The Annals of Probability
1987, Vol. 15, No. 3, 985–1008

ASYMPTOTIC PROPERTIES OF SOME MULTIDIMENSIONAL DIFFUSIONS

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Let $X_t \in \mathbf{R}^d$ be the solution to the stochastic differential equation

$$dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt, \qquad X_0 \in \mathbf{R}^d,$$

where B_t is a Brownian motion in $\mathbf{R}^d$. The aim of this paper is to make the following statement precise: "Let x_t be a solution of $\dot{x} = b(x)$. If $|x_t| \to \infty$ as $t \to \infty$ and the drift vector field $b(x)$ is well behaved near x_t then with positive probability, $X_t \to \infty$, and does so asymptotically like x_t." Examples are provided to illustrate the situations in which this theorem may be applied.

1. Introduction. Let X_t be the diffusion in $\mathbf{R}^d$ given by

$$X_t = X_0 + \int_0^t \sigma(X_s)\,dB_s + \int_0^t b(X_s)\,ds, \tag{1.1}$$

where X_0 is a point in $\mathbf{R}^d$ and B_t a standard Brownian motion in R^d. We suppose that σ and b are Lipschitz continuous with

$$|x^{\mathsf{T}} b(x)| + \operatorname{trace}\big(\sigma(x)\sigma(x)^{\mathsf{T}}\big) \le K\big(|x|^2 + 1\big)$$

so that (1.1) has a unique solution for which $|X_t| < \infty$ for all $t \ge 0$; see Durrett (1984) for this and the other facts about stochastic integrals that we shall use below.

Define the *flowline* $\{x_t, t \ge 0\}$ as the solution to the ordinary differential equation $\dot{x} = b(x)$ given by

$$x_t = x_0 + \int_0^t b(x_s)\,ds, \tag{1.2}$$

where x_0 is some point in $\mathbf{R}^d$, i.e., the process which results when we take $\sigma = 0$ and $X_0 = x_0$ in (1.1). Our objective is to give conditions on $b(x)$ so that $X_t \to \infty$ like x_t as $t \to \infty$.

The first result treats the case $d = 1$ with $\sigma(X_t)$ replaced by σ_t, a bounded predictable process, in (1.1). The result for non-Markov X_t is required in the proof of Theorem 2. Here and throughout, $\mathbf{P}_x$ refers to the probability measure induced on the space of continuous paths by the process X_t in (1.1) started at $X_0 = x$.

Received June 1985; revised November 1985.

AMS 1980 *subject classifications*. Primary 60H10; secondary 60J25.

Key words and phrases. Transience, asymptotic behavior, diffusion process, stochastic differential equation.

THEOREM 1. *Let*

$$X_t = x_0 + \int_0^t \sigma_s \, dB_s + \int_0^t b(X_s) \, ds,$$

a diffusion on $\mathbf{R}$, *where* B_t *is a standard Brownian motion on* $\mathbf{R}$, σ_t *is a bounded predictable process, and*

(i) $$\sigma_s^2 \le C \quad \textit{for all } s \ge 0,$$

(ii) $$b(x) > 0 \quad \textit{for } x \textit{ large},$$

(iii) $$\lim_{x \to \infty} \frac{b(cx)}{b(x)} = c^{\delta} \quad \textit{for some } -1 < \delta < 1.$$

Let

$$x(t, x_0) = x_0 + \int_0^t b(x(s)) \, ds,$$

a solution of $\dot{x} = b(x)$. *Then*

$$\mathbf{P}_{x_0}\left\{ \left| \frac{X_t}{x(t, x_0)} - 1 \right| < \varepsilon, \textit{ for all } t \ge 0 \right\} \to 1 \quad \textit{as } x_0 \to \infty$$

and consequently, on $\{X_t \to \infty\}$, *with* $X_0 = x_0$,

$$\frac{X_t}{x(t, x_0)} \to 1 \quad a.s.$$

Condition (iii) says that b is regularly varying with index δ. (See Feller (1971), Section VIII.8 for facts about such functions.) If $b(x) \ge c > 0$ then the above result is a consequence of the proof of Theorem 2 in Chapter 4, Section 17 of Gihman and Skorohod (1972).

To see what the theorem says we consider the special case $\sigma_s \equiv 1$ and $b(x) = x^{\delta}$ for $x > 1$. We can calculate (1.2) for $x_0 > 1$, $\delta < 1$,

(1.3) $$x(t) = \left((1 - \delta)t + x_0^{1-\delta}\right)^{1/(1-\delta)} \sim C_{\delta} t^{1/(1-\delta)}.$$

When $-1 < \delta < 1$ we can use Theorem 1 to conclude that as $t \to \infty$,

(1.4) $$\frac{X_t}{t^{1/(1-\delta)}} \to C_{\delta} \quad \text{a.s.}$$

and as we shall see below, this is (almost) the largest range for δ for which such a result can hold.

When $\delta < -1$, X_t is recurrent although $x_t \to \infty$. If we notice that the power $1/(1 - \delta) < \frac{1}{2}$ in this case this should not be surprising because B_t has standard deviation $t^{1/2}$ while x_t grows more slowly. For $\delta > 1$, (1.3) holds but $x(t) \to \infty$ as $t \to t^* = x_0^{1-\delta}/(\delta - 1)$. The process X_t has the same behavior; there is a finite random explosion time τ^* such that $X_t \to \infty$ as $t \to \tau^*$.

To treat dimensions $d \ge 2$ we must first quantify the statement "near the flowline"; Figure 1 illustrates the following definitions. Given $\{x_t, t \ge 0\}$ from

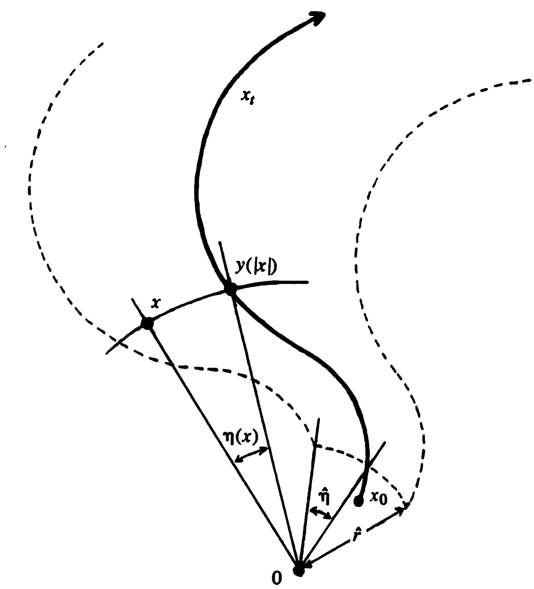

FIG. 1. *The region enclosed by dotted lines is* $C(\hat{r}, \hat{\eta})$.

(1.2), define $\tau(r) = \inf\{t \geq 0;\ |x_t| = r\}$. Later we make assumptions on b that ensure $|x_t|$ is strictly increasing in t, so that we may define

$$y(r) = x_{\tau(r)}, \tag{1.5}$$

$$\eta(x) = \cos^{-1}\left(\frac{x^{\mathsf{T}}y(|x|)}{|x|^2}\right), \tag{1.6}$$

or in words, $\eta(x)$ is the angle between x and the point $y(|x|)$ on the flowline at the same distance from 0.

We define a truncated conelike region about the flowline (see Figure 1) as

$$C(\hat{r}, \hat{\eta}) = \{x \in \mathbf{R}^d\colon |x| > \hat{r}, \eta(x) < \hat{\eta}\} \tag{1.7}$$

for $\hat{r} > 0$ and $0 < \hat{\eta} < \pi/2$. Note that if $\{x_t,\ t \geq 0\}$ is a ray starting from 0 then $C(\hat{r}, \hat{\eta})$ is a bona fide cone.

