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GRADIENT DYNAMICS OF INFINITE POINT SYSTEMS

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Nonequilibrium gradient dynamics of d-dimensional particle systems is investigated. The interaction is given by a superstable pair potential of finite range. Solutions are constructed in the well-defined set of locally finite configurations with a logarithmic order of energy fluctuations. If the system is deterministic and $d \leq 2$, then singular potentials are also allowed. For stochastic models with a smooth interaction we need $d \leq 4$. In order to develop some prerequisites for the theory of hydrodynamical fluctuations in equilibrium, we investigate smoothness of the Markov semigroup and describe some properties of its generator.

0. Introduction. The purpose of this paper is to study existence and regularity properties of solutions to the following infinite system of stochastic differential equations. Consider a countable set S of d-dimensional particles suspended in a fluid, where the interaction is given by a pair potential U: $\mathbb{R}^d \to (-\infty, +\infty]$. In a quasi-microscopic description of such systems the effect of collisions with the particles of the fluid can be represented by uncorrelated stochastic forces, and the soft resistance of the liquid medium reduces the order of the equations of motion from two to one, see [13], [23], and [24] for some further references. Configurations of these systems are countable subsets of $\mathbb{R}^d$ such that any bounded domain contains a finite number of points only. Particles of a configuration will be identified by labelling points by elements of S. Thus a labelled configuration is of the form $\omega = (\omega_k)_{k \in S}$ with $\omega_k \in \mathbb{R}^d$, i.e., $\omega \in (\mathbb{R}^d)^S$. Of course, not every element of this product space is locally finite. Suppose now that we are given a family $[w_k\colon k \in S]$ of independent standard Wiener processes in $\mathbb{R}^d$. Then the evolution law is given by

$$d\omega_k = -\sum_{j \neq k} \operatorname{grad} U(\omega_k - \omega_j)\, dt + \sigma\, dw_k, \qquad k \in S, \tag{0.1}$$

where $\sigma \geq 0$ is a constant. We assume that the interaction has a finite radius $R > 0$, i.e., $U(x) = 0$ if $|x| > R$. Then the right-hand side of (0.1) makes sense in the space of locally finite configurations. Nevertheless, the configuration space should be restricted in a much more radical way.

The study of system (0.1) has been initiated by Lang [13]. Since any Gibbs state μ with interaction U and temperature $\sigma^2/2$ is formally stationary and reversible, one could construct the associated diffusion process with initial distribution μ in a certain configuration space Ω_0. This Ω_0 has not been specified in an explicit form, but $\mu(\Omega_0) = 1$ holds true. The existence of nonequilibrium dynamics was known in the one-dimensional case only, see Lippner [15] and

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Rost [20] for smooth interactions and Lang [14] if $\sigma = 0$ but U is convex for $x > 0$. My work is motivated mainly by a recent paper of Spohn [24], where hydrodynamical fluctuations of the particle number in the equilibrium dynamics are described. The proof of this Gaussian central limit theorem is based on a fairly sophisticated essential self-adjointness property of the formal generator $\mathbb{L}$ associated to (0.1); similar technicalities appear in Guo and Papanicolau [10]. Although both the central limit problem as well as its hypothesis are formulated in the framework of equilibrium dynamics, for a proof of the self-adjointness of $\mathbb{L}$ a very good understanding of the nonequilibrium dynamics is needed; see Marchioro, Pellegrinotti and Pulvirenti [16] and Fritz [7] for a related problem of classical dynamics.

All problems will be posed in a relatively general form, see Fritz [6] for the case of lattice systems. For smooth interactions we prove existence and uniqueness of nonequilibrium solutions if $d \leq 4$; for a singular U we need $d \leq 2$ and $\sigma = 0$. Solutions are constructed as the a.s. limit of solutions to finite subsystems, the semigroup $\mathbb{P}^t$ of transition probabilities is defined in a σ-compact space Ω of allowed configurations. Let us remark that $\mathbb{P}^t$ is neither strongly continuous nor Feller continuous in the usual sense, but we introduce some spaces $\mathbb{C}_1 \subset \mathbb{C}$, $\mathbb{D}^3_{e1} \subset \mathbb{D}^2_e$ of smooth quasi-local functions such that $\mathbb{P}^t\mathbb{C}_1 \subset \mathbb{C}$ and $\mathbb{P}^t\mathbb{D}^3_{e1} \subset \mathbb{D}^2_e$ for each $t > 0$. Elements of $\mathbb{C}$ are continuous in a restricted sense and $\mathbb{L}$ is well defined on the set $\mathbb{D}^2_e$; thus we have some ingredients of semigroup theory. The technical tools developed here seem to be sufficient for the theory of equilibrium fluctuations (see the remarks at the end of Section 6), but we are not going to go into a complete discussion.

1. Generalized stochastic gradient systems. This section summarizes some basic notation and the main results. The problem will be formulated for the more general system (1.6); conditions on the coefficients c_k and σ_k of (1.6) will be given in terms of the pair potential U: $\mathbb{R}^d \to \mathbb{R}$ of (0.1). The phrase that U is singular means that $U(0) = +\infty$ and $\lim U(x) = +\infty$ as $x \to 0$, but U is twice continuously differentiable at $x \neq 0$. If U is not singular, then it is assumed to have two continuous derivatives at each $x \in \mathbb{R}^d$; the symmetry property $U(x) = U(-x)$, and $U(x) = 0$ if $|x| > R$ are assumed in both cases. The singularity of U, if any, cannot be too strong; we need

$$(1.1) \qquad |x||\operatorname{grad} U(x)| \leq a + bU(x),$$

with some positive a and b, i.e., $|x|^b U(x)$ is bounded even if U is singular; $|\cdot|$ denotes the usual norm in $\mathbb{R}^d$. The next basic property of the interaction is the so-called superstability; this condition is used to prove the existence of Gibbs random fields for U, see Ruelle [21] and [22]. In the present context the superstability of U means that we have some constants $A \geq 0$ and $B > 0$ such that

$$(1.2) \qquad nA + \sum_{k=1}^{n} \sum_{j \neq k} U(q_k - q_j) \geq BN,$$

for any finite sequence, $q_1, q_2, \ldots, q_n$ of not necessarily distinct points of $\mathbb{R}^d$;

here N denotes the number of pairs (j, k) such that $|q_k - q_j| \leq R$. These properties of the interaction U will be assumed throughout the paper.

Now we are in a position to describe the set Ω of allowed configurations. Consider first the quantity

$$(1.3) \qquad H(\omega, m, \rho) = \sum_{k:\, |\omega_k - m| \leq \rho} \left[1 + A + \sum_{j \neq k,\, |\omega_j - m| \leq \rho} U(\omega_k - \omega_j) \right];$$

it is a weighted sum of potential energy and particle number in the ball of center m and radius ρ. In view of (1.2) this H is nonnegative and dominates the number of points and the number of interacting pairs as well. Let $g(u) = [1 + \log(1 + u)]^{1/d}$ and

$$(1.4) \qquad \overline{H}(\omega) = \sup_{m \in \mathbb{Z}^d} \sup_{r \in \mathbb{N}} \left[(rg(|m|))^{-d} H(\omega, m, rg(|m|)) + 1 \right],$$

where $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{Z}^d$ is the integer lattice in $\mathbb{R}^d$; the set of allowed configurations is defined as $\Omega = [\omega \in (\mathbb{R}^d)^S\colon \overline{H}(\omega) < +\infty]$. Since (1.2) implies $U(0) > 0$, the elements of Ω are locally finite in the sense that each point can occur in a sequence $\omega \in \Omega$ with a finite multiplicity only; multiple points are excluded if U is singular. Notice that if U is bounded, then Ω does not depend on U anymore.

The topology we are introducing in Ω is the following mixture of the product topology and the weak topology of integer valued measures. Let $B(\rho)$ denote the open ball in $\mathbb{R}^d$ with center 0 and radius ρ, and define $G(\varepsilon, \rho)$ as the set of pairs $(\omega, \overline{\omega})$, $\omega, \overline{\omega} \in \Omega$ such that $|\omega_k - \overline{\omega}_k| < \varepsilon$ whenever at least one of ω_k and $\overline{\omega}_k$ lies in $B(\rho)$. This family of relations forms a base of a uniform structure, a neighborhood base at $\overline{\omega} \in \Omega$ of the associated topology consists of the sets $[\omega \in \Omega\colon (\omega, \overline{\omega}) \in G(\varepsilon, \rho)]$, $\varepsilon, \rho > 0$; we equip Ω with this separable and metrizable topology and the associated Borel structure. Notice that $\omega^{(n)} \to \omega$ in this topology means that $\omega_k^{(n)} \to \omega_k$ for each $k \in S$ and $\Sigma\varphi(\omega_k^{(n)}) \to \Sigma\varphi(\omega_k)$ whenever φ is continuous with compact support. If $\omega \in \Omega$, then $\operatorname{supp} \omega$ denotes the set of points of ω with multiplicity and $\Omega^{(s)} = [\operatorname{supp} \omega\colon \omega \in \Omega]$ is the symmetrized configuration space. The elements of this symmetrized space are identified with integer valued measures, $N_\omega(dx) = (\operatorname{supp} \omega)(dx)$ by the formula $\Sigma\varphi(\omega_k) = \int \varphi(x)(\operatorname{supp} \omega)(dx)$. We give $\Omega^{(s)}$ the weak topology of measures, then $\omega \to \operatorname{supp} \omega$ is a continuous mapping of Ω onto $\Omega^{(s)}$.

The space Ω of labelled configurations is very convenient when we have to identify the trajectories of the particles, e.g., in the construction of strong solutions. On the other hand, the most natural probability measures, the Gibbs states, are defined on the symmetrized space $\Omega^{(s)}$ only. Since the evolution law (0.1) does not depend on the enumeration of particles, there is no conflict as far as we are dealing with symmetric functions only; a function φ on Ω is symmetric if $\varphi(\omega) = \varphi(\overline{\omega})$ whenever $\operatorname{supp} \omega = \operatorname{supp} \overline{\omega}$. Such notational questions play a minor role in Section 6. It is more relevant that Ω and $\Omega^{(s)}$ are large enough to carry a wide class of probability measures. Indeed, if μ is a probability measure on $(\mathbb{R}^d)^S$ and for some $\lambda > 0$ and $c > 0$

$$\int \exp[\lambda H(\omega, m, \rho)] \mu(d\omega) \leq \exp(c\rho^d), \quad \text{if } m \in \mathbb{R}^d, \quad \rho > 1, \tag{1.5}$$

then the Borel–Cantelli lemma implies $\mu(\Omega) = 1$. The superstable estimates of Ruelle [22] imply (1.5) for a wide class of Gibbs states with different potentials including the equilibrium potential U, see [4], [6], [8], [11], [12] and [16].

The strategy of the proof of existence of solutions is the usual one. We consider finite subsystems of (0.1) and prove compactness of this family by means of an a priori bound for $\overline{H}$; further a priori bounds concern the first and second variational systems of (0.1). For the delicate estimates of Section 6 we need a family of partial dynamics such that the equilibrium state of (0.1) turns out to be a reversible measure, and moreover, the differentiability of solutions with respect to initial data remains in force. These requirements can be fulfilled only if we let the diffusion coefficient depend on the configuration. A fairly general form of evolution laws of this kind reads as

$$d\omega_k = c_k(\omega)\, dt + \sigma_k(\omega)\, dw_k, \qquad k \in S, \tag{1.6}$$

where $c_k \colon \Omega \to \mathbb{R}^d$ and $\sigma_k \colon \Omega \to \mathbb{R}$ are some measurable functions. The rest of the paper treats (1.6) rather than (0.1). Of course, this system cannot be investigated in a full generality, we need an additional structure mimicking that of (0.1). We are not going to consider the most general structure, it will be discussed elsewhere.

The first condition we need is the so-called locality of the interaction. This is relevant for all infinite models from the statistical physics category. For simplicity, we assume that the interaction has a finite radius $R > 0$, i.e., c_k and σ_k depend on ω_j only if $|\omega_j - \omega_k| \leq R$.

The most serious problem is the second one, namely, the problem of explosions. Explosions may appear if the drift is not bounded, they are especially characteristic for point systems in which a critical accumulation of infinitely many particles in a bounded domain may take place in a finite time. To avoid this phenomenon, in the one-dimensional case it is sufficient to control the magnitude of the drift and the diffusion coefficient, see [11], [15] and [20]. If $d \geq 2$, then the drift must be oriented in such a way that the particles have a global tendency to move from densely occupied domains towards desert areas. Since the potential energy dominates the number of particles in view of the superstability condition (1.2), it is natural to assume that the drift is not radically different from the negative gradient of potential energy, cf. (0.1). Let $S_k(\omega)$ denote the set of such $j \in S$ that $|\omega_j - \omega_k| \leq R$ and put $N_k(\omega) = \operatorname{card} S_k(\omega)$ and

$$H_k(\omega) = \sum_{j \neq k} U(\omega_k - \omega_j). \tag{1.7}$$

We say that (1.6) is a generalized stochastic gradient system (see [6]) if we have constants $C > 0$, $\delta > 0$ and measurable functions $\delta_k \colon \Omega \to \mathbb{R}$ such that $|\sigma_k(\omega)| \leq$

$\delta_k(\omega) \le C$, and

$$\langle \nabla_k H_k(\omega), c_k(\omega) \rangle \le -\delta\delta_k^2(\omega) |\nabla_k H_k(\omega)|^2 + CN_k(\omega), \tag{1.8}$$

$$|c_k(\omega)| \le C\delta_k(\omega) |\nabla_k H_k(\omega)| + CN_k^{1/2}(\omega), \tag{1.9}$$

where ∇_k is the vector of partial differential operators corresponding to coordinates of $\omega_k \in \mathbb{R}^d$, and $\langle\ ,\ \rangle$ is the usual scalar product in $\mathbb{R}^d$.

The third assumption is a kind of local Lipschitz condition, only its quantitative nature, the factor h_k^C below, might be a little bit surprising. Let $h_k = H(\omega, \omega_k, R) + H(\overline{\omega}, \overline{\omega}_k, R)$ and suppose that for two arbitrary configurations we have

$$\begin{aligned} &|c_k(\omega) - c_k(\overline{\omega})|^2 + |\sigma_k(\omega) - \sigma_k(\overline{\omega})|^2 \\ &\qquad \le Ch_k^C \sum_{j \in S_k(\omega) \cup S_k(\overline{\omega})} |\omega_j - \overline{\omega}_j|^2. \end{aligned} \tag{1.10}$$

The validity of conditions (1.8), (1.9) and (1.10) will be assumed throughout this paper; they can easily be verified in the case of system (0.1).

Finally, we have to restrict the configuration space as well as the concept of solution. Indeed, if the energy of particles of the initial configuration increases too rapidly with the distance from the origin, then the system as a whole can collapse in a finite time. This kind of explosion is excluded by the logarithmic growth condition of (1.4). In fact, our space Ω of allowed configurations is essentially the smallest one such that $\mu(\Omega) = 1$ for a reasonable class of probability measures. The additional condition of temperedness is needed to ensure uniqueness of solutions, see [6] and [12] for counterexamples. To define the concept of solution let $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$ denote the space of continuous mappings of $[0, +\infty)$ into $\mathbb{R}^d$ with the usual topology corresponding to uniform convergence on compacts, and let $\mathbb{W} = [\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)]^S$ with the product topology and the associated Borel field $\mathscr{A}$. The smallest σ-algebra on which all projections $w_k(s)$, $k \in S$, $s \le t$, of $w \in \mathbb{W}$ are measurable will be denoted by $\mathscr{A}_t$. Finally, suppose that we are given a probability measure P on $\mathscr{A}$ such that our Wiener processes are realized as components of the random element $w = (w_k)_{k \in S}$ of $\mathbb{W}$.

Definition 1. An $\mathscr{A}_t$-adapted mapping $\omega(t) = \omega(t, w)$ of $\mathbb{W}$ into itself is called a tempered solution to (1.6) with initial configuration $z \in \Omega$ if $\omega(0) = z$, almost each trajectory $\omega(\cdot, w)$ satisfies the integral form of (1.6), and $\overline{H}(\omega(t))$ is bounded on finite intervals of time with probability one.

In the deterministic case, i.e., when $\sigma_k = 0$, solutions do not depend on the random element $w \in \mathbb{W}$, and we have the following existence theorem, cf. Lang [14].

Theorem 1. *If $d \le 2$ and $\sigma_k = 0$ for all $k \in S$, then for each initial configuration $z \in \Omega$ there exists a unique tempered solution to* (1.6).

