepsilon)h.$$

This is just a power series in ρ, hence analytic in the interior of its domain of convergence. Also

$$\hat{b}_\varepsilon(\rho_0) = (1-\varepsilon)c \in (1, \infty),$$

for sufficiently small ε. But this is the definition of geometric recurrence of K_ε (with minorization $(1-\varepsilon)h \otimes \nu$). □

The next lemma is a converse of the above.

LEMMA 2.2. *If $K \geq h \otimes \nu$ and K_{ε_0} is recurrent, then K_ε is g-recurrent for all* $0 \leq \varepsilon < \varepsilon_0$.

PROOF. We have $K_{\varepsilon_0} \geq (1-\varepsilon_0)h \otimes \nu$, and hence letting R_{ε_0} denote the convergence parameter of K_{ε_0} and using (2.6) again, this time applied to K_{ε_0} with minorization $(1-\varepsilon_0)h \otimes \nu$, we see that

$$\begin{aligned} 1 &= \sum R_{\varepsilon_0}^n \nu\left(K_{\varepsilon_0} - (1-\varepsilon_0)h \otimes \nu\right)^{n-1}(1-\varepsilon_0)h \\ &= (1-\varepsilon_0)\sum R_{\varepsilon_0}^n \nu(K - h \otimes \nu)^{n-1}h = (1-\varepsilon_0)\hat{b}(R_{\varepsilon_0}). \end{aligned}$$

Thus

$$1 < \hat{b}(R_{\varepsilon_0}) = (1-\varepsilon_0)^{-1} < \infty,$$

and hence K is geometrically recurrent. The same argument applies to K_ε for $0 \leq \varepsilon < \varepsilon_0$. □

The next two lemmas give some special hypotheses under which g-recurrence can be verified. These will be referred to in examples later on. The first is a slight generalization of Theorem 3.10.1 of Harris (1963).

LEMMA 2.3. *Assume there exist constants $0 < a \leq c < \infty$, a function h: $\mathbb{E} \to \mathbb{R}$ and a measure ν on $(\mathbb{E}, \mathscr{E})$ such that*

$$ah(x)\nu(A) \leq K(x, A) \leq ch(x)\nu(A), \qquad x \in \mathbb{E}, A \in \mathscr{E}. \tag{2.11}$$

Then K is g-recurrent.

NOTATION. Sometime we abbreviate (2.11) by $K \cong h \otimes \nu$.

PROOF. (i) Recall that

$$R = \sup\left\{\rho \geq 0: \sum_n \rho^n \nu K^n h < \infty\right\}.$$

By (2.11)

$$(K - ah \otimes \nu) \leq \rho_0 K, \tag{2.12}$$

where $\rho_0 = 1 - a/c < 1$. Note that necessarily $h(x) < \infty$, or else by (2.11), $K(x, \cdot)$ would not be σ-finite for some $x \in \mathbb{E}$. Now

$$\sum_{n=1}^{\infty} R_0^n \nu (K - ah \otimes \nu)^{n-1} ah < \infty, \tag{2.13}$$

for $R_0 = \rho_0^{-1/2} R > R$. This implies K is g-recurrent. □

REMARKS. (i) The hypothesis (2.11) can be replaced by the somewhat weaker condition $K^{n_0} \cong \sum_{i=1}^N h^i \otimes \nu^i$, for some $N < \infty$, $n_0 \geq 1$ and functions h^i and measures ν^i. See the arguments in Section 5.7 of Nummelin (1984).

(ii) If $K(x, \mathbb{E}) = 0$, we simply throw away this trivial point x. Thus it means no loss of generality to assume $K(x, \mathbb{E}) > 0$ for all $x \in \mathbb{E}$.

If $\{K(x, A);\ x \in \mathbb{E},\ A \in \mathscr{E}\}$ is *stochastic*, then a useful recurrence criterion is the "Doeblin condition"

$$a\nu(A) \leq K(x, A), \qquad x \in \mathbb{E},\ A \in \mathscr{E}, \tag{2.14}$$

for some $0 < a$ and probability measure ν. (In fact, of course, this implies *uniform* recurrence.) Unfortunately, for general kernels, this fails. Namely, we may have (2.14) but not have any kind of recurrence.

The following lemma is a step in the direction of a Doeblin condition for kernels.

LEMMA 2.4. *Let $K(\cdot, \cdot)$ be an irreducible aperiodic kernel on $(\mathbb{E}, \mathscr{E})$. Assume that $a \leq K(x, \mathbb{E}) \leq b$, $0 < a \leq b < \infty$, $x \in \mathbb{E}$ and that there is a probability measure ν on $(\mathbb{E}, \mathscr{E})$ such that for some $c < a$*

$$K(x, A) \geq (K(x, \mathbb{E}) - c)\nu(A), \tag{2.15}$$

for all $A \in \mathscr{E}$. Then K is g-recurrent.

PROOF. Let $h(x) = K(x, \mathbb{E}) - a$. By (2.15)

$$\begin{aligned} K(x, dy) &- h(x)\nu(dy) \\ &= K(x, dy) - (K(x, \mathbb{E}) - c)\nu(dy) + (a - c)\nu(dy) \\ &\geq (a - c)\nu(dy), \end{aligned}$$

and hence

$$\tilde{K}(x, A) \equiv a^{-1}(K - h \otimes \nu)(x, A) \tag{2.16}$$

is a stochastic kernel with

$$\tilde{K}(x, A) \geq \left(1 - \frac{c}{a}\right)\nu(A), \qquad x \in \mathbb{E},\ A \in \mathscr{E}. \tag{2.17}$$

Thus $\tilde{K}$ is uniformly 1-recurrent. Hence by Lemma 2.2 K is g-recurrent. □

COROLLARY 2.1. *If*

$$K(x, A) \geq c\nu(A), \quad \text{for } x \in \mathbb{E},\ A \in \mathscr{E}, \tag{2.18}$$

for some $c > b - a$ and some probability measure ν, then K is g-recurrent.

3. Regeneration. We define a regenerative scheme for the MA-process which is similar to constructions in Iscoe, Ney and Nummelin (1985), Athreya, McDonald and Ney (1978) and Nummelin (1978). Since some of the limit theory also applies to general regenerative processes, we minimize the dependence on the Markov aspect of the process in this section.

Consider the following:

(M) *Minorization condition. There exists a family of measures $\{h(x, \Gamma);\ \Gamma \in \mathscr{R}^d\}$ on $\mathbb{R}^d$, for each $x \in \mathbb{E}$, and a probability measure $\{\nu(A \times \Gamma);\ A \in \mathscr{E},\ \Gamma \in \mathscr{R}^d\}$ on $\mathbb{E} \times \mathbb{R}^d$, such that for all $x \in \mathbb{E}$, $A \in \mathscr{E}$, $\Gamma \in \mathscr{R}^d$,*

$$h(x, \cdot) * \nu(A \times \cdot) \leq P(x, A \times \Gamma).$$

Let $\mathscr{F}_n$ = the σ-field generated by $(X_0, \dots, X_n, \xi_1, \dots, \xi_n)$. The consequence to be drawn from (M) is

LEMMA 3.1. *Under* (M) *there exist random variables $0 < T_0 < T_1 < \cdots$ and a decomposition $\xi_{T_i} = \xi'_{T_i} + \xi''_{T_i}$, $i = 0, 1, \dots$, with the following properties*:

(i) *$\{T_{i+1} - T_i;\ i = 0, 1, \dots\}$ are i.i.d. random variables*;

(ii) *the random blocks*

$$\left\{X_{T_i}, \dots, X_{T_{i+1}-1}, \xi''_{T_i}, \xi_{T_i+1}, \dots, \xi_{T_{i+1}-1}, \xi'_{T_{i+1}}\right\}, \qquad i = 0, 1, \dots,$$

are independent; *and*

(iii) *$\mathbb{P}_x\{(X_{T_i}, \xi''_{T_i}) \in A \times \Gamma'' | \mathscr{F}_{T_i - 1}, \xi'_{T_i}\} = \nu(A \times \Gamma'')$, for $A \in \mathscr{E}$, $\Gamma'' \in \mathscr{R}^d$.*

PROOF. See Ney and Nummelin (1984). □

COROLLARY 3.1. *If $h(x, \mathbb{R}^d) \geq c > 0$, then*

$$P\{T_{i+1} - T_i > k\} \leq c^k.$$

We note two special cases of (M). First, if $\xi''_n = 0$, $n = 1, 2, \dots,$ then (M) becomes

$$h(x, \Gamma)\nu(A) \leq P(x, A \times \Gamma), \tag{M_1}$$

where ν is now a measure on $(\mathbb{E}, \mathscr{E})$. In this case the independent blocks are

$$\left\{X_{T_i}, \dots, X_{T_{i+1}-1}, \xi_{T_i+1}, \dots, \xi_{T_i}\right\} \tag{3.1}$$

and

$$P\left\{X_{T_i} \in A | \mathscr{F}_{T_i - 1}, \xi_{T_i}\right\} = \nu(A). \tag{3.1'}$$

On the other hand, if $\xi_n' = 0$, then (M) reduces to

$$(\mathrm{M}_2) \qquad h(x)\nu(A \times \Gamma) \le P(x, A \times \Gamma),$$

where $h\colon \mathbb{E} \to \mathbb{R}'$.

HYPOTHESIS 1. *To avoid repetitive consideration of different special cases we will work with condition* (M_1) *in this paper* (*except for examples*). *We remark, however, that* (M_2), *or the apparently weaker* (M) *would work just as well.*

By a result of Niemi and Nummelin (1986), given in the following proposition, a mild nonsingularity condition on P is sufficient to assure a slightly weaker condition than (M_1) or (M_2).

PROPOSITION 3.1. *Assume that there exists a* φ*-positive set* $B \in \mathscr{E}$ *such that for every* $x \in B$, $P^n(x, \cdot \times \cdot)$ *is nonsingular with respect to* $\varphi \times$ *Lebesgue measure for some* $n = n(x) \ge 1$. *Then there exists an integer* n_0, *a function* $f(x, s)\colon \mathbb{E} \times \mathbb{R}^d \to \mathbb{R}_+$ *and a probability measure* ν *on* $(\mathbb{E}, \mathscr{E})$ *such that* $\int\int f(x,s)\varphi(dx)\,ds > 0$ *and*

$$f(x, s)\, ds\nu(dy) \le P^{n_0}(x, dy \times ds).$$

There also exists an integer n_1, *a function* $h(x)\colon \mathbb{E} \to \mathbb{R}^+$ *and a probability measure* $\nu(dy \times ds) = \nu(dy \times s)\,ds$, *such that*

$$h(x)\nu(dy \times ds) \le P^{n_1}(x, dy \times ds).$$

REMARK. Requiring that a minorization hold for P^{k_0} for *some* $0 \le k_0 < \infty$ is a weaker hypothesis. However, for purposes of large deviation theory it is no loss of generality to take $k_0 = 1$, since the required eigenvalue properties for P and P^{k_0} are the same. [See Ney and Nummelin (1984) for details of this argument.]

Under (M_1), we may define the generating function

$$\psi(\alpha, \zeta) = E_\nu e^{\langle \alpha, S_\tau\rangle - \zeta\tau}, \qquad \alpha \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^1, \tag{3.2}$$

where

$$\tau =_D T_{i+1} - T_i, \qquad S_\tau =_D S_{T_i+1} - S_{T_i}, \qquad i = 1, 2, \dots .$$

(If $P\{X_0 \in A\} = \nu(A)$, define $T_0 = 0$.)
Let

$$\mathscr{W} = \{(\alpha, \zeta)\colon \psi(\alpha, \zeta) < \infty\} \subset \mathbb{R}^{d+1}. \tag{3.3}$$

DEFINITION 3.1. *Assume* (M_1). *If*

$$\psi(\alpha, \Lambda) = 1, \tag{3.4}$$

for some $\Lambda < \infty$, *then define* $\Lambda = \Lambda(\alpha)$ *by* (3.4). *Since* $\psi(\alpha, \zeta)$ *is monotone in* ζ, *if such a* Λ *exists it is unique.*

In Section 4 we will prove that if $\mathscr{W}$ is open then $\Lambda(\cdot)$ exists and $e^{\Lambda(\alpha)}$ is an eigenvalue of $\hat{P}(\alpha)$. In this section we establish some properties of Λ which follow from the definition.