The object of Theorem 2 is to state conditions on σ and b in (1.1) to hold only in the region $C(\hat{r}, \hat{\eta})$ so that we get, with high probability,

(i) X_t remains inside $C(\hat{r}, \hat{\eta})$ forever,

(ii) $R_t = |X_t| \to \infty$ as $t \to \infty$, and

(iii) $\eta_t = \eta(X_t) \to 0$ as $t \to \infty$.

We give now the five assumptions required by Theorem 2, together with brief discussions of the nature of each.

A1. *Bounded variance.* Let $\mathbf{A}(x) = \sigma(x)\sigma(x)^{\mathrm{T}}$. Then there exists a $\lambda > 0$ so that

$$x^{\mathrm{T}}\mathbf{A}(x)x \le \lambda |x|^2. \tag{A1}$$

This condition can be guaranteed by a time change of the diffusion so by itself it entails no loss of generality.

A2. *Lower bound on outward drift.* Define

$$b_r(x) = \frac{x^{\mathrm{T}}b(x)}{|x|},$$

the radial component of $b(x)$. Then

$$b_r(x) \ge f(|x|), \tag{A2}$$

where $f(r) > 0$ is regularly varying with index δ, $-1 < \delta < 1$. In the discussion of the one-dimensional case we explained why we only consider this range.

A3. *Nontangential flowline.* For b_r as given in A2,

$$\frac{b_r(y)}{|b(y)|} \ge \rho > 0 \quad \text{for all } y \in \{x_t, t \ge 0\}. \tag{A3}$$

This says the flowline intersects spheres about the origin with angle bounded away from zero, and implies that the function $t \to |x_t|$ is strictly increasing. Without this assumption, examples can be easily constructed where relatively small random perturbations could cause X_t to skip over significant portions of the path of x_t, in which case we would have no hope of showing $X_t \sim x_t$.

A4. *Curvature of the flowline.* Let $\kappa(r)$ be the curvature of the flowline $\{x_t, t \ge 0\}$ at $|x_t| = r$, defined here as in elementary calculus by

$$\frac{d^2}{dt^2}x_t = \left(\frac{d^t}{dt^2}|x_t|\right)T + \kappa\left(\frac{d}{dt}|x_t|\right)^2\mathbf{N},$$

where $\mathbf{T} = x_t/|x_t|$, $\mathbf{N}$ is a unit vector orthogonal to $\mathbf{T}$, and $\kappa > 0$. Then

$$\frac{\kappa(r)}{f(r)} \to 0 \quad \text{as } r \to \infty. \tag{A4}$$

This is essentially a smoothness condition on the flowline and is implied by, but not equivalent to, the simpler one

$$|\partial_j b^i(x_t)| \le \beta(|x_t|) \quad \text{for } 1 \le i, j \le d,\ t \ge 0,$$

where $b^1, \dots, b^d$ are the components of the vector b and

$$\frac{\beta(r)}{f(r)} \to 0 \quad \text{as } r \to \infty.$$

We investigate condition A4 more closely in Examples D and E.

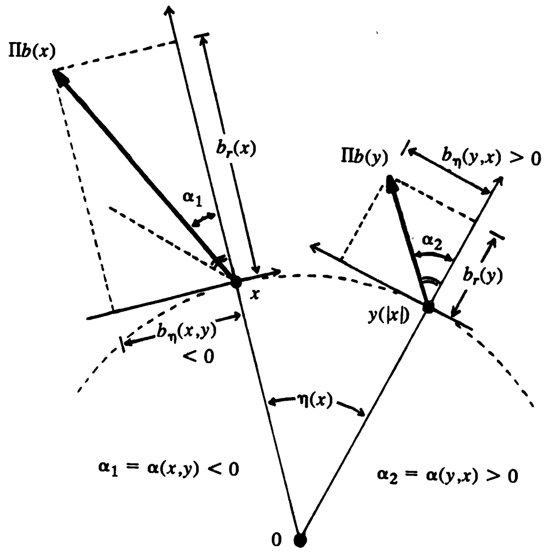

FIG. 2. *Notice that* $T(x) = \tan\alpha(y, x) + \tan\alpha(x, y) > 0$ *even though* $b_\eta(x, y) + b_\eta(y, x) < 0$.

A5. *Toe-in.* The last and most important condition is that the drift vectors $b(x)$ have positive components in the direction toward the flowline. This is easily formulated in two dimensions and when the flowline is a straight line through the origin but gets more complicated in general; refer to Figure 2 to illustrate the following definitions.

Recall the definition (1.5) of $y(r)$ and abbreviate $y(|x|)$ as y. Define Π to be the projection onto the plane containing 0, x, and y. Decompose $\Pi b(x)$ into its orthogonal components $b_r(x)e_r(x)$ and $b_\eta(x, y)e_\eta(x, y)$ where $e_r(x) = x/|x|$ and $e_\eta(x, y)$ is the unit vector in this plane orthogonal to $e_r(x)$ pointing toward y from x. Decompose $\Pi b(y)$ similarly into $b_r(y)e_r(y)$ and $b_\eta(y, x)e_\eta(y, x)$. Define $\alpha(x, y)$ as the angle $\Pi b(x)$ makes to x, oriented as is $e_\eta(x, y)$; likewise for $\alpha(y, x)$. We define

$$T(x) = \tan\alpha(x, y) + \tan\alpha(y, x) = \frac{b_\eta(x, y)}{b_r(x)} + \frac{b_\eta(y, x)}{b_r(y)}. \tag{1.8}$$

The toe-in condition, A5, is

$$T(x) \geq \varepsilon(\eta(x)), \quad \text{where } \varepsilon(\eta) \downarrow 0 \text{ as } \eta \downarrow 0 \tag{A5}$$

and is essential in that this is the driving force keeping X_t near the flowline, i.e., making $\eta(X_t) \to 0$. In Example B we briefly discuss a situation where A5 fails and indeed the conclusion of Theorem 2 does not hold.

With all of our assumptions introduced we can finally state

THEOREM 2. *Let X_t be defined as in* (1.1). *Suppose assumptions* A1–A5 *hold in the region $C(\hat{r}, \hat{\eta})$ for some $\hat{r} \geq 0$, $0 < \hat{\eta} < \pi/2$. Let $z(t)$ be the solution to the ordinary differential equation $\dot{z} = f(z)$ with $z(0) = 0$ and $f(r)$ as given in* A2. *Let*

$$\tau = \inf\{t \geq 0: X_t \notin C(\hat{r}, \hat{\eta})\}.$$

Then for $0 < \gamma < \hat{\eta}$ we have

(i) $\mathbf{P}_{X_0}(\tau = \infty) \to 1$ *as* $|X_0| \to \infty$ *uniformly in* $\eta(X_0) < \gamma$,

(ii) $$\liminf_{t\to\infty} \frac{|X_t|}{z(t)} \geq 1 \quad \textit{on } \{\tau = \infty\},$$

(iii) $\eta_t \to 0$ *as* $t \to \infty$ *on* $\{\tau = \infty\}$.