The proof of this result is based on an a priori bound for $\overline{H}(\omega(t))$, to estimate the gain of $\overline{H}$ the spatial flow of potential energy should be controlled. Heuristic ideas behind this proof are roughly as follows. Imagine a globally homogeneous system, and let H, N, V denote the energy, the number of particles, and the typical magnitude of the drift in a given volume Λ. Then the energy of particles in the shell of unit width at the boundary of Λ is proportional to $H^{1-1/d}$, thus the energy flow through the surface of Λ may be an order of $VH^{1-1/d}$. On the other hand, the dissipation of the energy of one particle is at least an order of V^2, thus $dH \leq O(VH^{1-1/d} - NV^2)\,dt \leq O(N^{-1}H^{2-2/d})\,dt$, whence by $N \geq 1$ we obtain $dH \leq O(H)\,dt$ if $d \leq 2$. Since a differential inequality $dH \leq O(H^{\lambda})\,dt$ has a global maximal solution if $\lambda \leq 1$, we can hope for an a priori bound if $d \leq 2$. If U is smooth then $N \geq O(H^{1/2})$, thus $dH \leq O(H^{3/2-2/d})\,dt$, whence $d \leq 4$ follows in the same way as above. The mathematical manifestation of these arguments will be presented in the next two sections; the second one also extends to stochastic systems. Singular potentials are very hard to tackle in the stochastic case, because then ΔU, the Laplacian of U, appears in the stochastic differential of H. Therefore, independently of the dimension, we can hope for a differential inequality $dH \leq O(H)\,dt$ only if $\Delta U \leq a + bU$ holds. If $d = 1$, then this inequality cannot be satisfied by a singular potential, but potentials with a logarithmic singularity are allowed if $d = 2$. We are not going to discuss this very particular case here, because the proof of uniqueness of solutions is rather difficult. Let us remark that (0.1) with $U(x) = -\log|x|$ plays a role in the vortex theory of the Navier–Stokes equation, see Marchioro and Pulvirenti [17]. However, this interaction is so strong at large distances that we do not have tools at all to handle the related infinite system. Another open problem is that of the existence of solutions to one-dimensional stochastic gradient systems with a singular potential. If U is smooth, then we have:

THEOREM 2. *If $d \leq 4$ and U is not singular, then for each initial configuration $z \in \Omega$ there exists a unique tempered solution, $\omega(t) = \omega(t, z, w)$ to (1.6). The general solution $\omega(t, z, w)$ is a jointly measurable function of its variables.*

The fundamental a priori bound has the following structure. Let $\Omega_h = [\omega \in \Omega: \overline{H}(\omega) \leq h]$, then for each tempered solution there exists a $\mathscr{A}$-measurable random variable N, and an explicitly given continuous function $\bar{h} = \bar{h}(h, T, N)$ such that $z \in \Omega_h$ implies $\omega(t, z, w) \in \Omega_{\bar{h}}$ for all $t \leq T$ with probability one. This N has an exponential tail, and the distribution of the tail can be bounded in terms of the universal constants d, δ and C of conditions (1.7)–(1.10). When we speak about universal constants or functions we always mean explicit expressions of d, δ, C; the function $\bar{h}$ is universal in this sense, too.

The fundamental a priori bound will be proven in the next two sections; first for finite systems satisfying (1.7)–(1.10), then we show convergence of solutions by means of an iteration technique which goes back to Lanford [11] and [12] and Fritz [6]. There are essentially two different methods to define the partial dynamics. We can pick a finite $I \subset S$ and put $c_k = \sigma_k = 0$ if $k \notin I$. Then particles outside I are frozen, while particles from I move according to (1.6) in the field of other particles. Another possibility is to let c_k and σ_k depend on ω_k in such a way that both turn into zero as ω_k is crossing the boundary of a bounded domain $D \subset \mathbb{R}^d$. For instance, we can multiply c_k and σ_k by smooth functions of ω_k vanishing outside D. In such cases particles cannot escape from D, outer particles are frozen; thus we can realize a continuous transition from vivid particles to the frozen ones. This trick allows us to save both reversibility and smoothness of the partial dynamics, cf. [13] and [16], where violation of this nice property causes some technical difficulties. The a priori bounds, of course, do not depend on the size of the finite subsystem we consider.

Now we turn to the problem of smooth dependence of solutions on the initial configuration. Since the modulus of Lipschitz continuity of the right-hand side of (1.6) does depend on the actual energy level $\overline{H}$, and the interior of each Ω_h is empty, we cannot hope for a continuous dependence on initial data. Because of the enormous complexity of the prototype (0.1) we are not able to present a counterexample, but continuity of $\omega(t, z)$ in $z \in \Omega$ could allow one to extend the general solution from Ω to $(\mathbb{R}^d)^S$, which seems absurd. To formulate the modified Feller property of the semigroup, let $\|\varphi\|_h = \sup[|\varphi(\omega)|: \omega \in \Omega_h]$ for measurable φ: $\Omega \to \mathbb{R}$, and define $\mathbb{P}^t\varphi = \mathbb{P}^t\varphi(z) = \mathsf{E}[\varphi(\omega(t, z))] = \int\varphi(\omega(t, z, w))\mathsf{P}(dw)$ whenever the expectation makes sense.

Definition 2. Let $\mathbb{C}(\Omega)$ denote the space of φ: $\Omega \to \mathbb{R}$ such that the restriction of φ to each nonempty Ω_h is uniformly continuous and bounded, $\mathbb{C}_b(\Omega)$ is the set of bounded elements of $\mathbb{C}(\Omega)$. The set of $\varphi \in \mathbb{C}(\Omega)$ such that $\|\varphi\|_h$ is bounded by a polynomial of h is denoted by $\mathbb{C}_p(\Omega)$, while elements of the space $\mathbb{C}_1(\Omega)$ are characterized by a subexponential growth condition $\|\varphi\|_h \le \exp(c_\varphi h^p)$ with $p < 1$. Each of these spaces is equipped with the scale $\|\cdot\|_h$, $h > 1$, of seminorms, convergence means convergence with respect to each of these norms.

Now we turn to continuity properties of $\mathbb{P}^t$; see [6].

Theorem 3. *Under conditions of Theorem* 2 *the operator* $\mathbb{P}^t$, $t > 0$, *is well defined and strongly continuous on* $\mathbb{C}_1(\Omega)$. *In fact, we have* $\mathbb{P}^t\mathbb{C}_1(\Omega) \subset \mathbb{C}(\Omega)$, $\mathbb{P}^t\mathbb{C}_b(\Omega) \subset \mathbb{C}_b(\Omega)$ *and* $\lim_{t\to 0}\|\mathbb{P}^t\varphi - \varphi\|_h = 0$ *for each* $h > 1$ *and* $\varphi \in \mathbb{C}_1(\Omega)$. *If* $d < 4$, *then* $\mathbb{P}^t\mathbb{C}_p(\Omega) \subset \mathbb{C}_p(\Omega)$.

The next question is that of differentiable dependence on initial data. Its solution is needed to clarify the relation of the semigroup $\mathbb{P}^t$ to its formal generator, $\mathbb{L}$, defined as

$$\mathbb{L}\varphi = \sum_{k\in S}\left(\langle c_k, \nabla_k\varphi\rangle + \tfrac{1}{2}\sigma_k^2\Delta_k\varphi\right), \tag{1.11}$$

on the space $\mathbb{D}_e^2(\Omega)$ specified below, where Δ_k denotes the Laplace operator for the coordinates of ω_k. Here and in what follows, $\nabla_j\nabla_k\varphi$ and $\nabla_i\nabla_j\nabla_k\varphi$ denote the $d \times d$ matrix and the $d \times d \times d$ hypermatrix of the second and third partial derivatives of φ with respect to the coordinates of ω_k, ω_j and ω_i for any $k, j \in S$

and $k, j, i \in S$. Even if A is a matrix or hypermatrix, $|A|^2$ denotes the sum of the squares of its elements.

DEFINITION 3. Let $\mathbb{D}_e^q(\Omega)$, $q = 1,2,3$, denote the space of $\varphi \in \mathbb{C}_1(\Omega)$ possessing derivatives of order less than or equal to q, each of these derivatives belongs to $\mathbb{C}_1(\Omega)$. Moreover, there exists an $\varepsilon = \varepsilon_\varphi > 0$ such that for all $h > 0$ and $i, j, k \in S$, we have

$$\text{(i)} \qquad \left\| |\nabla_k \varphi(\omega)| \exp(\varepsilon|\omega_k|) \right\|_h \le C_\varphi(\varepsilon, h), \quad \text{for } q = 1,2,3,$$

$$\text{(ii)} \qquad \left\| |\nabla_j \nabla_k \varphi(\omega)| \exp(\varepsilon|\omega_k| + \varepsilon|\omega_j|) \right\|_h \le C_\varphi(\varepsilon, h), \quad \text{for } q = 2,3,$$

$$\text{(iii)} \quad \left\| |\nabla_i \nabla_j \nabla_k \varphi(\omega)| \exp(\varepsilon|\omega_k| + \varepsilon|\omega_j| + \varepsilon|\omega_i|) \right\|_h \le C_\varphi(\varepsilon, h), \quad \text{for } q = 3.$$

If $\varphi \in \mathbb{D}_e^q(\Omega)$ and we have a $p < 1$ such that the first q inequalities from the triplet (i)–(iii) are satisfied with some $C_\varphi(\varepsilon, h) = \exp[c_\varphi(\varepsilon)(1 + h)^p]$, then we say that $\varphi \in \mathbb{D}_{e1}^q(\Omega)$.

Since $\|c_k\|_h$ is bounded by a polynomial of h, see (1.9) and (1.10), $L\varphi$ is a well-defined element of $\mathbb{C}_1(\Omega)$ whenever $\varphi \in \mathbb{D}_{e1}^2(\Omega)$. For explicit calculations, cf. [10], [23], and [24], one usually needs the Kolmogorov equations

$$\mathbb{P}^t\varphi = \varphi + \int_0^t \mathbb{P}^s \mathbb{L}\varphi \, ds = \varphi + \int_0^t \mathbb{L}\mathbb{P}^s\varphi \, ds, \tag{1.12}$$

as well as the commutation relation $\mathbb{L}\mathbb{P}^t\varphi = \mathbb{P}^t\mathbb{L}\varphi$ for some nice $\varphi\colon \Omega \to \mathbb{R}$.

THEOREM 4. *Suppose that U has four continuous derivatives and $d \le 4$, then the semigroup defined by* (0.1) *satisfies $\mathbb{P}^t\mathbb{D}_{e1}^3(\Omega) \subset \mathbb{D}_e^2(\Omega)$ for all $t > 0$.*

Such results are based on some a priori bounds for the variational systems of (0.1). Theorem 4 and its extension to (1.6) will be proven in Sections 5 and 6. In this part the particular structure of our system plays a minor role and will be exploited only in the derivation of the a priori bounds of Sections 2 and 3.

In the theory of equilibrium fluctuations some further, more quantitative information is needed, but the problems are posed in the more familiar space $\mathbb{L}^2(\mu)$ of square-integrable functions with respect to the reversible equilibrium state of our system. If μ is a Gibbs state for U with unit temperature, then a formal condition of reversibility of (1.6) in the state μ reads as

$$c_k(\omega) = \tfrac{1}{2} e^{H_k(\omega)} \nabla_k \left[\sigma_k^2(\omega) e^{-H_k(\omega)} \right], \tag{1.13}$$

see [7] and [13] for some further references. In this case $\mathbb{L}$ turns into

$$\mathbb{L}\varphi = \tfrac{1}{2} \sum_{k \in S} e^{H_k} \nabla_k \left[\sigma_k^2 e^{-H_k} \nabla_k \varphi \right], \tag{1.14}$$

and the bilinear form associated to $\mathbb{L}$ in $\mathbb{L}^2(\mu)$ reduces to

$$\int \varphi_1 \mathbb{L}\varphi_2 \, d\mu = -\tfrac{1}{2} \sum_{k \in S} \int \langle \nabla_k \varphi_1, \nabla_k \varphi_2 \rangle \sigma_k^2 \, d\mu; \tag{1.15}$$

thus $\mathbb{P}^t$ will be self-adjoint in $\mathbb{L}^2(\mu)$. The missing step in Spohn's approach to equilibrium fluctuations for (0.1) is that a certain space $\mathbb{D}_0^\infty$ of smooth local functions forms a core for the closure of the quadratic form $Q(\varphi) = -\langle\varphi|\mathbb{L}\varphi\rangle$ defined by

$$\langle\varphi_1|\varphi_2\rangle = \int_{\mathbb{R}^d}\left[\int\varphi_1 T_u\varphi_2\,d\mu - \int\varphi_1\,d\mu\int\varphi_2\,d\mu\right]du, \tag{1.16}$$

where T_u denotes the shift by $u \in \mathbb{R}^d$. Because of its very technical character, we postpone the discussion of this question to Section 6, where some other results are also presented.

2. The deterministic problem. Now we start the proof of the fundamental a priori bound for $\overline{H}$. In this section we consider the contribution of the drift to the spatial flow of the energy. The energy flow will be controlled by a partial differential inequality, see [4]–[8]. This differential inequality will be proven for a smooth version $Q(\omega, m, \rho)$ of $H(\omega, m, \rho)$. The definition of this Q is based on the following—fairly sophisticated—modification of the indicator function of the ball of center m and radius ρ, see [4] and [6–8].

Let $q \in (0,1]$ and consider a nonincreasing and twice continuously differentiable f_q: $[0, +\infty) \to (0,1)$ such that

$$f_q(u) = e^{q-qu} \quad \text{if } u \geq 2,$$
$$f_q(u) = (1 + q + q^2/2)e^{-q} \quad \text{if } u \leq 1;$$

f_q is a convex function if $u \geq \frac{3}{2}$ and it is concave if $u \leq \frac{3}{2}$. Finally, $0 \leq -f_q'(u) \leq qf_q(u) \leq qe^{q-qu}$, $f_q(u) \geq e^{-q-qu}$ and $|f_q''(u)| \leq f_q(u)$ for all $u > 0$. Then, for $x, m \in \mathbb{R}^d$ and $\rho \geq 1$

$$f(x, \rho) = q^d\int_{\mathbb{R}^d} f_q(|x-y|/\rho)e^{-2q|y|}\,dy, \tag{2.1}$$

$$Q(\omega, m, \rho) = \rho^d + \sum_{k\in S} f(\omega_k - m, \rho)[1 + A + B + H_k(\omega)]. \tag{2.2}$$

The following elementary properties of f are more or less direct consequences of the definition, see [7]. We have

$$c_1(d)\exp(-q|x|/\rho) \leq f(x, \rho) \leq c_2(d)\exp(-q|x|/\rho), \tag{2.3}$$

$$f(x, \rho) \leq f(y, \rho)e^{2q|x-y|}, \qquad f'(x, \rho) \leq f'(y, \rho)e^{2q|x-y|}, \tag{2.4}$$

$$|\operatorname{grad} f(x, \rho)| \leq \min\left[f'(x, \rho), \frac{1}{\rho}f(x, \rho)\right], \tag{2.5}$$

$$|\Delta f(x, \rho)| \leq df(x, \rho), \tag{2.6}$$

$$g^p(|x|)|\operatorname{grad} f(x - m, \rho)| \leq 4g^p(|m| + \rho)f'(x - m, \rho), \tag{2.7}$$

where $0 < c_1(d) < c_2(d) < +\infty$, $0 < p \leq d$, f' is the derivative of f with respect to ρ, grad f and Δf denote the gradient and the Laplacian of f with respect to x. It is less trivial, see Lemma 2.13 in [7], that the superstability of U

and (2.4) imply that if $q > 0$ is small enough, then

$$Q(\omega, m, \rho) \geq \frac{B}{4} \sum_{k\in S} f(\omega_k - m, \rho) N_k(\omega), \tag{2.8}$$

$$Q'(\omega, m, \rho) \geq \frac{B}{4} \sum_{k\in S} f'(\omega_k - m, \rho) N_k(\omega), \tag{2.9}$$

where Q' is the derivative of Q with respect to ρ.

Now we are in a position to calculate and estimate the temporal derivative, $\dot{Q}$ of Q along a tempered solution $\omega(t)$ of (1.6). Here and in what follows, the abbreviations $f_k = f(\omega_k - m, \rho)$, $f_k' = f'(\omega_k - m, \rho)$, $\operatorname{grad} f_k = \operatorname{grad} f(\omega_k - m, \rho)$ will be used.