First some more notation. Let

$$\begin{aligned}
\mathscr{U} &= \{\alpha \in \mathbb{R}^d: \psi(\alpha, 0) < \infty\},\\
\mathscr{U}_r &= \{\alpha: \psi(\alpha, \Lambda) = 1 \text{ for some } \Lambda = \Lambda(\alpha) < \infty\},\\
\mathscr{U}_p &= \{\alpha \in \mathscr{U}_r: -(\partial\psi/\partial\zeta)^+(\alpha, \Lambda(\alpha)) < \infty\},\\
\mathscr{U}_g &= \{\alpha: 1 < \psi(\alpha, \zeta) < \infty \text{ for some } \zeta \in \mathbb{R}\},\\
\mathscr{D} &= \{\alpha: \psi(\alpha, \zeta) < \infty \text{ for some } \zeta < \infty\},\\
\mathscr{S} &= \operatorname{Supp}_\nu(S_\tau/\tau),
\end{aligned}$$

where $\operatorname{Supp}_\nu(Y) \equiv$ the convex hull of the support of the measure $P_\nu\{Y \in \cdot\}$.

Note that $\mathscr{U}_g \subseteq \mathscr{U}_p \subseteq \mathscr{U}_r \subseteq \mathscr{D}$ and $\mathscr{U} \subseteq \mathscr{D}$ always.

HYPOTHESIS 2. *We assume from now on that $\mathring{\mathscr{S}} \neq \varnothing$, thereby assuring that S_n is genuinely d-dimensional. This will not always be explicitly stated.*

LEMMA 3.2. *Assume* (M_1) *and that $\mathscr{W}$ is open. Then*

$$\mathscr{D} = \mathscr{U} = \mathscr{U}_r = \mathscr{U}_p = \mathscr{U}_g = \text{an open set}.$$

COROLLARY 3.2. *Under the above hypothesis* $(\alpha, \Lambda(\alpha)) \in \mathscr{W}$ *for all* $\alpha \in \mathscr{D}$. *Hence* $\Lambda(\alpha)$ *is analytic on* $\mathscr{D}$.

PROOF OF LEMMA 3.2. Take any $\alpha \in \mathscr{D}$. Since $\tau \geq 1$

$$\lim_{\zeta \nearrow \infty} \psi(\alpha, \zeta) = 0. \tag{3.5}$$

Now let $\zeta_0 \in \{\zeta: (\alpha, \zeta) \in \partial\mathscr{W}\} \cup \{-\infty\}$. Then

$$\lim_{\zeta \to \zeta_0} \psi(\alpha, \zeta) = \infty, \tag{3.6}$$

since by Fatou's lemma, any generating function with open domain diverges along any sequence approaching a finite boundary point of the domain. Since $\psi(\alpha, \zeta)$ is continuous on $\mathscr{W}$, (3.5) and (3.7) imply that there exists a $\zeta = \Lambda(\alpha)$ such that $\psi(\alpha, \zeta) = 1$, i.e., $(\alpha, \zeta) \in \mathscr{W}$, $\alpha \in \mathscr{U}_r$. The fact that $\mathscr{W}$ is open then also implies that there is a ζ' such that $1 < \psi(\alpha, \zeta') < \infty$, i.e., $\alpha \in \mathscr{U}_g$. Since clearly $\mathscr{U}_g \subset \mathscr{D}$, we have $\mathscr{U}_g = \mathscr{D}$. Since $\mathscr{D}$ is the projection of $\mathscr{W}$ on the α-space, it is also open. □

PROOF OF COROLLARY 3.2. If $\alpha \in \mathscr{D}$, then the above proof implies $(\alpha, \Lambda(\alpha)) \in \mathscr{W}$. The rest of the corollary then follows from the strict convexity of $\psi(\alpha, \zeta)$, its monotonicity in ζ, and the implicit function theorem for analytic functions [see, e.g., Gunning and Rossi (1965), page 2, or Dieudonne (1960), page 268]. □

NOTATION. Write

$$E_\nu Y e^{\langle\alpha, S_\tau\rangle - \Lambda(\alpha)\tau} \overset{\text{def}}{=} E_\nu^{(\alpha)} Y. \tag{3.7}$$

LEMMA 3.3 (Differentiation). *If* $\alpha \in \mathscr{U}_r$ *and* $(\alpha, \Lambda(\alpha)) \in \mathscr{W}$, *then* $\nabla\Lambda(\alpha)$ *and* $\nabla\nabla'\Lambda(\alpha)$ *exist, and*

$$\nabla\Lambda(\alpha) = \left(E_\nu^{(\alpha)}\tau\right)^{-1} E_\nu^{(\alpha)} S_\tau, \tag{3.8}$$

and

$$\nabla'\nabla\Lambda(\alpha) = \left(E_\nu^{(\alpha)}\tau\right)^{-1} \operatorname{cov}_\nu^{(\alpha)}(S_\tau - \nabla\Lambda(\alpha)\tau). \tag{3.9}$$

[∇' = *transp.* ∇, $\nabla'\nabla\Lambda$ = *Hessian matrix of* Λ, *and* $\operatorname{cov}(\cdot)$ *means covariance matrix.*]

PROOF. The hypothesis allows us to differentiate through (3.7) with $Y \equiv 1$, yielding

$$\begin{aligned} 0 &= E_\nu\left[(S_\tau - \tau\nabla\Lambda(\alpha))e^{\langle\alpha, S_\tau\rangle - \Lambda(\alpha)\tau}\right] \\ &= E_\nu^{(\alpha)}(S_\tau - \tau\nabla\Lambda(\alpha)) \end{aligned}$$

and hence (3.8). Differentiating again,

$$0 = E_\nu^{(\alpha)}\left\{\left[(S_\tau' - \tau\nabla'\Lambda(\alpha))(S_\tau - \tau\nabla\Lambda(\alpha)) - \tau\nabla\nabla'\Lambda(\alpha)\right]e^{\langle\alpha, S_\tau\rangle - \Lambda(\alpha)\tau}\right\},$$

which is (3.9). □

COROLLARY 3.3. $\Lambda(\alpha)$ *is strictly convex.*

PROOF. Since $\mathring{\mathscr{S}} \neq \varnothing$ by hypothesis the support of $S_\tau - \nabla\Lambda(\alpha)\tau$ under $P_\nu^{(\alpha)}$ has nonempty interior, and by the lemma $\nabla'\nabla\Lambda(\alpha)$ is positive definite. □

DEFINITION 3.2. A convex function f with domain D ($D \neq \varnothing$) is called *essentially smooth* if it is differentiable throughout D, and if for any sequence $\{\alpha_n\} \subset D$ such that $\alpha_n \to \alpha_0 \in \partial D$ one has $\|\nabla f(\alpha_n)\| \to \infty$.

LEMMA 3.4. *If* (M_1) *holds and* $\mathscr{W}$ *is open, then* Λ *is essentially smooth on* $\mathscr{D}$.

PROOF. Choose $\{\alpha_n;\ n = 1, 2, \dots\} \subset \mathscr{D}$ such that $\alpha_n \to \alpha_0 \in \partial\mathscr{D}$. Now $\alpha_n \in \mathscr{D}$ implies $\alpha_n \in \mathscr{U}_r$ (by Lemma 3.2) and hence there exist $\zeta_n = \Lambda(\alpha_n)$, $n = 1, 2\dots$, such that $\psi(\alpha_n, \Lambda(\alpha_n)) = 1$. If $|\Lambda(\alpha_n)| \nrightarrow \infty$, then there is a subsequence $\{n'\}$ such that $\Lambda(\alpha_{n'}) \to$ some ζ_0, and thus $(\alpha_{n'}, \zeta_{n'}) \to (\alpha_0, \zeta_0)$. If $(\alpha_0, \zeta_0) \in \mathscr{W}$, then there is a neighborhood $\mathscr{N}(\alpha_0, \zeta_0) \subset \mathscr{W}$ (since $\mathscr{W}$ is open); hence also a neighborhood of α_0, $\mathscr{N}(\alpha_0) \subset \mathscr{D}$, contradicting $\alpha_0 \in \partial\mathscr{D}$. Hence $(\alpha_0, \zeta_0) \notin \mathscr{W}$. But $(\alpha_n, \zeta_n) \in \mathscr{W}$ since $\psi(\alpha_n, \zeta_n) = 1 < \infty$. Hence $(\alpha_0, \zeta_0) \in \overline{\mathscr{W}}$ and thus $\in \partial\mathscr{W}$. Now, using Fatou's lemma again, this implies $\psi(\alpha_n, \zeta_n) \to \infty$, contradicting $\psi(\alpha_n, \zeta_n) = 1$ for all n. Thus we must have $|\Lambda(\alpha_n)| \to \infty$ as $\alpha_n \to \alpha_0$, in fact the convexity forces $\Lambda(\alpha_n) \to +\infty$. This in turn implies $\|\nabla\Lambda(\alpha_n)\| \to \infty$ as $\alpha_n \to \alpha_0 \in \partial\mathscr{D}$, i.e., Λ is essentially smooth. □

The range of the gradient map of Λ plays a role in some large deviation theorems, so it is desirable to characterize it in terms of the parameters of the problem. To this end we prove:

LEMMA 3.5. *Assume* (M_1) *and that* $\mathscr{W}$ *is open. Then*

$$\mathring{\mathscr{S}} = \left(\operatorname{Supp}_\nu\left(\frac{S_\tau}{\tau}\right)\right)^0 = \left\{\frac{E_\nu^{(\alpha)}S_\tau}{E_\nu^{(\alpha)}\tau};\ \alpha \in \mathscr{D}\right\} = \nabla\Lambda(\mathscr{D}). \tag{3.10}$$

EXAMPLE 3.1. When $S_n = \Sigma_1^n \xi_i$ is a sum of i.i.d. r.v.'s, $\Lambda(\alpha) = \log$(generating function of ξ_i), and it is intuitively clear (and is easily proved) that as α ranges over $\mathscr{D}$, the mean of the α-conjugate distribution, $E^{(\alpha)}\xi$, sweeps out the convex hull of the support of ξ_i. The appropriate generalization of this statement to MA-processes is (3.10).

PROOF. The equality on the right of (3.10) is just Lemma 3.3.

(i) $\mathring{\mathscr{S}} \subset \nabla\Lambda(\mathscr{D})$.

Assume first that $O \in \mathring{\mathscr{S}}$. Let $\{\alpha_i;\ i = 1, 2, \dots\} \subset \mathscr{D}$. Recall $\mathscr{W}$ open implies $\mathscr{D}$ open (Lemma 3.2). Hence by the essential smoothness of $\{\Lambda(\alpha);\ \alpha \in \mathscr{D}\}$ (Lemma 3.4), if $\alpha_i \to \alpha_0 \in \partial\mathscr{D}$, then $\|\nabla\Lambda(\alpha_i)\| \to \infty$ and in fact by the proof of Lemma 3.4 $\Lambda(\alpha_i) \to \infty$. Suppose that $\|\alpha_i\| \to \infty$ and note that

$$O \in \left(\operatorname{Supp}_\nu\left(\frac{S_\tau}{\tau}\right)\right)^0, \quad \text{if and only if } O \in (\operatorname{Supp}_\nu(S_\tau))^0 \tag{3.11}$$

(since $1 \le \tau$, and $\tau < \infty$ since $\mathscr{D} = \mathscr{U}_r$ by Lemma 3.2). Hence if $\|\alpha_n\| \to \infty$ and $\Lambda(\alpha_n) \not\to \infty$, then $\Lambda(\alpha_n)$ remains bounded on a subsequence $\{\alpha_{n'}\}$, and then $O \in (\operatorname{Supp}_\nu S_\tau)^0$ implies that $\psi(\alpha_{n'}, \Lambda(\alpha_{n'})) = E_\nu e^{\langle \alpha_{n'}, S_\tau\rangle - \Lambda(\alpha_{n'})\tau} \to \infty$. But $\psi(\alpha_{n'}, \Lambda(\alpha_{n'})) = 1$, a contradiction. Thus $\Lambda(\alpha_n) \to \infty$ as either $\alpha_n \to \alpha_0 \in \partial\mathscr{D}$ or as $\|\alpha_n\| \to \infty$, and together with the strict convexity and differentiability of Λ, it has a unique minimum at some $\alpha^* \in \mathscr{D}$ and $\nabla\Lambda(\alpha^*) = 0$, namely $O \in \nabla\Lambda(\mathscr{D})$.

