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HYDRODYNAMICAL LIMIT FOR THE ASYMMETRIC SIMPLE EXCLUSION PROCESS

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We prove a strong law of large numbers for the rescaled asymmetric simple exclusion process. By a coupling procedure we show that the density profile is a weak solution to a first order quasilinear partial differential equation. Moreover, the monotonicity of the process allows us to show that it is the unique solution satisfying the entropy condition. The local equilibrium is then an easy consequence.

Introduction. It is well known that the simple exclusion process with translation invariant transition probabilities preserves stochastic order and has a family $\{\nu^c\}$ of equilibrium measures indexed by a continuous parameter ranging from 0 to 1. We recall this notation in Sections 1 and 2.

In this article we are interested in the asymptotic behavior of the one-dimensional asymmetric simple exclusion process in which only transitions to nearest neighbors are allowed. We consider this process starting from a product measure $\nu^{a,b}$ corresponding to two half-spaces in equilibrium with different parameters a and b.

After a suitable space and time rescaling, the distribution of particles at time t defines a random measure on the real line, each particle contributing an equal mass. We show that this measure converges weakly almost surely to a deterministic measure, as the rescaling parameter goes to zero. In other words, we obtain a strong law of large numbers for the system (Section 2).

Moreover, the limiting measure has a density, called the density profile, which is a weak solution of a nonlinear hyperbolic P.D.E.; this is shown in Section 3, under the nearest-neighbor assumption, by a coupling argument.

The identification of the density profile as the only weak solution satisfying the entropy condition relies strongly on the properties of the limiting equation summarized in the Appendix.

In Section 4 we deduce from the preceding result the local equilibrium at each point of continuity of the density profile.

When the process starts from the particular configuration corresponding to $a = 1$ and $b = 0$, this problem is solved by Rost [9] when the particles move only in one direction to the nearest neighbor and by Liggett [7] for general transition probabilities.

1. The simple exclusion process.

1.1. *Generator and initial distribution.* Consider on the space $E = \{0,1\}^{\mathbb{Z}}$ with elements $e = \{e(k),\ k \in \mathbb{Z}\}$, the Markovian process of a simple random

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walk with exclusion which can be described intuitively in the following way: Particles are distributed initially on $\mathbb{Z}$ in such a way that there is at most one particle per site. Each particle waits an exponential time with parameter one and attempts a transition of one unit to the right with probability p and one unit to the left with probability $1-p$; it makes the transition if that site is vacant, while if the site is occupied, the particle remains where it was.

A general result of existence and uniqueness [7] shows that there exists a unique Markov process corresponding to this description. Its *generator* is defined on cylindrical functions on E by

$$(1)\qquad \begin{aligned} Lf(e) = &\sum_{k\in\mathbb{Z}} pe(k)(1-e(k+1))\left[f(e^{k,k+1}) - f(e)\right] \\ &+ \sum_{k\in\mathbb{Z}} (1-p)e(k)(1-e(k-1))\left[f(e^{k,k-1}) - f(e)\right], \end{aligned}$$

where $e^{k,l}$ is obtained from e by permuting the coordinates at k and l and keeping the rest fixed. Denote by (T_t) the semigroup generated by L. Let $\{X_t = (X_t(k),\ k\in\mathbb{Z}),\ t\ge 0\}$ be the right-continuous version with left limits of the Markov process with semigroup (T_t). Observe that $X_t(k) = 0$ or 1 for every $t \ge 0$ and $k \in \mathbb{Z}$.

The *initial distribution* $\nu^{a,b}$ will be a product measure on E such that

$$(2)\qquad \nu^{a,b}\{e\in E\colon e(k)=1\} = \begin{cases} a, & \text{if } k\le 0,\\ b, & \text{if } k>0,\end{cases}$$

for given a and b, $0\le b\le a\le 1$. Since a product of Bernoulli measures with the same parameter is an invariant measure for (T_t) (see [7], Chapter 8), we may see this initial distribution as the juxtaposition of two half-spaces in equilibrium and then observe its evolution according to the exclusion process. Note that $p=1$, $a=1$ and $b=0$ is the case studied in [9].

1.2. *The process as a solution of a system of stochastic equations.* Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space equipped with a right-continuous filtration $(\mathscr{F}_t)_{t\ge 0}$, $\mathscr{F}_0$ complete, supporting the following *independent* objects:

—the initial distribution $\nu^{a,b}$;
—a family $\{P_t(k),\ t\ge 0\}_{k\in\mathbb{Z}}$ of independent Poisson processes with intensity p;
—a family $\{Q_t(k),\ t\ge 0\}_{k\in\mathbb{Z}}$ of independent Poisson processes with intensity $1-p$.

$P(k)$ [resp. $Q(k)$] will be the bell attached to site k which indicates transition times to the right (resp. to the left). Consider the following infinite system of stochastic differential equations:

$$(3)\qquad \begin{aligned} dX_t(k) = &\ X_{t-}(k-1)[1-X_{t-}(k)]\,dP_t(k-1) \\ &-X_{t-}(k)[1-X_{t-}(k+1)]\,dP_t(k) \\ &+X_{t-}(k+1)[1-X_{t-}(k)]\,dQ_t(k+1) \\ &-X_{t-}(k)[1-X_{t-}(k-1)]\,dQ_t(k), \end{aligned}$$

$$\operatorname{law}(X_0) = \nu^{a,b}.$$

It is easy to check that $(X_t)_{t\geq 0}$ is a Markov process with generator L defined by (1). Consequently, (3) has a unique solution in law concentrated on the right-continuous trajectories having left limits.

1.3. *Rescaling.* If $I_{[c,d)}$ is the indicator function of the real interval $[c,d)$, for $h>0$ and $x\in\mathbb{R}$ we define

$$(4)\qquad \begin{aligned} X_t^h(x) &= \sum_{k\in\mathbb{Z}} X_{t/h}(k) I_{[hk,\,h(k+1))}(x),\\ P_t^h(x) &= \sum_{k\in\mathbb{Z}} P_{t/h}(k) I_{[hk,\,h(k+1))}(x),\\ Q_t^h(x) &= \sum_{k\in\mathbb{Z}} Q_{t/h}(k) I_{[hk,\,h(k+1))}(x). \end{aligned}$$

We want to prove almost sure weak convergence of the measure $X_t^h(x)\,dx$ as h goes to 0.