LEMMA 1. *There exists a universal constant $K > 0$ such that along any tempered solution $\omega(t)$ to* (1.6) *with $\sigma_k = 0$, $k \in S$,*

$$\dot{Q}(\omega(t), m, \rho) + \delta \sum_{k\in S} f_k \delta_k^2(\omega) |\nabla_k H_k(\omega)|^2$$
$$\leq KQ(\omega(t), m, \rho) + \frac{K}{\rho} g^2(|m| + \rho) \overline{H}(\omega(t)) Q'(\omega(t), m, \rho).$$

PROOF. A direct calculation yields $\dot{Q}(\omega(t), m, \rho) = I_1 + I_2 + I_3$, where

$$I_1 = \sum_{k\in S} \langle \operatorname{grad} f_k, c_k \rangle (1 + A + B + H_k), \tag{2.10}$$

$$I_2 = 2 \sum_{k\in S} f_k \langle \nabla_k H_k, c_k \rangle, \tag{2.11}$$

$$I_3 = \sum_{k\in S} \sum_{j\neq k} (f_k - f_j) \langle \operatorname{grad} U(\omega_k - \omega_j), c_j \rangle. \tag{2.12}$$

Using (1.8) and (1.9) and $xy \leq Cx^2/2\delta f_k + \delta f_k y^2/2C$, we obtain

$$\begin{aligned} I_1 &\leq C \sum_{k\in S} |\operatorname{grad} f_k|\, |1 + A + B + H_k| \left[\delta_k |\nabla_k H_k| + N_k^{1/2}\right] \\ &\leq C^2 \delta^{-1} \sum_{k\in S} |\operatorname{grad} f_k|^2 f_k^{-1} (1 + A + B + H_k)^2 \\ &\quad + \frac{\delta}{2} \sum_{k\in S} f_k \delta_k^2 |\nabla_k H_k|^2 + \frac{\delta}{2} \sum_{k\in S} f_k N_k, \end{aligned} \tag{2.13}$$

$$\begin{aligned} I_3 &\leq C \sum_{k\in S} \sum_{j\neq k} |f_k - f_j| \left|\operatorname{grad} U(\omega_k - \omega_j)\right| \left[\delta_j |\nabla_j H_j| + N_j^{1/2}\right] \\ &\leq C^2 \delta^{-1} \sum_{j\in S} f_j^{-1} \left[\sum_{k\neq j} |f_k - f_j| \left|\operatorname{grad} U(\omega_k - \omega_j)\right|\right]^2 \\ &\quad + \frac{\delta}{2} \sum_{j\in S} f_j \delta_j^2 |\nabla_j H_j|^2 + \frac{\delta}{2} \sum_{j\in S} f_j N_j, \end{aligned} \tag{2.14}$$

while

$$I_2 \le -2\delta \sum_{k\in S} f_k \delta_k^2 |\nabla_k H_k|^2 + 2C \sum_{k\in S} F_k N_k. \tag{2.15}$$

Now we can exploit (1.1) to derive

$$(1 + A + B + H_k)^2 \le K_1 g^d(|\omega_k|)\overline{H}(\omega)[1 + A + B + H_k + K_2 N_k], \tag{2.16}$$

$$\left[\sum_{k\neq j} |f_k - f_j| \left|\operatorname{grad} U(\omega_k - \omega_j)\right|\right]^2$$

$$\le \left[\sum_{k\in S_j(\omega)} \big(a + bU(\omega_k - \omega_j)\big)\right]\left[\sum_{k\in S_j(\omega)} |\operatorname{grad} f_{kj}|^2 \big(a + bU(\omega_k - \omega_j)\big)\right] \tag{2.17}$$

$$\le K_3 g^d(|\omega_j|)\overline{H}(\omega)\frac{1}{\rho} f_j f_j' \left[aN_j + bH_j\right],$$

where grad f_{kj} is the intermediate value of grad f in the Lagrange theorem, in the second step (2.4) and (2.5) were used. Taking into account (2.8) and (2.9), the statement follows by an easy calculation. □

The partial differential inequality of Lemma 1 can be solved by the method of characteristics. Namely, if $\rho = \rho(t)$ decreases fast enough, then $e^{-Kt}Q(\omega(t), m, \rho(t))$ turns into a decreasing function of time. Indeed, let

$$Z(t) = \int_0^t \overline{H}(\omega(s))\, ds, \tag{2.18}$$

$$\rho_r(t, m) = g(|m|)\left[r^2 - 2Kg^2(r)Z(t)\right]^{1/2}, \qquad r \in \mathbb{N}, \quad t \le T_r, \tag{2.19}$$

where $[0, T_r)$ is the maximal interval such that the difference under the square-root exceeds one. Since $g(|m| + \rho) \le g(|m|)g(\rho)$, for $t \le T_r$ we have

$$Q(\omega(t), m, \rho_r(t, m)) \le e^{Kt}Q(\omega(0), m, rg(|m|)). \tag{2.20}$$

To solve (2.20) let $n \in \mathbb{N}$ and denote $r(t, n, m)$ the first $r \in \mathbb{N}$ such that $\rho_r(t, m) \ge ng(|m|)$. Since both T_r and ρ_r go to infinity with r, $r(t, n, m)$ is well defined and $T_{r(t,n,m)} > t$. Therefore, choosing $r = r(t, n, m)$ in (2.20) and using (2.3) we obtain

$$\overline{H}(\omega(t)) \le K_4 e^{Kt}\overline{H}(\omega(0)) \sup_{m\in\mathbb{Z}^d} \sup_{n\in\mathbb{N}} \left[\frac{1}{n} r(t, n, m)\right]^d. \tag{2.21}$$

In view of the definition of $r = r(t, n, m)$ we have

$$(r - 1)^2 \le n^2 + 2Kg^2(r - 1)Z(t) \le K_5\left[n^2 + (r - 1)Z(t)\right]. \tag{2.22}$$

At the last step the elementary inequality

$$1 + \log(1 + u) \le \frac{1}{\varepsilon}(1 + u)^{\varepsilon}, \quad \text{if } u > 0, \quad 0 < \varepsilon \le 1, \tag{2.23}$$

has been used with $\varepsilon = \frac{1}{2}$. Since $r \ge n$, we obtain $r - 1 \le 2K_5 n(1 + Z)$. Thus substituting this rough bound again into (2.22),

$$(r-1)^2 \le K_6 n^2\left[1+Z(t)g^2(Z(t))\right] \tag{2.24}$$

follows directly and (2.21) results in

$$\overline{H}(\omega(t)) \le Me^{Kt}\overline{H}(\omega(0))\left[1+Z(t)g^2(Z(t))\right]^{d/2}, \tag{2.25}$$

with some universal constant M. In view of (2.18), (2.25) is a differential inequality for Z, and the variables of (2.25) can be separated by dividing by $[1+Zg^2(Z)]^{d/2}$. Since

$$\int_0^{\infty}\left[1+zg^2(z)\right]^{-d/2}dz = +\infty, \quad \text{if } d \le 2, \tag{2.26}$$

(2.25) has a continuous maximal solution on $[0,+\infty)$, i.e., we have an a priori bound for $\overline{H}$.

Proposition 1. *Under conditions of Theorem* 1 *we have a universal constant* $\lambda \ge \frac{1}{2}$, *and a universal function* $\overline{h} = \overline{h}(h,T)$ *such that for all* $t \le T$ *and* $k \in S$, *we have* $\overline{H}(\omega(t)) \le \overline{h}(h,T)$ *and*

$$\begin{aligned}&|\omega_k(0)| - \overline{h}^{\lambda}(h,T)\log^{\lambda}\left(e+|\omega_k(0)|\right)\\ &\quad\le |\omega_k(t)|\\ &\quad\le |\omega_k(0)| + \overline{h}^{\lambda}(h,T)\log^{\lambda}\left(e+\overline{h}(h,T)\right)\log^{\lambda}\left(e+|\omega_k(0)|\right),\end{aligned} \tag{2.27}$$

whenever $\omega(t)$ *is a tempered solution with* $\overline{H}(\omega(0)) \le h$.

Proof. Only (2.27) needs a proof. Observe that (1.9) and (1.10) imply

$$|c_k(\omega)| \le C_1 H^{\lambda}(\omega,\omega_k,R), \tag{2.28}$$

with some C_1 and $\lambda = \max[\frac{1}{2}, \frac{1}{2}C]$, where C is the exponent of h_k on the right-hand side of (1.10). Hence the first part of (2.27) follows directly, the factor $C_1 T$ can be absorbed into $\overline{h}$. On the other hand, if $q = \max[|\omega_k(t)|: t \le T]$, then

$$q \le |\omega_k(0)| + C_2 T\overline{h}^{\lambda}(h,T)\left(1+\log(1+q)\right)^{\lambda}, \tag{2.29}$$

whence by (2.23) we obtain $(1+q)^{1/2} \le (1+|\omega_k(0)|)^{1/2} + C_3\overline{h}^{\lambda}(h,T)$. Substituting this bound into (2.29) and using $\log(e+u+v) \le \log(e+u)\log(e+v)$ if $u, v \ge 0$, we obtain (2.27). □

Observe now that all conditions of Proposition 1 remain in force if we let some coefficients vanish. For each finite $I \subset S$ consider $c_k^I = c_k$ if $k \in I$, $c_k^I = 0$ if $k \notin I$; $\sigma_k = \sigma_k^I = 0$ for all $k \in S$ in this section. The corresponding solution with initial configuration $z \in \Omega$ will be denoted by $\omega^I = \omega^I(t,z)$. It is obviously a tempered one, and the a priori bound does not depend on I. Consequently, (2.28), Proposition 1 and the Arzela–Ascoli theorem imply that the family ω^I is precompact in $\mathbb{W}$, consider the limiting trajectories as $I \to S$. In view of (1.10) and the a priori bound, the integrated form of (1.6) shows that any limit point ω of ω^I in $\mathbb{W}$ satisfies (1.6). Since $\overline{H}$ is a lower semicontinuous function of ω, these limiting solutions are tempered solutions. Thus Proposition 1 also holds for the limiting solutions.

The uniqueness of the tempered solution can be proven by means of the iteration technique of Lanford [11] and [12]. Suppose that $\omega(t)$ and $\bar{\omega}(t)$ are tempered solutions with $\omega(0) = \bar{\omega}(0) = z \in \Omega$, let $S(z, r) = [k \in S\colon\ |z_k| < r]$ and introduce

$$D(t, r) = \sum_{k \in S(z, r)} \sup_{s \le t} |\omega_k(s) - \bar{\omega}_k(s)|^2. \tag{2.30}$$

In view of (2.27) we have $r < r' < r' + R < \bar{r} = (r^{1/2} + K_T)^2$ such that, at least as long as $t \le T$, the particles with $k \in S(z, r)$ cannot escape from the ball $B(r')$, and the particles from the set $S \setminus S(z, \bar{r})$ cannot hit the ball $B(r' + R)$. Therefore by (1.10)

$$D(t, r) \le D(0, r) + L_T[\log(e + r)]^{\lambda+1} \int_0^t D(s, \bar{r})\, ds \tag{2.31}$$

follows for $t \le T$ with some universal L_T. Since $D(t, r) \le 4\bar{r}^2\bar{h}(h, T)\bar{r}^d$ if $t \le T$ and $z \in \Omega_h$, (2.31) can be iterated infinitely many times, thus $D(0, r) = 0$ implies $D(t, r) = 0$ for all $r > 0$ and $t > 0$, which completes the proof of Theorem 1.

REMARK 1. If $d = 1$, then $\bar{h}(h, T)$ can be bounded by a polynomial of h and T. Thus the finiteness of all moment measures of the initial point process is conserved for all times; this property is not known if $d = 2$.

REMARK 2. The above proof of uniqueness works without any essential change in the case of deterministic systems of second order, cf. [11] and [12] and [4] and [8]. The case of stochastic systems is more complex, see [13], [15] and [6].

3. A priori bounds for stochastic systems. The notation of the previous section is used with the exception that

$$Z(t) = \int_0^t \bar{H}^{1/2}(\omega(s))\, ds, \tag{3.1}$$

in the definition (2.19) of ρ_r. If ω is tempered, then the following stochastic differentials make sense, and

$$\begin{aligned} &d\left[e^{-Kt}Q(\omega(t), m, \rho_r(t, m))\right] \\ &\quad= -Ke^{-Kt}Q\,dt - \frac{K}{\rho_r}g^2(|m|)g^2(r)\bar{H}^{1/2}e^{-Kt}Q'\,dt \\ &\qquad+e^{-Kt}(I_1 + I_2 + I_3)\,dt \\ &\qquad+e^{-Kt}(dI_4 + dI_5 + dI_6) \\ &\qquad+e^{-Kt}(I_7 + I_8 + I_9)\,dt, \end{aligned} \tag{3.2}$$

where I_1, I_2, I_3 are the same as in Section 2, and

$$dI_4 = \sum_{k\in S} \langle \operatorname{grad} f_k, \sigma_k\, dw_k\rangle (1 + A + B + H_k), \tag{3.3}$$

$$dI_5 = 2\sum_{k\in S} f_k \langle \nabla_k H_k, \sigma_k\, dw_k\rangle, \tag{3.4}$$

$$dI_6 = \sum_{k\in S}\sum_{j\neq k} (f_k - f_j)\big\langle \operatorname{grad} U(\omega_k - \omega_j), \sigma_j\, dw_j\big\rangle, \tag{3.5}$$

$$I_7 = \tfrac{1}{2}\sum_{k\in S} \Delta f_k \sigma_k^2 (1 + A + B + H_k) \le K_7 Q(\omega, m, \rho_r), \tag{3.6}$$

$$I_8 = \sum_{k\in S} \langle \operatorname{grad} f_k, \nabla_k H_k\rangle \sigma_k^2 \le K_8 Q(\omega, m, \rho_r), \tag{3.7}$$

$$I_9 = \tfrac{1}{2}\sum_{k\in S} f_k \sum_{j\neq k} \Delta U(\omega_k - \omega_j)\big(\sigma_k^2 + \sigma_j^2\big) \le K_9 Q(\omega, m, \rho_r). \tag{3.8}$$

The last three inequalities are direct consequences of the properties of the cut-off function f; see the beginning of Section 2. Estimation of the deterministic integrals I_1, I_2 and I_3 is essentially the same as in Section 2. The only difference is that, due to the boundedness of U, we have

$$\sum_{k\in S_j(\omega)} \big(a + bU(\omega_k - \omega_j)\big) \le K_{10} N_j(\omega) \le K_{11} H^{1/2}(\omega, \omega_j, R). \tag{3.9}$$

The last bound is a consequence of the superstability of U,

$$\big[1 + A + B + H_j(\omega)\big]^2 \le K_{12} g^{d/2}(|\omega_j|) \bar{H}^{1/2}(\omega)(1 + A + B + H_k(\omega)), \tag{3.10}$$

and

$$\Big[\sum_{k\neq j} |f_k - f_j| \big|\operatorname{grad} U(\omega_k - \omega_j)\big|\Big]^2 \le K_{13} g^{d/2}(|\omega_j|) \bar{H}^{1/2}(\omega) f_j f_j' \rho_r^{-1}\big[aN_j(\omega) + bH_j(\omega)\big], \tag{3.11}$$

cf. (2.16) and (2.17).

The stochastic integrals I_4, I_5, I_6 can be estimated by means of the maximal inequality

$$\mathsf{P}\Big[\sup_{t>0} \sum_{k\in S} \int_0^t \Big(p_k\, dw_k - \frac{\lambda}{2} p_k^2\, ds\Big) > u\Big] \le e^{-\lambda u}, \tag{3.12}$$

provided that the processes p_k are $\mathscr{A}_t$-adapted and

$$\mathsf{P}\Big[\sum_{k\in S} \int_0^t p_k^2\, ds < +\infty\Big] = 1,$$

for all $t > 0$; see, e.g., McKean [18] for finite sums, whence (3.12) follows by continuity. The squares of the coefficients of dw_j in dI_4, dI_5 and dI_6 can be estimated by means of (3.9), (2.15) and (3.11), respectively. Introduce now

$$N(m,r) = \sup_{t\le T_r} \int_0^t e^{-Ks}\left[dI_4 - Q(\omega(s), m, \rho_r(s,m))\, ds\right]$$

$$(3.13) \qquad + \sup_{t\le T_r} \int_0^t e^{-Ks}\left[dI_5 - \delta \sum_{k\in S} f_k \delta_k^2 |\nabla_k H_k|^2\, ds\right]$$

$$+ \sup_{t\le T_r} \int_0^t e^{-Ks}\left[dI_6 - \rho_r^{-1} g^2(|m|) g^2(r) \overline{H}(\omega(s)) Q'(\omega(s), m, \rho_r)\, ds\right].$$

Then following the proof of (2.20) we obtain such universal constants $K > 0$ and $c > 0$ such that

$$(3.14) \quad \sup_{t\le T_r} e^{-Kt} Q(\omega(t), m, \rho_r(t,m)) \le Q(\omega(0), m, rg(|m|)) + N(m,r),$$

where $\mathsf{P}[N(m,r) > u] \le 3e^{-cu}$ for all $m \in \mathbb{Z}^d$, $r \in \mathbb{N}$ and $u > 0$. Now we are in a position to prove the fundamental a priori bound for (1.6).

Proposition 2. *Under the conditions of Theorem 2 we have a universal constant $c > 0$ and universal function $q = q_\varepsilon(h, T)$ with the following properties. For any tempered solution $\omega = \omega(t)$, there exists an $\mathscr{A}$-measurable random variable $N \ge 0$ such that $\mathsf{P}[N > u] < e^{-cu}$ and $\overline{H}(\omega(0)) \le h$ implies $\sup_{t<T} \overline{H}(\omega(t)) \le q_\varepsilon(h,T)(1+N)^{1+\varepsilon}$ for all $\varepsilon > 0$ with probability one. If $d < 4$, then $q_\varepsilon(h,T) \le p_\varepsilon(T)(1+h)^{(d+\varepsilon)/4-d}$ with some universal p_ε.*

Proof. Introduce

$$(3.15) \qquad N = \max\left[0, \sup_{r\in\mathbb{N}} \sup_{m\in\mathbb{Z}^d} \left[N(m,r) - vr^d g^d(|m|)\right]\right].$$

Then from (3.14) we have

$$(3.16) \qquad \sup_{t\le T_r} e^{-Kt} Q(\omega(t), m, \rho_r(t,m)) \le Q(\omega(0), m, rg(|m|)) + N + vr^d g^d(|m|),$$

simultaneously for all $r \in \mathbb{N}$ and $m \in \mathbb{Z}^d$ with probability one. Finally, $\mathsf{P}[N > u] < e^{-cu}$ if v is large enough. Using again (2.3) we obtain, along almost every realization, that

$$(3.17) \qquad \overline{H}(\omega(t)) \le K_{14} e^{Kt}\left[N + h \sup_{m\in\mathbb{Z}^d} \sup_{n\in\mathbb{N}} \left[\frac{1}{n} r(t,n,m)\right]^d\right]$$

holds for all t; here $r(t,n,m)$ is the same as in (2.21), and the same estimation method yields the differential inequality

$$(3.18) \qquad \dot{Z} \le K_{15} e^{Kt/2}\left[N + h(1 + Zg^2(Z))^{d/2}\right]^{1/2}, \qquad Z(0) = 0,$$

where $\dot{Z}$ denotes the temporal derivative of Z, i.e., $\dot{Z} = \overline{H}^{1/2}$. Since the realizations of Z are absolutely continuous, the method of separation of variables can be applied to solve (3.18). Indeed, let $V = 1 + (N/h)^{2/d} + Z$, then an easy calculation yields

$$\dot{Z} = \dot{V} \le K_{16} e^{Kt/2} h^{1/2} V^{d/4} (1 + \log V)^{1/2}. \tag{3.19}$$

If $d = 4$, then we have an explicit maximal solution, namely

$$\begin{aligned} 1 + \log V(t) &\le \left[(1 + \log V(0))^{1/2} + q(t)\right]^2 \\ &\le (1 + \lambda)(1 + \log V(0)) + (1 + \lambda^{-1}) q^2(t), \end{aligned} \tag{3.20}$$

for all $\lambda > 0$ with some universal function q, whence the first bound follows directly. If $d < 4$, then (2.23) can be used to find an explicit bound, see [7]. □

A localization bound like (2.27) can be proven in a similar way, see the next section. Notice that if $d < 4$, then the existence of all moment measures at $t = 0$ is conserved for all positive times.