REMARK. If we have in addition, in $C(\hat{r}, \hat{\eta})$,

(A2′) $$b_r(x) \leq \bar{f}(|x|),$$

where $\bar{f}$ is regularly varying with index $\delta' < 1$, and define $w(t)$ by $\dot{w} = \bar{f}(w)$, $w_0 = 0$, we get the additional result

(ii′) $$\limsup_{t\to\infty} \frac{|X_t|}{w(t)} \leq 1 \quad \text{on } \{\tau = \infty\}.$$

In particular, if $f(r) \sim \bar{f}(r)$ then $|X_t|$ has asymptotic growth rate equal to $z(t) \sim w(t)$.

The proof of Theorem 2 depends only on the semimartingale properties of X_t and hence $\sigma(X_t)$ may be replaced by σ_t, a bounded predictable process on $\mathbf{R}^{d\times d}$. Although all of the bounds in the proof carry through in the same way, they become more cumbersome to present without the aid of the function $\mathbf{A}(x) = \sigma(x)\sigma(x)^{\mathsf{T}}$. This generalization can be checked by inspection and is therefore left to the reader.

Kesten (1976) proved similar theorems for Markov chains on $\mathbf{Z}^d$ with applications to birth–death chains. Theorem 2 is modeled after his results which required essentially the same toe-in condition and in which the flowline x_t was replaced by a straight line and the radial drifts were bounded below and above ($0 < c \leq b(x) \leq C$). The proofs relied heavily on the latter two facts; in particular, the martingale bounds used were not sensitive enough to allow $b_r(x) \to 0$. Similar questions concerning a different class of diffusions are treated by Pinsky (1987).

2. Examples. To illustrate the theorem we will now consider a number of examples. We assume throughout that $\sigma(x)$ satisfies A1 and X_t is well defined for all starting points X_0, and therefore give only the drift functions $b(x)$.

EXAMPLE A. THE SADDLE. Let

(2.1) $$b(x) = \nabla F(x_1^2 - x_2^2) = 2F'(x_1^2 - x_2^2)\begin{pmatrix} x_1 \\ -x_2 \end{pmatrix},$$

where $F' \geq 0$ is regularly varying with index α. Let $x_0 = (1, 0)$ so that $\{x_t, t \geq 0\}$

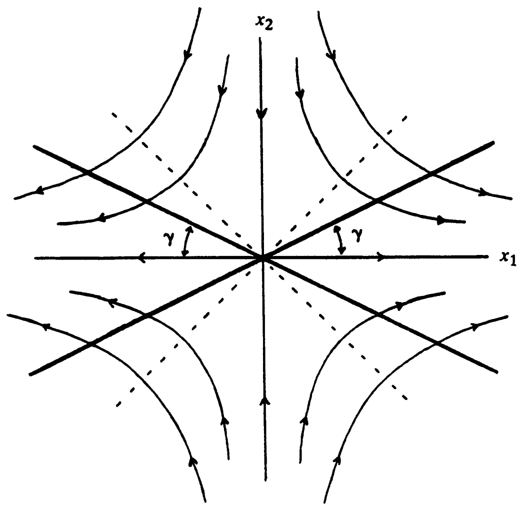

FIG. 3.

is the positive x_1 axis. Consider the cone $\{x\colon x_1 > 0,\ |\theta(x)| < \gamma\}$ for some $0 < \gamma < \pi/4$. Clearly, inside this cone, we have

$$f(|x|) \equiv C_\gamma |x| F'(|x|^2) < b_r(x) < 2|x|F'(|x|^2) \equiv \bar{f}(|x|), \tag{2.2}$$

where $C_\gamma \uparrow 2$ as $\gamma \downarrow 0$ so condition A2 is satisfied for $-1 < 1 + 2\alpha < 1$ or $-1 < \alpha < 0$. Since the flowline is the $+x_1$ axis (see Figure 3) the nontangential and curvature conditions A3 and A4 are trivially satisfied. Finally it is easy to see from the formula above (by substituting $x_2 = cx_1$ and looking at the (constant) angle $b(x)$ makes to x along this ray) that the toe-in condition A5 holds so Theorem 2 implies that with positive probability,

$$\theta_t = \theta(X_t) \to 0. \tag{2.3}$$

Applying the theorem and the remark for small γ and letting $\gamma \to 0$, we get that

$$R_t = |X_t| \sim r(t), \tag{2.4}$$

where $r(t)$ is the solution to $\dot{r} = \bar{f}(r)$.

Similarly, we can get $\theta_t \to \pi$, $R_t \sim r(t)$ with positive probability. In fact it is not difficult to show that X_t follows either one or the other of these paths from any starting point; for any X_0, X_t must eventually enter one of the two cones $\{|\theta(x)| < \gamma\}$ or $\{|\theta(x) - \pi| < \gamma\}$.

EXAMPLE B. Consider

$$b(x) = \nabla F(x_1^3 + x_2^3) = 3F'(x_1^3 + x_2^3)\begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix}, \tag{2.5}$$

where $F' \geq 0$ is regularly varying with index α. (See Figure 4.) If we apply the

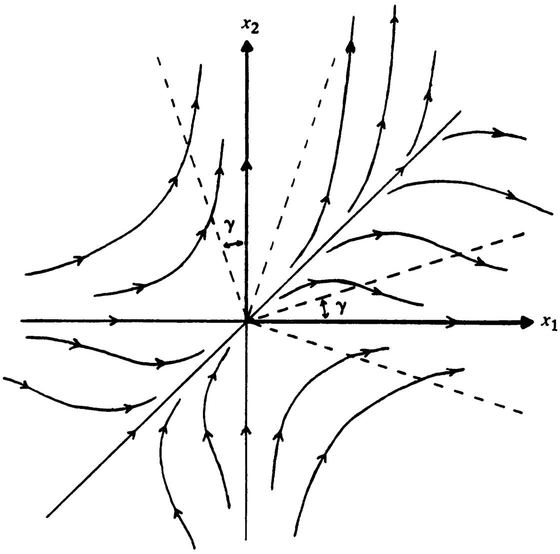

FIG. 4.

theorem with $x_0 = (1,0)$ or $(0,1)$ then as in Example A we can show that if $-1 < 1 + 3\alpha < 1$ then X_t has positive probability of going to infinity in the cones $\{|\theta(x)| < \gamma\}$ and $\{|\theta(x) - \pi/2| < \gamma\}$ when $0 < \gamma < \pi/4$. Our result cannot be applied to the flowline starting at $(1,1)$ but there is a good reason for this: X_t exits the cone $\{|\theta(x) - \pi/4| < \gamma\}$ eventually for any starting point within this cone. Cranston (1983) proved a lemma to this effect on his way to defining the invariant σ field for a considerably smaller class of diffusions in $\mathbf{R}^2$, where $b_r(x) \equiv 1$ and $\mathbf{R}^2$ can be divided into (bona fide) cones within which we have either toe-in or toe-out; $T(x) \leq -\varepsilon(\eta(x))$. His argument can be generalized as long as the lower and upper bound functions of $b_r(x)$ given in (A2) and (A2′) are asymptotically comparable.