Observe that for $t=0$ the strong law of large numbers holds: The random measure on the real line defined by $X_0^h(x)\,dx = \sum_{k\in\mathbb{Z}} X_0(k) I_{[hk,\,h(k+1))}(x)\,dx$ converges weakly almost surely to the deterministic measure $u_0(x)\,dx = (aI_{(-\infty,0]} + bI_{(0,+\infty)})(x)\,dx$ [i.e. for every continuous function φ with compact support on $\mathbb{R}$, $\lim_{h\to 0}\int_{\mathbb{R}}\varphi(x)X_0^h(x)\,dx = \int_{\mathbb{R}}\varphi(x)u_0(x)\,dx$ a.s.].

In order to guess the limit we need an equation for $(X_t^h)_{t\geq 0}$. On the set of functions $u\colon\mathbb{R}\to\mathbb{R}$ we define the following functionals:

$$(5)\qquad \begin{aligned} F_{\pm h}u(x) &= u(x)[1-u(x\pm h)],\\ Fu(x) &= u(x)[1-u(x)],\\ D_{\pm h}u(x) &= \frac{\pm 1}{h}[u(x\pm h)-u(x)]. \end{aligned}$$

From (3), (4) and (5) we deduce the equation satisfied by $(X_t^h)_{t\geq 0}$,

$$(6)\qquad dX_t^h = -D_{-h}\left[F_h(X_{t-}^h)\,d(hP_t^h)\right] + D_h\left[F_{-h}(X_{t-}^h)\,d(hQ_t^h)\right],$$

initial condition X_0^h.

Intuitively, as $h\to 0$, $hP_t^h\to pt$, $hQ_t^h\to(1-p)t$, $D_{\pm h}\to\partial/\partial x$ and $F_{\pm h}\to F$, we should have that $X_t^h(x)\,dx$ converges weakly almost surely to $u(x,t)\,dx$ where $u(x,t)$ is a solution of the first-order nonlinear hyperbolic P.D.E.

$$(7)\qquad \frac{\partial u}{\partial t} + (2p-1)\frac{\partial F(u)}{\partial x} = 0,\qquad u(\cdot,0)=u_0.$$

We will suppose $p\neq\frac{1}{2}$ (i.e., the exclusion is asymmetric); for $p=\frac{1}{2}$ the right time scaling is t/h^2 and the limit is a solution of the linear parabolic equation $\partial u/\partial t = \frac{1}{2}(\partial^2 u/\partial x^2)$ (see [2] and [7], Chapter 8).

In the Appendix we give the essential properties of (7), the main result about entropy and a complete description of the density profile.

Note that for $p < \frac{1}{2}$ the discontinuity of u_0 propagates while for $p > \frac{1}{2}$ it disappears.

2. Monotonicity properties and convergence. In this chapter we show a law of large numbers for the family of processes $\{(X_t^h)_{t\geq 0},\ h \searrow 0\}$ defined in Section 1 by (3) and (4). We will use a subadditive ergodic theorem due to Liggett ([7], Chapter 6, Theorem 2.6). In Section 1 we recall the basic tools for the study of the exclusion process.

2.1. *Stochastic order, monotonicity, coupling and priority.* The notion of *stochastic order* for probability measures on E is defined as follows: For e and e' in E we say that $e \leq e'$ if $e(k) \leq e'(k)$ for every $k \in \mathbb{Z}$; a continuous real-valued function f defined on E is called monotone ($f \in \mathscr{M}$) if $f(e) \leq f(e')$ whenever $e \leq e'$; ν and μ being probability measures on E, we say that ν is *stochastically larger* than μ ($\mu \leq \nu$) if $f\mu \leq f\nu$ for every f in $\mathscr{M}$.

It is well known that (T_t) preserves stochastic order: $\mu \leq \nu$ implies $\mu T_t \leq \nu T_t$ for all $t \geq 0$ which is equivalent to $T_t f \in \mathscr{M}$ for every $f \in \mathscr{M}$ (see [7], Chapter 2). The underlying Markov process $(X_t)_{t\geq 0}$ is called *monotone* or attractive in this context.

This notion is connected to *coupling* which allows us to construct simultaneously on the same probability space versions of the exclusion process starting from an arbitrary configuration at an arbitrary time (cf. [7], Chapter 8, Section 2). We will use coupling in the particular situation where the initial distributions are stochastically ordered.

We will also need the *priority procedure* which can be described by the following generator defined on the cylindrical functions on $\{(e_1, e_2) \in E \times E \mid e_1(k)e_2(k) = 0,\ \forall\ k \in \mathbb{Z}\}$:

$$\begin{aligned}\tilde{L}_p f(e_1, e_2) = \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \Big\{ & p(k,m)e_1(k)(1 - e_1(m)) \\ & \times \left\{ e_2(m)\left[f\left(e_1^{k,m}, e_2^{m,k}\right) - f(e_1, e_2)\right] \right. \\ & \quad \left. + (1 - e_2(m))\left[f\left(e_1^{k,m}, e_2\right) - f(e_1, e_2)\right]\right\} \\ & + p(k,m)e_2(k)(1 - e_2(m) - e_1(m)) \\ & \quad \times \left[f\left(e_1, e_2^{k,m}\right) - f(e_1, e_2)\right]\Big\},\end{aligned}$$

with

$$p(k,m) = \begin{cases} p, & \text{if } m = k + 1, \\ 1 - p, & \text{if } m = k - 1, \\ 0, & \text{elsewhere,} \end{cases}$$

for every $k \in \mathbb{Z}$.

On the same space one can say that: Given a configuration formed by two types of particles X_1 and X_2, the system $X_1 + X_2$ evolves according to the simple exclusion process but each time an X_1-particle decides to jump to a site occupied by an X_2-particle, the two particles exchange; we say that X_1 has priority over X_2 ($X_1 \vdash X_2$) and we know that the global system starting from $(X_{1,0} + X_{2,0})$ and the subsystem starting from $X_{1,0}$ evolve according to simple exclusion processes.