REMARK 3. If $d < 4$, then the a priori bound of Proposition 2 for $\overline{H}$ is actually a linear function of N, see [6]; presumably the same holds even if $d = 4$. We conjecture that in all stochastic cases the variable N can be replaced by an $\mathscr{A}_T$-adapted process N_T having a T-dependent normal tail, i.e., the a priori bound for $\overline{H}$ also has a normal tail, see Remark 6 in Section 5.

REMARK 4. If $d < 4$, then the function g in the definition (1.4) of $\overline{H}$ can be replaced by a power law depending on d, which results in a much larger set of allowed configurations, see [4].

4. Passing to the thermodynamical limit. Here we combine the iteration method of [11] and [12] with that of [6] to construct limiting solutions to (1.6), see the proof of the uniqueness part of Theorem 1. The very same technique yields continuous dependence on initial data. First, we need a localization bound like (2.27). Consider a tempered solution $\omega(t)$ with $\omega(0) = z$, and define

$$Y(t) = \sup_{k \in S} \left[\log(e + |z_k|)\right]^{-1} \max_{s \le t} \left| \int_0^s \sigma_k(\omega(s))\, dw_k \right|, \tag{4.1}$$

$$X(t) = t + Y(t) + \int_0^t \overline{H}^{1/2}(\omega(s))\, ds. \tag{4.2}$$

Then we have:

LEMMA 2. *If U is smooth and $\omega(t)$ is a tempered solution to* (1.6) *with $\omega(0) = z$, then for all $t > 0$ and $k \in S$,*

$$\begin{aligned} |z_k| - MX(t)\log(e + |z_k|) &\le \min_{s \le t} |\omega_k(s)| \le \max_{s \le t} |\omega_k(s)| \\ &\le |z_k| + MX(t)\log(e + X(t))\log(e + |z_k|), \end{aligned}$$

with probability one, provided that M is large enough.

PROOF. Since (2.28) holds with $\lambda = \frac{1}{2}$ in this case, the statement follows in the same way as (2.27). □

We need some information on the tail of $X(t)$.

LEMMA 3. *We have a universal constant M such that*

$$\mathsf{P}\big[Y(t) > \tfrac{1}{2}C^2t + u\big] \leq M\overline{H}^{1/2}(z)e^{-u}.$$

PROOF. Since $|\sigma_k| \leq C$, the maximal inequality (3.12) yields

$$\mathsf{P}\left[\max_{s\leq t}\int_0^s \sigma_k\,dw_k > \tfrac{1}{2}C^2t + u\right] \leq e^{-u},$$

but the superstability of U implies

$$\sum_{k\in S}\exp\big[-u\log(e+|z_k|)\big] \leq M_1\overline{H}^{1/2}(z)\sum_{r=1}^{\infty}(e+r)^{d-1-u}\log(e+r),$$

which completes the proof. □

Now we develop an estimate for the difference of two solutions. Suppose that we are given two systems of coefficients, $(c_k, \sigma_k)_{k\in S}$ and $(\bar{c}_k, \bar{\sigma}_k)$ satisfying (1.8)–(1.10) with the same U, δ, C. Here and in what follows, the bar refers to the other system $d\bar{\omega}_k = \bar{c}_k(\bar{\omega})\,dt + \bar{\sigma}_k(\bar{\omega})\,dw_k$. It is very important that the Wiener trajectories are the same in both cases. Let $\omega = \omega(t)$ and $\bar{\omega} = \bar{\omega}(t)$ be tempered solutions to the corresponding systems with $\omega(0) = z$ and $\bar{\omega}(0) = \bar{z}$. Following [6] we introduce

$$d_k(t,\lambda) = \mathsf{E}\left[\sup_{s\leq t} I(s,\lambda)|\omega_k(s) - \bar{\omega}_k(s)|^2\right], \tag{4.3}$$

where $I(t,\lambda) = 1$ if $\Lambda(t) \leq \lambda$, $I(t,\lambda) = 0$ if $\Lambda(t) > \lambda$, with

$$\Lambda(t) = 1 + X(t) + \overline{X}(t) + \sup_{s\leq t}\big[\overline{H}^{1/2}(\omega(s)) + \overline{H}^{1/2}(\bar{\omega}(s))\big],$$

and $\overline{X}$ is the process defined for $\bar{\omega}(t)$ by (4.2). Notice that $I(t,\lambda)$ is a nonanticipating process, and $I(t,\lambda) = 1$ implies $I(s,\lambda) = 1$ for all $s \leq t$. Consequently,

$$\begin{aligned} I(t,\lambda)|\omega_k(t) - \bar{\omega}_k(t)|^2 &\leq 3I(0,\lambda)|\omega_k(0) - \bar{\omega}_k(0)|^2 \\ &\quad + 3t\int_0^t I(s,\lambda)\big[c_k(\omega(s)) - \bar{c}_k(\bar{\omega}(s))\big]^2\,ds \\ &\quad + 3\left[\int_0^t I(s,\lambda)\big[\sigma_k(\omega(s)) - \bar{\sigma}_k(\bar{\omega}(s))\big]\,dw_k\right]^2. \end{aligned}$$

Thus $I^2(t,\lambda) = I(t,\lambda)$ and the maximal inequality

$$\mathsf{E}\left[\sup_{s\leq t}\left(\int_0^s p_k\,dw_k\right)^2\right] \leq 4\mathsf{E}\left[\int_0^t p_k^2\,ds\right],$$

see, e.g., [25], imply that

$$
(4.4)\qquad \begin{aligned} d_k(t,\lambda) \le 3d_k(0,\lambda) &+ 3t\int_0^t \mathsf{E}\left[I(s,\lambda)|c_k(\omega(s)) - \bar{c}_k(\bar{\omega}(s))|^2\right] ds \\ &+12\int_0^t \mathsf{E}\left[I(s,\lambda)|\sigma_k(\omega(s)) - \bar{\sigma}_k(\bar{\omega}(s))|^2\right] ds; \end{aligned}
$$

that is, we may exploit (1.10).

LEMMA 4. *Let* $\rho_0 = \rho$, $\rho_{n+1} = (\rho_n^{1/2} + \lambda L)^2$ *if* $n > 0$ *and suppose that* $(z, \bar{z}) \in G(R, \rho_{r+1})$. *Moreover,* $c_k = \bar{c}_k$ *and* $\sigma_k = \bar{\sigma}_k$ *if* $k \in S(\rho_r)$, *where* $S(\rho) = [k \in S\colon \min[|z_k|, |\bar{z}_k|] < \rho]$. *Define*

$$
D(t,\lambda,\rho) = \sum_{k\in S(\rho)} d_k(t,\lambda)
$$

and suppose that L is large enough. Then

$$
\begin{aligned} D(t,\lambda,\rho) \le 3\sum_{n=0}^{r} D(0,\lambda,\rho_n)\frac{L^n}{n!}(t+t^2)^n(\lambda\log(e+\rho_n))^{Ln} \\ +4\frac{L^{r+2}}{(r+1)!}(t+t^2)^{r+1}(\lambda\log(e+\rho_{r+1}))^{Lr+L+1}\rho_{r+1}^{d+2}. \end{aligned}
$$

PROOF. In view of Lemma 2 we have $\rho_n < \rho_n' < \rho_n' + R < \rho_{n+1}$ such that, as long as $I(t,\lambda) \ne 0$, the particles with $k \in S(\rho_n)$ cannot escape from the ball $B(\rho_n')$, while the particles from $S \setminus S(\rho_{n+1})$ are not able to hit the ball $B(\rho_n' + R)$. Therefore from (4.4) by (1.10) and Lemma 2 we obtain

$$
D(t,\lambda,\rho_n) \le 3D(0,\lambda,\rho_n) + L(1+t)(\lambda\log(e+\rho_n))^L\int_0^t D(s,\lambda,\rho_{n+1})\,ds,
$$

at least if $n \le r$. Starting with $n = 0$, this inequality can be iterated r times; the last term contains the integral of $D(s,\lambda,\rho_{r+1})$. Since $|\omega_k - \bar{\omega}_k| \le 2\rho_{r+1}$ in this sum, and the number of summands of $D(s,\lambda,\rho_{r+1})$ is bounded by a multiple of $\lambda\rho_{r+1}^d\log(e+\rho_{r+1})$, we obtain Lemma 4 by an easy calculation. □

Now we are in a position to conclude the existence of a unique tempered solution, $\omega = \omega(t,z)$ for each initial configuration $z \in \Omega$. This solution will be constructed as the a.s. limit in $\mathbb{W}$ of the following sequence of partial solutions. Let $z \in \Omega$ be fixed, and put $c_k^{(n)} = \sigma_k^{(n)} = 0$ if $|z_k| \ge e^n$, while $c_k^{(n)} = c_k$, $\sigma_k^{(n)} = \sigma_k$ if $|z_k| < e^n$; the corresponding solution to (1.6) will be denoted by $\omega^{(n)} = \omega^{(n)}(t,z)$. For the pair $(\omega,\bar{\omega}) = (\omega^{(n)},\omega^{(n+1)})$ we use $I_n(t,\lambda) = I(t,\lambda)$ and $D_n(t,\lambda,\rho) = D(t,\lambda,\rho)$ as defined above. Comparing Proposition 2 and Lemmas 2 and 3 we obtain

$$
(4.5)\qquad \sum_{n=1}^{\infty} \mathsf{P}[I_n(t,n) \ne 1] < +\infty,
$$

for all $t > 0$. Since $\omega^{(n)}(0, z) = \omega^{(n+1)}(0, z) = z$, Lemma 4 yields

$$(4.6)\qquad \sum_{n=1}^{\infty} \left[D_n(t, n, \rho)\right]^{1/2} < +\infty.$$

Thus for each $t > 0$ and $\rho > 0$ we have

$$(4.7)\qquad \mathsf{P}\left[\sum_{n=1}^{\infty} \sum_{k \in S(\rho)} \sup_{s \le t} \left|\omega_k^{(n)}(s, z) - \omega_k^{(n+1)}(s, z)\right| < +\infty\right] = 1;$$

put $\omega = \omega(t, z) = \lim \omega^{(n)}(t, z)$ and notice that the limit exists in the topology of $\mathbb{W}$ as well as in that of Ω. As a limit of finite-dimensional partial solutions, ω is a jointly measurable function of its variables, and the a priori bound and the lower semicontinuity of $\overline{H}$ imply that ω is a tempered solution to (1.6) with initial configuration z. The uniqueness of this tempered solution follows simply by letting r go to infinity in Lemma 4, which completes the proof of Theorem 2.

The restricted Feller continuity of $\mathbb{P}^t$ follows again from Lemma 4 by means of the a priori bound. Proposition 2 shows that $\mathbb{P}^t\varphi(z) = \mathsf{E}[\varphi(\omega(t, z))]$ is well defined and $\|\mathbb{P}^t\varphi\|_h < +\infty$ for each $h > 0$ if $\varphi \in \mathbb{C}_1(\Omega)$. Moreover, if $d < 4$ and $\varphi \in \mathbb{C}_p(\Omega)$, then the a priori bound is a polynomial of h, thus $\|\mathbb{P}^t\varphi\|_h$ is also polynomially bounded. To prove the restricted continuity property, let $\varphi \in \mathbb{C}_1(\Omega)$ and $h > 1$, $t > 0$; we have to find an $\bar{\varepsilon} > 0$ and a $\bar{\rho} > 0$ such that $|\mathbb{P}^t\varphi(z) - \mathbb{P}^t\varphi(\bar{z})| < \varepsilon$ whenever $z, \bar{z} \in \Omega_h$ and $(z, \bar{z}) \in G(\bar{\varepsilon}, \bar{\rho})$. First, we choose $\bar{h}$ in such a way that

$$(4.8)\qquad \mathsf{E}\left[\bar{I}_{\bar{h}}(\omega(t, z))|\varphi(\omega(t, z))|\right] < \frac{\varepsilon}{6}, \quad \text{if } z \in \Omega_h,$$

where $\bar{I}_h = \bar{I}_h(\omega)$ denotes the indicator of the set $\Omega \setminus \Omega_h$. In view of the definition of $\mathbb{C}_1(\Omega)$ we have $\varepsilon' > 0$ and $\rho' < +\infty$ such that

$$(4.9)\qquad |\varphi(\omega) - \varphi(\bar{\omega})| < \frac{\varepsilon}{3}, \quad \text{if } \omega, \bar{\omega} \in \Omega_{\bar{h}} \text{ and } (\omega, \bar{\omega}) \in G(\varepsilon', \rho').$$

Now we can apply Lemma 4 to find $\bar{\varepsilon} > 0$ and $\bar{\rho} < +\infty$ such that

$$(4.10)\qquad \|\varphi\|_{\bar{h}}\mathsf{P}[(\omega(t, z), \omega(t, \bar{z})) \notin G(\varepsilon', \rho')] < \frac{\varepsilon}{3},$$

whenever $z, \bar{z} \in \Omega_h$ and $(z, \bar{z}) \in G(\bar{\varepsilon}, \bar{\rho})$, which completes the proof of the Feller property of Theorem 3.

The intrinsic relationship of the transition semigroup $\mathbb{P}^t$ to its formal generator $\mathbb{L}$ is revealed by:

PROPOSITION 3. *If $\varphi \in \mathbb{C}_1(\Omega)$, then $\mathbb{P}^t\varphi$ draws a continuous trajectory in $\mathbb{C}(\Omega)$. Let $\varphi \in \mathbb{D}^2_{e1}(\Omega)$, then $\mathbb{L}\varphi \in \mathbb{C}_1(\Omega)$ and*

$$\mathbb{P}^t\varphi = \varphi + \int_0^t \mathbb{P}^s\mathbb{L}\varphi\, ds.$$

Thus

$$\mathbb{L}\varphi = \lim_{t\to 0} \frac{1}{t}(\mathbb{P}^t\varphi - \varphi),$$

where the integral and the limit make sense in $\mathbb{C}(\Omega)$.

PROOF. Let $\varphi \in \mathbb{C}_1(\Omega)$ and let $\varepsilon, h, \bar{h}, \varepsilon', \rho'$ be as above, then (4.8) and (4.9) may be assumed for all $t < T$. Since

$$\mathsf{E}\Big[|\omega_k(t,z) - \omega_k(s,z)|^2\Big] \le 2(t-s)\int_s^t \mathsf{E}\big[c_k^2(\omega(u,z))\big]\,du + 2C^2(t-s),$$

in view of (1.9) and the a priori bounds we have a $T_\varepsilon > 0$ such that $z \in \Omega_h$, $s < t < T$ and $t - s < T_\varepsilon$, imply

$$\|\varphi\|_{\bar{h}}\mathsf{P}[(\omega(t,z),\omega(s,z)) \notin G(\varepsilon',\rho')] < \frac{\varepsilon}{3}, \tag{4.11}$$

which completes the proof of the strong continuity of $\mathbb{P}^t\varphi$.

Suppose now that $\varphi \in \mathbb{D}^2_{e1}(\Omega)$ and let $\mathbb{P}^t_n$ and $\mathbb{L}_n$ denote the semigroup and its formal generator for the partial dynamics $\omega^{(n)}$. Since $\lim \omega^{(n)} = \omega$, we have $\lim \mathbb{P}^t_n\varphi = \mathbb{P}^t\varphi$, at least if φ is continuous and bounded. However, Proposition 2 permits us to extend $\lim \mathbb{P}^t_n\varphi = \mathbb{P}^t\varphi$ to $\varphi \in \mathbb{C}_1(\Omega)$, while the Itô lemma yields

$$\mathbb{P}^t_n\varphi = \varphi + \int_0^t \mathbb{P}^s_n\mathbb{L}_n\varphi\,ds, \quad \text{if } \varphi \in \mathbb{D}^2_{e1}. \tag{4.12}$$

On the other hand, each term $c_k\nabla_k\varphi + \sigma_k^2/2\Delta_k\varphi$ of $\mathbb{L}_n\varphi$ belongs to $\mathbb{C}_1(\Omega)$, and the remainder is uniformly bounded in view of our a priori bounds, thus the dominated convergence theorem implies the final statement by letting n go to infinity in (4.12). □

REMARK 5. Lemma 4 yields a rate for $\mathbb{P}^t_n \to \mathbb{P}^t$, and $\rho_n = O(n^2)$ can be relaxed to $\rho_n = O(n \log n)$, which improves this rate. In the next section we prove an exponential rate for this convergence by different methods.