Next suppose that $v \in \mathring{\mathscr{S}}$. Then

$$O \in \left(\operatorname{Supp}_\nu\left(\frac{S_\tau}{\tau} - v\right)\right)^0 \Rightarrow O \in (\operatorname{Supp}_\nu(S_\tau - v\tau))^0.$$

Now if we translate the original MA-process by v, namely replace all the ξ_i's by $\xi_i^{(v)} = \xi_i - v$, then we get a new MA-process with transform kernel $\hat{P}_v(\alpha) = \hat{P}(\alpha)e^{-\langle\alpha, v\rangle}$, and with eigenvalue $e^{\Lambda_v(\alpha)}$, $\Lambda_v(\alpha) = \Lambda(\alpha) - \langle\alpha, v\rangle$; and with $S_n^{(v)} = S_n - nv$, $S_\tau^{(v)} = S_\tau - \tau v$. The regeneration structure for the translated process remains the same, $\Lambda_v(\alpha)$ remains essentially smooth, and

$$\psi_v(\alpha, \Lambda_v(\alpha)) = E_\nu E^{\langle\alpha, S_\tau^{(v)}\rangle - \Lambda_v(\alpha)\tau} = 1.$$

Now arguing as before $O \in (\operatorname{Supp}_\nu S_\tau^{(v)})^0$ implies $O = \nabla\Lambda_v(\alpha) = \nabla(\Lambda(\alpha) -$

$\langle \alpha, v\rangle)$ has a unique solution α_v, i.e., $\nabla\Lambda(\alpha_v) = v$. Hence $v \in \nabla\Lambda(\mathscr{D})$. This proves (i).

(ii) $\nabla\Lambda(\mathscr{D}) \subset \mathring{\mathscr{S}}$.

We thank the referee for the following proof.

Let $(S_{\tau_n}^{(n)}, \tau_n)$ be i.i.d. copies of (S_τ, τ). By the SLLN

$$\frac{\sum_{n=1}^m S_{\tau_n}^{(n)}}{\sum_{n=1}^m \tau_n} = \frac{\sum_{n=1}^m \tau_n \left(S_{\tau_n}^{(n)}/\tau_n\right)}{\sum_{n=1}^m \tau_n} \to \frac{E_\nu^{(\alpha)} S_\tau}{E_\nu^{(\alpha)} \tau} \quad \text{a.s.} \left[P_\nu^{(\alpha)}\right].$$

The left side is a convex combination of $(S_{\tau_n}^{(n)}/\tau_n)$, $n = 1, \dots, m$, and hence the limit $E_\nu^{(\alpha)} S_\tau / E_\nu^{(\alpha)} \tau$ must be in the closed convex hull of the support of S_τ/τ, hence also in its interior. □

4. Transform kernels. In this section we study the properties of the transform kernel $\hat{P}(\alpha)$. We start by proving that $\Lambda(\alpha)$, as defined in the last section by the equation

$$\psi(\alpha, \Lambda(\alpha)) = 1, \tag{4.1}$$

is in fact an eigenvalue of $\hat{P}(\alpha)$. Thus (4.1) is a *characteristic equation* for Λ.

With $\Lambda(\alpha)$ givenby (4.1), define the functions

$$r(x; \alpha) = E_x\left[e^{\langle \alpha, S_{T_0}\rangle - \Lambda(\alpha) T_0}\right], \qquad x \in \mathbb{E}, \alpha \in \mathscr{U}_r, \tag{4.2}$$

and the measures

$$l(A; \alpha) = E_\nu\left[\sum_{n=0}^{\tau-1} e^{\langle \alpha, S_n\rangle - \Lambda(\alpha) n};\ X_n \in A\right], \qquad A \in \mathscr{E}, \alpha \in \mathscr{U}_r. \tag{4.3}$$

We assume (M_1) throughout this section.

LEMMA 4.1. (i) *For $\alpha \in \mathscr{U}_r$, $\lambda(\alpha) = e^{\Lambda(\alpha)}$ is an eigenvalue of $\hat{P}(\alpha)$, with associated eigenfunction $r(\alpha)$ and measure $l(\alpha)$ given by the representations* (4.2) *and* (4.3).

(ii) $l(\alpha) r(\alpha) = e^{-\Lambda(\alpha)} E_\nu^{(\alpha)} \tau$ *and if also $\alpha \in \mathscr{U}_p$, then $l(\alpha) r(\alpha) < \infty$. In this case we always multiply l or r by a constant* (*depending on* α) *so that* $l(\alpha) r(\alpha) = 1$.

PROOF. Clearly,

$$\mathbb{P}_x\{T_0 = n, S_n \in ds\} = (P - h \otimes \nu)^{n-1} * h(x, ds). \tag{4.4}$$

Hence

$$\begin{aligned} r(x; \alpha) &= \sum_{n=1}^\infty \int e^{\langle \alpha, s\rangle - \Lambda(\alpha) n} (P - h \otimes \nu)^{n-1} * h(x, ds) \\ &= \sum_{n=0}^\infty e^{-\Lambda(\alpha)(n+1)} \left(\hat{P}(\alpha) - \hat{h}(\alpha) \otimes \nu\right)^n \hat{h}(\alpha)(x). \end{aligned} \tag{4.5}$$

But this shows that $r(\cdot; \alpha)$ is just the essentially unique minimal invariant function h_ν for the kernel $K = \hat{P}(\alpha)$ with minorization $\hat{P}(\alpha) \geq \hat{h}(\alpha) \otimes \nu$ [as in Theorem 5.1 of Nummelin (1984), page 70].

Similarly,

$$l(A; \alpha) = \sum_{n=0}^{\infty} e^{-\Lambda(\alpha)(n+1)} \nu\big(\hat{P}(\alpha) - \hat{h}(\alpha) \otimes \nu\big)^n (A) \tag{4.6}$$

is the essentially unique $e^{-\Lambda(\alpha)}$-invariant measure π_s given by Theorem 5.2 of Nummelin (1984).

Finally, by formula (5.6) of Nummelin (1984), we have

$$\begin{aligned} e^{\Lambda(\alpha)} l(\alpha) r(\alpha) &= \sum_{n=1}^{\infty} n e^{-\Lambda(\alpha)n} \nu\big(\hat{P}(\alpha) - \hat{h}(\alpha) \otimes \nu\big)^{n-1} \hat{h}(\alpha) \\ &= \sum_{n=1}^{\infty} \int n e^{\langle \alpha, s\rangle - \Lambda(\alpha)n} \Big(\nu\big(P - h \otimes \nu\big)^{n-1} * h\Big)(ds) \\ &= E_\nu^{(\alpha)} \tau. \end{aligned} \tag{4.7}$$

□

We will now relate the recurrence properties of $\hat{P}(\alpha)$ to those of the imbedded regenerative process. Letting Λ be as above, define the sets

$$\tilde{\mathscr{U}}_r = \big\{\alpha \in \mathbb{R}^d\colon \hat{P}(\alpha) \text{ is } e^{-\Lambda(\alpha)}\text{-recurrent}\big\},$$

$$\tilde{\mathscr{U}}_p = \big\{\alpha \in \tilde{\mathscr{U}}_r\colon \hat{P}(\alpha) \text{ is positive } e^{-\Lambda(\alpha)}\text{-recurrent}\big\},$$

$$\tilde{\mathscr{U}}_g = \big\{\alpha \in \tilde{\mathscr{U}}_p\colon \hat{P}(\alpha) \text{ is geometrically } e^{-\Lambda(\alpha)}\text{-recurrent}\big\}.$$

LEMMA 4.2. *Let $\alpha \in \mathscr{U}_r$ (so that $\Lambda(\alpha)$ is well defined). Then $e^{-\Lambda(\alpha)}$ is the convergence parameter for $\hat{P}(\alpha)$. Furthermore,*

$$\mathscr{U}_r = \tilde{\mathscr{U}}_r, \qquad \mathscr{U}_p = \tilde{\mathscr{U}}_p \quad \text{and} \quad \mathscr{U}_g = \tilde{\mathscr{U}}_g. \tag{4.8}$$

PROOF. Let

$$\begin{aligned} b_n^{(\alpha)} &= P_\nu^{(\alpha)}\{\tau = n\} \\ &= E_\nu\big[e^{\langle \alpha, S_n\rangle - \Lambda(\alpha)n};\ \tau = n\big] \\ &= e^{-\Lambda(\alpha)n} \nu\big(\hat{P}(\alpha) - \hat{h} \otimes \nu\big)^{n-1} \hat{h} \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} u_n^{(\alpha)} &= E_\nu\Big[e^{\langle \alpha, S_n\rangle - \Lambda(\alpha)n};\ Y_{n-1} = 1\Big] \\ &= e^{-\Lambda(\alpha)n} \nu \hat{P}^{n-1} \hat{h}. \end{aligned} \tag{4.10}$$

(Recall $\{Y_{n-1} = 1\} = \bigcup_{i=0}^{\infty}\{T_i = n\}$.) The sequences $\{b_n^{(\alpha)}\}$ and $\{u_n^{(\alpha)}\}$ are the analogs, for the present setup, of $\{b_n\}$ and $\{u_n\}$ in (2.5).

Now $\{u_n^{(\alpha)}\}$ is the renewal sequence generated by $\{b_n^{(\alpha)}\}$. Recall that the convergence parameter $R(\alpha)$ of $\hat{P}(\alpha)$ satisfies

$$R(\alpha) = \sup\Big\{\rho\colon \sum \rho^n \big(\hat{P}(\alpha) - \hat{h} \otimes \nu\big)^{n-1} \hat{h} \leq 1\Big\}. \tag{4.11}$$

But by (4.9)

$$\sum_n b_n^{(\alpha)} = \sum_n e^{-\Lambda(\alpha)n} \nu\big(\hat{P}(\alpha) - \hat{h} \otimes \nu\big)^{n-1} \hat{h} \tag{4.12}$$
$$= E_\nu e^{\langle \alpha, S_\tau \rangle - \Lambda(\alpha)\tau} = 1.$$

Hence $R(\alpha) = e^{-\Lambda(\alpha)}$.

The last identity also tells us that $\alpha \in \mathscr{U}_r$ if and only if $\Sigma b_n^{(\alpha)} = 1$, while R-recurrence means $\Sigma R^n(\alpha)(\hat{P} - \hat{h} \otimes \nu)^{n-1}\hat{h} = 1$, i.e., again $\Sigma b_n^{(\alpha)} = 1$. Thus $\mathscr{U}_r = \tilde{\mathscr{U}}_r$.

Similarly, $\alpha \in \mathscr{U}_p$ means $|(\partial\psi^+/\partial\zeta)(\alpha, \Lambda(\alpha))| < \infty$; namely,

$$\sum_n E_\nu\big[ne^{\langle \alpha, S_n \rangle - \Lambda(\alpha)n}; \tau = n\big] = \sum nb_n^{(\alpha)} < \infty.$$

But this means that the associated renewal sequence $\{u_n^{(\alpha)}\}$ is positive recurrent, which by (4.10) is equivalent to $\hat{P}(\alpha)$ being positive recurrent, i.e, $\alpha \in \tilde{\mathscr{U}}_p$.

Finally, $\alpha \in \mathscr{U}_g$ means $\Sigma r^n b_n^{(\alpha)} < \infty$ for some $r > 1$, i.e.,

$$\sum_n \rho^n \nu(\hat{P} - \hat{h} \otimes \nu)^{n-1}\hat{h} < \infty$$

for some $\rho > R(\alpha) = e^{-\Lambda(\alpha)}$, which is the definition of $\hat{P}(\alpha)$ being geometrically $e^{-\Lambda(\alpha)}$-recurrent. Thus $\mathscr{U}_g = \tilde{\mathscr{U}}_g$. □

In Section 3 we saw that the hypothesis "$\mathscr{W}$ open" played an important role in determining analyticity and other regularity properties of Λ. The following lemma gives a sufficient condition for this in terms of $\hat{P}(\alpha)$.

If $\hat{h} \otimes \nu$ is a minorization for $\hat{P}(\alpha)$, then obviously $\varepsilon\hat{h} \otimes \nu$, for $0 < \varepsilon \le 1$, is also. Let $\psi_\varepsilon(\alpha, \zeta)$ be the generating function defined in (3.2), with the minorization $\varepsilon\hat{h} \otimes \nu$, and let $\mathscr{W}_\varepsilon$ be its domain of convergence. Let $\hat{P}_\varepsilon(\alpha) = \hat{P}(\alpha) - \varepsilon\hat{h} \otimes \nu$.

LEMMA 4.3. *If $\mathscr{D}$ is open, $0 < \varepsilon_0 < 1$, and $\hat{P}_{\varepsilon_0}(\alpha)$ is recurrent for all $\alpha \in \mathscr{D}$, then $\mathscr{W}_\varepsilon$ is open for $0 < \varepsilon < \varepsilon_0$.*

PROOF. (i) $\mathscr{U}_g = \mathscr{D}$. Apply Lemma 2.2 to $\hat{P}_{\varepsilon_0}(\alpha)$ with $\varepsilon = 0$. This implies $\alpha \in \mathscr{U}_g$ for all $\alpha \in \mathscr{D}$, i.e., $\mathscr{U}_g \supseteq \mathring{\mathscr{D}}$. The converse $\mathscr{U}_g \subseteq \mathscr{D}$ always holds.