2.2. *Convergence*. Let ν^b be a product measure on E such that $\nu^b\{e \in E; e(k) = 1\} = b$ for every $k \in \mathbb{Z}$; we call B-particles the corresponding system in equilibrium. We add Y-particles to B-particles at negative sites in such a way that the distribution of $(B + Y)$-particles, denoted by X-particles, is our initial distribution $\nu^{a,b}$. Finally, we add Z-particles to X-particles at positive sites in such a way that the distribution of $(X + Z)$-particles is ν^a and we impose the following priorities: $B \vdash Y \vdash Z$. We have $\text{law}(B_t) = T_t\nu^b = \nu^b$, $\text{law}((B + Y)_t) = T_t\nu^{a,b} = \text{law}(X_t)$ and $\text{law}((B + Y + Z)_t) = T_t\nu^a = \nu^a$ for every $t \geq 0$. Let $u > 0$ and m, n be integers such that $0 \leq m \leq n$; let the system (B, Y, Z) evolve up to time m/u; we call Y^m the $(Y + Z)$-particles which are at time m/u at sites less or equal than m and Z^m the $(Y + Z)$-particles which are at the same time at sites greater than m; we call B^m the B-particles at that time. After time m/u we let the system (B^m, Y^m, Z^m) evolve according to priorities $B^m \vdash Y^m \vdash Z^m$ up to time n/u. We denote by $S_{m,n}$ the number of Y^m-particles which are at time n/u at sites greater than n: $S_{m,n} = \Sigma_{j>n} Y^m_{n/u}(j)$. Similarly, we define $S(k, t)$ as the number of Y-particles which are at time t at sites greater than k: $S(k, t) = \Sigma_{j>k} Y_t(j)$. With these definitions we have $S_{0,n} = S(n, n/u)$. We denote by $[x]$ the integer part of x.

PROPOSITION 1. *For every u in $\mathbb{R}$, $t^{-1}S([ut], t)$ converges, as t goes to $+\infty$, almost surely to a random variable $G(u)$.*

PROOF. We prove the convergence along the sequence of times $t = n/u$ for $u > 0$; the extension to convergence along all t is routine, the case $u < 0$ is handled similarly and for $u = 0$ we define $S_{0,n}$ as $\Sigma_{j>0} Y_n(j)$ and $S_{m,n}$ as $\Sigma_{j>0} Y^m_n(j)$ where relabelling is performed at time m.

We prove the following properties:

(a) $S_{0,n} \leq S_{0,m} + S_{m,n}$ and $S_{0,0} = 0$ a.s.;
(b) for fixed $k \in \mathbb{N}$, $(S_{k(n-1),kn}, n \in \mathbb{N}^*)$ is a stationary sequence;
(c) for fixed m, $\text{law}(S_{m,m+k}, k \in \mathbb{N}) = \text{law}(S_{m+1,m+1+k}, k \in \mathbb{N})$;
(d) $E(S_{0,n}) < +\infty$ for every $n \in \mathbb{N}$.

Theorem 2.6 in Chapter 6 of [7] will then give the almost sure convergence of $n^{-1}S_{0,n}$, as n goes to $+\infty$, to a *random variable* $G(u)$.

Let $\overline{X}_t$ for $t \geq m/u$ be the process which starts at time m/u in the configuration $(B^m + Y^m)_{m/u} \vee X_{m/u}$ and define $\overline{Y}$ by $\overline{X} = B + \overline{Y}$. Since $\overline{X}_{m/u}(j) \geq X_{m/u}(j)$ for every $j \in \mathbb{Z}$, we have $\overline{X}_{n/u}(j) \geq X_{n/u}(j)$ for every $j \in \mathbb{Z}$. Subtracting $B_{n/u}(j)$ we get $\overline{Y}_{n/u}(j) \geq Y_{n/u}(j)$ and $S_{0,n} = \Sigma_{j>n} Y_{n/u}(j) \leq \Sigma_{j>n} \overline{Y}_{n/u}(j)$. By priority we have

$$\sum_{j>n} \overline{Y}_{n/u}(j) \leq S_{m,n} + \text{card}\{j \in \mathbb{Z}/Y_{m/u}(j) = 1 \text{ and } Y^m_{m/u}(j) = 0\},$$

where by definition the last term is $S_{0,m}$. Therefore we get (a). (b) follows from the definition of $S_{m,n}$ and Lemma 2.1 in [8]. (c) is easily obtained by a simple shift of the space E. (d) is proved as in [7] (Theorem 5.3, Chapter 8) by comparison of the exclusion process with independent particles moving only to the right. The subadditive ergodic Theorem 2.6 in Chapter 6 in [7] shows that $n^{-1}S_{0,n}$ converges almost surely to a *random variable* $G(u)$. □

REMARK. The ergodicity under time and space translation of the exclusion process starting from a measure ν^c, $0 < c < 1$, and Theorem 3.1 in [8] would have given the ergodicity of the sequence $(S_{(n-1)k,nk},\ n \in \mathbb{N}^*)$ and by Theorem 2.6 in Chapter 6 in [7] the nonrandomness of $G(u)$. In the case $a = 1$ and $b = 0$, studied in [9] with $p = 1$ and in [7], Chapter 8, with general p, this sequence is obviously independent and identically distributed.

Unfortunately, in our situation $(0 < c < 1)$, we are unable to prove this ergodicity, but we have the following proposition:

PROPOSITION 2. *For every $t \geq 0$ and real numbers x and y such that $x < y$, $h\sum_{j=[x/h]+1}^{[y/h]} X_{t/h}(j)$ converges almost surely to a nonrandom limit as h goes to 0.*

PROOF. Setting $s = t/h$,

$$h \sum_{j=[x/h]+1}^{[y/h]} X_{t/h}(j) = ts^{-1} \sum_{j=[s\cdot x/t]+1}^{[s\cdot y/t]} X_s(j)$$
$$= ts^{-1} \sum_{j=[s\cdot x/t]+1}^{[s\cdot y/t]} B_s(j) + ts^{-1} \sum_{j=[s\cdot x/t]+1}^{+\infty} Y_s(j)$$
$$- ts^{-1} \sum_{j=[s\cdot y/t]+1}^{+\infty} Y_s(j).$$

By Proposition 1

$$ts^{-1} \sum_{j=[s\cdot x/t]+1}^{+\infty} Y_s(j) - ts^{-1} \sum_{j=[s\cdot y/t]+1}^{+\infty} Y_s(j)$$
$$= t\big(s^{-1}S([s\cdot x/t], s) - s^{-1}S([s\cdot y/t], s)\big)$$

converges almost surely as s goes to $+\infty$ to the random variable $t(G(x/t) - G(y/t))$.