5. A priori bounds for the variational systems. The main task of this section is to understand the structure of the expression of $\mathbb{L}\mathbb{P}^t$. We are going to develop estimates in $\mathbb{L}^2(\mathsf{W},\mathsf{P})$ for the derivatives of the trajectories with respect to some parameters. These a priori bounds will then be used to derive Theorem 4 and some further results on the equilibrium dynamics. For this second purpose it will be very convenient to let the coefficients depend on a continuous parameter characterizing the size of the system with which we are dealing. Of course, we are given an infinite system $(c_k, \sigma_k)_{k\in S}$, but for each $\rho > 1$ we introduce a finite subsystem $(c_k(\cdot,\rho), \sigma_k(\cdot,\rho))_{k\in S}$ in such a way that $c_k(\omega,\rho) = c_k(\omega)$ and $\sigma_k(\omega,\rho) = \sigma_k(\omega)$ if $|\omega_k| < \rho$, while $c_k(\omega,\rho) = 0$ and $\sigma_k(\omega,\rho) = 0$ if $|\omega_k| \ge \rho + R$. We assume that these coefficients satisfy all conditions of Section 1 with the same U, δ, C. The interaction U is smooth and additional conditions are listed in the following propositions.

Let $\omega = \omega(t, z, \rho)$ denote the general solution to (1.6) with the above coefficients, i.e., $\omega(0, z, \rho) = z \in \Omega$, and let $\overline{\nabla}_n^{(r)}$, $n \in S$, $r = 1, 2, \ldots, d$, denote the operator of differentiating with respect to $z_n^{(r)} = \langle z_n, e_r \rangle$, and $\overline{\nabla}_n = (\overline{\nabla}_n^{(1)}, \overline{\nabla}_n^{(2)}, \ldots, \overline{\nabla}_n^{(d)})$, where $e_1, e_2, \ldots, e_d$ are the elements of our orthonormal base in $\mathbb{R}^d$. We are interested in the following quantities:

$$u_k = \overline{\nabla}_n^{(r)} \omega_k(t, z, \rho), \qquad \bar{u}_k = \overline{\nabla}_m^{(s)} \omega_k(t, z, \rho), \qquad \omega_k' = \frac{\partial}{\partial \rho} \omega_k(t, z, \rho),$$

$$v_k = \overline{\nabla}_m^{(s)} u_k = \overline{\nabla}_m^{(s)} \overline{\nabla}_n^{(r)} \omega_k(t, z, \rho),$$

$$u_k' = \frac{\partial}{\partial \rho} u_k = \overline{\nabla}_n^{(r)} \omega_k' = \frac{\partial}{\partial \rho} \overline{\nabla}_n^{(r)} \omega_k(t, z, \rho),$$

$$v_k' = \frac{\partial}{\partial \rho} v_k = \overline{\nabla}_m^{(s)} u_k' = \frac{\partial}{\partial \rho} \overline{\nabla}_m^{(s)} \overline{\nabla}_n^{(r)} \omega_k(t, z, \rho),$$

where n, m, r, s are arbitrary, but fixed. For finite systems with smooth coefficients the existence of these processes follows by standard methods, see Section 16 in [9] or Chapter 7 in [1].

Since the variational systems are not dissipative like (1.6), we need a new trick to control boundary effects. For $x \in \mathbb{R}^d$, $\lambda \geq 0$ and $\sigma > 0$, define

$$\theta(x, \lambda) = \exp\Big[-\lambda\big(1 + \log(2 + \lambda^2 + |x|^2)\big) - \sigma\big(1 + |x|^2\big)^{1/2}\Big], \tag{5.1}$$

and set $\theta' = \partial\theta/\partial\lambda$; the operators grad and Δ are acting on θ as a function of x. It is easy to find a universal constant, C such that for all x, λ, σ we have

$$(1 + \log(1 + |x|))\theta(x, \lambda) \leq -\theta'(x, \lambda), \tag{5.2}$$

$$|\operatorname{grad} \theta(x, \lambda)| + |\Delta\theta(x, \lambda)| \leq C(1 + \sigma)^2 \theta(x, \lambda), \tag{5.3}$$

$$\theta(x, \lambda) \leq \theta(y, \lambda) \exp[(1 + \sigma)|x - y|]. \tag{5.4}$$

The following a priori bounds are formulated in terms of

$$Z_p(t, z) = t + \int_0^t \overline{H}^{p/2}(\omega(s, z, \rho))\, ds$$

and

$$V_p(t, z) = 1 + Z_p(t, z) + \overline{H}^{p/2}(\omega(t, z, \rho)).$$

The abbreviations

$$q_k(\omega) = \overline{H}^{1/2}(\omega)\big(1 + \log(1 + |\omega_k|)^{1/2}\big), \qquad \dot{Z}_p = \partial Z_p/\partial t$$

are also useful.

Proposition 4. *Let* $c_{kj} = \nabla_j c_k(\omega, \rho)$, $\sigma_{kj} = \nabla_j \sigma_k(\omega, \rho)$, *and suppose that*

(i) $$\sum_{j\in S} |c_{kj}(\omega,\rho)| \le Cq_k(\omega), \qquad \sum_{k\in S} |c_{kj}(\omega,\rho)| \le Cq_j(\omega),$$

(ii) $$\sum_{j\in S} |\sigma_{kj}(\omega,\rho)| \le Cq_k^{1/2}(\omega), \qquad \sum_{k\in S} |\sigma_{kj}(\omega,\rho)| \le Cq_j^{1/2}(\omega).$$

Then for each $\varphi \in \mathbb{D}^1_{e1}(\Omega)$ *we have some* $p \in (1,2)$, $\sigma > 0$ *and* $K < +\infty$, *depending only on* φ, *such that*

$$\left|\overline{\nabla}_n \mathbb{P}^t_\rho \varphi(z)\right| \le \exp(-\sigma|z_n|)\mathsf{E}\left[\exp\left(KV_p(t,z)\right)\right],$$

where $\mathbb{P}^t_\rho\varphi(z) = \mathsf{E}[\varphi(\omega(t,z,\rho))]$.

Proof. Observe that $\overline{\nabla}^{(r)}_n\varphi(\omega) = \Sigma\langle \nabla_k\varphi(\omega), u_k\rangle$ if $\omega = \omega(t,z,\rho)$. Thus $\varphi \in \mathbb{D}^1_{e1}(\Omega)$ and the Cauchy inequality yield

$$\begin{aligned} |\overline{\nabla}^{(r)}_n\varphi| &\le \exp\left[c_\varphi(\varepsilon)\dot{Z}_p\right] \sum_{k\in S} e^{-\varepsilon|\omega_k|}|u_k| \\ &\le \exp\left[c_\varphi(\varepsilon)\dot{Z}_p\right]\left[\sum_k e^{-\varepsilon|\omega_k|}\right]^{1/2}\left[\sum_k e^{-\varepsilon|\omega_k|}u_k^2\right]^{1/2}, \end{aligned} \tag{5.5}$$

with some $\varepsilon > 0$ and $p \in (1,2)$. On the other hand, from (1.2)

$$\sum_{k\in S} e^{-\varepsilon|\omega_k|} \le K_1\overline{H}^{1/2}(\omega)\sum_{r=1}^{\infty} e^{-\varepsilon r}r^{d-1}\log(e+r), \tag{5.6}$$

while $(1+c^2)^{1/2} \le 1 + c$ and $\log(1+c) \le c$ if $c > 0$ result in

$$e^{-\varepsilon|x|} \le \theta(x,\lambda)\exp\left[\sigma + \lambda\left(1 + \log(1+\lambda^2)\right) + \lambda\log\frac{\varepsilon-\sigma+2}{\varepsilon-\sigma}\right] \tag{5.7}$$

for all $x \in \mathbb{R}^d$, $\lambda \ge 0$ and $0 < \sigma < \varepsilon$. Consequently,

$$|\overline{\nabla}^{(r)}_n\varphi| \le \exp\left[K_2\dot{Z}_p + K_2Z_1\log(e+Z_1)\right]R_1^{1/2}(t, KZ_1), \tag{5.8}$$

where

$$R_1(t,\lambda) = \sum_{k\in S}\theta(\omega_k(t,z,\rho),\lambda)|u_k(t,z,\rho)|^2. \tag{5.9}$$

Thus it is sufficient to show that

$$\mathsf{E}\left[R_1(t, KZ_1(t,z))\right] \le \exp(-\sigma|z_n|). \tag{5.10}$$

We have to calculate the stochastic differential of R_1 when $d\omega_k$ is given by (1.6), while

$$du_k = \sum_{j\in S} c_{kj}u_j\,dt + \sum_{j\in S}\langle\sigma_{kj}, u_j\rangle\,dw_k, \qquad k \in S. \tag{5.11}$$

The following abbreviations will also be very useful in the forthcoming proofs. If an $\mathscr{A}_t$-adapted process has a stochastic differential $dI = a\,dt + \Sigma b_k\,dw_k$, then

$d_d I = a\,dt$ denotes the deterministic part of dI. For R_1 we have $d_d R_1(t, KZ_1) = -\dot{Z}_1 I_0\,dt + (I_1 + I_2 + I_3 + I_4)\,dt$, where

$$I_q = \sum_{k \in S} I_{qk}, \quad \text{for } q = 0,1,2,\dots \quad \text{and} \quad I_{0k} = -K\theta_k' u_k^2,$$

$$I_{1k} = \left(\langle \operatorname{grad} \theta_k, c_k \rangle + \tfrac{1}{2}\Delta\theta_k \sigma_k^2\right) u_k^2,$$

$$I_{2k} = 2\theta_k \sum_{j \in S} \langle u_k, c_{kj} u_j \rangle,$$

$$I_{3k} = d\theta_k \left[\sum_{j \in S} \langle \sigma_{kj}, u_j \rangle \right]^2,$$

$$I_{4k} = \langle \operatorname{grad} \theta_k, u_k \rangle \sigma_k \sum_{j \in S} \langle \sigma_{kj}, u_j \rangle,$$

and, as in Section 2, $\theta_k = \theta(\omega_k, KZ_1)$, $\theta_k' = \theta'(\omega_k, KZ_1)$, and so on. Notice that $I_0 = -K\,\partial R_1/\partial\lambda$, thus $I_q \le \dot{Z}_1 I_0/4$ implies (5.10) as $R_1(0,0) = \exp(-\sigma|z_n|)$. This bound is trivial for I_1, the other terms here and below will be estimated by means of the following patterns. If x_k, y_j, a_{kj}, b_{kj} are positive numbers and $|\omega_k - \omega_j| \le R$, then from (5.4) we obtain

$$\text{(A)} \qquad 2\theta_k x_k a_{kj} y_j \le \theta_k a_{kj}\left(x_k^2 + y_j^2\right) \le \theta_k x_k^2 a_{kj} + C_1 a_{kj}\theta_j y_j^2,$$

$$\text{(B)} \qquad \theta_k \left[\sum_j b_{kj} y_j\right]^2 \le C_1 \left[\sum_j b_{kj}\right]\left[\sum_j b_{kj}\theta_j y_j^2\right],$$

with some C_1 depending only on σ. Indeed, applying (A) to I_2 and I_4, and (B) to I_3, the desired bounds follow from (i), (ii) and (5.2)–(5.4) by a direct calculation, which proves (5.10). Thus Proposition 4 follows by the Cauchy inequality. □

The next problem is the rate of convergence of $\mathbb{P}_\rho^t\varphi \to \mathbb{P}^t\varphi$.

PROPOSITION 5. *Let* $c_k' = \partial/\partial\rho\, c_k(\omega, \rho)$ *and* $\sigma_k' = \partial/\partial\rho\, \sigma_k(\omega, \rho)$, *and in addition to* (i) *and* (ii) *of Proposition* 4, *suppose that* $|c_k'(\omega, \rho)| \le C q_k(\omega)$, $|\sigma_k'(\omega, \rho)| \le C$. *If* $\varphi \in \mathbb{D}_{el}^1(\Omega)$, *then we have some* $p \in (1,2)$, $\sigma > 0$ *and* $K < +\infty$, *such that*

$$\left| \frac{\partial}{\partial\rho} \mathbb{P}_\rho^t \varphi(z) \right| \le \exp(-\sigma\rho) \mathsf{E}\left[\exp\left(K V_p(t, z)\right)\right].$$

PROOF. Since $\partial/\partial\rho\, \varphi(\omega) = \Sigma\langle \nabla_k \varphi(\omega), \omega_k' \rangle$, this bound follows from

$$(5.12) \qquad \mathsf{E}\left[R_2(t, KZ_1(t, z))\right] \le M e^{-\sigma\rho} \rho^{d-1} \log(e + \rho) \mathsf{E}\left[Z_2(t, z)\right],$$

in the same way as Proposition 4 has been derived, where

$$(5.13)\qquad R_2(t,\lambda) = \sum_{k\in S} \theta(\omega_k(t,z,\rho),\lambda)|\omega_k'(t,z,\rho)|^2,$$

$$(5.14)\quad d\omega_k' = c_k'\,dt + \sum_{j\in S} c_{kj}\omega_j'\,dt + \sigma_k'\,dw_k + \sum_{j\in S}\langle \sigma_{kj},\omega_j'\rangle\,dw_k, \qquad k\in S.$$

To prove (5.12) observe that $d_dR_2 = (-\dot{Z}_1I_0 + I_1 + \cdots + I_4 + I_5)\,dt$, where $I_0,\dots,I_4$ are as in d_dR_1, but with ω_k' in the place of u_k, while $I_5 = \Sigma I_{5k}$ and

$$I_{5k} = 2\theta_k\langle c_k',\omega_k'\rangle + 2\langle \operatorname{grad}\theta_k,\omega_k'\rangle\sigma_k\sigma_k' + d\theta_k\left[\sigma_k'^2 + 2\sigma_k'\sum_{j\in S}\langle\sigma_{kj},\omega_j'\rangle\right].$$

In view of the proof of (5.12) we have $I_q \le \dot{Z}_1I_0/5$ if $q = 1,2,3,4$ and K is large, while I_5 can be estimated by applying $|\omega_k'| \le 1 + \omega_k'^2$ to each term which is linear in ω_k'. Indeed, all quadratic terms arising in this way can be dominated by $\dot{Z}_1I_0/5$ via (A) and (5.2)–(5.4). Consequently, as $R_2(0,0) = 0$,

$$(5.15)\qquad \mathsf{E}[R_2(t,KZ_1(t,z))] \le K_1\sum_{k\in S}\int_0^t \mathsf{E}[\theta_k l_k(\omega_k)q_k(\omega)]\,ds,$$

where l_k denotes the indicator function of $B(\rho+R)\setminus B(\rho)$. This means that $\theta_k \le e^{-\rho\sigma}$ if $l_k \ne 0$, whence (5.12) follows by the superstability of U, which completes the proof. □

In the second variational system $\sigma_{kji} = \nabla_i\nabla_j\sigma_k(\omega,\rho)$ is a matrix, while $c_{kji} = \nabla_i\nabla_jc_k(\omega,\rho)$ is a vector formed by d matrices, namely $c_{kji}^{(q)} = \nabla_i\nabla_j\langle c_k, e_q\rangle$, $q = 1,2,\dots,d$. We denote

$$(5.16)\qquad [c_{kji}u_j,\bar{u}_i] = \sum_{q=1}^{d} e_q\langle c_{kji}^{(q)}u_j,\bar{u}_i\rangle,\qquad |c_{kji}|^2 = \sum_{q=1}^{d}|c_{kji}^{(q)}|^2,$$

i.e., $[c_{kji}\,\cdot\,,\cdot\,]$ is an $\mathbb{R}^d$-valued bilinear form.