EXAMPLE C. A THREE-DIMENSIONAL SPIRAL SADDLE. Let $f(r)$ be regularly varying with index $-1 < \delta < 1$ and

$$b(x) = \frac{f(|x|)}{|x|}\{(x_1, -x_2, -x_3) + g(x)(0, -x_3, x_2)\}, \tag{2.6}$$

with $x_0 = (1,0,0)$, so that the flowline $\{x_t, t \geq 0\}$ is the $+x_1$ axis. Note that $b(x)$ is the sum of a three-dimensional generalization of the saddle (Example A) and a vector field of pure rotation about the x_1 axis (see Figure 5). We check first the toe-in condition A5 by expressing the coordinates as

$$\begin{aligned} x_1 &= r\cos\eta, \\ x_2 &= r\sin\eta\cos\theta, \\ x_3 &= r\sin\eta\sin\theta, \end{aligned} \tag{2.7}$$

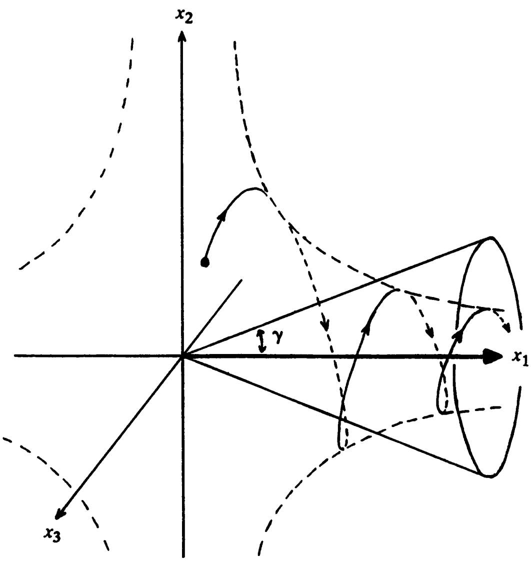

FIG. 5.

to get the unit vectors

$$e_r = \frac{x}{|x|} = (\cos\eta, \sin\eta\cos\theta, \sin\eta\sin\theta), \tag{2.8}$$
$$e_\eta = (\sin\eta, -\cos\eta\cos\theta, -\cos\eta\sin\theta),$$

where $e_r \cdot e_\eta = 0$ and $e_\eta(r)$ points from x toward the $+x_1$ axis. (We can compute $e_\eta = \nu/|\nu|$ from $\nu = x \times (e_1 \times x)$, where $e_1 = (1,0,0)$ and $\times$ denotes the vector cross product in $\mathbf{R}^3$.) As described in the discussion of A5, we get the components of $b(x)$,

$$b_r(x) = b \cdot e_r = f(r)\cos 2\eta, \tag{2.9}$$
$$b_\eta(x) = b \cdot e_\eta = f(r)\sin 2\eta,$$

and because the flowline is on a straight line through the origin, the toe-in function of (1.8) reduces to

$$T(x) = \frac{b_\eta(x)}{b_r(x)} = \tan 2\eta, \tag{2.10}$$

and so (A5) is satisfied with $\varepsilon(\eta) = 2\eta$ for $\hat{\eta} < \pi/2$.

Because of our choice for the flowline x_t, conditions A4 and A3 are satisfied trivially as in the previous two examples. Furthermore, since $b_r(x)/f(|x|) = \cos 2\eta(x)$, (A2) is satisfied when $\hat{\eta} < \pi/4$, as with the (two-dimensional) saddle (Example A). Therefore Theorem 2 implies $X_t \to \infty$ along the $\pm x_1$ axis with $|X_t| \sim r(t)$, the solution to $\dot{r} = f(r)$.

Note that the conditions A1–A5 require nothing whatsoever of the function $g(x)$; we only need the process X_t to be well defined in the first place. The choice of flowlines x_t is very important here as it is clear that the curvature condition A4 certainly need not be satisfied by any other flowlines without imposing severe restrictions on $g(x)$. Freidlin (1966) proved a theorem about existence and uniqueness of solutions to the exterior Dirichlet problem for a special class of diffusions which implies this result provided $b_r(x) \geq c > 0$ for large x and $g(x) \leq h(|x|)$ with $h(r)/r$ integrable.

EXAMPLE D. A WIGGLY FLOWLINE. In the above examples, A4 was satisfied trivially since the relevant flowlines were straight. In this example the flowline oscillates and the process follows the oscillations. Suppose the flowline $\{x_t, t \geq 0\}$ may be written in polar coordinates as

$$\theta(r) = \sin \log r. \tag{2.11}$$

This is all we need to define $\eta(x)$ and the cones $C(\hat{r}, \hat{\eta})$ and to check conditions A3 and A4. Then, for some $\hat{r}, \hat{\eta}$, we can fill in the drift vectors $b(x)$ so as to satisfy A5 and A2 with, say, $f(r) = r^{\delta}$, for some $-1 < \delta < 1$.

If $\gamma(r)$ is the tangent of the angle the flowline makes to the radial outward direction at distance r from 0, then it is easily verified that $r\dot{\theta}(r) = \gamma(r)$, where the dot denotes d/dr. We have $\dot{\theta} = (1/r)\cos \log r$, so $|\gamma(r)| = |\cos \log r| \leq 1$ and the nontangential condition A3 is satisfied.

An easy but tedious calculation shows that the curvature of this flowline is

$$\kappa(r) = \frac{2\dot{\theta} + r\ddot{\theta} + r^2\dot{\theta}^3}{(1 + r^2\dot{\theta}^2)^{3/2}} \leq \frac{\text{const.}}{r}. \tag{2.12}$$

Hence $\kappa(r)/f(r) = r^{\delta - 1} \to 0$ for all $-1 < \delta < 1$ and A4 is satisfied, so by Theorem 2, X_t follows the oscillations of the flowline $\{x_t, t \geq 0\}$.

If $\log r$ is replaced with r^{β} above, we see that A3 fails for $\beta > 0$ even though A4 may hold for some $-1 < \delta < 1$. Similarly, if we consider examples where the flows spiral out from the origin, A3 will fail unless $\dot{\theta}(r) \leq c/r$, which implies they are exponential spirals; $\theta(r) \leq c \log r$. We also note that by a similar triangles argument, if the flowline oscillates from one side of a cone to the other, and A3 is satisfied, then the radial distance between successive intersections with the cone must grow exponentially.

EXAMPLE E. One last example, to illustrate the relevance of the curvature condition A4, is constructed as follows. Let $x_n = n^{\alpha}$. Join the points $(x_n, 0)$ with $\frac{1}{6}$ circles. The nth circle has radius $x_{n+1} - x_n \sim \alpha n^{\alpha - 1}$; hence $\kappa_n \sim (1/\alpha) r^{(1/\alpha - 1)}$ and, with $f(r) = r^{\delta}$,

$$\frac{\kappa(r)}{f(r)} \sim \frac{1}{\alpha} r^{1/\alpha - 1 - \delta}, \tag{2.13}$$

so A4 requires $1/\alpha - 1 - \delta < 0$, or, $\alpha > 1/(1 + \delta)$.

This same inequality comes up in the proof of Theorem 1 [(3.16)], and denotes the lower bound of α so that the radial bound process (with drift $f(r)$) exits the successive overlapping intervals $[x_{n-1}, x_{n+1}]$ at x_{n+1} for all $n > N(\omega)$. For smaller α the ($d = 1$) process exits at x_{n-1} infinitely often and our proof breaks down even though $x_t \to \infty$. Since $1/r\kappa(r) \sim \alpha r^{-1/\alpha} \to 0$, these flowlines are asymptotically straight in that $\theta(x_t) \to 0$. The breakdown of the proof is due to the way in which we control the stochastic drift term of $\cos \eta_t$ and not that the conclusion fails for $\alpha > 1/(1 + \delta)$. Similarly, regular variation is not required for the conclusion of Theorem 1 to hold.

3. Proof of Theorem 1. Facts about regularly varying functions not proven here may be found in Feller (1971), Section VIII.8. Hereafter we shall write "$f(x)$ is regularly varying with index α" as "$f(x) \in \mathrm{RV}(\alpha)$" and this means that $f(x)$ is continuous and, for $\alpha \in \mathbf{R}$,

$$\lim_{x \to \infty} \frac{f(cx)}{f(x)} = c^{\alpha}. \tag{3.0}$$

Equivalently we may define RV(0) as above, otherwise known as the set of slowly varying functions, and define $f(x) \in \mathrm{RV}(\alpha)$ iff $f(x) = x^{\alpha} g(x)$ for some $g(x) \in \mathrm{RV}(0)$.