On the other hand, $ts^{-1}\sum_{j=[s\cdot x/t]+1}^{[s\cdot y/t]} B_s(j)$ converges almost surely to $(y-x)b$ as s goes to $+\infty$. In order to see that, for $u > 0$, $(B_{n/u}(j),\ j = 1,\dots,n)$ is i.i.d. with moments of all orders (actually bounded); therefore

$$E\left[n^{-1} \sum_{j=1}^{n} \big(B_{n/u}(j) - b\big)\right]^4 = C/n^2$$

and consequently,

$$\sum_{n=1}^{+\infty} P\left\{\left[n^{-1} \sum_{j=1}^{n} \big(B_{n/u}(j) - b\big)\right]^2 > \varepsilon\right\} < +\infty, \quad \text{for every } \varepsilon > 0.$$

By the Borel–Cantelli lemma $n^{-1}\Sigma_{j=1}^{n}(B_{n/u}(j) - b)$ converges almost surely to 0. We get $\lim_{n \to +\infty} n^{-1}\Sigma_{j=1}^{n} B_{n/u}(j) = b$ a.s. and it is routine to obtain $\lim_{s \to +\infty} s^{-1}\Sigma_{j=1}^{[sy]} B_s(j) = by$ a.s. for $y > 0$, the case $y < 0$ being handled similarly.

Combining these two convergence results we get that $h\Sigma_{j=[x/h]+1}^{[y/h]} X_{t/h}(j)$ converges almost surely, as h goes to 0, to the *random variable* $b(y - x) + t(G(x/t) - G(y/t))$.

In order to prove the nonrandomness of this variable it is enough to prove that for every $y > 0$, setting $u = y/t$, $\lim_{n \to +\infty} n^{-1}\Sigma_{j=1}^{n} X_{n/u}(j)$ is nonrandom, the case $y < 0$ being handled similarly. u being fixed, we denote this limit by I.

Let σ_t be the σ-algebra generated by X_t; we define $\mathscr{F}_t$, $\mathscr{G}_t$, $\mathscr{F}_\infty$ and $\mathscr{G}_\infty$, respectively, by $\bigvee_{s \le t}\sigma_s$, $\bigvee_{s \ge t}\sigma_s$, $\bigvee_{t \ge 0}\mathscr{F}_t$ and $\bigcap_{t \ge 0}\mathscr{G}_t$. By definition I is measurable with respect to $\mathscr{G}_\infty \cap \mathscr{F}_\infty = \mathscr{G}_\infty$. Observe that $E(I|\mathscr{F}_t) = E(I|\sigma_t)$ a.s. by the Markov property. We will prove that for any fixed $t \ge 0$, $E(I|\sigma_t) = E(I)$ a.s.; I being bounded (by 1), the martingale convergence theorem will then give $I = E(I)$ a.s., in other words, the *nonrandomness of I*.

It is enough to prove that for any cylindrical function f on E with $\|f\| = \sup_{e \in E}|f(e)| \le 1$, we have $E(If(X_t)) = E(I)E(f(X_t))$. f being fixed, we have the following property:

(8) for every $\varepsilon > 0$, there exists an integer N^ε such that for every function h in $C(E)$ with $\|h\| \le 1$ and support$(h) \subset \{0,1\}^{[-N^\varepsilon, N^\varepsilon]^c}$, $|E(h(X_t)f(X_t)) - E(h(X_t))E(f(X_t))| < \varepsilon$.

(8) follows from 1.4.6. and part (c) of 1.3.9. in [7] and from the fact that $\nu^{a,b}$ is a product measure.

At time t we define (X_t^1, X_t^2) by $(X_t, (X_t)^{(N^\varepsilon)})$ where $(X_t)^{(N^\varepsilon)}(j) = 0$ if $-N^\varepsilon \le j \le N^\varepsilon$ and $X_t(j)$ elsewhere.

Let $\tilde{X}$ be the coupled process starting at time t from (X_t^1, X_t^2). If $\tilde{\nu}$ denotes the law of (X_t^1, X_t^2) and $\tilde{E}$ the expectation under the law of the coupled process starting from $\tilde{\nu}$, we have

(8′) for every h in $C(E)$ with $\|h\| \le 1$,

$$\left|\int h(X_t^2) f(X_t^1)\,d\tilde{\nu} - \int h(X_t^2)\,d\tilde{\nu} \int f(X_t^1)\,d\tilde{\nu}\right| < \varepsilon,$$

since $h(X_t^2) = h_1(X_t^1)$ with $h_1 \in C(E)$, $\|h_1\| \le 1$ and support$(h_1) \subset \{0,1\}^{[-N^\varepsilon, N^\varepsilon]^c}$, so that by (8)

$$\left|\int h(X_t^2) f(X_t^1)\,d\tilde{\nu} - \int h(X_t^2)\,d\tilde{\nu} \int f(X_t^1)\,d\tilde{\nu}\right|$$
$$= |E(h_1(X_t)f(X_t)) - E(h_1(X_t))E(f(X_t))| < \varepsilon.$$

On the other hand,

$$\left|\sum_{j=1}^{n} X_{n/u}^1(j) - \sum_{j=1}^{n} X_{n/u}^2(j)\right| \le 2N^\varepsilon + 1,$$

for $n \geq ut$ shows that $I(X^1) = I(X^2)$ $P_{\tilde{\nu}}$ a.s. Therefore $E(If(X_t)) = \tilde{E}(I(X^1)f(X_t^1)) = \tilde{E}(I(X^2)f(X_t^1))$. Setting $I_n(e) = n^{-1}\Sigma_{j=1}^n e(j)$, we get

$$\begin{aligned} E(If(X_t)) &= \tilde{E}\Big(\lim_{n\to+\infty} I_n\big(X_{n/u}^2\big)f\big(X_t^1\big)\Big) \\ &= \lim_{n\to+\infty} \tilde{E}\Big(I_n\big(X_{n/u}^2\big)f\big(X_t^1\big)\Big) \\ &= \lim_{n\to+\infty} \int f(e_1)T_{(n/u)-t}I_n(e_2)\,d\tilde{\nu} \\ &= \lim_{n\to+\infty} \tilde{E}\Big(T_{(n/u)-t}I_n\big(X_t^2\big)f\big(X_t^1\big)\Big). \end{aligned}$$