Proposition 6. *In addition to* (i) *and* (ii) *of Proposition* 4, *let*

(i) $\sum_{j,i}|c_{kji}(\omega,\rho)| \le Cq_k(\omega),\qquad \sum_k |c_{kji}(\omega,\rho)| \le Cq_i^{1/2}(\omega)q_j^{1/2}(\omega),$

(ii) $\sum_{j,i}|\sigma_{kji}(\omega,\rho)| \le Cq_k^{1/2}(\omega),\qquad \sum_k|\sigma_{kji}(\omega,\rho)| \le Cq_i^{1/4}(\omega)q_j^{1/4}(\omega).$

If $\varphi\in\mathbb{D}_{el}^2(\Omega)$, *then with some* $p\in(1,2)$, $\sigma>0$ *and* $K<+\infty$,

$$|\overline{\nabla}_m\overline{\nabla}_n\varphi(z)| \le \exp(-\sigma|z_n| - \sigma|z_m|)\mathsf{E}\big[\exp(KV_p(t,z))\big].$$

Proof. Here we start with

$$\overline{\nabla}_m^{(s)}\overline{\nabla}_n^{(r)}\varphi(\omega) = \sum_{k\in S}\langle\nabla_k\varphi(\omega),v_k\rangle + \sum_{k\in S}\sum_{j\in S}\langle\nabla_j\nabla_k\varphi(\omega)u_k,\bar{u}_j\rangle;$$

estimation of the first sum follows (5.5). For the second one we apply the Cauchy inequality to derive

$$(5.17)\qquad \left|\left\langle \nabla_j\nabla_k\varphi(\omega)u_k, \bar{u}_j\right\rangle\right| \le \exp\left(c_\varphi \dot{Z}_p\right)e^{-\varepsilon|\omega_k|}|u_k|e^{-|\omega_j|}|\bar{u}_j|,$$

whence we obtain a factorizable bound. Therefore, if $0 < \sigma < \varepsilon/2$, then using (5.6) and (5.7) and $a^{1/2} + b^{1/2} \le (2a + 2b)^{1/2}$ we obtain

$$(5.18)\quad |\overline{\nabla}_m^{(s)}\overline{\nabla}_n^{(r)}\varphi| \le \exp\left[K_3\dot{Z}_p + K_3 Z_1 \log(e + Z_1)\right] R_3^{1/2}(t, KZ_1(t, z)),$$

where $\bar{R}_1$ is defined by (5.9) with $\bar{u}_k$ in the place of u_k,

$$(5.19)\quad R_3(t, \lambda) = \sum_{k\in S} \theta^2(\omega_k(t, z, \rho), \lambda)|v_k(t, z, \rho)|^2 + R_1(t, \lambda)\bar{R}_1(t, \lambda)$$

and

$$(5.20)\qquad \begin{aligned} dv_k = {} & \sum_{j\in S} c_{kj}v_j\,dt + \sum_{j\in S}\sum_{i\in S}\left[c_{kji}u_j, \bar{u}_i\right]dt \\ & + \sum_{j\in S}\langle \sigma_{kj}, v_j\rangle\,dw_k + \sum_{j\in S}\sum_{i\in S}\langle \sigma_{kji}u_j, \bar{u}_i\rangle\,dw_k, \qquad k\in S; \end{aligned}$$

therefore it is sufficient to prove

$$(5.21)\qquad \mathsf{E}\left[R_3(t, KZ_1(t, z))\right] \le \exp(-\sigma|z_n| - \sigma|z_m|).$$

The summands of $d_d R_3$ can be cast into three groups. The first consists of

$$J_{1k} = 2\left[\theta_k\langle \operatorname{grad}\theta_k, c_k\rangle + \theta_k\Delta\theta_k\sigma_k^2 + |\operatorname{grad}\theta_k|^2\theta_k^2\right]v_k^2,$$

$$J_{2k} = 2\theta_k^2\sum_j\langle v_k, c_{kj}v_j\rangle + 2\theta_k^2\sum_j\sum_i\left\langle v_k, \left[c_{kji}u_j, \bar{u}_i\right]\right\rangle,$$

$$J_{3k} = d\theta_k^2\left[\sum_j\langle \sigma_{kj}, v_j\rangle + \sum_j\sum_i\langle \sigma_{kji}u_j, \bar{u}_i\rangle\right]^2,$$

$$J_{4k} = 4\theta_k\langle \operatorname{grad}\theta_k, v_k\rangle\sigma_k\sum_j\left(\langle \sigma_{kj}, v_j\rangle + \sum_i\langle \sigma_{kji}u_j, \bar{u}_i\rangle\right)$$

and $J_q = \sum J_{qk}$. The second group consists of the terms $J_{4+q} = I_q\bar{R}_1 + \bar{I}_q R_1$, $q = 1, 2, 3, 4$, where the bar refers to $\bar{u}$, cf. the proof of Proposition 4. We have four mixed terms, namely

$$J_{9k} = |\operatorname{grad}\theta_k|^2\sigma_k^2 u_k^2\bar{u}_k^2,$$

$$J_{10k} = 4\theta_k^2\langle u_k, \bar{u}_k\rangle\left[\sum_j\langle \sigma_{kj}, u_j\rangle\right]\left[\sum_j\langle \sigma_{kj}, \bar{u}_j\rangle\right],$$

$$J_{11k} = 2\theta_k\langle \operatorname{grad}\theta_k, \bar{u}_k\rangle u_k^2\sigma_k\sum_j\langle \sigma_{kj}, \bar{u}_j\rangle,$$

and J_{12} is obtained from J_{11} by interchanging u and $\bar{u}$.

We have $d_d R_3 = -\dot{Z}_1 J_0\,dt + (J_1 + J_2 + \cdots + J_{12})\,dt$ and we need $J_q \le \dot{Z}_1 J_0/12$, where $J_0 = -K\,\partial R_3/\partial\lambda$, which follow by the patterns (A) and (B) as well as by their extensions (C) and (D):

$$\text{(C)} \qquad 2\theta_k^2 x_k a_{kji} y_j \bar{y}_i \le \theta_k^2 x_k^2 a_{kji} + C_1 a_{kji} \theta_j y_j^2 \theta_i \bar{y}_i^2,$$

$$\text{(D)} \qquad \theta_k^2 \left[\sum_{j,i} b_{kji} y_j \bar{y}_i \right]^2 \le C_1^2 \left[\sum_{j,i} b_{kji} \right] \left[\sum_{j,i} b_{kji} \theta_j y_j^2 \theta_i \bar{y}_i^2 \right],$$

where a_{kji}, b_{kji}, x_k, y_j, $\bar{y}_i$ are nonnegative numbers. The cases of J_1 and J_9 are trivial, the estimation of J_q with $q = 5, 6, 7, 8$, reduces to that of I_q and $\bar{I}_q$, see the proof of (5.10). The desired bounds for the first sums in J_2 and J_4 follow by (A), for the first sum in J_3 we use (B). The second sums of J_2 and J_4 can be bounded by means of (C), while for the second sum in J_3 we need (D); the sums over three indices can be factorized by means of (i) and (ii). Notice that $c_{kji} \neq 0$ or $\sigma_{kji} \neq 0$ imply $|\omega_k - \omega_j| \le R$ and $|\omega_k - \omega_i| \le R$, the factors q_k, q_j, q_i are of the same order. Finally, using $\Sigma a_k b_k \le (\Sigma a_k)(\Sigma b_k)$ if $a_k, b_k \ge 0$, the desired bounds of $J_{10}, \ldots, J_{12}$ follow directly by (A), which completes the proof. □

A rate for $\overline{\nabla}_n \mathbb{P}_\rho^t \varphi \to \overline{\nabla}_n \mathbb{P}^t \varphi$ as $\rho \to +\infty$ is given by:

Proposition 7. *Let $c'_{kj} = \partial/\partial\rho\, c_{kj}(\omega, \rho)$, $\sigma'_{kj} = \partial/\partial\rho\, \sigma_{kj}(\omega, \rho)$, and in addition to all conditions of Propositions 4–6, suppose that*

$$\text{(i)} \qquad \sum_{j \in S} \left| c'_{kj}(\omega, \rho) \right| \le C q_k(\omega), \qquad \sum_{k \in S} \left| c'_{kj}(\omega, \rho) \right| \le C q_j(\omega),$$

$$\text{(ii)} \qquad \sum_{j \in S} \left| \sigma'_{kj}(\omega, \rho) \right| \le C q_k^{1/2}(\omega), \qquad \sum_{k \in S} \left| \sigma'_{kj}(\omega, \rho) \right| \le C q_j^{1/2}(\omega).$$

Then, for each $\varphi \in \mathbb{D}_{el}^2(\Omega)$ we have some $p \in (1,2)$, $\sigma > 0$ and $K < +\infty$, such that

$$\left| \frac{\partial}{\partial \rho} \overline{\nabla}_n \varphi(z) \right| \le \exp(-\sigma\rho - \sigma|z_n|) \mathsf{E}\left[\exp\left(K V_p(t, z) \right) \right].$$

Proof. Since we have

$$\frac{\partial}{\partial \rho} \overline{\nabla}_n^{(r)} \varphi(\omega) = \sum_{k \in S} \left\langle \nabla_k \varphi(\omega), u'_k \right\rangle + \sum_{k \in S} \sum_{j \in S} \left\langle \nabla_j \nabla_k \varphi(\omega) u_k, \omega'_j \right\rangle,$$

following the proof of Proposition 6 we see that the statement reduces to

$$\begin{aligned} (5.22) \qquad & \mathsf{E}\left[R_4(t, K Z_1(t, z)) \right] \\ & \le M \exp(-\sigma\rho - \sigma|z_n|) \rho^{d-1} \log(e + \rho) \left[t \mathsf{E}\left[Z_4(t, z) \right] \right]^{1/2}, \end{aligned}$$

where K and M are universal constants,

$$(5.23) \qquad R_4(t, \lambda) = \sum_{k \in S} \theta^2(\omega_k(t, z, \rho), \lambda) \left| u'_k(t, z, \rho) \right|^2 + R_1(t, \lambda) R_2(t, \lambda)$$

and for all $k \in S$,

$$du'_k = \sum_j c'_{kj} u_j\, dt + \sum_j \sum_i [c_{kji} u_j, \omega'_i]\, dt + \sum_j c_{kj} u'_j\, dt$$
$$+ \sum_j \langle \sigma'_{kj}, u_j \rangle\, dw_k + \sum_j \sum_i \langle \sigma_{kji} u_j, \omega'_i \rangle\, dw_k + \sum_j \langle \sigma_{kj}, u'_j \rangle\, dw_k. \tag{5.24}$$

Most terms of $d_d R_4$ have appeared already in $d_d R_3$ with the notational difference that now u'_k plays the role of v_k and ω'_k that of $\bar{u}_k$. We have $d_d R_4 = (J_1 + J_2 + \cdots + J_{12} - \dot{Z}_1 J_0)\, dt + (J_{13} + J_{14} + R_1 I_5 + J_{15})\, dt$, where the additional terms are those containing some of the coefficients $c'_k, \sigma'_k, c'_{kj}, \sigma'_{kj}$, namely

$$J_{13k} = 2\theta_k \sum_j \left[\theta_k \langle u'_k, c'_{kj} u_j \rangle + 2\langle \operatorname{grad} \theta_k, u'_k \rangle \sigma_k \langle \sigma'_{kj}, u_j \rangle \right],$$

$$J_{14k} = d\theta_k^2 \left[\sum_j \langle \sigma'_{kj}, u_j \rangle \right]^2$$
$$+ 2d\theta_k^2 \left[\sum_j \langle \sigma'_{kj}, u_j \rangle \right] \left[\sum_j \left(\langle \sigma_{kj}, u'_j \rangle + \sum_i \langle \sigma_{kji} u_j, \omega'_i \rangle \right) \right],$$

$$J_{15k} = 2\theta_k \langle \operatorname{grad} \theta_k, \omega'_k \rangle u_k^2 \sigma_k \sigma'_k + 4\theta_k^2 \langle u_k, u'_k \rangle \sum_j \langle \sigma_{kj}, u_j \rangle \sigma'_k,$$

and $J_q = \Sigma J_{qk}$ for $q = 13, 14, 15$. We know that $J_q \le \dot{Z}_1 J_0 / 16$ if $q \le 12$. For $q > 12$ including $J_{16} = R_1 I_5$ we prove that

$$J_q \le \tfrac{1}{16} \dot{Z}_1 J_0 + M_1 e^{-\sigma\rho} \rho^{d-1} \log(e + \rho) \dot{Z}_2 R_1; \tag{5.25}$$

but $\mathsf{E}[R_1^2] \le e^{-2\sigma|z_n|}$ from (5.21), which implies (5.22) by the Cauchy inequality. To prove (5.25) for J_{13} it is sufficient to separate u'_k from u_j by (A) in both sums and observe that $\theta_k \le e^{-\sigma\rho}$ whenever $c'_{kj} \ne 0$ or $\sigma'_{kj} \ne 0$. For $q = 14$ observe that J_{14k} can be written as $J_{14k} = A_k^2 + 2A_k B_k + 2A_k C_k \le 3A_k^2 + B_k^2 + C_k^2$, where ΣA_k^2 and ΣB_k^2 can be estimated by means of (B), while for ΣC_k^2 we need (D). For the first sum of J_{15} we use $|\omega'_k| \le 1 + \omega_k'^2$ and $\Sigma a_k b_k \le (\Sigma a_k)(\Sigma b_k)$ if $a_k, b_k \ge 0$, for the second one (C) yields the desired bound. Finally, the estimation of J_{16} reduces to that of I_5, see the proof of (5.12), which completes the proof of Proposition 7. □

The convergence of second derivatives of $\mathbb{P}_\rho^t \varphi$ is controlled by the third variational system involving four new quantities and two trilinear forms. Let $\sigma'_{kji} = \partial/\partial\rho\, \sigma_{kji}(\omega, \rho)$ and $c'_{kji} = \partial/\partial\rho\, c_{kji}(\omega, \rho)$; $\sigma_{kjih} = \nabla_h \sigma_{kji}(\omega, \rho)$ and $c_{kjih} = \nabla_h c_{kji}(\omega, \rho)$, where σ_{kjih} is an $r \times r \times r$ matrix, while c_{kjih} is a vector consisting of such matrices. The associated trilinear forms will be denoted as $\langle \sigma_{kjih} | u_j, \bar{u}_i, \omega'_h \rangle$ and $[c_{kjih} | u_j, \bar{u}_i, \omega'_h] = \Sigma e_q \langle \nabla_h \nabla_i \nabla_j c_k^{(q)} | u_j, \bar{u}_i, \omega'_h \rangle$. Of course, by the Cauchy inequality

$$\left|\langle \sigma_{kjih} | u_j, \bar{u}_i, \omega'_h \rangle\right| \le |\sigma_{kjih}|\, |u_j|\, |\bar{u}_i|\, |\omega'_h|,$$

$$\left|\left[c_{kjih} | u_j, \bar{u}_i, \omega'_h \right]\right| \le |c_{kjih}|\, |u_j|\, |\bar{u}_i|\, |\omega'_h|.$$

PROPOSITION 8. *In addition to all conditions of this section, suppose that we have a universal C such that*

(i) $\sum_j \sum_i \left| c'_{kji}(\omega, \rho) \right| \le C q_k(\omega), \qquad \sum_k \left| c'_{kji}(\omega, \rho) \right| \le C q_j^{1/2}(\omega) q_i^{1/2}(\omega),$

(ii) $\sum_j \sum_i \left| \sigma'_{kji}(\omega, \rho) \right| \le C q_k^{1/2}(\omega), \qquad \sum_k \left| \sigma'_{kji}(\omega, \rho) \right| \le C q_j^{1/4}(\omega) q_i^{1/4}(\omega),$

(iii) $\sum_{j,i,h} \left| c_{kjih}(\omega, \rho) \right| \le C q_k(\omega), \qquad \sum_{j,i,h} \left| \sigma_{kjih}(\omega, \rho) \right| \le C q_k^{1/2}(\omega),$

(iv) $\sum_k \left| c_{kjih}(\omega, \rho) \right| \le C q_j^{1/3}(\omega) q_i^{1/3}(\omega) q_h^{1/3}(\omega),$

(v) $\sum_k \left| \sigma_{kjih}(\omega, \rho) \right| \le C q_j^{1/6}(\omega) q_i^{1/6}(\omega) q_h^{1/6}(\omega).$

Let $\varphi \in \mathbb{D}^3_{el}(\Omega)$. *Then with some* $p \in (1,2)$, $\sigma > 0$, $K < +\infty$ *we have*

$$\left| \frac{\partial}{\partial \rho} \nabla_m \nabla_n \varphi(z) \right| \le \exp(-\sigma\rho - \sigma|z_n| - \sigma|z_m|) \mathsf{E}\left[\exp\left(K V_p(t, z)\right)\right].$$

PROOF. We are faced with a fairly involved expression, namely

$$\begin{aligned} \frac{\partial}{\partial \rho} \overline{\nabla}^{(s)}_m \overline{\nabla}^{(r)}_n \varphi(\omega) = {} & \sum_{k \in S} \left\langle \nabla_k \varphi(\omega), v'_k \right\rangle + \sum_{k \in S} \sum_{j \in S} \left\langle \nabla_j \nabla_k \varphi(\omega) v_k, \omega'_j \right\rangle \\ & + \sum_{k \in S} \sum_{j \in S} \left\langle \nabla_j \nabla_k \varphi(\omega) u'_k, \bar{u}_j \right\rangle + \sum_{k \in S} \sum_{j \in S} \left\langle \nabla_j \nabla_k \varphi(\omega) u_k, \bar{u}'_j \right\rangle \\ & + \sum_{k \in S} \sum_{j \in S} \sum_{i \in S} \left\langle \nabla_i \nabla_j \nabla_k \varphi(\omega) \middle| u_k, \bar{u}_j, \omega'_i \right\rangle. \end{aligned}$$

Fortunately, we do not need any new trick to estimate this sum. Using $\varphi \in \mathbb{D}^3_{el}(\Omega)$ and (5.6) and (5.7), an easy calculation results in

$$\left| \frac{\partial}{\partial \rho} \overline{\nabla}^{(s)}_m \overline{\nabla}^{(r)}_n \varphi \right| \le \exp\left[K_4 \dot{Z}_p + K_4 Z_1 \log(e + Z_1)\right] R_5^{1/2}(t, K Z_1(t, z)), \tag{5.26}$$

where $\sigma < \varepsilon/3$ and

$$\begin{aligned} R_5(t, \lambda) = {} & \sum_{k \in S} \theta^3(\omega_k(t, z, \rho), \lambda) \left| v'_k(t, z, \rho) \right|^2 \\ & + R_3(t, \lambda) R_2(t, \lambda) + R_4(t, \lambda) \overline{R}_1(t, \lambda) + \overline{R}_4(t, \lambda) R_1(t, \lambda), \end{aligned} \tag{5.27}$$

where the bar refers to $\bar{u}$ and $\bar{u}'$, respectively, and

$$\begin{aligned} dv_k' = {} & \sum_j c_{kj} v_j' \, dt + \sum_j c_{kj}' v_j \, dt + \sum_j \sum_i [c_{kji} v_j, \omega_i'] \, dt \\ & + \sum_j \sum_i [c_{kji}' u_j, \bar{u}_i] \, dt + \sum_j \sum_i [c_{kji} u_j', \bar{u}_i] \, dt + \sum_j \sum_i [c_{kji} u_j, \bar{u}_i'] \, dt \\ & + \sum_j \sum_i \sum_h [c_{kjih} | u_j, \bar{u}_i, \omega_h'] \, dt + \sum_j \langle \sigma_{kj}, v_j' \rangle \, dw_k + \sum_j \langle \sigma_{kj}', v_j \rangle \, dw_k \\ & + \sum_j \sum_i \langle \sigma_{kji} v_j, \omega_i' \rangle \, dw_k + \sum_j \sum_i \langle \sigma_{kji}' u_j, \bar{u}_i \rangle \, dw_k + \sum_j \sum_i \langle \sigma_{kji} u_j, \bar{u}_i' \rangle \, dw_k \\ & + \sum_j \sum_i \langle \sigma_{kji} u_j', \bar{u}_i \rangle \, dw_k + \sum_j \sum_i \sum_h \langle \sigma_{kjih} | u_j, \bar{u}_i, \omega_h' \rangle \, dw_k. \end{aligned} \tag{5.28}$$

Just like Proposition 7, it is sufficient to show that

$$d_d R_5 \le M e^{-\sigma\rho} \rho^{d-1} \log(e + \rho) \dot{Z}_2 R_3, \tag{5.29}$$

where $\lambda = KZ_1(t, z)$ and K, M are big constants. Indeed, using

$$\mathsf{E}\big[R_3^2(t, KZ_1(t, z))\big] \le \exp(-2\sigma|z_n| - 2\sigma|z_m|) \tag{5.30}$$

and the Cauchy inequality, we obtain Proposition 8 by comparing (5.26) and (5.29).