(ii) $(\alpha, \Lambda(\alpha)) \in \mathring{\mathscr{W}}$ for all $\alpha \in \mathscr{D}$. If $(\alpha, \Lambda(\alpha)) \in \partial\mathscr{W}$, then (since $\alpha \in \mathscr{U}_g$) $(\alpha, \Lambda(\alpha) - \delta) \in \partial\mathscr{W}$ for some $\delta > 0$. Hence $L_\alpha = \{(\alpha, \zeta): \zeta \ge \Lambda(\alpha) - \delta\} \in \partial\mathscr{W}$. But this is impossible since $\mathscr{D}$ is open.

(iii) $\Lambda(\cdot)$ is analytic on $\mathscr{D}$. This follows from (ii) and the implicit function theorem.

(iv) Let $\exp(-\Lambda_\varepsilon(\alpha))$ be the convergence parameter of $\hat{P}_\varepsilon(\alpha)$. Note that the domain of $\hat{P}_\varepsilon(\alpha) \equiv \mathscr{D}_\varepsilon \equiv \mathscr{D}$. Applying the same argument to $\hat{P}_\varepsilon$ as to $\hat{P}$ in (i), (ii) and (iii), conclude that $\Lambda_\varepsilon(\alpha)$ is analytic on $\mathscr{D}$.

(v) Show that $\mathscr{W}_\varepsilon$ is open. Take any $(\alpha_0, \zeta_0) \in \mathscr{W}_\varepsilon$. Then

$$\sum e^{-\zeta_0 n}\nu(\hat{P} - \varepsilon\hat{h} \otimes \nu)^{n-1}\hat{h}(\alpha_0) < \infty.$$

But $\hat{P}_\varepsilon(\alpha_0)$ recurrent implies

$$\sum e^{-\Lambda_\varepsilon(\alpha_0)n}\nu(\hat{P}-\varepsilon\hat{h}\otimes\nu)^{n-1}\hat{h}(\alpha_0)=\infty.$$

Hence $\zeta_0 > \Lambda_\varepsilon(\alpha_0)$. Let $\delta = \zeta_0 - \Lambda_\varepsilon(\alpha_0) > 0$. By (iv), there is a neighborhood $\mathscr{N}(\alpha_0) \subset \mathscr{D}$ such that

$$\Lambda_\varepsilon(\alpha) < \Lambda_\varepsilon(\alpha_0) + \tfrac{1}{3}\delta, \quad \text{for } \alpha \in N(\alpha_0).$$

Then for

$$(\alpha,\zeta) \in \mathscr{N}(\alpha_0) \times \left(\zeta_0 - \frac{\delta}{3}, \zeta_0 + \frac{\delta}{3}\right),$$

we have

$$\zeta > \Lambda_\varepsilon(\alpha),$$

and hence

$$(\alpha,\zeta) \in \mathscr{W}_\varepsilon. \qquad \square$$

Finally, we investigate the properties of $r(\alpha)$ and $l(\alpha)$. There is a technical problem about the finiteness of r and l which is illustrated by the following example. Let $\mathbb{E} = [0,1]$, and the Markov chain $\{X_n\}$ be i.i.d. uniform random variables on $\mathbb{E}$. Let $P_x\{\xi_1 \geq t\} = e^{-\varepsilon t}$ when $x = 0$, or in fact for $x \in$ any set M_0 of measure 0, and $P_x\{\xi_1 \geq t\} = e^{-t}$ for $x \notin M_0$. Then $\nu(dy) = l(dy) =$ the Lebesgue measure on $[0,1]$; $h(x, ds) = P_x\{\xi_1 \in ds\}$; $\tau \equiv 1$; $\psi(\alpha,\zeta) = (1-\alpha)^{-1}e^{-\zeta}$; $\mathscr{W} = (-\infty,1)\times(-\infty,\infty) =$ open; $\Lambda(\alpha) = -\log(1-\alpha)$, $\alpha \in \mathscr{D} = (-\infty, 1)$; $r(x;\alpha) = 1$ if $x \notin M_0$, $= \varepsilon(\varepsilon-\alpha)^{-1}(1-\alpha)$ if $x \in M_0$, $\alpha < \varepsilon$, $= \infty$ for $x \in M_0$, $\alpha \in [\varepsilon; 1) \subset \mathscr{D}$.

In other words, the measure $\mathbb{P}_\nu(\cdot)$ has been smoothed out and may have better convergence properties than individual $\mathbb{P}_x(\cdot)$'s. The following lemma shows that this can happen only on a set of φ-measure 0, and this set can be chosen *independent of* α. For $x \in$ the complement of this set, $r(x,\cdot)$ will be seen to be analytic on $\mathscr{D}$. A similar question arises for $l(A;\alpha)$ and is the object of Lemma 4.6.

LEMMA 4.4. *Assume that $\mathscr{W}$ is open. Then there exists a "full" set $\mathbb{F} \in \mathscr{E}$ such that $\varphi(\mathbb{F}^c) = 0$, and for each $x \in \mathbb{F}$, $r(x,\cdot) < \infty$ on $\mathscr{D}$, and is analytic on $\mathscr{D}$.*

PROOF. Fix $(\alpha,\zeta) \in \mathscr{W}$ and let

$$\tilde{r}(x;\alpha,\zeta) = E_x e^{\langle\alpha, S_\tau\rangle - \zeta\tau}. \tag{4.13}$$

[Note $\tilde{r}(x;\alpha,\Lambda(\alpha)) = r(x;\alpha)$.]

Let

$$z_n = \begin{cases} 1, & \text{when } n \text{ is a regeneration time,} \\ 0, & \text{otherwise,} \end{cases}$$

$$Z_n = z_0 + \cdots + z_n,$$

$$\bar{S}_n = (S_n, Z_n) \in \mathbb{R}^{d+1},$$

$$\bar{\alpha} = (\alpha, \alpha_{d+1}) \in \mathbb{R}^{d+1}.$$

Then $\{(X_n, \bar{S}_n)\}$ is an MA-process on $\mathbb{E} \times \mathbb{R}^{d+1}$, with a transition function $\bar{P}$, and transform $\hat{\bar{P}}(\bar{\alpha})$.

Let

$$\bar{\psi}(\bar{\alpha}, \zeta) = E_\nu e^{\langle \bar{\alpha}, \bar{S}_\tau \rangle - \zeta\tau} = E_\nu e^{\langle \alpha, S_\tau \rangle - \alpha_{d+1} - \zeta\tau}$$

and $\bar{R}(\bar{\alpha}) = e^{-\bar{\Lambda}(\bar{\alpha})}$ be the convergence parameter of $\hat{\bar{P}}(\alpha)$. Now for $(\alpha, \zeta) \in \mathscr{W}$, take $\alpha_{d+1} = -\log \psi(\alpha, \zeta)$. Then $\bar{\psi}(\bar{\alpha}, \zeta) = 1$, namely, $\zeta = \bar{\Lambda}(\bar{\alpha})$. The invariant function for $\hat{\bar{P}}(\alpha)$ with eigenvalue $e^{\bar{\Lambda}(\bar{\alpha})}$ is

$$\bar{r}(x; \bar{\alpha}) = E_x^{\langle \bar{\alpha}, \bar{S}_\tau \rangle - \bar{\Lambda}(\bar{\alpha})\tau} = \tilde{r}(x; \alpha, \zeta)/\psi(\alpha, \zeta),$$

and

$$\nu\bar{r}(\bar{\alpha}) = \bar{\psi}(\bar{\alpha}, \psi(\alpha, \zeta)) = 1.$$

But then since $\hat{\bar{P}}^n(\bar{\alpha})\bar{r}(\bar{\alpha}) = e^{\zeta n}\bar{r}(\bar{\alpha})$,

$$\begin{aligned} \infty > \nu\bar{r}(\bar{\alpha}) &= e^{-\zeta n}\nu\hat{\bar{P}}^n\bar{r}(\bar{\alpha}) \\ &\geq e^{-\zeta n}\int_{\mathbb{F}^c(\alpha, \zeta)} (\nu\hat{\bar{P}}^n)(dy)\bar{r}(y; \bar{\alpha}), \end{aligned}$$

where

$$\mathbb{F}(\alpha, \zeta) = \{x\colon \bar{r}(x; \bar{\alpha}) = \tilde{r}(x; \alpha, \zeta)/\psi(\alpha, \zeta) < \infty\} = \{x\colon \tilde{r}(x; \alpha, \zeta) < \infty\}.$$

Now, for fixed (α, ζ), if $\varphi(\mathbb{F}^c(\alpha, \zeta)) > 0$, then the φ-irreducibility of $\hat{\bar{P}}(\alpha)$ (which is apparent from that of $\{X_n\}$) implies that $\nu\hat{\bar{P}}^n(\mathbb{F}^c(\alpha, \zeta);\ \bar{\alpha}) > 0$ for some $n < \infty$; the integral above is ∞, and we have a contradiction. Hence $\varphi(\mathbb{F}^c(\alpha, \zeta)) = 0$.

Now let $\mathscr{W}_c$ be a countable dense subset of $\mathscr{W}$, and set $\mathbb{F} = \bigcap_{(\alpha, \zeta) \in \mathscr{W}_c} \mathbb{F}(\alpha, \zeta)$, and fix $x \in \mathbb{F}$. Note that $\tilde{r}(x; \alpha, \zeta)$ is a generating function and let $\mathscr{D}_x$ be its domain of convergence. Thus $\tilde{r}(x, \cdot, \cdot)$ is analytic on $\mathscr{D}_x$. Then since $\tilde{r}(x; \alpha, \zeta) < \infty$ for $(\alpha, \zeta) \in \mathscr{W}_c$, we have $\mathscr{W}_c \subset \mathscr{D}_x$, and since $\mathscr{D}_x$ is convex, $\mathscr{W} \subset \mathring{\mathscr{D}}_x$. Thus $\tilde{r}(x; \alpha, \zeta)$ is analytic on $\mathscr{W}$. But $\Lambda(\cdot)$ is analytic on $\mathscr{D}$ and $(\alpha, \Lambda(\alpha)) \in \mathscr{W}$ for $\alpha \in \mathscr{D}$. (Recall $\mathscr{W}$ and $\mathscr{D}$ are open.) Hence $r(x; \alpha, \Lambda(\alpha)) = r(x, \alpha)$ is analytic on $\mathscr{D}$. □

The positivity of $r(x; \alpha)$ is also of interest. Though easily $r(x; \alpha) > 0$ for all x, it need not be uniformly positive. However, we will show that there is a countable partition of the state space $\mathbb{E} = \bigcup \mathbb{E}_i$, independent of α, such that $r(x; \alpha)$ is uniformly positive on each $\mathbb{E}_i$. We emphasize that this is a very general fact and requires no hypothesis like $\mathscr{W}$ open.

LEMMA 4.5. *Let $\hat{P}(\alpha)$ be the transform kernel of an irreducible* MA-*process satisfying* (M_1), *$R(\alpha)$ be its convergence parameter and $\{r(x; \alpha);\ x \in \mathbb{E}\}$ the associated subinvariant function. Then there exists a partition $\mathbb{E} = \bigcup_{i=1}^\infty \mathbb{E}_i$ and a sequence of functions $f_i(\cdot)\colon \mathbb{R}^d \to (0, \infty)$, $i = 1, 2, \ldots,$ such that*

$$r(x; \alpha) \geq f_i(\alpha)1_{\mathbb{E}_i}(x), \qquad x \in \mathbb{E}, \alpha \in \mathscr{D}, i = 1, 2, \ldots. \tag{4.14}$$

PROOF. An R-subinvariant function is given by

$$r(x;\alpha) = \sum_{n=0}^{\infty} R^n(\alpha)\hat{P}^n(\alpha)\hat{h}(\alpha)(x), \quad \text{if } \hat{P} \text{ is } R\text{-transient,} \tag{4.15}$$

and by

$$r(x;\alpha) = \sum_{n=0}^{\infty} R^n(\alpha)\big[\hat{P}(\alpha) - \hat{h}(\alpha)\otimes\nu\big]^n\hat{h}(x;\alpha), \quad \text{if } \hat{P} \text{ is } R\text{-recurrent.} \tag{4.16}$$

In the former case

$$r(x;\alpha) \geq \sum_{n=0}^{N} \big(R^n\hat{P}^n(\alpha)\hat{h}(\alpha)\big)(x).$$

Now

$$\big(\hat{P}^n(\alpha)\hat{h}(\alpha)\big)(x) \geq \int_{[-N,N]^d} (P^n * h)(x, ds)e^{\langle\alpha, s\rangle}$$
$$\geq e^{-N\|\alpha\|}\big(P^n * h\big)\big(x, [-N,N]^d\big),$$

and hence

$$r(x;\alpha) \geq \big[R^N(\alpha)\wedge 1\big]e^{-N\|\alpha\|}\sum_{n=0}^{N}(P^n * h)\big(x,[-N,N]^d\big). \tag{4.17}$$

Now let

$$\mathbb{E}^N = \left\{y\colon \sum_{n=0}^{N}(P^n * h)\big(y,[-N,N]^d\big) \geq \frac{1}{N}\right\}. \tag{4.18}$$

Hence if $x \in \mathbb{E}^N$, then

$$r(x,\alpha) \geq \big[R^N(\alpha)\wedge 1\big]e^{-N\|\alpha\|}\frac{1}{N}, \qquad 1 < N < \infty.$$

By irreducibility

$$\sum_{n=0}^{\infty}(P^n * h)\big(y,[-N,N]\big) > 0, \quad \text{for } N \text{ sufficiently large.}$$

Hence $\mathbb{E}^N \nearrow \mathbb{E}$. Let $\mathbb{E}_N = \mathbb{E}^N - E^{N-1}$, and set

$$f_N(\alpha) = \big[R^N(\alpha)\wedge 1\big]e^{-N\|\alpha\|}\frac{1}{N}. \tag{4.19}$$

Then (4.14) is satisfied.