Setting $h_n(e) = T_{(n/u)-t}I_n(e)$ for $n/u \geq t$, we get by (8′)

$$\left|\tilde{E}\big(h_n(X_t^2)f(X_t^1)\big) - \tilde{E}\big(h_n(X_t^2)\big)\tilde{E}\big(f(X_t^1)\big)\right| < \varepsilon, \quad \text{for every } n \geq ut$$

and therefore $|E(If(X_t)) - E(I)E(f(X_t))| < \varepsilon$ for every $\varepsilon > 0$ which implies $E(I|\sigma_t) = E(I)$ a.s. □

REMARK. The same argument shows that in fact I is independent of $\mathscr{F}_t$ for every $t \geq 0$.

COROLLARY 1. *There exists a nonrandom decreasing convex function H such that for all $x < y$ in $\mathbb{R}$ and $t \geq 0$*

$$\lim_{h\to 0} h \sum_{j=[x/h]+1}^{[y/h]} X_{t/h}(j) = t\{H(x/t) - H(y/t)\} \quad a.s. \tag{9}$$

PROOF. By Proposition 2, $h\Sigma_{j=[x/h]+1}^{[y/h]}X_{t/h}(j)$ converges almost surely, as h goes to 0, to the *nonrandom* variable $b(y-x) + t\{G(x/t) - G(y/t)\}$ which is then equated to its expectation, $b(y-x) + t\{E(G(x/t)) - E(G(y/t))\}$.

Setting $H(u) = -bu + E(G(u))$, it remains to prove that H is decreasing and convex; this is satisfied as soon as the function $E(G(\cdot))$ is itself decreasing and convex.

Going back to Proposition 1, if $u \leq v$, by definition $S([ut], t) \geq S([vt], t)$; dividing by t and taking the limit as t goes to $+\infty$ we get $G(u) \geq G(v)$ a.s. which implies $E(G(u)) \geq E(G(v))$.

A simple generalization of (a) gives that for every k and m in $\mathbb{Z}$ and $r > 0$, $s > 0$, $S(k+m, r+s) \leq S(k, r) + \tilde{S}(m, s)$, where $\text{law}(\tilde{S}(m,s)) = \text{law}(S(m,s))$; taking the expectation with $k = [\alpha ut]$, $m = [(1-\alpha)vt]$, $r = \alpha t$ and $s = (1-\alpha)t$ $(0 < \alpha < 1)$ we get

$$\begin{aligned} E\{S([(\alpha u + (1-\alpha)v)t], t)\} &\leq E\{S([\alpha ut], \alpha t)\} \\ &\quad + E\{S([(1-\alpha)vt], (1-\alpha)t)\}. \end{aligned}$$

Dividing by t and taking the limit as t goes to $+\infty$ gives

$$E(G(\alpha u + (1-\alpha)v)) \leq \alpha E(G(u)) + (1-\alpha)E(G(v)). \qquad \square$$

REMARK. We did not prove the nonrandomness of $G(u)$ but Corollary 1 is sufficient for the sequel. In fact $G(u)$ is nonrandom and one way to see that is to observe that $G(u) - G(v)$ is nonrandom and $G(v)$ converges almost surely to 0 as v goes to $+\infty$.

COROLLARY 2. *There exists a function $u(x, t)$, decreasing and right continuous in the x-variable such that $b \le u(x, t) \le a$ and $X_t^h(x)\,dx$ converges weakly a.s. to $u(x, t)\,dx$.*

PROOF. With (4) we have

$$\int_x^y X_t^h(z)\,dz = h \sum_{j=[x/h]+1}^{[y/h]} X_{t/h}(j) + O(h), \qquad \lim_{h\to 0} O(h) = 0 \quad \text{a.s.},$$

which shows

$$\lim_{h\to 0} \int_x^y X_t^h(z)\,dz = t\{H(x/t) - H(y/t)\} \quad \text{a.s.} \tag{10}$$

For $t = 0$ we have $u(x,0) = u_0(x) = (aI_{(-\infty,0)} + bI_{[0,+\infty)})(x)$ and for $t > 0$ we define $u(x, t) = -H_d'(x/t)$ where H_d' is the derivative on the right of H. Then (10) and $-\int_x^y H_d'(z/t)\,dz = t\{H(x/t) - H(y/t)\}$ give the convergence part. The inequality $b \le u(x, t) \le a$ follows from $\nu^b \le \nu^{a,b} \le \nu^a$ and the monotonicity of the process. □

REMARK. In order to identify the density profile u it will be enough to study $u_t^h(x) = E\{X_t^h(x)\} = E\{X_{t/h}([x/h])\}$ as h goes to 0, since by dominated convergence $\lim_{h\to 0}\int_x^y E\{X_t^h(z)\}\,dz = \int_x^y u(z, t)\,dz$ and therefore $u(z, t) = \lim_{h\to 0} E\{X_t^h(z)\}$ for almost every z.

3. Identification of the density profile.

PROPOSITION 3. *u is a weak solution to* (7).

PROOF. From (6) we deduce

$$\begin{aligned} du_t^h &= -pD_{-h}\big[\mathbb{E}\big\{F_h\big(X_{t-}^h\big)\big\}\big]\,dt + (1-p)D_h\big[\mathbb{E}\big\{F_{-h}\big(X_{t-}^h\big)\big\}\big]\,dt, \\ u_0^h &= aI_{(-\infty,h)} + bI_{[h,+\infty)}, \end{aligned} \tag{11}$$

which is understood in the following weak sense: For every smooth function φ: $\mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, with compact support, we have

$$\begin{aligned} &\int_{\mathbb{R}^+}\int_{\mathbb{R}} \left[u_t^h \frac{\partial\varphi}{\partial t} + p\mathbb{E}\big\{F_h\big(X_t^h\big)\big\}D_h\varphi - (1-p)\mathbb{E}\big\{F_{-h}\big(X_t^h\big)\big\}D_{-h}\varphi\right] dx\,dt \\ &\qquad = -\int_{\mathbb{R}} u_0^h(x)\varphi(x,0)\,dx. \end{aligned} \tag{12}$$

The main difficulty is to prove that terms as $\mathbb{E}\{X_t^h(x)X_t^h(x+h)\}$ and $\mathbb{E}\{X_t^h(x)X_t^h(x-h)\}$, appearing in (12), converge to $(u(x, t))^2$.