First we prove (5.30). We need $d_d R_3^2 \le 0$ for large K. For every stochastic differential $dy_k = c_k^y \, dt + \sigma_k^y \, dw_k$, $y_k \in \mathbb{R}^d$, $k \in S$, we introduce the differential operator $D_k^y = \sigma_k^y \nabla_k^y$, where ∇_k^y is the gradient operator associated with y_k. Then we have

$$\begin{aligned} d_d R_3^2 &= 2R_3 d_d R_3 + \sum_{k\in S} |D_k^\omega R_3 + \bar{R}_1 D_k^u R_1 + R_1 D_k^{\bar{u}} \bar{R}_1 + D_k^v R_3|^2 \, dt \\ &\le 2R_3 d_d R_3 + 4 \sum_{k\in S} \big(|D_k^\omega R_3|^2 + |\bar{R}_1 D_k^u R_1|^2 + |R_1 D_k^{\bar{u}} \bar{R}_1|^2 + |D_k^v R_3|^2\big) \, dt; \end{aligned} \tag{5.31}$$

each term here should be bounded by a multiple of $R_3 \dot{Z}_1 J_0$. The case of $D_d R_3$ is known from the proof of Proposition 6, the next three sums are trivial, while $|D_k^v R_3|^2 = 4\theta_k^4 v_k^2 (\sigma_k^v)^2$, and the right bound of $|v_k|\,|\sigma_k^v|$ follows by (A) and (C). This completes the proof of (5.30) via $(\Sigma a_k^2 b_k^2) \le (\Sigma a_k^2)(\Sigma b_k^2)$.

Now we turn to the proof of (5.29), and most terms are quite familiar from previous calculations. The sums over four indices will be estimated by means of the patterns

$$2\theta_k^3 x_k a_{kjih} y_j \bar{y}_i z_h \le \theta_k^3 x_k^2 a_{kjih} + C_1^3 a_{kjih} \theta_j y_j^2 \theta_i \bar{y}_i^2 \theta_h z_h^2, \tag{E}$$

$$\theta_k^3 \Big[\sum b_{kjih} y_j \bar{y}_i z_h\Big]^2 \le C_1^3 \Big[\sum b_{kjih}\Big] \Big[\sum b_{kjih} \theta_j y_j^2 \theta_i \bar{y}_i^2 \theta_h z_h^2\Big], \tag{F}$$

where $a_{kjih} > 0$, $b_{kjih} \ge 0$, $|\omega_k - \omega_f| \le R$ for $f = i, j, h$, and the sums in (F) are over the triplet j, i, h. The majority of the summands of $d_d R_5$ will be estimated by a multiple of $\dot{Z}_1 L_0$, where $L_0 = -\partial R_5/\partial\lambda$. In the remainder the symbols c' and σ' are involved; for such terms $\theta_k \le e^{-\sigma\rho}$; and the quantity on the right-hand side of (5.29) also appears in the bound. New terms of $d_d R_5$ are $L_q = \Sigma L_{qk}$,

$q = 1, 2, 3, 4$, see (5.28) for $c_k^{v'}$ and $\sigma_k^{v'}$ defined by $dv'_k = c_k^{v'}\,dt + \sigma_k^{v'}\,dw_k$.

$$L_{1k} = \left(d_d\theta_k^3\right)v_k'^2, \qquad L_{2k} = 2\theta_k^3\langle v'_k, c_k^{v'}\rangle,$$

$$L_{3k} = 2d\theta_k^3\left(\sigma_k^{v'}\right)^2, \qquad L_{4k} = 3\theta_k^2\langle \operatorname{grad}\theta_k, v'_k\rangle\sigma_k\sigma_k^{v'}.$$

The case of L_1 is trivial, while $2\langle \operatorname{grad}\theta_k, v'_k\rangle\sigma_k^{v'} \le C\theta_k v_k'^2 + C\theta_k(\sigma_k^{v'})^2$ reduces the treatment of L_4 to that of L_3. The first two sums in the expression of L_2 can be estimated by means of (A), to the next four terms we apply (C), while the seventh sum can be treated by means of (E), in this last case a threefold product appears. The treatment of L_3 is analogous, but here (B), (D) and (F) should be used, respectively.

The second, and last, group of new expressions is formed by the terms of $d_dR_2R_3 - R_2d_dR_3 - R_3d_dR_2$, $d_d\overline{R}_1R_4 - \overline{R}_1d_dR_4 - R_4d_d\overline{R}_1$ and $d_dR_1\overline{R}_4 - R_1d_d\overline{R}_4 - \overline{R}_4d_dR_1$. Similarly as in (5.31), we have

$$\begin{aligned}(5.32)\qquad & d_dR_2R_3 - R_2d_dR_3 - R_3d_dR_2\\ &= \sum_{k\in S}\langle D_k^{\omega}R_2 + D_k^{\omega'}R_2, D_k^{\omega}R_3 + R_1D_k^{\bar u}\overline{R}_1 + \overline{R}_1D_k^{u}R_1 + D_k^{v}R_3\rangle.\end{aligned}$$

The cases of the second and third expressions are completely analogous, therefore we have to estimate $D_k^yR_q$ and $D_k^y\overline{R}_q$ for $y = \omega, u, \bar u, \omega', v, u', \bar u'$ and for $q = 1, 2, 3, 4$; then we can apply $(\Sigma a_k^2b_k^2) \le (\Sigma a_k^2)(\Sigma b_k^2)$. The cases with $y = \omega$ are trivial, in all other cases we have to apply (A) and (C) to conclude (5.29), which completes the proof of Proposition 8. □

REMARK 6. In view of Proposition 2, the Laplace method can be applied to evaluate $\mathsf{E}[\exp(KV_p(t, z))]$, but the explicit form of this expectation is useless, we need only its finiteness. In the case of the equilibrium dynamics the initial configuration, z is also random, and the nonequilibrium a priori bound of Proposition 2 yields $+\infty$ for the expectation of $\exp(KV_p(t, z))$ if $p > 1$. Therefore it is very important that $\sup_{s\le t}\overline{H}(\omega(s, z))$ is not involved in the expression of $V_p(t, z)$. Another class of problems is implied by the dependence of the diffusion coefficient on the configuration because then the variational systems are also stochastic. These two problems are responsible for the sophisticated technicalities of this section; we could not apply a more standard iteration method, cf. [7] and [16].

REMARK 7. The results of this section are very general. Concerning the structure of (1.6) we have only used

$$\text{(i)}\qquad |c_k(\omega, \rho)| \le Cq_k(\omega), \qquad |\sigma_k(\omega, \rho)| \le C.$$

This means that as soon as we have an a priori bound for V_p we can conclude that $\mathbb{P}_\rho^t\varphi \to \mathbb{P}^t\varphi$, $\overline{\nabla}_n\mathbb{P}_\rho^t\varphi \to \overline{\nabla}_n\mathbb{P}^t\varphi$ and $\overline{\nabla}_m\overline{\nabla}_n\mathbb{P}_\rho^t\varphi \to \overline{\nabla}_m\overline{\nabla}_n\mathbb{P}^t\varphi$ as $\rho \to +\infty$, provided that φ belongs to the corresponding functional space $\mathbb{D}_{el}^q(\Omega)$, $q = 1, 2, 3$.

REMARK 8. Since

$$\mathbb{P}^t\varphi = \mathbb{P}_\rho^t\varphi + \int_\rho^{+\infty}\frac{\partial}{\partial s}\mathbb{P}_s^t\varphi\,ds,$$

Proposition 5 yields a new construction of the transition semigroup $\mathbb{P}^t$, but the conditions for this approach are more restrictive than those for the iteration technique of Section 4. This iteration method is needed also to prove the uniqueness of tempered solutions. On the other hand, the results of Section 5 imply explicit exponential bounds for the rate of convergence of the partial dynamics $\mathbb{P}_\rho^t$ to $\mathbb{P}^t$.

6. On the generator of the transition semigroup. Now we are in a position to conclude Theorem 4 and some further results on $\mathbb{L}\mathbb{P}^t$; the general framework related to (1.6) will be adopted, thus Theorem 5 below will be a proper generalization of Theorem 4. The interaction potential U is assumed to be smooth and superstable. This U is used in the definition of V_p, but the particular structure of (1.6) will not be exploited any more. All the specific information we need on the dynamics is contained in the a priori bound

$$\mathsf{E}\big[\exp\big(KV_p(t,z)\big)\big] \le E_p(t,z), \tag{6.1}$$

where either E_p or its distribution does not depend on the partial dynamics we consider. We are assuming that we are given a set of measurable coefficients $c_k\colon \Omega \to \mathbb{R}^d$ and $\sigma_k\colon \Omega \to \mathbb{R}$, $k \in S$, such that $\nabla_j c_k(\omega) = 0$ and $\nabla_j \sigma_k(\omega) = 0$ whenever $|\omega_k - \omega_j| \ge R$, i.e., the interaction has a finite radius R. Moreover,

$$|c_k(\omega)| \le Cq_k(\omega), \qquad |\sigma_k(\omega)| \le C, \tag{6.2}$$

$$\sum_{j\in S} |\nabla_j c_k(\omega)| \le Cq_k(\omega), \qquad \sum_{k\in S} |\nabla_j c_k(\omega)| \le Cq_j(\omega), \tag{6.3}$$

$$\sum_{j\in S} |\nabla_j \sigma_k(\omega)| \le Cq_k^{1/2}(\omega), \qquad \sum_{k\in S} |\nabla_j \sigma_k(\omega)| \le Cq_j^{1/2}(\omega), \tag{6.4}$$

$$\sum_i \sum_j |\nabla_i \nabla_j c_k(\omega)| \le Cq_k(\omega), \qquad \sum_k |\nabla_i \nabla_j c_k(\omega)| \le Cq_i^{1/2}(\omega) q_j^{1/2}(\omega), \tag{6.5}$$

$$\sum_i \sum_j |\nabla_i \nabla_j \sigma_k(\omega)| \le Cq^{1/2}(\omega), \qquad \sum_k |\nabla_i \nabla_j \sigma_k(\omega)| \le Cq_i^{1/4}(\omega) q_j^{1/4}(\omega), \tag{6.6}$$

$$\begin{aligned} &\sum_h \sum_i \sum_j |\nabla_h \nabla_i \nabla_j c_k(\omega)| \le Cq_k(\omega), \\ &\sum_h \sum_i \sum_j |\nabla_h \nabla_i \nabla_j \sigma_k(\omega)| \le Cq_k^{1/2}(\omega), \end{aligned} \tag{6.7}$$

$$\sum_k |\nabla_h \nabla_i \nabla_j c_k(\omega)| \le Cq_h^{1/3}(\omega) q_i^{1/3}(\omega) q_j^{1/3}(\omega), \tag{6.8}$$

$$\sum_k |\nabla_h \nabla_i \nabla_j \sigma_k(\omega)| \le Cq_h^{1/6}(\omega) q_i^{1/6}(\omega) q_j^{1/6}(\omega), \tag{6.9}$$

with some universal constant C, the functions q_k, $k \in S$, have been defined before Proposition 4. It is easy to verify that if U is not singular and it has four continuous derivatives, then the coefficients of (0.1) satisfy the above set of conditions.

Now we introduce the coefficients defining the partial dynamics by means of a cut-off function $f = f(x, \rho)$, $x \in \mathbb{R}^d$, $\rho \geq 1$; this f differs from that used in Section 2. Let $f(x, \rho) = f_0(|x| - \rho)$, where f_0: $\mathbb{R} \to [0,1]$ is nonincreasing and three times continuously differentiable, and $f_0(y) = 1$ if $y \leq 0$, $f_0(y) = 0$ if $y \geq R$, then $\sigma_k(\omega, \rho) = f(\omega_k, \rho)\sigma_k(\omega)$ and $c_k(\omega, \rho) = f^2(\omega_k, \rho)c_k(\omega) + \sigma_k^2(\omega)f(\omega_k, \rho)\mathrm{grad}\, f(\omega_k, \rho)$ coincide with $\sigma_k(\omega)$ and $c_k(\omega)$, respectively, if $|\omega_k| \leq \rho$, and both vanish if $|\omega_k| \geq \rho + R$. Since the bounds of the derivatives of f do not depend on ρ, it is quite easy to verify the conditions of Propositions 4–8. The reason for this artificial definition of $c_k(\omega, \rho)$ consists in the requirement that the reversibility condition (1.13) should remain in force if it was presupposed for the original coefficients c_k and σ_k.

Consider now the transition semigroup $\mathbb{P}_\rho^t$ defined by

$$d\omega_k = c_k(\omega, \rho)\, dt + \sigma_k(\omega, \rho)\, dw_k, \qquad k \in S, \tag{6.10}$$

with initial condition $\omega(0) = z \in \Omega$, i.e., $\mathbb{P}_\rho^t\varphi(z) = \mathsf{E}[\varphi(\omega(t, z))]$ if $\omega(\cdot, z)$ denotes the solution to (6.10) with initial condition $\omega(0, z) = z$. Since the elements of Ω are locally finite, (6.10) is a finite system, thus we have no problems concerning the smoothness of $\mathbb{P}_\rho^t$, see Chapter 4 of [6] or Chapter 7 of [1]. More exactly, $\mathbb{P}_\rho^t\varphi$, $\nabla_n\mathbb{P}_\rho^t\varphi$ and $\nabla_m\nabla_n\mathbb{P}_\rho^t\varphi$ are all uniformly continuous on each Ω_h if $\varphi \in \mathbb{C}_1(\Omega)$, $\varphi \in \mathbb{D}_{el}^1(\Omega)$ and $\varphi \in \mathbb{D}_{el}^2(\Omega)$, respectively. The results of the previous section imply:

Theorem 5. *Suppose that the a priori bound $E_p(t, z)$ of* (6.1) *does not depend on the parameter $\rho > 1$ of the partial dynamics* (6.10), *and $\|E_p(t,\cdot)\|_h < +\infty$ for each $h > 1$, $p \in (1,2)$ and $\sigma \in (0,1)$. If $\varphi \in \mathbb{D}_{el}^1(\Omega)$, then $\mathbb{P}_\rho^t\varphi$ converges to some limit $\mathbb{P}^t\varphi \in \mathbb{C}(\Omega)$ as $\rho \to +\infty$, the convergence is uniform on each Ω_h. Similarly, we have $\nabla_n\mathbb{P}_\rho^t\varphi \to \nabla_n\mathbb{P}^t\varphi$ if $\varphi \in \mathbb{D}_{el}^2(\Omega)$, $n \in S$, while $\nabla_m\nabla_n\mathbb{P}_\rho^t\varphi \to \nabla_m\nabla_n\mathbb{P}^t\varphi$ if $\varphi \in \mathbb{D}_{el}^3(\Omega)$, $n, m \in S$, again in the topology of $\mathbb{C}(\Omega)$ as $\rho \to +\infty$. Moreover, $\mathbb{P}^t\mathbb{D}_{el}^{q+1}(\Omega) \subset \mathbb{D}_e^q(\Omega)$ for $q = 1,2$, and for each $\varphi \in \mathbb{D}_{el}^3(\Omega)$ we have some $p \in (1,2)$, $\sigma > 0$ and $K < +\infty$, such that*

$$\left|\mathbb{P}_\rho^t\varphi(z) - \mathbb{P}^t\varphi(z)\right| + \left|\mathbb{L}\mathbb{P}_\rho^t\varphi(z) - \mathbb{L}\mathbb{P}^t\varphi(z)\right| \leq \mathsf{E}\left[\exp\left(KV_p(t, z)\right)\right]e^{-\sigma\rho},$$

where $\mathbb{L}$ denotes the formal generator of $\mathbb{P}^t$, see (1.11).