In the recurrent case simply replace $\hat{P}(\alpha)$ by $[\hat{P}(\alpha) - \hat{h}(\alpha)\otimes\nu]$ in the above argument. Then take

$$\mathbb{E}^N = \left\{y\colon \sum_{n=0}^{N}(P - h\otimes\nu)^n * h\big(y,[-N,N]^d\big) \geq \frac{1}{N}\right\}$$

and $\mathbb{E}^N \nearrow \mathbb{E}$ since

$$\sum_{n=0}^{\infty} (P - h \otimes \nu)^n * h(y, \mathbb{R}^d) = \mathbb{P}_y\{\tau < \infty\} > 0$$

(in fact = 1 by recurrence). The rest of the argument goes through for $P - h \otimes \nu$ as for P before. □

DEFINITION 4.1. We call any set contained in a finite union of $\mathbb{E}_i$'s an s-set (for "sufficiently small").

EXAMPLE 4.1. If the state space $\mathbb{E}$ is countable then every finite set is an s-set.

EXAMPLE 4.2. Suppose that the minorization satisfies (M_2) with $h(x) \geq \delta > 0$ for $x \in A$. Then for φ-positive A and $x \in A$

$$\begin{aligned}(P^n h)(x, \Gamma) &\geq \delta P^n(x, A \times \Gamma) \geq \delta(h \times \nu)^n(x, A \times \Gamma)\\ &\geq \delta^n(\nu(A \times \cdot))^{*n}(\Gamma).\end{aligned}$$

Hence

$$\sum_{n=0}^{N} (P^n h)(x, (-N, N]^d) \geq \frac{1}{N},$$

for N sufficiently large, namely $A \subset \mathbb{E}^N$.

COROLLARY 4.1. *Assume $\mathscr{W}$ is open, If A is an s-set then $l(A; \alpha) < \infty$ for all $\alpha \in \mathscr{D}$.*

PROOF. The hypotheses imply that $\hat{P}(\alpha)$ is geometrically $e^{-\Lambda(\alpha)}$-recurrent. Hence (as we have seen) $l(\alpha)r(\alpha) < \infty$ (and we normalized to $lr = 1$). Now if A is an s-set, then $r(x; \alpha) \geq f_i(\alpha)$ for $x \in A$ and for some i. Hence if $A \subset \mathbb{E}^i$

$$1 \geq \int l(dx; \alpha) r(x, \alpha) \geq f_i(\alpha) l(A; \alpha), \tag{4.20}$$

with $0 < f_i(\alpha) < \infty$. □

LEMMA 4.6. *If $\mathscr{W}$ is open, and A is a φ-positive s-set. Then $\{l(A; \alpha); \alpha \in \mathscr{D}\}$ is analytic.*

PROOF. Let $l_n(A \times \Gamma) = \nu(P - h \otimes \nu)^{n-1}(A \times \Gamma)$: For fixed A, this is a measure on $\mathbb{N} \times \mathbb{R}^d$. Define the generating function

$$\tilde{l}(A; \alpha, \zeta) = \int_{\mathbb{R}^d} \sum_{n=1}^{\infty} \nu(P - \hat{h} \otimes \nu)^{n-1}(A \times ds) e^{\langle \alpha, s \rangle - \zeta n},$$

and note that $\tilde{l}(A; \alpha, \Lambda(\alpha)) = l(A; \alpha)$. [Recall that $\hat{P}(\alpha)$ is positive

$e^{-\Lambda(\alpha)}$-recurrent.] For $x \in \mathbb{F}$ write out $\tilde{r}(x; \alpha, \zeta)$ as defined in (4.13) as a series, and we get

$$
\begin{aligned}
\tilde{r}(x;\alpha,\zeta) &= \sum_{n=0}^{\infty} \big(\hat{P}(\alpha) - \hat{h}(\alpha)\otimes\nu\big)^n \hat{h}(\alpha) e^{-\zeta n} \\
&= \int_{\mathbb{R}^d} \sum_{n=0}^{\infty} (P - h\otimes\nu)^n * h(x, ds) e^{\langle\alpha, s\rangle - \zeta n} \\
&\geq \int_{[-N,N]^d} \sum_{n=0}^{N} (P - h\otimes\nu)^n * h(x, ds) e^{\langle\alpha, s\rangle - \zeta n} \\
&\geq e^{-N\|\alpha\| - |\zeta| N} \sum_{n=0}^{N} (P - h\otimes\nu)^n * h\big(x, [-N,N]^d\big),
\end{aligned} \tag{4.21}
$$

as in (4.17). Define $\mathbb{E}^N$ as in (4.18). Then

$$\tilde{r}(x;\alpha,\zeta) \geq e^{-N(\|\alpha\| + \zeta)} 1_{\mathbb{E}_N}(x), \qquad x \in \mathbb{E}.$$

Now with $\alpha_{d+1} = -\log\psi(\alpha,\zeta)$,

$$\bar{r}(x;\bar{\alpha}) = \frac{\tilde{r}(x;\alpha,\zeta)}{\psi(\alpha,\zeta)} \quad \text{and} \quad \bar{l}(A;\alpha) = \frac{\tilde{l}(A;\alpha,\zeta)}{\psi(\alpha,\zeta)}$$

are an $e^{-\bar{\Lambda}(\bar{\alpha})} = e^{-\zeta}$-invariant function and measure for $\hat{\bar{P}}(\bar{\alpha})$. [Note that $\bar{\psi}(\bar{\alpha},\zeta) = e^{\alpha_{d+1}}\psi(\alpha,\zeta) < \infty$ on $\mathscr{W} \times \mathbb{R}^1$, i.e., the domain of $\bar{\psi}$ is open; hence $\hat{\bar{P}}(\bar{\alpha})$ is geom. $e^{-\zeta}$-recurrent.] Hence $\tilde{l}\tilde{r}(\alpha,\zeta) < \infty$ for $\alpha, \zeta \in \mathscr{W}$ and $\int \tilde{l}(dx;\alpha,\zeta) r(x;\alpha,\zeta) < \infty$. Now if A is an s-set, then $\tilde{l}(A;\alpha,\zeta) < \infty$ for $(\alpha,\zeta) \in \mathscr{W}$. But $\tilde{l}(A;\cdot,\cdot)$ is a generating function with some domain of converge $\mathscr{D}_A$, and hence $\mathscr{W} \subset \mathring{\mathscr{D}}_A$. But $\tilde{l}(A;\cdot,\cdot)$ is analytic on $\mathring{\mathscr{D}}_A$, hence on $\mathscr{W}$. □

We summarize the results about Λ, r and l in the next theorem.

THEOREM 4.1. *Assume that* (M_1) *holds,* $\mathscr{W}$ *is open, and* $\mathring{\mathscr{S}} \neq \varnothing$. *Let* $\Lambda(\cdot)$ *be defined by the characteristic equation* (4.1). *Then*

(a) (i) $\mathscr{D} = \{\alpha: \Lambda(\alpha) < \infty\} = \{\alpha: (\alpha,\zeta) \in \mathscr{W}$ *for some* $\zeta\}$ *is an open set. For* $\alpha \in \mathscr{D}$, $\hat{P}(\alpha)$ *has convergence parameter* $R(\alpha) = e^{-\Lambda(\alpha)}$, *and* Λ *is analytic, strictly convex and essentially smooth on* $\mathscr{D}$. (ii) *For* $\alpha \in \mathscr{D}$, $e^{\Lambda(\alpha)}$ *is an eigenvalue of* $\hat{P}(\alpha)$ *with* (*right*) *eigenfunction* $\{r(x;\alpha): x \in \mathbb{E}\}$ *and* (*left*) *eigenmeasure* $\{l(A;\alpha): A \in \mathscr{E}\}$ *having the representations* (4.2) *and* (4.3). (iii) *There is a fixed set* $\mathbb{F} \subset \mathbb{E}$ *with* $\varphi(\mathbb{F}^c) = 0$, *such that for each* $x \in \mathbb{F}$, $r(x;\cdot) < \infty$ *and is analytic on* $\mathscr{D}$. *If* A *is an s-set, then* $l(A;\cdot) < \infty$ *and is analytic on* $\mathscr{D}$. (iv) $\hat{P}(\alpha)$ *is geometrically* $e^{-\Lambda(\alpha)}$*-recurrent for* $\alpha \in \mathscr{D}$.

(b) *Conversely, if* $\mathscr{D}$ *is open and* $\hat{P}_{\varepsilon_0}(\alpha)$ *is recurrent for some* $\varepsilon_0 > 0$ *and all* $\alpha \in \mathscr{D}$, *then* $\mathscr{W}_\varepsilon$ *is open for some* $\varepsilon > 0$, *and hence* (i), (ii) *and* (iii) *hold.*

5. Limit theorems. Assume (M_1) throughout this section. Let $\Lambda(\alpha)$, $r(x,\alpha)$, $x \in \mathbb{E}$, $\alpha \in \mathscr{D}$ be as in Section 3, define

$$Q(x, dy \times ds; \alpha) = e^{-\Lambda(\alpha) + \langle\alpha, s\rangle} P(x, dy \times ds) r(y;\alpha)/r(x;\alpha), \tag{5.1}$$

and note that for each $\alpha \in \mathscr{D}$ this is an MA-transition function. Denote by $\{(X_n^{(\alpha)}, S_n^{(\alpha)})\}$ the MA-process associated with $Q(\alpha)$. Call this the *α-conjugate process*. We also use the same symbol Q for

$$(5.2)\qquad Q(x, dy; \alpha) \overset{\text{def}}{=} Q(x, dy \times \mathbb{R}^d; \alpha).$$

Let

$$(5.3)\qquad \pi(dx; \alpha) = l(dx; \alpha) r(x; \alpha)$$

and note that

$$(5.4)\qquad \pi(\alpha) Q(\alpha) = \pi(\alpha).$$

Denote the measure induced by the kernel $Q(\alpha)$ by $\mathbb{Q}^{(\alpha)}$, and expectation with respect to this measure by $E^{\mathbb{Q}^{(\alpha)}}(\cdot)$.