One fact about regularly varying functions is essential to both theorems and is easily shown:

LEMMA 0. *Let* $0 < f \in \mathrm{RV}(\alpha)$. *Then*

(i) *if* $\int_0^{\infty} f(u)\,du < \infty$ *then* $\int_r^{\infty} f(u)\,du \in \mathrm{RV}(\alpha + 1)$;

(ii) *if* $\int_0^{\infty} f(u)\,du = \infty$ *then* $\int_0^{r} f(u)\,du \in \mathrm{RV}(\alpha + 1)$.

PROOF. In the first case, let

$$F(x) = \int_x^{\infty} f(y)\,dy;$$

then

$$\frac{F(cx)}{F(x)} = \frac{\int_{cx}^{\infty} f(y)\,dy}{\int_x^{\infty} f(y)\,dy} = \frac{c\int_x^{\infty} f(cy)\,dy}{\int_x^{\infty} f(y)\,dy}.$$

By the definition (3.0) of $f \in \mathrm{RV}(\alpha)$, for x large and $y > x$,

$$(1 - \varepsilon)c^{\alpha} f(y) \le f(cy) \le (1 + \varepsilon)c^{\alpha} f(y),$$

which upon integration gives

$$(1 - \varepsilon)c^{\alpha} \int_x^{\infty} f(y)\,dy \le \int_x^{\infty} f(cy)\,dy \le (1 + \varepsilon)c^{\alpha} \int_x^{\infty} f(y)\,dy,$$

which rearranges to

$$(1 - \varepsilon)c^{\alpha+1} \le \frac{F(cx)}{F(x)} \le (1 + \varepsilon)c^{\alpha+1},$$

where $\varepsilon > 0$ is arbitrary. Hence $F(x)$ satisfies the definition (3.0) of $\mathrm{RV}(\alpha + 1)$. The second case works in the same way. □

We prove Theorem 1 by establishing that, along some sequence of points $y_n \to \infty$, each sample path X_t eventually hits the points y_n sequentially without ever backing up to $y_n - 1$. Also the transit time from y_{n-1} to y_n approaches the transit time for the associated deterministic process x_t. The first claim holds if the points are spread out far enough apart, but the proof for the second claim fails unless they are close enough together that the drift function $b(x)$ is essentially constant over the intervals $[y_{n-1}, y_{n+1}]$. It turns out that the proper growth rate for the sequence y_n is

$$y_n = n^\alpha \quad \text{for some } \alpha > \frac{1}{1+\delta}. \tag{3.1}$$

First, we find the solution $f(t)$ to the ordinary differential equation (1.2) by defining its inverse,

$$f^{-1}(x) = \int_0^x \frac{dy}{b(y)} \in \mathrm{RV}(1-\delta) \quad \text{(by Lemma 0)}. \tag{3.2}$$

Then $x(t) = f(t + f^{-1}(x_0))$, and $f(t) \in \mathrm{RV}(1/(1-\delta))$. Since $b(x) > 0$, we see that $f^{-1}(x)$ is strictly increasing, so $f(t)$ and $x(t)$ are as well.

Let M_t be the martingale part of X_t, given by

$$M_t = \int_0^t \sigma_s \, dB_s. \tag{3.3}$$

Then the variance process of M_t is (see Durrett (1984), Sections 2.4 and 2.5)

$$\langle M \rangle_t = \int_0^t \sigma_s^2 \, ds. \tag{3.4}$$

The condition that $|\sigma_s| < C$ implies that M_t grows at most like a Brownian motion; for convenience we take, without any loss of generality, $C = 1$. We define the hitting time of y_n for X_t,

$$T_n = \inf\{t \geq 0 \colon X_t \geq y_n\}, \tag{3.5}$$

and we begin by bounding the martingale M_t for $T_n \leq t \leq T_{n+1}$.

LEMMA 1. *For any stopping time T and positive constants K and ε, let*

$$\Lambda = \Lambda(T, K, \varepsilon) = \{|M_{t+T} - M_T| < K + \varepsilon t \text{ for all } t \geq 0;\ T < \infty\}.$$

Then $\mathbf{P}_{x_0}(\Lambda^c; T < \infty) \leq 2\exp(-2K\varepsilon)$.

PROOF. Using the exponential martingale it is easy to show this (see Durrett (1984), page 27) with $T = 0$ and M_t a Brownian motion. For finite stopping times T, we define $N_t = M_{t+T} - M_T$. By the optional stopping theorem, N_t is a martingale, and hence is a time change of a (different) Brownian motion W_t. This time change is given by its variance process, $\langle N \rangle_t$,

$$N_t = \int_0^t \sigma_{s+T} \, dB_{s+T}, \tag{3.6}$$

$$\langle N \rangle_t = \int_0^t \sigma_{s+T}^2 \, ds, \tag{3.7}$$

$$N_t = W_{\langle N \rangle_t}. \tag{3.8}$$

By hypothesis, $\sigma_{s+T}^2 \le 1$; hence $\langle N \rangle_t \le t$ and

$$|M_{t+T} - M_T| = |N_t| = |W_{\langle N \rangle_t}| \le K + \varepsilon \langle N \rangle_t \le K + \varepsilon t,$$

with probability greater than $1 - 2\exp(-2K\varepsilon)$.

If $\mathbf{P}_{x_0}(T = \infty) > 0$, we apply the lemma with the stopping times $T \wedge n$ to get $\mathbf{P}_{x_0}(\Lambda^c; T < n) \le 2\exp(-2K\varepsilon)$ and let $n \to \infty$. □

We now define, for positive sequences K_n and ε_n to be chosen later, the sets

$$\Lambda_n = \Lambda(T_n, K_n, \varepsilon_n), \tag{3.9}$$

$$\Omega_N = \bigcap_{n \ge N} \Lambda_n. \tag{3.10}$$

For the moment we choose $x_0 = y_N$ for some large N; thus $T_n = 0$, $n \le N$. We will prove that the conclusion of Theorem 1 holds on Ω_N and that $\mathbf{P}_{y_N}(\Omega_N) \to 1$ as $N \to \infty$, using the Borel–Cantelli lemma.

LEMMA 2. *If N is large, and $x_0 = y_N$, then for $n \ge N$, we have*

(i) *on Λ_n, $T_{n+1} < \infty$ and $X_t > y_{n-1}$ for $T_n \le t \le T_{n+1}$*

(ii) *and on Ω_N,* $$\left| \frac{T_n}{x^{-1}(y_n)} - 1 \right| \le r_N$$

for some postive sequence $r_N \to 0$ as $N \to \infty$.