Proof. The convergence of $\mathbb{P}_\rho^t\varphi$ and that of its derivatives is a direct consequence of Propositions 5, 7 and 8, respectively. To identify the limits of the derivatives it is enough to notice that

$$\int_{\mathbb{R}^d}\varphi_1(z_n)\nabla_n\mathbb{P}_\rho^t\varphi(z)\, dz_n = -\int_{\mathbb{R}^d}[\nabla_n\varphi_1(z_n)]\mathbb{P}_\rho^t\varphi(z)\, dz_n$$

and

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi_2(z_n, z_m)\nabla_m\nabla_n\mathbb{P}_\rho^t\varphi(z)\, dz_n\, dz_m$$

$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\nabla_n\nabla_m\varphi_2(z_n, z_m))\mathbb{P}_\rho^t\varphi(z)\, dz_n\, dz_m$$

hold true for smooth test functions φ_1 and φ_2 of compact support. The exponential bound is now a direct consequence of Propositions 5, 7, and 8, while Propositions 4 and 6 extend to the infinite system by letting ρ go to infinity, which completes the proof of Theorem 5. □

REMARK 9. If $d \le 4$, then the conditions of Theorem 2 imply the a priori bound we need for Theorem 5, which proves Theorem 4. In the case of the equilibrium dynamics below a probability estimate will be used. Since we do not have any closed subspace $\overline{\mathbb{C}}$ of $\mathbb{C}(\Omega)$ such that $\mathbb{P}^t\overline{\mathbb{C}} \subset \overline{\mathbb{C}}$ and $\mathbb{P}^t$ happens to be strongly continuous on $\overline{\mathbb{C}}$, it does not make too much sense to talk about the infinitesimal generator of $\mathbb{P}^t$ in the spirit of semigroup theory. Notice, however, that $\mathbb{P}^t\mathbb{L}\varphi = \mathbb{L}\mathbb{P}^t\varphi$ whenever $\varphi \in \mathbb{D}^3_{el}(\Omega)$.

For systems in a Gibbsian equilibrium state μ we have a stationary a priori bound, namely the consequence (1.5) of Ruelle's [22] superstable probability estimate, see Lang [13] and Marchioro, Pellegrinotti and Pulvirenti [16]. In addition to (6.1)–(6.9) suppose that the coefficients c_k and σ_k do not depend on the enumeration of the particles and satisfy the reversibility condition (1.13), the dimension d of the space is arbitrary, the partial dynamics $\mathbb{P}^t_\rho$ is the very same as above. Let μ denote a tempered Gibbs state with interaction U at unit temperature and arbitrary activity; this μ is a probability measure on the space $\Omega^{(s)} = [\text{supp}\ \omega\colon\ \omega \in \Omega]$ of unlabelled configurations (integer valued measures on $\mathbb{R}^d$), see [2], [3], [21] and [22]. The elements of the Hilbert space $\mathbb{L}^2(\Omega^{(s)}, \mu)$ are symmetric functions on Ω, each $\mathbb{P}^t_\rho$ preserves symmetry as the dynamics does not depend on the enumeration of particles. In view of (1.13) the Gibbs state μ is a stationary and reversible measure for each partial dynamics. Thus for all $\varphi_1, \varphi_2 \in \mathbb{L}^2(\Omega^{(s)}, \mu)$ we have

$$\int \varphi_1 \mathbb{P}^t_\rho \varphi_2 \, d\mu = \int \varphi_2 \mathbb{P}^t_\rho \varphi_1 \, d\mu, \tag{6.11}$$

while (1.5) implies for each partial dynamics the bound

$$\int \mathsf{E}\big[\exp\big(KV_p(t,z)\big)\big]\mu(dz) \le C_p(t,K,\sigma) < +\infty, \tag{6.12}$$

where $p \in (1,2)$, $K > 0$, $\sigma > 0$, $t > 0$; C_p does not depend on ρ, and it is a monotonic function of its variables.

It is easy to check that each $\mathbb{P}^t_\rho$ is a strongly continuous semigroup of self-adjoint contractions in $\mathbb{L}^2(\Omega^{(s)}, \mu)$. Applying the a priori bound (6.12) to the estimate of Proposition 5 we see that $\mathbb{P}^t_\rho\varphi$ converges in $\mathbb{L}^2(\Omega^{(s)}, \mu)$ for each $\varphi \in \mathbb{L}^2(\Omega^{(s)}, \mu)$. The limit, denoted by $\mathbb{P}^t = \mathbb{P}^t\varphi$, is again a strongly continuous contraction semigroup in $\mathbb{L}^2(\Omega^{(s)}, \mu)$. Since (6.11) remains in force also for $\mathbb{P}^t$, the infinitesimal generator, $\mathbb{G}$ of $\mathbb{P}^t$ is a self-adjoint operator in $\mathbb{L}^2(\Omega^{(s)}, \mu)$, $\mathbb{G}$ is an extension of the formal generator $\mathbb{L}$ defined by (1.14). Let $\mathbb{D}^q_0(\Omega^{(s)})$ denote the space of symmetric and local functions from $\mathbb{D}^q_{el}(\Omega)$, $q = 1,2,3$; that is, $\varphi \in \mathbb{D}^q_0(\Omega^{(s)})$ means that $\varphi \in \mathbb{D}^q_{el}(\Omega)$ and there exists a $\rho > 0$ such that $\varphi(\omega) = \varphi(\overline{\omega})$ whenever $\sum f(\omega_k) = \sum f(\overline{\omega}_k)$ for all continuous $f\colon \mathbb{R}^d \to \mathbb{R}$ vanishing outside the ball $B(\rho)$. Notice that $\mathbb{L}$ is well defined on $\mathbb{D}_0^2(\Omega^{(s)})$ and $\mathbb{L}\varphi = \mathbb{G}\varphi \in \mathbb{L}^2(\Omega^{(s)}, \mu)$ if $\varphi \in \mathbb{D}_0^2(\Omega^{(s)})$, but $\mathbb{P}^t\mathbb{D}_0^3(\Omega^{(s)}) \subset \mathbb{D}_e^2(\Omega)$ is known only if $d \le 4$.

THEOREM 6. *If the space* $\mathbb{D}_0 \subset \mathbb{D}_0^3(\Omega^{(s)})$ *is dense in the sense that for each* $F \in \mathbb{D}_0^2(\Omega^{(s)})$ *there exists a sequence* $\varphi_n \in \mathbb{D}_0$ *such that* $\varphi_n \to F$ *weakly in* $\mathbb{L}^2(\Omega^{(s)}, \mu)$ *and* $\mathbb{L}\varphi_n$ *is bounded in* $\mathbb{L}^2(\Omega^{(s)}, \mu)$, *then* $\mathbb{D}_0$ *is a core for* $\mathbb{G}$, *that is* $\mathbb{L}$ *is essentially self-adjoint with domain* $\mathbb{D}_0$ *and its closure equals* $\mathbb{G}$.

PROOF. We have to show that for each φ from the domain of $\mathbb{G}$ there is a sequence $\varphi_n \in \mathbb{D}_0$ such that both $\varphi_n \to \varphi$ and $\mathbb{L}\varphi_n \to \mathbb{G}\varphi$ in $\mathbb{L}^2(\Omega^{(s)}, \mu)$, see Section X.8 of [19]. Let $\|f\| = [\int f^2\,d\mu]^{1/2}$ denote the norm in $\mathbb{L}^2(\Omega^{(s)}, \mu)$, then $\lambda\|f\|^2 \le \int f(\lambda f - \mathbb{L}f)\,d\mu$ if $f \in \mathbb{D}_0$ as $\mathbb{L} \le 0$, see (1.15). Therefore $\lambda\|f\| \le \|\lambda f - \mathbb{L}f\|$ if $\lambda > 0$. This means that $\lambda\varphi_n - \mathbb{L}\varphi_n \to \lambda F - \mathbb{G}F$ implies both $\varphi_n \to F$ and $\mathbb{L}\varphi_n \to \mathbb{G}F$. Consequently, it is sufficient to show that $[\lambda\varphi - \mathbb{L}\varphi\colon\ \varphi \in \mathbb{D}_0]$ is dense in $\mathbb{L}^2(\Omega^{(s)}, \mu)$ for some $\lambda > 0$. Suppose not, then we have a nonzero $g \in \mathbb{L}^2(\Omega^{(s)}, \mu)$ such that

$$\int g\mathbb{L}\varphi\,d\mu = \lambda\int g\varphi\,d\mu, \quad \text{if } \varphi \in \mathbb{D}_0, \tag{6.13}$$

and a contradiction is obtained as soon as we manage to extend (6.13) to functions of the type $F = \mathbb{P}^t\varphi$ with $\varphi \in \mathbb{D}_0$. Indeed, then

$$\frac{d}{dt}\int g\mathbb{P}^t\varphi\,d\mu = \int g\mathbb{P}^t\mathbb{G}\varphi\,d\mu$$

$$= \int g\mathbb{G}\mathbb{P}^t\varphi\,d\mu = \lambda\int g\mathbb{P}^t\varphi\,d\mu,$$

whence $\int g\mathbb{P}^t\varphi\,d\mu = e^{\lambda t}\int g\varphi\,d\mu$, which is possible only if $g = 0$. Let $F_\rho = \mathbb{P}_\rho^t\varphi$; it is obvious that $F_\rho \in \mathbb{D}_0^2(\Omega^{(s)})$, thus we have a sequence $\varphi_n \in \mathbb{D}_0$ such that $\varphi_n \to F_\rho$ weakly, while $\mathbb{L}\varphi_n$ remains bounded. Since $\int f\mathbb{L}\varphi_n\,d\mu = \int \varphi_n\mathbb{L}f\,d\mu$ if $f \in \mathbb{D}_0$, but $\mathbb{D}_0$ is dense, we see that $\mathbb{L}\varphi_n \to \mathbb{L}F_\rho$. Consequently,

$$\int g\mathbb{L}\mathbb{P}_\rho^t\varphi\,d\mu = \lambda\int g\mathbb{P}_\rho^t\varphi\,d\mu, \quad \text{if } \varphi \in \mathbb{D}_0. \tag{6.14}$$

On the other hand, the superstable estimate (6.1) and Proposition 8 imply the fundamental estimate

$$\|\mathbb{P}_\rho^t\varphi - \mathbb{P}^t\varphi\| + \|\mathbb{L}\mathbb{P}_\rho^t\varphi - \mathbb{G}\mathbb{P}^t\varphi\| \le C_\varphi e^{-\sigma\rho}, \quad \text{for } \varphi \in \mathbb{D}_0^3(\Omega^{(s)}), \tag{6.15}$$

as $\mathbb{G}$ is closed, where $\sigma > 0$ depends on φ. Thus letting ρ go to infinity in (6.14) we obtain

$$\int g\mathbb{G}\mathbb{P}^t\varphi\,d\mu = \lambda\int g\mathbb{P}^t\varphi\,d\mu, \quad \text{for } \varphi \in \mathbb{D}_0, \tag{6.16}$$

which completes the proof. □

REMARK 10. The most favored candidate for $\mathbb{D}_0$ is certainly the space $\mathbb{D}_0^\infty$ of symmetric functions of the type

$$\varphi(\omega) = F(\omega(f_1), \omega(f_2), \ldots, \omega(f_n)),$$

where $n \in \mathbb{N}$, $\omega(f) = \Sigma f(\omega_k)$, and $F\colon \mathbb{R}^n \to \mathbb{R}$ and $f_k\colon \mathbb{R}^d \to \mathbb{R}$ are infinitely differentiable with compact supports. The condition that $\mathbb{D}_0^\infty$ is dense is a direct consequence of (1.5) and the Stone–Weierstrass theorem, but to prove that $\mathbb{L}\varphi_n$ remains bounded, a very complicated construction is needed. This is a question of approximation theory we do not want to discuss here.

REMARK 11. It is very important that $\mathbb{P}_\rho^t\varphi$ is smooth if $\varphi \in \mathbb{D}_0$. This property is due to the continuity of the transition between the frozen and the living parts of the system. Such a construction is possible even in the case of Hamiltonian systems, see [16] for a different approach and the related difficulties.

REMARK 12. The hypothesis of Spohn's [24] theory of equilibrium fluctuations is closely related to Theorem 6, but instead of the familiar space $\mathbb{L}^2(\Omega^{(s)}, \mu)$, it is formulated in terms of the scalar product (1.16) and the associated Hilbert space $\mathbb{H}$. The strong form of this hypothesis, namely Proposition 2 of [24], claims that $\mathbb{D}_0^\infty$ is a core for $\mathbb{G}$ in $\mathbb{H}$; this assertion is used then to extend the inequality

$$|\langle F|g\rangle|^2 \le K_g\langle F|\mathbb{L}F\rangle, \qquad F \in \mathbb{D}_0^\infty, \tag{6.17}$$

to functions of the type $F = \mathbb{P}^t\varphi$, $\varphi \in \mathbb{D}_0^\infty$, where $g \in \mathbb{D}_0^2(\Omega^{(s)})$ is fixed and K_g depends only on g. For this purpose the full power of Proposition 2 of [24] is certainly not needed and it is sufficient to select a sequence $\varphi_n \in \mathbb{D}_0^\infty$ in such a way that $\langle\varphi_n|g\rangle \to \langle\mathbb{P}^t\varphi|g\rangle$ and $\langle\varphi_n|\mathbb{L}\varphi_n\rangle \to \langle\mathbb{P}^t\varphi|\mathbb{G}\mathbb{P}^t\varphi\rangle$. Just as in the procedure of passing from (6.13) to (6.16), we first have to extend (6.17) to $F = \mathbb{P}_\rho^t\varphi$, which is mainly a question of approximation theory, cf. Remark 10. The second step is to let ρ go to infinity and (6.15) can be used here. More exactly, we can find a sequence $\varphi_n \in \mathbb{D}_0^\infty$ such that

$$\int\varphi_n T_u g\,d\mu \to \int\mathbb{P}^t\varphi T_u g\,d\mu \quad\text{and}\quad \int\varphi_n T_u\mathbb{L}\varphi_n\,d\mu \to \int\mathbb{P}^t\varphi T_u\mathbb{G}\mathbb{P}^t\varphi\,d\mu,$$

for each $u \in \mathbb{R}^d$. To conclude the statement implying Proposition 3 of [24] it is sufficient to show that both convergence relations above are dominated, that is we can interchange the limit and the integral over $u \in \mathbb{R}^d$ defining $\langle\,\cdot\,|\,\cdot\,\rangle$ by (1.16). This is really true in the high-temperature–low-density domain, then μ has a very strong, exponential mixing property, see Lemma 4 of [24]. Using the trick that

$$\mathbb{P}_\rho^t\varphi = \mathbb{P}_1^t\varphi + \int_1^\rho \frac{\partial}{\partial s}\mathbb{P}_s^t\varphi\,ds,$$

and the analogous representation of the derivatives of $\mathbb{P}_\rho^t\varphi$, we can easily dominate $\int\mathbb{P}_\rho^t\varphi T_u g\,d\mu$ and $\int\mathbb{P}_\rho^t\varphi T_u\mathbb{L}\mathbb{P}_\rho^t\varphi\,d\mu$ by integrable functions of $u \in \mathbb{R}^d$, see Propositions 5, 7, and 8 and the proof of Lemma 5 in [24]. Using the symmetric form (1.15) of $\int F\mathbb{L}F\,d\mu$, we can reduce the problem to the control of first derivatives only, i.e., Proposition 8 involving the third variational system is not needed here. Of course, this is only a draft of the proof.

REMARK 13. The proof of the essential self-adjointness property stated in Proposition 2 of [24] is very similar. Following the proof of Theorem 6 we obtain the identity

$$\langle g|\mathbb{L}F\rangle = \lambda\langle g|F\rangle, \quad \text{for } F \in \mathbb{D}_0^\infty, \tag{6.18}$$

where $g \in \mathbb{H}$ is fixed; we have to extend (6.18) to functions of type $F = \mathbb{P}^t\varphi$, $\varphi \in \mathbb{D}_0^\infty$. In contrast to the problem of Remark 12, here we have also to control the second derivatives of $\mathbb{P}_\rho^t\varphi$, i.e., Proposition 8 and the third variational system cannot be abandoned. Since the function g of (6.18) is not a local one, we cannot apply the dominated convergence theorem directly; we have to show that $\langle f|\mathbb{P}_\rho^t\varphi\rangle \to \langle f|\mathbb{P}^t\varphi\rangle$ for smooth local f, as $\rho \to +\infty$, while $\mathbb{L}\mathbb{P}_\rho^t\varphi$ remains bounded in $\mathbb{H}$. This proof is essentially the same as in Remark 12.

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