Let $f\colon \mathbb{R}^{d+1} \to \mathbb{R}^1$ be a measurable function, and recall (from Section 3) the expectation operator

$$(5.5)\qquad E_\nu^{(\alpha)} f \overset{\text{def}}{=} E_\nu^{\mathbb{P}}\left[f e^{\langle \alpha, S_\tau \rangle - \Lambda(\alpha)\tau} \right],$$

where $\mathbb{P} = \mathbb{Q}^{(0)}$ = the measure induced by $P(x, dy \times ds)$. Let

$$(5.6a)\qquad \nu^{(\alpha)}(dy) = \frac{\nu(dy) r(y; \alpha)}{(\nu r)(\alpha)} = \text{a probability measure,}$$

and

$$(5.6b)\qquad h^{(\alpha)}(x, ds) = e^{-\Lambda(\alpha)} \nu r(\alpha) r^{-1}(x; \alpha) e^{\langle \alpha, s \rangle} h(x, ds),$$

where $\nu(\cdot)$ and $h(x, \cdot)$ are the measures in the minorization

$$(\mathrm{M}_1)\qquad P(x, dy \times ds) \ge h(x, ds) \nu(dy).$$

LEMMA 5.1. *Assume that $\mathscr{W}$ is open. Then for $\alpha \in \mathscr{D}$*

$$(5.7)\qquad E_{\nu^{(\alpha)}}^{\mathbb{Q}^{(\alpha)}} f(S_\tau, \tau) = E_\nu^{(\alpha)} f(S_\tau, \tau).$$

REMARK. It is *not* true that for *fixed* n,

$$E_{\nu^{(\alpha)}}^{\mathbb{Q}^{(\alpha)}} f(S_n, n) = E_\nu^{(\alpha)} f(S_n, n).$$

PROOF. By (M_1)

$$(\mathrm{M}_1^{(\alpha)})\qquad \begin{aligned} Q(x, dy \times ds; \alpha) &\ge e^{-\Lambda(\alpha)} \nu r(\alpha) \frac{e^{\langle \alpha, s \rangle} h(x, ds)}{r(x; \alpha)} \cdot \frac{\nu(dy) r(y; \alpha)}{\nu r(\alpha)} \\ &= h^{(\alpha)}(x, ds) \nu^{(\alpha)}(dy). \end{aligned}$$

Now in general, for any MA-process satisfying (M_1),

$$(5.8)\qquad \mathbb{P}_\nu\{\tau = n, S_n \in ds\} = \nu(P - h \times \nu)^{n-1} * h(ds),$$

where τ is the inter-regeneration time. Thus for the α-process, we have by $(\mathrm{M}_1^{(\alpha)})$,

$$\mathbb{Q}_{\nu^{(\alpha)}}^{(\alpha)}\{\tau = n,\ S_n \in ds\} = \nu^{(\alpha)}\big(Q^{(\alpha)} - h^{(\alpha)} \otimes \nu^{(\alpha)}\big)^{n-1} * h^{(\alpha)}(ds). \tag{5.9}$$

Applying the definitions of the above quantities, after some calculating, this

$$\begin{aligned}
&= e^{-\Lambda(\alpha)n}\nu\big(e^{\langle\alpha,\cdot\rangle}(P - h \otimes \nu)\big)^{n-1} * \big(e^{\langle\alpha,\cdot\rangle}h(ds)\big)\\
&= e^{-\Lambda(\alpha)n}e^{\langle\alpha,\cdot\rangle}\nu(P - h \otimes \nu)^{n-1} * h(ds)\\
&= e^{\langle\alpha, s\rangle - \Lambda(\alpha)n}\mathbb{P}_\nu\{\tau = n,\ S_n \in ds\}\\
&= E_\nu\big[e^{\langle\alpha, S_n\rangle - \Lambda(\alpha)n};\ \tau = n,\ S_n \in ds\big].
\end{aligned}$$

Hence

$$\begin{aligned}
E_{\nu^{(\alpha)}}^{\mathbb{Q}^{(\alpha)}} f(S_\tau, \tau) &= \int_{\mathbb{R}^d}\sum_n f(s, n)\mathbb{Q}_{\nu^{(\alpha)}}^{(\alpha)}\{\tau = n,\ S_n \in ds\}\\
&= \int\sum f(s, n)E_\nu\big[e^{\langle\alpha, S_n\rangle - n\Lambda(\alpha)};\ \tau = n,\ S_n \in ds\big]\\
&= E_\nu e^{\langle\alpha, S_\tau\rangle - \Lambda(\alpha)\tau}f(S_\tau, \tau) = E_\nu^{(\alpha)}f(S_\tau, \tau). \qquad \square
\end{aligned}$$

LEMMA 5.2. *Assume that* $\mathscr{W}$ *is open. If* $E_\pi S_1$ *exists then*

$$\frac{E_\nu S_\tau}{E_\nu\tau} = E_\pi S_1. \tag{5.10}$$

REMARK. It may be that the left side of (5.10) exists, but the right side does not. See Example 6.3.

PROOF. Since $\mathscr{W}$ is open, $\{X_n\}$ is a geometrically recurrent Harris process, with invariant probability measure π (Theorem 4.1). If $\pi(f)$ exists, then [see, e.g., Athreya and Ney (1978) and Nummelin (1978)]

$$\frac{1}{E_\nu\tau}E_\nu\sum_{n=0}^{\tau-1} f(X_n) = \pi(f).$$

Now take $f(x) = E_x S_1$. By hypothesis this is π-integrable. Then

$$\begin{aligned}
\frac{E_\nu S_\tau}{E_\nu\tau} &= \frac{E_\nu\sum_{n=1}^{\tau}\xi_n}{E_\nu\tau} = \frac{E_\nu\sum_1^\infty\xi_n I\{\tau \ge n\}}{E_\nu\tau}\\
&= (E_\nu\tau)^{-1}E_\nu\left(\sum_1^\infty E_\nu[\xi_n|\mathscr{F}_{n-1}]I\{\tau \ge n\}\right).
\end{aligned}$$

Now $E_\nu[\xi_n|\mathscr{F}_{n-1}] = E_{X_{n-1}}S_1 = f(X_{n-1})$. Thus

$$\frac{E_\nu S_\tau}{E_\nu\tau} = \frac{E_\nu\sum_{n=1}^{\tau} f(X_{n-1})}{E_\nu(\tau)} = \pi f = E_\pi S_1. \qquad \square$$

The important part of the following lemma is (5.13), which is the *twisting formula*.

LEMMA 5.3. *Assume that $\mathscr{W}$ is open. Then*

$$\nabla\Lambda(\alpha) = \left(E_{\nu(\alpha)}^{\mathbf{Q}(\alpha)}\tau\right)^{-1}\left(E_{\nu(\alpha)}^{\mathbf{Q}(\alpha)}S_\tau\right) \tag{5.11}$$

$$= E_{\pi(\alpha)}^{\mathbf{Q}(\alpha)}S_1, \tag{5.12}$$

provided the latter exists. If $\nabla\Lambda(\alpha) = v$ has solution α_v for some $v \in \mathbb{R}^d$, then obviously

$$E_{\pi(\alpha_v)}^{\mathbf{Q}(\alpha_v)}S_1 = v. \tag{5.13}$$

PROOF. By Lemma 3.3

$$\nabla\Lambda(\alpha) = \frac{E_\nu^{(\alpha)}S_\tau}{E_\nu^{(\alpha)}\tau},$$

and by Lemma 5.1 this = the right side of (5.11). If $\alpha = \alpha_v$, the above = v.
If $E_{\pi(\alpha)}^{\mathbf{Q}(\alpha)}S_1$ exists, then Lemma 5.2 applied to the α-process implies (5.12). □

Note also that

$$E_{\pi(\alpha)}^{\mathbf{Q}(\alpha)}S_n = E_{\pi(\alpha)}^{\mathbf{Q}(\alpha)}\xi_1 + \cdots + E_{\pi(\alpha)}^{\mathbf{Q}(\alpha)}\xi_n = nE_{\pi(\alpha)}^{\mathbf{Q}(\alpha)}\xi_1 = n\nabla\Lambda(\alpha). \tag{5.14}$$

We note that (5.12) is equivalent to the following differentiation formula:

COROLLARY 5.1.

$$\nabla\big(l(\alpha)\hat{P}(\alpha)r(\alpha)\big) = l(\alpha)\big(\nabla\hat{P}(\alpha)\big)r(\alpha). \tag{5.15}$$

PROOF. The left side = $\nabla e^{\Lambda(\alpha)} = (\nabla\Lambda(\alpha))e^{\Lambda(\alpha)}$, and the right side

$$= \int\int\int l(dx;\alpha)se^{\langle\alpha,s\rangle}P(x, dy\times ds)r(y;\alpha)$$

$$= e^{\Lambda(\alpha)}\int\int\int \pi(dx;\alpha)sQ(x, dy\times ds;\alpha) \tag{5.16}$$

$$= e^{\Lambda(\alpha)}E_{\pi(\alpha)}^{\mathbf{Q}(\alpha)}S_1.$$

□

REMARK. This formula was proved in Iscoe, Ney and Nummelin (1985) under a strong recurrence hypothesis, using a tedious direct calculation. (Elsewhere it has also been the object of lengthy calculations, under special conditions.) It is a simple consequence of the regenerative approach, and is seen to hold quite generally.

Applying the strong LLN to $S_n^{(\alpha_v)}$, (5.11) implies:

COROLLARY 5.2. *If* $v \in \nabla\Lambda(\mathscr{D})$, *then*

$$\lim_{n\to\infty} \frac{1}{n} S_n^{(\alpha_v)} = v \quad \left[\mathbb{Q}_x^{(\alpha_v)}\right] \quad a.s., \qquad x \in \mathbb{E}. \tag{5.17}$$

We turn to some *large deviation theorems*. Applying Theorem 4.1 and Lemma 5.3, the following results are proved very similarly to their analogs in Iscoe, Ney and Nummelin (1985).

THEOREM 5.1. *Assume that* $\mathscr{W}$ *is open, let* $A \in \mathscr{E}$ *be a* φ*-positive set, and let* $x \in \mathbb{F}$. *Let* G *be any open set and* F *any closed set in* $\mathbb{R}^d$. *Write* Λ^* *for the convex conjugate of* Λ *and* $\overline{\Lambda}(\Gamma) = \inf\{\Lambda^*(v) \colon v \in \Gamma\}$. *Then*

$$\liminf \frac{1}{n} \log P^n(x, A \times nG) \geq -\overline{\Lambda}(G). \tag{5.18}$$

If also A *is an s-set, then*

$$\limsup \frac{1}{n} \log P^n(x, A \times nF) \leq -\overline{\Lambda}(F). \tag{5.19}$$

PROOF. The geometric ergodicity of $\hat{P}(\alpha)$ implies

$$\hat{P}^n(x, A; \alpha) = l(A; \alpha) e^{n\Lambda(\alpha)} r(x; \alpha)\left[1 + O(\delta^n(\alpha))\right], \tag{5.20}$$

where $0 \leq \delta(\alpha) < 1$, $\alpha \in \mathscr{D}$. Under the hypothesis of the theorem then

$$\lim \frac{1}{n} \log \hat{P}^n(x, A; \alpha) = \Lambda(\alpha), \tag{5.21}$$

where by Theorem 4.1 $\Lambda(\cdot)$ is differentiable and essentially smooth on $\mathscr{D}$. Thus

$$\lim \frac{1}{n} \log E_x\left[e^{\langle \alpha, S_n \rangle} I_n\right] = \Lambda(\alpha),$$

where I_n is the indicator function of the set $[X_n \in A]$. Our hypothesis also implies that $0 \in \mathscr{D}^0(\Lambda)$. The conclusions (5.18) and (5.19) then follow directly from Theorem II of Ellis (1984). □

The sharper large deviation results of Iscoe, Ney and Nummelin (1985) can also be extended to the present setting. By using the essential smoothness of Λ (Lemma 3.4), and the identification of the range of the gradient of Λ as $\mathring{\mathscr{S}} =$ the interior of the convex hull of the support of S_τ/τ (Lemma 3.5), exactly as in Iscoe, Ney and Nummelin (1985), one proves:

THEOREM 5.2. *If* $\mathscr{W}$ *is open and* B *is a convex set with* $(B \cap \mathring{\mathscr{S}}) \neq \varnothing$, *and* $E_\pi S_1 \notin \overline{B}$, *then there exists a point* $v_B \in \partial B \cap \mathring{\mathscr{S}}$ *such that* $B \subset \{v \colon \langle (v - v_B), \alpha_{v_B} \rangle \geq 0\}$, *where* $\alpha_v = (\nabla\Lambda)^{-1}(v)$. *Also* $\Lambda^*(v_B) = \overline{\Lambda}(B) = \inf\{\Lambda^*(v);\ v \in B\}$.

This implies [as in Iscoe, Ney and Nummelin (1985)] the representation formula

$$\begin{aligned}(5.22)\quad &\mathbb{P}_x\left\{X_n \in A, \frac{S_n}{n} \in B\right\} \\ &= e^{-n\bar{\Lambda}(B)} r(x; \alpha_{v_B}) \int_{A \times n(B - v_B)} \frac{e^{-\langle \alpha_{v_b}, s\rangle}}{r(y; \alpha_{v_B})} \tilde{P}_x\left\{\tilde{X}_n \in dy, \frac{\tilde{S}_n}{n} \in ds\right\},\end{aligned}$$

where $\{(\tilde{X}_n, \tilde{S}_n)\}$ is a centered MA-process with measure $\tilde{\mathbb{P}}_x$. Namely, $E_{\tilde{\pi}} \tilde{S}_n = 0$, $\tilde{\pi}$ being the invariant measure of $\{\tilde{X}_n\}$. Also note that the exponent $\langle \alpha_{v_B}, s\rangle$ is ≥ 0 over the range of integration.