PROOF. Define, for $n \ge N$,

$$\underline{b}_n = \inf_{y_{n-1} \le x \le y_{n+1}} b(x), \tag{3.11}$$

$$\overline{b}_n = \sup_{y_{n-1} \le x \le y_{n+1}} b(x), \tag{3.12}$$

$$\Delta y_n = y_{n+1} - y_n. \tag{3.13}$$

Since $b(x)$ is regularly varying and $y_n = n^\alpha$ grows only polynomially fast, we have $\underline{b}_n \sim \overline{b}_n \sim b(y_n)$ as $n \to \infty$. Estimating the diffusion equation for X_t for times after T_n,

$$X_{t+T_n} = X_{T_n} + M_{t+T_n} - M_{T_n} + \int_0^t b(X_{s+T_n})\, ds, \tag{3.14}$$

we have, for $0 \le t \le \inf\{t \ge 0: X_{t+T_n} \notin [y_{n-1}, y_{n+1}]\}$,

$$y_n - K_n - \varepsilon_n t + \underline{b}_n t \le X_{t+T_n} \le y_n + K_n + \varepsilon_n t + \overline{b}_n t. \tag{3.15}$$

Note that $\Delta y_n \sim \alpha n^{\alpha - 1} \in \mathrm{RV}(\alpha - 1)$ in n, and $b(y_n) = b(n^\alpha) \in \mathrm{RV}(\alpha\delta)$. We now choose K_n and ε_n so that they are asymptotically dominated by Δy_n and $b(y_n)$, respectively. Let

$$\alpha > \beta > \frac{1}{1+\delta}, \tag{3.16}$$

$$K_n = n^{\beta - 1}, \tag{3.17}$$

$$\varepsilon_n = n^{\beta\delta}. \tag{3.18}$$

We now see from (3.15) that X_t hits y_{n+1} before y_{n-1}, for $n > N$, N sufficiently large, on Λ_n, proving (i). Also we have $T_{n+1} < \infty$ and, by solving (3.15) for t, with $t = T_{n+1} - T_n$, we get

$$\frac{\Delta y_n - K_n}{\bar{b}_n + \varepsilon_n} \le T_{n+1} - T_n \le \frac{\Delta y_n + K_n}{\underline{b}_n - \varepsilon_n}. \tag{3.19}$$

By our choice of K_n, ε_n we see that for some sequence of numbers $R_n \to 0$ as $n \to \infty$, on Λ_n we have

$$(1 - R_n)\frac{\Delta y_n}{b(y_n)} \le T_{n+1} - T_n \le (1 + R_n)\frac{\Delta y_n}{b(y_n)}. \tag{3.20}$$

Sum the telescoping series starting from N to get, on Ω_N,

$$\sum_{k=N}^{n-1}(1 - R_k)\frac{\Delta y_k}{b(y_k)} \le T_n - T_N = T_n \le \sum_{k=N}^{n-1}(1 + R_k)\frac{\Delta y_k}{b(y_k)}. \tag{3.21}$$

Take the sequence R_n to be decreasing; then

$$(1 - R_N)\sum_{k=N}^{n-1}\frac{\Delta y_k}{b(y_k)} \le T_n \le (1 + R_N)\sum_{k=N}^{n-1}\frac{\Delta y_k}{b(y_k)}. \tag{3.22}$$

Since $b(x)$ is regularly varying, it is easy to see that

$$\frac{\Delta y_k}{b(y_k)} \sim \int_{y_k}^{y_{k+1}}\frac{dy}{b(y)} \tag{3.23}$$

along any sequence y_n of polynomial growth; thus, since

$$\int^{\infty}\frac{dy}{b(y)} = \infty,$$

for some other sequence $r_n \to 0$ as $n \to \infty$,

$$(1 - r_N)\int_{x_0}^{y_n}\frac{dy}{b(y)} \le T_n \le (1 + r_N)\int_{x_0}^{y_n}\frac{dy}{b(y)}, \tag{3.24}$$

which may be rearranged to get (ii). □

Notice now that $K_n\varepsilon_n = n^{\nu}$, with $\nu = \beta(1 + \delta) - 1 > 0$, by (3.16), so $2\exp(-2K_n\varepsilon_n)$ is summable. Lemma 1 and the Borel–Cantelli lemma imply that

$$\mathbf{P}_{x_0}(\Lambda_n^c \text{ i.o.}, X_t \to \infty) = 0 \tag{3.25}$$

and hence the sets Ω_N (given by (3.10)) increase to the set $\{X_t \to \infty\}$ a.s. as $n \to \infty$.

We can prove rather easily now that $X_t/x(t) \to 1$ a.s. on $\{X_t \to \infty\}$ by noting first that in the proof of Lemma 2, equations (3.21) and (3.23) imply that T_n is asymptotic to $x^{-1}(y_n)$ (expressed by (3.28)) on $\{X_t \to \infty\}$ for any starting point x_0. In general, for regularly varying g, the mapping g preserves asymptotic equivalence; $x(t)$ is regularly varying so $T_n \sim x^{-1}(y_n)$ implies $x(T_n) \sim y_n = X_{T_n}$.

Hence, for $n \geq N(\omega) < \infty$, and for $T_n \leq t \leq T_{n+1}$,

$$\frac{y_{n-1}}{y_{n+1}}\frac{y_{n+1}}{x(T_{n+1})} = \frac{y_{n-1}}{x(T_{n+1})} \leq \frac{X_t}{x(t)} \leq \frac{y_{n+1}}{x(T_n)} = \frac{y_n}{x(T_n)}\frac{y_{n+1}}{y_n}. \tag{3.26}$$

As $n \to \infty$, both sides of (3.26) go to 1, proving $X_t \sim x_t$. However, for applications we want the stronger result of Theorem 1, which is most easily obtained by keeping all the hidden epsilons out in the open in the above argument, as follows.

Lemma 2 implies, on Ω_N, with $x_0 = y_N$,

$$T_n = (1 + a_n(\omega))x^{-1}(y_n), \tag{3.27}$$

where $a_n < r_N$. Also, from (3.2),

$$x^{-1}(y_n) = f^{-1}(y_n) - f^{-1}(y_N), \tag{3.28}$$

so

$$\begin{aligned} x(T_n) &= f(T_n + f^{-1}(y_N)) \\ &= f((1 + a_n)f^{-1}(y_n) - a_n f^{-1}(y_N)). \end{aligned} \tag{3.29}$$

Since f^{-1} is monotone increasing,

$$|a_n(f^{-1}(y_n) - f^{-1}(y_N))| \leq |a_n| f^{-1}(y_n),$$

and hence (3.29) implies

$$x(T_n) = f((1 + \hat{a}_n)f^{-1}(y_n)) \tag{3.30}$$

for some $|\hat{a}_n(\omega)| < r_N$.

To see that regularly varying mappings preserve asymptotic equivalence, let $g \in \mathrm{RV}(\alpha)$; then given $\eta > 0$, there exists an $M > 0$ so that for $x > M$, $1 - \eta < c < 1 + \eta$,

$$\left|\frac{g(cx)}{g(x)} - c^{\alpha}\right| < \varepsilon,$$

and hence, in fact, given ε, choose η and then M to get

$$\left|\frac{g(cx)}{g(x)} - 1\right| < \varepsilon \quad \text{for } 1 - \eta < c < 1 + \eta. \tag{3.31}$$

Thus, since f is regularly varying,

$$f((1 + \hat{a}_n)f^{-1}(y_n)) \sim f(f^{-1}(y_n)) = y_n, \tag{3.32}$$

so for N large enough, we have

$$\left|\frac{x(T_n)}{y_n} - 1\right| < \varepsilon. \tag{3.33}$$

Now $y_n = X_{T_n}$ and, perhaps for N yet larger,

$$\left|\frac{y_{n+1}}{y_n} - 1\right| < \varepsilon. \tag{3.34}$$

We plug (3.33) and (3.34) into (3.26) and solve to get the conclusion of Theorem 1, albeit with ε replaced by $2\varepsilon + \varepsilon^2$.