We will compare our process (X_t) with the same exclusion process (B_t^c) starting from the invariant measure ν^c with $0 \le c \le 1$. Initially we impose $X_0(k) \ge B^c(k)$ for $k \le 0$ and $X_0(k) \le B^c(k)$ for $k > 0$; then we let the processes evolve according to the *coupling procedure* and *priorities* $XB^c \vdash X$ and $XB^c \vdash B^c$ where $(XB^c)(k) = X(k)B^c(k)$ for every $k \in \mathbb{Z}$. With these conditions X and B evolve like exclusion processes starting, respectively, from $\nu^{a,b}$ and ν^c. At time t, $k(c,t) = \max\{k \in \mathbb{Z};\ X_t(k) > B_t^c(k)\}$ defines a random number on $\mathbb{Z} \cup \{-\infty, +\infty\}$; moreover, we have $X_t(k) \ge B_t^c(k)$ for every $k \le k(c,t)$ and $X_t(k) \le B_t^c(k)$ for every $k > k(c,t)$.

LEMMA 1. *For any c outside a countable set C, $hk(c, t/h)$ converges a.s. to a deterministic number $\gamma(c,t)$ in $\mathbb{R} \cup \{-\infty, +\infty\}$.*

PROOF. H being the decreasing convex function obtained in Corollary 1 for $d > 0$, $C_d = \{C_d(n) = td^{-1}[H(nd/t) - H((n+1)d/t)],\ n \in \mathbb{Z}\}$ is a decreasing sequence between b and a; therefore, if $c \notin C_d$, there is one and only one number $n_d(c,t)$ such that $C_d(n) > c$ if $n < n_d(c,t)$ and $C_d(n) < c$ if $n \ge n_d(c,t)$. By Corollary 1, the definition of $k(c,t)$ and the convergence $hd^{-1}\sum_{j=[nd/h]+1}^{[(n+1)d/h]} B_{t/h}^c(j) \to c$ a.s. as h goes to 0 (obtained at the beginning of the proof of Proposition 2), we deduce

$$(13)\qquad \begin{aligned} d(n_d(c,t) - 1) \le \liminf_{h\to 0} hk(c,t/h) &\le \limsup_{h\to 0} hk(c,t/h) \\ &\le d(n_d(c,t)+1) \quad \text{a.s.} \end{aligned}$$

By construction $[d(n_d(c,t) - 1), d(n_d(c,t)+1)]$ is a decreasing family of intervals as $d \searrow 0$ with $d = 2^{-m}$, $m \nearrow +\infty$, converging to one point denoted by $\gamma(c,t)$: the number of intervals where the function H is linear being countable, Lemma 1 holds for every c outside a countable set C (the set of the slopes of linear pieces of H). □

In order to finish the proof of Proposition 3 we observe that for almost every x such that $\gamma(c+d, t) < x < \gamma(c,t)$, we have

$$(14)\qquad \begin{aligned} c &\le \lim_{h\to 0} E\{X_t^h(x)\} = u(x,t) \le c + d, \\ c^2 &\le \liminf_{h\to 0}, \sup E\{X_t^h(x)\cdot X_t^h(x+h)\} \le (c+d)^2, \\ c^2 &\le \liminf_{h\to 0}, \sup E\{X_t^h(x)\cdot X_t^h(x-h)\} \le (c+d)^2. \end{aligned}$$

Therefore, for almost every $x \in \mathbb{R}$ we have

$$(15)\qquad \limsup_h \left| E\{X_t^h(x)\cdot X_t^h(x \mp h)\} - \left(u_t^h(x)\right)^2 \right| = 0.$$

From (12), (15) and the limit as h goes to 0 we get

$$(16)\quad \int_{\mathbb{R}^+}\int_{\mathbb{R}} \left\{ u\frac{\partial\varphi}{\partial t} + (2p-1)u(1-u)\frac{\partial\varphi}{\partial x} \right\} dx\,dt + \int_{\mathbb{R}} u_0(x)\varphi(x,0)\,dx = 0,$$

which implies that u is a weak solution to (7). □

THEOREM 1. *u is the weak solution to* (7) *satisfying the entropy condition.*

We refer to the Appendix for this notion and a complete description of u.

PROOF. $p < \frac{1}{2}$: Since u is decreasing in x, the entropy condition ($u^+ \le u^-$) is satisfied. Therefore, from the Appendix, we get

$$u(x,t) = \begin{cases} a, & \text{if } x < (2p-1)(1-a-b)t, \\ b, & \text{if } x > (2p-1)(1-a-b)t, \quad \text{a.s.} \end{cases}$$

This a.s. equality, the monotonicity (in x) of $u(x,t)$ and the fact that $u(x,t) = u(x/t, 1)$ imply that the equality holds for all (x,t) such that $x \neq (2p-1)(1-a-b)t$.

$p > \frac{1}{2}$: Let $\bar{u}$ be the solution to (7) satisfying the entropy condition starting from u_0 (see the Appendix). The monotonicity with respect to the initial conditions for these solutions (see [3]) shows that $a \ge \bar{u} \ge b$ and therefore $(2p-1)(1-2a) \le F'(\bar{u}) \le (2p-1)(1-2b)$; then the method of characteristics gives

$$x > (2p-1)(1-2b)t \Rightarrow x > F'(\bar{u})t \Rightarrow \bar{u}(x,t) = u_0(x - F'(\bar{u})t) = b$$

and

$$x < (2p-1)(1-2a)t \Rightarrow x < F'(\bar{u})t \Rightarrow \bar{u}(x,t) = u_0(x - F'(\bar{u})t) = a$$

(see the Appendix for a complete description of this solution). We know that our weak solution u satisfies $a \ge u \ge b$. The inequality $u - b \ge 0$ and the method used to obtain the a priori estimate for (7) (see [3] or [4]) show that $u = b$ a.e. in the region $\{(x,t)/x > (2p-1)(1-2b)t\}$. The inequality $a - u \ge 0$ and the same techniques show that $u = a$ a.e. in the region $\{(x,t)/x < (2p-1)(1-2a)t\}$. The argument given in the case $p < \frac{1}{2}$ shows that these equalities hold everywhere in these regions.