For an upper bound, we conclude from (5.22) that if $x \in \mathbb{F}$ and A is an s-set, then under the hypothesis of Theorem 5.2

$$(5.23)\qquad \mathbb{P}_x\left\{X_n \in A; \frac{S_n}{n} \in B\right\} \leq c'' e^{-n\bar{\Lambda}(B)}.$$

In the other direction the local limit theorem [Theorem 5.2 of Iscoe, Ney and Nummelin (1985)] for S_n applies. Clearly $r(\cdot\,; \alpha_{v_B})$ will also be bounded on some subset of A. We conclude that if the S_n are lattice valued and A is an s-set, then under the hypothesis of Theorem 5.2

$$(5.24)\qquad P_x\left\{X_n \in A; \frac{S_n}{n} \in B\right\} \geq c' n^{-d/2} e^{-n\bar{\Lambda}(B)}.$$

Thus:

Theorem 5.3. *Assume the hypothesis of Theorem 5.2, and that A is an s-set. Let $\{S_n\}$ be lattice valued. Then there exist constants $0 < c' \leq c'' < \infty$ such that*

$$(5.25)\qquad c' n^{-d/2} e^{-n\bar{\Lambda}(B)} \leq \mathbb{P}_x\left\{X_n \in A; \frac{S_n}{n} \in B\right\} \leq c'' e^{-n\bar{\Lambda}(B)}.$$

Similar estimates hold in the nonlattice cases, and for continuous parameter processes $\{(X_t, S_t)\}$, with n replaced by t in (5.25). See Iscoe, Ney and Nummelin (1985) for further discussion of such results under the more stringent hypotheses of that paper

Remark. The CLT for MA-processes is well known [see, e.g., Nagaev (1957) and Keilson and Wishart (1964)]. We note that it also follows trivially from (5.20) (though under excessive hypothesis). Let Φ be the Gaussian measure with mean 0 and covariance $\Sigma_\tau = (E_\nu \tau)^{-1} \operatorname{cov}_\nu S_\tau$. Just replace α by $\alpha/\sqrt{n}$ in (5.20) to get

$$\lim_n P^n\left(x, A\frac{\alpha}{\sqrt{n}}\right) = \pi(A) \lim \lambda^n\left(\frac{\alpha}{\sqrt{n}}\right) = \pi(A)\exp\left[\frac{1}{2}\alpha\Sigma_\tau\alpha'\right],$$

and hence $\mathbb{P}_x(X_n \in A, S_n/\sqrt{n} \in \Gamma) \to \pi(A)\Phi(\Lambda)$.

6. Special cases and examples. We describe several MA-processes which illustrate the nature of our hypotheses and also some that point to unanswered questions.

EXAMPLE 6.1. *The "bounded" process.* This simple example will play an important role in Part II of the paper. Suppose that the summands $\{\xi_i; i = 1, 2, \dots\}$ of the additive component are uniformly bounded random variables, and that the inter-regeneration times $\{T_{i+1} - T_i; i = 0, 1, \dots\}$ are also bounded. Then $\psi(\alpha, \zeta) < \infty$ for all $\alpha \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^1$ and $\mathscr{W} = \mathbb{R}^{d+1}$. Furthermore, from the representation formulas for $r(x; \alpha)$, $l(A; \alpha)$ [(4.2) and (4.3)], we see that for each $\alpha \in \mathbb{R}^d$, $\{r(x; \alpha); x \in \mathbb{E}\}$ is bounded and uniformly positive, and $\{l(A; \alpha); A \in \mathscr{E}\}$ is a finite measure. Thus the conditions required for Theorem 4.1 and for the limit theorems of Section 5 are satisfied.

EXAMPLE 6.2. *Finite space space.* $\mathbb{E} = \{1, \dots, N\}$. It has already been observed in Iscoe, Ney and Nummelin (1985) that all the large deviation theorems work in this case, but we point out that the present hypotheses are especially easily checked. Namely, with

$$p_{ij}(\Gamma) = P\{X_{n+1} = j, \xi_{n+1} \in \Gamma | X_n = i\}, \qquad \hat{p}_{ij}(\alpha) = \int p_{ij}(ds) e^{\langle \alpha, s \rangle},$$

$\mathscr{D}_{ij} = \{\alpha\colon \hat{p}_{ij}(\alpha) < \infty\}$, $\mathscr{D} = \bigcap_{i,j} \mathscr{D}_{ij}$; we see that $\Lambda(\alpha)$ exists and is $< \infty$ if and only if $p_{ij}(\alpha) < \infty$ for all i, j; namely, $\mathscr{D}(\Lambda) = \mathscr{D}$. Assume $P = \{p_{ij}(\mathbb{R}^d)\}$ is irreducible. Then for all $\alpha \in \mathscr{D}$, $\hat{P}(\alpha)$ is geometrically $\lambda^{-1}(\alpha)$-recurrent (Perron–Frobenius theorem). If $\mathscr{D}_{ij}$ are all open, then so is $\mathscr{D}$. By part (b) of Theorem 4.1 we conclude that $\mathscr{W}$ is open. [The minorization (M_1) can be taken as $h(i, \Gamma) = \delta_{i0} p_{00}(\Gamma)$, $\nu(j) = \delta_{j0}$.] Thus all the conclusions of that theorem hold.

EXAMPLE 6.3. *A modified "sawtooth" chain.* The following example of an integer valued MA-process on a countable state space M.C. illustrates a case where $\hat{P}(\alpha)$, considered as an operator on the space of bounded sequences with sup norm, is unbounded. However, the convergence parameter $\Lambda(\alpha)$ is well defined and finite for all $\alpha \in \mathbb{R}$. The eigenmeasure associated with $e^{\Lambda(\alpha)}$ is not a finite measure, and the (right) eigenfunction is not bounded away from 0. Thus, we can use our results to determine the rate function of $\mathbb{P}_i(S_n \geq an, X_n = j)$, but not of $\mathbb{P}_i(S_n \geq an)$, since the set $\mathbb{E} = \{0, 1, \dots\}$ is not an s-set. An approach via the spectral radius also does not work here.

Let $\{X_n\}$ be a M.C. on $\mathbb{E} = \{0, 4, 5, 6, \dots\}$ with transition function

$$P(0, 4i) = p_i, \qquad i = 0, 1, \dots,$$

$$P(4i, 4i+1) = P(4i+1, 4i+2) = P(4i+2, 4i+3) = P(4i+3, 0) = 1,$$

for $i = 1, 2, \dots$. Define, for $i \geq 1$, $f(4i) = 0$, $f(4i+1) = a_i$, $f(4i+2) = -f(4i+3) = i$, $f(0) = 0$, where a_i is a given sequence of numbers. Then

$$\hat{P}(\alpha) = \{P(i, j) e^{\alpha f(j)}, i, j \in \mathbb{E}\}.$$

The spectral radius of $\hat{P}(\alpha)$ is ∞ for $\alpha \neq 0$.

Now $P[(\tau, S_\tau) = (1,0)] = p_0$, $P[(\tau, S_\tau) = (5, a_i)] = p_i$, $i \geq 1$. Thus $\psi(\alpha, \zeta) = p_0 e^{-\zeta} + (1 - p_0)e^{-5\zeta}\varphi(\alpha)$, where $\varphi(\alpha) = (1 - p_0)^{-1}\sum_{i=1}^{\infty} p_i e^{\alpha a_i}$ is a generating function.

Thus clearly $\Lambda(\alpha)$ is well defined by $\psi(\alpha, \Lambda(\alpha)) = 1$. Suppose the sequence $\{a_i\}$ is bounded. Then $\varphi(\alpha) < \infty$ for $\alpha \in \mathbb{R}$ and hence $\Lambda(\alpha) < \infty$ for all $\alpha \in \mathbb{R}$. The invariant measure for $e^{\Lambda(\alpha)}$ is computed from (4.3):

$$l(0, \alpha) = 1, \qquad l(4i, \alpha) = p_i \exp(-\Lambda(\alpha)),$$
$$l(4i + 1; \alpha) = p_i \exp(\alpha a_i - 2\Lambda(\alpha)),$$
$$l(4i + 2; \alpha) = p_i \exp(\alpha(a_i + i) - 3\Lambda(\alpha)),$$
$$l(4i + 3; \alpha) = p_i \exp(\alpha a_i - 4\Lambda(\alpha)), \qquad i = 1, 2, \dots .$$

The right eigenfunction $r(i, \alpha)$ is easily computed from (4.2). If $p_i = ce^{-\gamma i}$, $i \geq 0$, then for $\alpha > \gamma$, $l(\mathbb{E}, \alpha) = \infty$. Also for $\alpha > 0$, $\liminf r(i; \alpha) = 0$. Now for all fixed (i, j) the sequence of probability measures

$$\mathbb{P}_i\left\{\frac{S_n}{n} \in \cdot, X_n = j\right\}$$

satisfies the LD principle with the same rate function $\Lambda^*(\cdot)$. But one easily checks that $P_0\{S_n \geq an\}$ may decay at a rate other than $\Lambda^*(a)$.

EXAMPLE 6.4. *Strong recurrence.* Markov chains on a general state space with transition function P satisfying

$$(6.1) \qquad a\nu(A) \leq P(x, A) \leq b\nu(A), \qquad A \in \mathscr{E},$$

for some probability measure ν, and some $0 < a \leq b < \infty$, act very similar to finite state space chains. Harris [(1963), Theorem 2.10.1] has proved a complete analog of the Perron–Frobenius theorem for this case.

In Iscoe, Ney and Nummelin (1985) we studied MA-processes satisfying a natural extension of (6.1). Namely,

$$(6.2) \qquad a\nu(A \times \Gamma) \leq P(x, A \times \Gamma) \leq b\nu(A \times \Gamma),$$

where ν is now a measure on $(\mathbb{E} \times \mathbb{R}^d, \mathscr{E} \times \mathscr{R}^d)$. We will show that a somewhat more general form of this condition also fits nicely into the scheme of the present paper; namely, that "$\mathscr{W}$ open" can be verified. The hypothesis consists of the minorization (M) together with a similar *upper* bound for P. Namely, that

(M) *There exist measures* $\{h(x, \cdot), x \in \mathbb{E}\}$ *on* $\mathbb{R}^d$ *and* ν *on* $\mathbb{E} \times \mathbb{R}^d$, *such that for some* $0 < a \leq b < \infty$,

$$(6.3) \qquad ah \circledast \nu(x, A \times \Gamma) \leq P(x, A \times \Gamma) \leq bh \circledast \nu(x, A \times \Gamma),$$

for all $x \in \mathbb{E}$, $A \in \mathscr{E}$, $\Gamma \in \mathbb{R}^d$, *where*

$$(h \circledast \nu)(x, A \times \Gamma) = h(x, \cdot) * \nu(A \times \cdot)(\Gamma) = \int_{\mathbb{R}^d} h(x, ds)\nu(A \times \Gamma - s).$$

The α-transform of (6.3) is

$$(6.4) \qquad a\hat{h} \otimes \hat{\nu}(\alpha) \leq \hat{P}(\alpha) \leq b\hat{h} \otimes \hat{\nu}(\alpha).$$

For the following proposition, (6.4) is sufficient (and slightly weaker), with only the *lower* bound in (6.3) needed to assure existence of the regenerative structure.

Take $0 < \varepsilon < a$, and consider the regeneration structure and associated generating function $\psi = \psi_\varepsilon(\alpha, \zeta)$ and domain $\mathscr{W} = \mathscr{W}_\varepsilon$ determined by the minorization $\varepsilon h \otimes \nu$. Let

$$\mathscr{D}' = \{\alpha: \hat{\nu}\hat{h}(\alpha) < \infty\},$$

$$\mathscr{S}' = \text{convex hull of support of } \nu * h(\cdot).$$

PROPOSITION 6.1. *Assume* (6.4), $\mathscr{D}'$ *open, and* $\mathring{\mathscr{S}}' \neq \varnothing$. *Then for some* $\varepsilon > 0$, $\mathscr{W}_\varepsilon$ *is open,* $\mathring{\mathscr{S}}' = \mathring{\mathscr{S}}$, $\mathscr{D}' = \mathscr{D}(\lambda)$. *Hence all the conclusions of Theorem* 4.1 *hold.*

PROOF. Starting with (6.4), conclude by Lemma 2.3 that $\hat{P}(\alpha)$ is g-recurrent. Let $\lambda^{-1}(\alpha) = e^{-\Lambda(\alpha)}$ be its associated convergence parameter. By (6.4)

$$a^n(\hat{\nu}\hat{h})^{n+1}(\alpha) \le \hat{\nu}\hat{P}^n\hat{h}(\alpha) \le b^n(\bar{\nu}\hat{h})^{n+1}(\alpha), \qquad \alpha \in \mathscr{D}'. \tag{6.5}$$

Now $\lambda(\alpha) < \infty$ if and only if $\Sigma\delta^n\hat{\nu}\hat{P}^n\hat{h}(\alpha) < \infty$ for some $\delta > 0$, which by (6.5) is equivalent to $\hat{\nu}\hat{h}(\alpha) < \infty$. Hence $\mathscr{D}' = \mathscr{D}(\lambda)$, and $\mathscr{D}(\lambda)$ is open. The conditions of Lemma 4.3 are satisfied and hence there is an $\varepsilon > 0$ such that $\mathscr{W}_\varepsilon$ is open.