Finally, we extend to general x_0, which was chosen to be a y_N for some N only to avoid overly technical details in computing the estimates for the first interval $[y_{n-1}, y_{n+1}]$ exited by X_t. Given α and x_0, let $N = \min(n: n^\alpha > x_0)$. Set $y_N = x_0$ and $y_{N-1} = (N-2)^\alpha$. The quantities $\underline{b}_N$ and $\overline{b}_N$ change only insignificantly for large N, hence all the above estimates hold for x_0 large enough, and with ε replaced by 2ε. □

4. Proof of Theorem 2. Let $R_t = |X_t|$ and $\eta_t = \eta(X_t)$ as defined by (1.6). In brief, we show first that as long as X_t stays inside the cone $C(\hat{r}, \hat{\eta})$, $R_t \sim r(t)$ by Theorem 1. We use this to estimate the martingale part of η_t (we actually use $\cos \eta_t$ for the calculations) and find that the martingale "runs out of gas," i.e., it converges to a finite limit and hence the drifts dominate and X_t indeed remains inside the cone, and $\eta_t \to 0$. The proof comes in three parts: In the first we get the lower bound we need on R_t and also prove (ii). In the second part we compute some formulas and estimates for $\cos \eta_t$, which we use in the third part to prove (i) and (iii). Throughout this section, $\mathbf{A}(x) = \sigma(x)\sigma(x)^{\mathrm{T}}$ is the $d \times d$ matrix of second-order coefficients for the generator of X_t, as mentioned in (A1).

PROOF OF (ii). Applying Itô's formula to R_t we get

$$R_t = |X_0| + N_t + \int_0^t b_r(X_s)\,ds + \int_0^t h_r(X_s)\,ds, \tag{4.1}$$

where

$$N_t = \int_0^t \frac{X_s^{\mathrm{T}}\sigma(X_s)}{R_s^2}\,dB_s, \tag{4.2}$$

which is a martingale with variance process given by (see Durrett (1984), Sections 2.4 and 2.5)

$$\langle N\rangle_t = \int_0^t \frac{X_s^{\mathrm{T}}\mathbf{A}(X_s)X_s}{R_s^2}\,ds, \tag{4.3}$$

$b_r(x)$ is the radial component of the drift term $b(x)$ for X_t, and

$$h_r(x) = \frac{1}{2|x|}\left(\operatorname{trace}\mathbf{A}(x) - \frac{x^{\mathrm{T}}\mathbf{A}(x)x}{|x|^2}\right) \tag{4.4}$$

is the "stochastic drift" for R_t, i.e., the drift term due to the quadratic variation of X_t.

We define a process Z_t on $\mathbf{R}^+$ by

$$Z_t = |X_0| + N_t + \int_0^t f(Z_s)\,ds \tag{4.5}$$

and establish, with $\tau = \inf\{t \geq 0: X_t \notin C(\hat{r}, \hat{\eta})\}$,

$$R_t \geq Z_t, \qquad 0 \leq t \leq \tau. \tag{4.6}$$

Since $\mathbf{A}(x)$ is positive definite, $h_r(x) \geq 0$; hence $f(|x|) \leq b_r(x) + h_r(x)$, the drift term for R_t. It seems that there should be a simple comparison theorem that we could apply here to get (4.6), but unfortunately I haven't been able to find one that works as stated for non-Markovian processes; however, the proof of Theorem 1.1 of Ikeda and Watanabe (1981), Chapter 6, works in the following simplified form. Without loss of generality, $f(r)$ is Lipschitz continuous with Lipschitz constant K;

$$
\begin{aligned}
(Z_t - R_t)^+ &= \int_0^t |f(Z_s) - b_r(X_s) - h_r(X_s)| 1_{\{Z_s > R_s\}}\, ds \\
&\leq \int^t |f(Z_s) - f(R_s)| 1_{\{Z_s > R_s\}}\, ds \\
&\leq K \int_0^t |Z_s - R_s| 1_{\{Z_s > R_s\}}\, ds \\
&= K \int_0^t (Z_s - R_s)^+\, ds = 0,
\end{aligned} \tag{4.7}
$$

since

$$\alpha(t) \leq K \int_0^t \alpha(s)\, ds$$

together with $\alpha \geq 0$ and α continuous implies that $\alpha = 0$.

We now apply Theorem 1 to get a lower bound on the process Z_t. We characterize the function $z(t)$ by its inverse, which satisfies (by Lemma 0 of Section 3),

$$t(z) = \int_0^z \frac{du}{f(u)} \in \mathrm{RV}(1 - \delta). \tag{4.8}$$

We can represent solutions $\hat{z}$ with $\hat{z}(0) = r_0 > 0$ by $\hat{z}(t) = z(t + z^{-1}(r_0))$. By A1,

$$\sigma_t^2 = \frac{X_t^{\mathsf{T}} \mathbf{A}(X_t) X_t}{R_t^2} \leq \lambda, \tag{4.9}$$

so (see Durrett (1984), Section 2.11) there is a unique one-dimensional Brownian motion W such that, for some continuous σ_s consistent with (4.9), we may rewrite (4.2) as

$$N_t = \int_0^t \sigma_s\, dW_s. \tag{4.10}$$

Since $f \in \mathrm{RV}(\delta)$, we can apply Theorem 1 to the process Z_t to get

$$\mathbf{P}_{X_0}\big(Z_t \geq \tfrac{1}{2} z\big(t + z^{-1}(|X_0|)\big), 0 \leq t \leq \tau\big) \geq 1 - p(|X_0|), \tag{4.11}$$

where $p(r) \to 0$ as $r \to \infty$. We also get $Z_t \sim z(t)$ as $t \to \infty$, which together with (4.6) proves (ii) of Theorem 2. □

Having considered lower bounds we now prove the remark to Theorem 2 referring to the upper bound on R_t. We note, by A1, that $h_r(x) \leq C/|x|$, so $h_r(x)/\bar{f}(|x|) \to 0$. Compute w from $\bar{f}$ and $\bar{z}$ from

$$\bar{f}_1 = \bar{f} + \sup\{h_r(x) \colon x = r, x \in C(\hat{r}, \hat{\eta})\}$$

as in (4.8), and define $\bar{Z}_t$ from $\bar{f}_1$ as in (4.5). Since $\bar{f} \sim \bar{f}_1$, we have $w \sim \bar{z}$ and the above arguments now give $\bar{Z}_t \geq R_t$, $0 \leq t \leq \tau$, with $\bar{Z}_t \sim \bar{z}(t) \sim w(t)$. □

Some estimates for $\cos\eta_t$. Applying Itô's formula to $\cos\eta_t$ we can write

$$\cos\eta_t = \cos\eta_0 + M_t + C_t + D_t + E_t, \tag{4.12}$$

where

$$M_t = \int_0^t \nabla \cos\eta(X_s)^{\mathsf{T}} \sigma(X_s)\, dB_s \tag{4.13}$$

is a martingale with variance process (see Durrett (1984), Sections 2.4 and 2.5) given by

$$\langle M \rangle_t = \int_0^t \nabla \cos\eta(X_s)^{\mathsf{T}} \mathbf{A}(X_s) \nabla \cos\eta(X_s)\, ds, \tag{4.14}$$

$$C_t = \int_0^t \nabla \cos\eta(X_s)^{\mathsf{T}} b(X_s)\, ds \tag{4.15}$$

and

$$D_t + E_t = \int_0^t \mathbf{H} \cos\eta(X_s) \cdot \mathbf{A}(X_s)\, ds, \tag{4.16}$$

where $(\mathbf{H}\cos\eta)_{ij} = \partial_i \partial_j \cos\eta$, the matrix of second partial derivatives of $\cos\eta$, and $\mathbf{H}\cos\eta \cdot \mathbf{A}$ is the matrix dot product formed by componentwise multiplication and summing; for example,

$$xx^{\mathsf{T}} \cdot \mathbf{A} = \sum_{ij} x^i x^j \mathbf{A}_{ij} = x^{\mathsf{T}} \mathbf{A} x.$$