If c is a real number such that $b \le c \le a$ and $u^{a,c}$ (resp. $u^{c,b}$) denotes the solution starting from $aI_{(-\infty,0]} + cI_{(0,+\infty)}$ (resp. $cI_{(-\infty,0]} + bI_{(0,+\infty)}$) obtained as a limit of a particle system as $u = u^{a,b}$, the monotonicity of the process implies $u^{c,b} \le u \le u^{a,c}$. Applying the previous result to $u^{a,c}$ and $u^{c,b}$ we get

$$x > (2p-1)(1-2c)t \Rightarrow u^{a,c}(x,t) = c$$

and

$$x < (2p-1)(1-2c)t \Rightarrow u^{c,b}(x,t) = c.$$

Then if u has a jump at (x_0, t), one has $u(x_0+, t) < u(x_0-, t)$ and $x_0 = (2p-1)(1-2c)t$ for every c such that $u(x_0+, t) < c < u(x_0-, t)$ which is impossible. Then $u(\cdot, t)$ is continuous and u is the unique solution to (7) satisfying the entropy condition. □

From the Appendix, we get

$$u(x,t) = \begin{cases} a, & \text{if } x \le (2p-1)(1-2a)t, \\ \frac{1}{2}\left(1 - \frac{x}{(2p-1)t}\right), & \text{if } (2p-1)(1-2a)t \le x \\ & \quad \le (2p-1)(1-2b)t, \\ b, & \text{if } x \ge (2p-1)(1-2b)t. \end{cases}$$

4. Local equilibrium. We deduce from Theorem 1 the limiting behavior of the particle process seen by a travelling observer (i.e., the weak limit of the distribution of $\{X_t([xt]+k),\ k \in \mathbb{Z}\}$ as t goes to $+\infty$ for fixed $x \in \mathbb{R}$).

THEOREM 2. *For all points of continuity of* $u(x,1)$ (*for all values of* $p \neq \frac{1}{2}$) *we have*

$$\operatorname{w.}\lim_{t \to +\infty} \operatorname{law}\{X_t([xt]+k),\ k \in \mathbb{Z}\} = \nu^{u(x,1)}.$$

REMARK. Except for one case ($p < \frac{1}{2}$, $x = (2p-1)(1-a-b)$), the limiting distribution is a product measure invariant under the action of the semigroup (T_t). We say that *propagation of chaos* holds and that the system is in *local equilibrium.*

PROOF. It is easily shown that for any finite set F of sites in $\mathbb{Z}$ we have

$$\lim_{t \to +\infty} \mathbb{E}\left\{\prod_{k \in F} X_t([xt]+k)\right\} = (u(x,1))^{\operatorname{card}(F)}. \tag{17}$$

Substitute 1 for t and let $h = 1/t$ in the proof of Theorem 1. We get $\gamma(u(x,1) + d_n, 1) < x < \gamma(u(x,1) - d_n, 1)$ for every x such that $u(\cdot,1)$ is continuous at x and for a sequence (d_n) decreasing to 0. Since for every $k \in F$

$$[(x + h(\min F - 1))/h] \le [x/h] + k \le [(x + h(\max F))/h]$$

and for h small enough $\gamma(u(x,1) + d_n, 1) < x + h(\min F - 1) \le x + h \max F < \gamma(u(x,1) - d_n, 1)$, we obtain

$$\begin{aligned}(u(x,1) - d_n)^{\operatorname{card}(F)} &\le \lim_{h \to 0} (\inf, \sup) \mathbb{E}\left\{\prod_{k \in F} X_{h^{-1}}([x/h]+k)\right\} \\ &\le (u(x,1) + d_n)^{\operatorname{card}(F)},\end{aligned}$$

which shows (17) as d_n goes to 0. □

REMARK. For $x = 0$, Theorem 2 is contained in [5] and [6]. Denote by η_∞ the weak limit of the distribution of X_t as t goes to $+\infty$; then $\eta_\infty = \nu^{u(0,1)}$

whenever $u(\cdot,1)$ is continuous at 0.

$$\begin{array}{lll} p>\frac{1}{2}: & 0\le b\le a\le \frac{1}{2}: & \eta_\infty = \nu^a, \\ & 0\le b\le \frac{1}{2}\le a\le 1: & \eta_\infty = \nu^{1/2}, \\ & \frac{1}{2}\le b\le a\le 1: & \eta_\infty = \nu^b, \\ p<\frac{1}{2}: & a+b<1: & \eta_\infty = \nu^b, \\ & a+b>1: & \eta_\infty = \nu^a, \end{array}$$

$a + b = 1$ is the only case where $u(\cdot,1)$ is not continuous at 0 and the identification of η_∞ remains an open problem (see Conjecture 1.6. in [5]: $\eta_\infty = \frac{1}{2}\nu^a + \frac{1}{2}\nu^b$).

Local equilibrium has been studied for the zero-range process by Andjel and Kipnis [1].

One may think of possible generalizations of the method presented in this article.

(1) *On the process itself*: This method seems to be suitable for monotone systems having a continuous family of equilibrium measures; in particular it should apply to the asymmetric zero-range processes, generalizing [1].

(2) *On the nearest-neighbor assumption*: In order to remove this assumption we need a proof of (14) which does not use the interface $k(c, t)$.

(3) *On the dimension*: In higher dimension, for suitable initial distributions, the difficult part of the problem is the law of large numbers; heuristically the limiting equation is of the same nature as (7) and the notion of entropy condition is still available (cf. [3]).

(4) *On the initial distribution*: It is not difficult to replace $\nu^{a,b}$ by a product measure ν_0 such that $\lim_{k\to-\infty}\int e(k)\,d\nu_0 = a$ and $\lim_{k\to+\infty}\int e(k)\,d\nu_0 = b$; it is also possible to obtain more general decreasing initial conditions u_0 by taking ν_0 dependent on the rescaling parameter. The real difficulty is to obtain initial conditions which are not monotone (for instance $u_0 = I_{[-1,0]}$).