Finally, note that

$$P_\nu\{S_\tau \in \Gamma\} = \sum_{n=1}^{\infty} \nu * (P - \varepsilon h \otimes \nu)^{n-1} * h(\Gamma). \tag{6.6}$$

Hence

$$\begin{aligned} O \in \mathring{\mathscr{S}} &\Leftrightarrow O \in \left(\text{Supp}_\nu\left(\frac{S_\tau}{\tau}\right)\right)^0 \Leftrightarrow O \in (\text{Supp}_\nu(S_\tau))^0 \\ &\Leftrightarrow O \in \text{Supp}\{\nu * [P - \varepsilon(h * \nu)]^{n-1} * h\}^0, \quad \text{for some } n \ge 1 \\ &\Leftrightarrow O \in \text{Supp}\{(\nu * h)^{*n}\}^0, \quad \text{for some } n \ge 1 \quad [\text{by (6.3)}] \\ &\Leftrightarrow O \in \mathring{\mathscr{S}}'. \end{aligned}$$

To argue that $v \in \mathring{\mathscr{S}}$, translate the process by v and argue as in the proof of Lemma 3.5. □

REMARK. The hypothesis (6.4) can be replaced by the somewhat weaker condition:

For some integer $k \ge 1$, there exist kernels

$$\{h^i(x, \Gamma): x \in \mathbb{E}, \Gamma \in \mathscr{R}^d\}, \qquad i \in I = \{1, \ldots, N\},$$

and probability measures $\nu^i(\cdot \times \cdot)$ on $(\mathbb{E} \times \mathbb{R}^d, \mathscr{E} \times \mathbb{R}^d)$ with transforms $\hat{h}^i(x; \alpha)$, $\hat{\nu}^i(A; \alpha)$, such that $\nu^i * h^j(\mathbb{R}^d) > 0$ for $i, j \in I$, and

$$a \sum_{i \in I} \hat{h}(\alpha) \otimes \hat{\nu}^i(\alpha) \le \hat{P}^k(\alpha) \le b \sum_{i \in I} \hat{h}^i(\alpha) \otimes \hat{\nu}^i(\alpha), \quad \text{for some } 0 < k < \infty. \tag{6.7}$$

Similarly the minorization (M) can be replaced by

$$\sum_{i \in I} h^i(x, \cdot) * \nu^i(A \times \cdot)(\Gamma) \le P^k(x, A; \Gamma). \tag{6.8}$$

COROLLARY 6.1. *Under* (6.3)

$$c_1\hat{h}(x,\alpha) \le r(x;\alpha) \le c_2\hat{h}(x,\alpha), \qquad x \in \mathbb{R}^d, \tag{6.9}$$

for some $0 < c_1(\alpha) \le c_2(\alpha) < \infty$, $\alpha \in \mathscr{D}$, *and*

$$\gamma_1\hat{\nu}(A,\alpha) \le l(A;\alpha) \le \gamma_2\hat{\nu}(A;\alpha), \qquad A \in \mathscr{E}, \tag{6.10}$$

for some $0 < \gamma_1(\alpha) \le \gamma_2(\alpha) < \infty$, $\alpha \in \mathscr{D}$.

PROOF. Just multiply through (6.4) on the right by $r(\alpha)$, then on the left by $l(\alpha)$. Recall that $\hat{\nu}r = 1$ and $l\hat{h} = 1$. □

Thus bounds on $\hat{h}$ or $\hat{\nu}$ imply similar ones on r and l. If $\hat{\nu}(E;\alpha) < \infty$, then $l(\cdot;\alpha)$ is a finite measure, and by Corollary 4.1 $l(A;\alpha)$ is analytic on $\mathscr{D}$ for each α.

EXAMPLE 6.5. *A "Doeblin"-type condition.* Here is an example where (6.4) may fail, but $\hat{P}(\alpha)$ satisfies (2.17), which is weaker. However, this is compensated by a strong restriction on $\{\xi_n\}$.

PROPOSITION 6.2. *Suppose* $P(x, dy \times ds) = \tilde{P}(x, dy)h(x, ds)$, *where* $\tilde{P}$ *is stochastic. Assume* $a\hat{g}(\alpha) \le \hat{h}(x;\alpha) \le b\hat{g}(\alpha)$ *for some fixed probability measure g on* $\mathbb{R}^d$, *and that* $c\hat{h}(x;\alpha)\nu(A) \le \hat{P}(x, A; \alpha)$ *for some* $c > b - a$ *and probability measure* ν *on* $\mathbb{E}$. *Then* $\hat{P}(\alpha)$ *is g-recurrent for* $\alpha \in \mathscr{D}(\lambda) = \mathscr{D}(\hat{g})$. *If this is an open set, then all the conclusions of Theorem* 4.1 *hold.*

PROOF. Verify the conditions of Lemma 2.4 for $\hat{P}(\alpha)$. Also $a\hat{g}(\alpha)\tilde{P}(x, dy) \le \hat{P}(x, dy; \alpha) \le b\hat{g}(\alpha)\tilde{P}(x, dy)$ and hence $a\hat{g}(\alpha) \le \lambda(\alpha) \le b\hat{g}(\alpha)$, and $\mathscr{D}(\lambda) = \mathscr{D}(\hat{g})$. □

For example, if $\mathbb{E} = \{0,1,2,\dots\}$ and $P = \{p_{ij}\}$ with $p_{i0} \ge \delta > 0$, and if $h(i,\Gamma) = P\{\xi_{n+1} \in \Gamma | X_n = i\}$ satisfies $a\hat{g}(\alpha) \le \hat{g}(i,\alpha) \le b\hat{g}(\alpha)$ for some fixed $\hat{g}$, with $b - a < \delta$, then the conditions of the proposition are satisfied.

EXAMPLE 6.6. *Another "sawtooth-type" process.* Consider a chain $\{X_n\}$ with transition kernel $P = \{p_{ij};\ i, j = 0,1,2,\dots\}$ satisfying $p_{ij} = 0$ for $i \ge 1$, $j \ge i$; namely, it is lower triangular except for the first row. Thus starting at 0, the process jumps to the right, then moves to the left by arbitrarily distributed steps until it hits 0 and jumps out again. This process has nice properties which illustrate some aspects of the applicability of our method. [Note that a condition like (6.4) cannot be satisfied here, whatever $\{S_n\}$ may be.]

Let us take S_n to be the occupation time vector of a finite number d of states. [$S_n = \sum_1^n f(X_{i-1}, X_i)$, with f having finite support would work just as well.] Let $\{T_i,\ i = 0,1,\dots\}$ be the hitting times of $\{0\}$. Then $\|S_\tau\| \le d$. If $\varphi(t) = \sum p_{0j}e^{tj} < \infty$ for all $t \in \mathbb{R}^1$, then

$$\psi(\alpha,\zeta) = Ee^{\langle\alpha, S_\tau\rangle - \zeta\tau} < \infty,$$

for all $(\alpha, \zeta) \in \mathbb{R}^{d+1}$. Thus $\mathscr{W} = \mathbb{R}^{d+1}$, and all the ensuing theory goes through.

However, this depends delicately on $\{p_{0j}\}$. Take, for example, the simple sawtooth process, namely $p_{i,i-1} = 1$ for $i \geq 1$, but drop the requirement that $\varphi(t) < \infty$ for all t. Simplify further to S_n = the occupation time of a single state, say $\{0\}$. Then

$$\hat{p}(i, j; \alpha) = p_{ij}(\delta_{i0}e^{\alpha} + (1 - \delta_{i0})]$$

and one can check that

$$r(i; \alpha) = (\lambda(\alpha))^{-i}, \qquad i \geq 1;\ r(0; \alpha) = 1 \text{ (say)}.$$

Also it is necessary that

$$e^{\alpha}\sum p_{0i}r(i) = \sum_{i=0}^{\infty} p_{0i}\lambda^{-i} = \lambda r(0) = \lambda.$$

Thus letting $\rho = \lambda^{-1}$, $\varphi(\rho) = \sum_{i=0}^{\infty} p_{0i}\rho^i$, we require that $\rho\varphi(\rho) = e^{-\alpha}$ have a solution. If $\alpha \geq 0$ this will always work, but if $\alpha < 0$ it may fail. A sufficient condition would be $\varphi(\cdot)$ essentially smooth. Let $\mu = \sum i p_{0i}$ and $\pi P = \pi$.

Now $\pi_k = \mu^{-1}\sum_{i=k}^{\infty} p_{0i}$ and hence $E_{\pi}\xi_1 = \Lambda'(\alpha)|_{\alpha=0} = \pi_0 = 1/\mu$. Thus if $v < 1/\mu$, then $\Lambda'(\alpha_v) = v$ would imply α_v is negative. We can therefore choose the probabilities $\{p_{0i}\}$ so that $\rho\varphi(\rho) = e^{-\alpha}$ has no solution, and the above methods cannot be applied to estimating $P\{S_n/n \leq v\}$. However, if $\varphi(\cdot)$ is essentially smooth the required roots always exist.

A similar situation prevails in somewhat less trivial cases like

$$p_{0j} = p_j; \qquad p_{10} = 1; \qquad p_{i0} = p_{i,i-1} = \tfrac{1}{2}, \quad \text{for } i \geq 2.$$

Here $r(i; \alpha)$ also has geometric growth, and the essential smoothness of the generating function of $\{p_i\}$ is sufficient, as above. One can construct other more complicated examples of this kind.

EXAMPLE 6.7. *Random sums of i.i.d. r.v.'s.* Let $0 < T_1 < T_2 < \cdots$ be the renewal epochs of a renewal process with increment distribution G. Let $\{\xi_i\}$ be a sequence of i.i.d. r.v.'s with distribution F and define the process $S_t = \sum_{\{i:\, T_i \leq t\}} \xi_i$; namely, $\xi_i = S_{T_i} - S_{T_i - 1}$ with $S_0 = 0$. Also assume $\{\xi_i;\ i = 1, 2, \dots\}$ are independent of $\{T_i;\ i = 1, 2, \dots\}$. Then in the notation of Section 3,

$$\psi(\alpha, \zeta) = Ee^{\langle \alpha, \xi_1 \rangle - \zeta T_1} = \hat{F}(\alpha)\hat{G}(-\zeta),$$

and

$$\mathscr{W} = \{\alpha \in \mathbb{R}^d\colon \hat{F}(\alpha) < \infty\} \times \{\zeta \in \mathbb{R}^1\colon \hat{G}(-\zeta) < \infty\},$$

$$\mathscr{U}_r = \left\{\alpha \in \mathbb{R}^d\colon G(-\Lambda) = \left(\hat{F}(\alpha)\right)^{-1}, \text{for some } \Lambda = \Lambda(\alpha)\right\}.$$

Thus if $\{\hat{F} < \infty\}$ and $\{\hat{G} < \infty\}$ are open, then so is $\mathscr{W}$.

If, for example, $\{S_t;\ t \geq 0\}$ is a compound Poisson process, i.e., $G = \exp(\beta)$, $\beta > 0$, then $\hat{G}(-\zeta) = \beta/(\beta + \zeta)$ and $\Lambda(\alpha) = \beta(\hat{F}(\alpha) - 1) > -\beta$ for all $\alpha \in \mathbb{R}^1$.

Hence $\psi(\alpha, \zeta) < \infty$ for some $\zeta < \Lambda(\alpha)$ whenever $\hat{F}(\alpha) < \infty$, and therefore

$$\mathscr{U}_g = \{\alpha: \hat{F}(\alpha) < \infty\} = \mathscr{D}.$$

Thus if $\mathscr{D}$ is open then Theorem 4.1 is satisfied.

EXAMPLE 6.8.. *Birth–death chain.* Take $\mathbb{E} = \{0, 1, \dots\}$,

$$P\{X_{n+1} = i + 1 | X_n = i\} = p_i, \qquad i \geq 1,$$
$$P\{X_{n+1} = i - 1 | X_n = i\} = 1 - p_i,$$
$$P\{X_{n+1} = 1 | X_n = 0\} = 1,$$

and let $S_n = \sum_{i=0}^{n} f(X_i)$. It seems difficult to directly verify any of the criteria of this paper for $\{(X_n, S_n)\}$ (even with restrictions on f). However, we will see in Part II by an approximation argument that the logarithmic large deviation theorem still holds.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN 53706

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HELSINKI
HELSINKI
FINLAND