(5) *On the dynamics*: Let p depend on the rescaling parameter in the following way: $p = \frac{1}{2} + h$. Heuristically the limiting equation will be the nonlinear parabolic P.D.E.:

$$\frac{\partial u}{\partial t} + 2\frac{\partial[u(1-u)]}{\partial x} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}.$$

Again the law of large numbers is the main difficulty in this asymptotically symmetric system.

We are presently working on these generalizations, (1) and (2) being the subject of a coming paper.

APPENDIX

Equation (7),

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0, \qquad u(\cdot,0) = u_0,$$

with $F(u) = (2p-1)u(1-u)$, $0\le p\le 1$, $p\ne\frac{1}{2}$ and $u_0 = aI_{(-\infty,0]} + bI_{(0,+\infty)}$ is a first-order quasilinear hyperbolic P.D.E. well studied in [3] and [4]. The existence of a weak solution is proved by a viscosity method which consists in the addition of a term $\varepsilon\Delta u$ where ε is going to 0. In general, we do not have uniqueness of weak solution for this kind of equation. However, Kruckov [3] introduced the notion of *entropy*; we give an equivalent definition [4], which describes the permissible jumps.

DEFINITION A.1. *A weak solution to* (7) *such that*

$$u^+(x,t) = \lim_{y \downarrow x} u(y,t) \quad \text{and} \quad u^-(x,t) = \lim_{y \uparrow x} u(y,t)$$

exist, satisfies the entropy condition if

$$\frac{F(u^+) - F(u^-)}{u^+ - u^-} = \max_{c \in [u^+ \wedge u^-,\, u^+ \vee u^-]} \frac{F(u^+) - F(c)}{u^+ - c}.$$

REMARK. In our context: If $p > \frac{1}{2}$, F is concave and A.1 $\Leftrightarrow u^+ \geq u^-$; if $p < \frac{1}{2}$, F is convex and A.1 $\Leftrightarrow u^+ \leq u^-$. The main result about entropy can be summarized as follows:

THEOREM A.1. *There exists, in the almost everywhere sense, a unique weak solution to* (7) *satisfying the entropy condition.*

The existence is given by the viscosity method and uniqueness is obtained via an a priori estimate ([3] or [4]).

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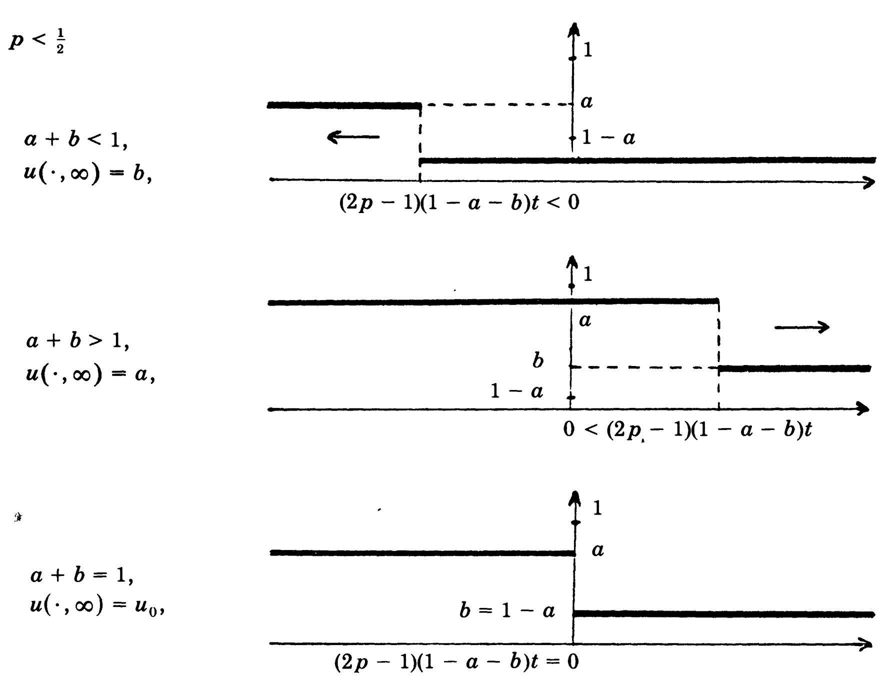
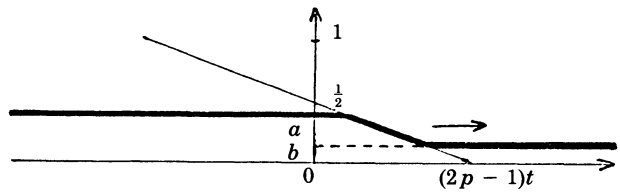
The Profile $u(\cdot, t)$

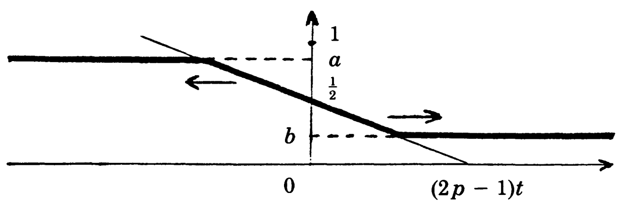


$p > \frac{1}{2}$, $u(\cdot, t)$ is continuous and the nonconstant piece $= \dfrac{1}{2}\left(1 - \dfrac{x}{(2p-1)t}\right)$.

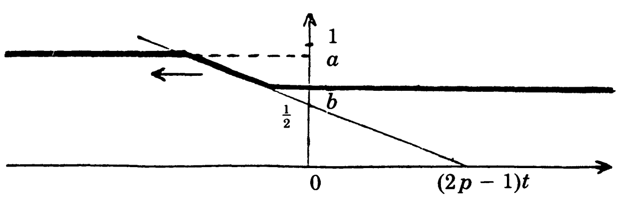
$0 \le b \le a \le \frac{1}{2}$,
$u(\cdot, \infty) = a$,



$0 \le b \le \frac{1}{2} \le a \le 1$,
$u(\cdot, \infty) = \frac{1}{2}$,



$\frac{1}{2} \le b \le a \le 1$,
$u(\cdot, \infty) = b$,



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