We bound M_t by writing it as a time change of a (pathwise uniquely defined) Brownian motion, W_t, given by (see Durrett (1984), Section 2.11)

$$M_t = W_{\langle M \rangle_t}.$$

Using the exponential martingale it is easy to show (Durrett (1984), page 27) that for $K, \varepsilon > 0$,

$$\mathbf{P}(|W_t| \leq K + \varepsilon t \text{ for all } t \geq 0) \geq 1 - \exp(-2K\varepsilon), \tag{4.17}$$

and hence

$$\mathbf{P}(|M_t| \leq K + \varepsilon\langle M \rangle_t, t \geq 0) \geq 1 - 2\exp(-2K\varepsilon). \tag{4.18}$$

We now bound (4.14), for $0 \leq t \leq \tau$. To facilitate computations, write (as in A5) for $x \in \mathbf{R}^d$: $r = |x|$, $y = y(|x|)$, $\eta = \eta(x)$, and $\mathbf{A} = \mathbf{A}(x) = \sigma(x)\sigma(x)^{\mathsf{T}}$. Recall the definition (1.5) of $y(r) = x_{t(r)}$ where $r(t) = |x_t|$, and x_t is the flowline starting at x_0 of $\dot{x} = b(x)$, and the definition (1.6) of $\eta(x)$. When we differentiate $\cos\eta$ we get

$$\nabla \cos\eta = \frac{y}{r^2} + (Q - 2\cos\eta)\frac{x}{r^2}, \tag{4.19}$$

where

$$\mathbf{Q} = \mathbf{Q}(x) = \frac{x^{\mathsf{T}} b(y)}{y^{\mathsf{T}} b(y)}, \tag{4.20}$$

and hence,

$$(\nabla \cos \eta)^{\mathsf{T}} \mathbf{A} \nabla \cos \eta = \frac{1}{r^2} \frac{y^{\mathsf{T}} \mathbf{A} y}{r^2} + \frac{1}{r^2} (\mathbf{Q} - 2 \cos \eta) \frac{2 y^{\mathsf{T}} \mathbf{A} x}{r^2} + \frac{1}{r^2} (\mathbf{Q} - 2 \cos \eta)^2 \frac{x^{\mathsf{T}} \mathbf{A} x}{r^2}. \tag{4.21}$$

Using A3 we get, for $x \in C(\hat{r}, \hat{\eta})$,

$$|\mathbf{Q}| = \frac{|x^{\mathsf{T}} b(y)|}{|x|\,|b(y)|} \frac{|b(y)|}{b_r(y)} \le \frac{1}{\rho} \tag{4.22}$$

and this with A1 implies (recall $0 < \cos \eta \le 1$)

$$\left| \left((\nabla \cos \eta)^{\mathsf{T}} \mathbf{A} \nabla \cos \eta \right)(x) \right| \le \frac{C_1}{r^2}, \tag{4.23}$$

for $\mathbf{C}_1 = \lambda(3 + 1/\rho)^2$. Hence, on the set (see (4.11))

$$\Omega_1 = \left\{ R_1 \ge \tfrac{1}{2} z\left(t + z^{-1}(|X_0|)\right), 0 \le t \le \tau \right\} \tag{4.24}$$

for $0 \le t \le \tau$, we have

$$\begin{aligned} \langle M \rangle_t &\le \int_0^t \mathbf{C}_1 R_s^{-2}\, ds \\ &\le \int_0^t 4 C_1 z\left(s + z^{-1}(|X_0|)\right)^{-2} ds \\ &= \int_{z^{-1}(|X_0|)}^{t + z^{-1}(|X_0|)} 4 \mathbf{C}_1 z(s)^{-2}\, ds \\ &\le \int_{z^{-1}(|X_0|)}^{\infty} 4 \mathbf{C}_1 z(s)^{-2}\, ds \\ &= h\left(z^{-1}(|X_0|)\right), \end{aligned} \tag{4.25}$$

where

$$h(t) = 4 \mathbf{C}_1 \int_t^{\infty} z(s)^{-2}\, ds. \tag{4.26}$$

We establish some asymptotic properties of $z(t)$ and $h(t)$. By (4.8) and some basic properties of regular variation,

$$z(t) \in \mathrm{RV}\left(\frac{1}{1 - \delta} \right) \tag{4.27}$$

and hence, by (4.26) and Lemma 0 of Section 3,

$$h(t) \in \mathrm{RV}\left(\frac{-1 - \delta}{1 - \delta} \right) \tag{4.28}$$

and

$$h\left(z^{-1}(r)\right) \in \mathrm{RV}(-1 - \delta), \tag{4.29}$$

which goes to zero like a negative power of r as $r \to \infty$.

We finally bound M_t with high probability by plugging (4.25) into (4.18);

$$\begin{aligned}&\{|M_t| < K + \varepsilon\langle M\rangle_t, t \geq 0\} \cap \Omega_1 \\ &\quad\subseteq \left\{|M_t| < K + \varepsilon h\left(z^{-1}(|X_0|)\right), 0 \leq t \leq \tau\right\} \cap \Omega_1,\end{aligned} \tag{4.30}$$

so if we choose

$$K = h\left(z^{-1}(|X_0|)\right)^{1/3},$$
$$\varepsilon = h\left(z^{-1}(|X_0|)\right)^{-2/3},$$

then for the set

$$\Omega_2 = \left\{|M_t| \leq 2h\left(z^{-1}(|X_0|)\right)^{1/3}, \quad 0 \leq t \leq \tau\right\} \tag{4.31}$$

we have

$$\mathbf{P}_{X_0}(\Omega_1 \cap \Omega_2^c) \leq 2\exp\left(-2h\left(z^{-1}(|X_0|)\right)^{-1/3}\right). \tag{4.32}$$

We estimate the probability of $\Omega_1 \cap \Omega_2$ using (4.6), (4.11) and (4.32),

$$\mathbf{P}_{X_0}\left((\Omega_1 \cap \Omega_2)^c\right) \leq p(|X_0|) + 2\exp\left(-2h\left(z^{-1}(|X_0|)\right)^{-1/3}\right). \tag{4.33}$$

Both quantities on the right-hand side go to 0 as $|X_0| \to \infty$, which is what we want; we will be showing $\tau = \infty$ on this set. We now bound the other terms of (4.12) on this set.

The expression for $\mathrm{H}\cos\eta$ is lengthy and uninteresting by itself, so we will only present

$$H\cos\eta \cdot \mathbf{A}(x) = \mathbf{D}(x) + \mathbf{E}(x) \tag{4.34}$$

broken up into the terms included in D_t and E_t, respectively. Let $b'(y)$ denote the matrix of first partial derivatives with entries $\partial_j b^i(y)$,

$$\mathbf{D}(x) = \frac{(x^{\mathsf{T}} - \mathbf{Q}y^{\mathsf{T}})b'(y)b(y)}{\left(y^{\mathsf{T}}b(y)\right)^2}\frac{x^{\mathsf{T}}\mathbf{A}x}{r^2}, \tag{4.35}$$

$$\begin{aligned}\mathbf{E}(x) = {}& \frac{1}{r^2}\frac{|b(y)|}{b_r(y)}\frac{2b(y)^{\mathsf{T}}\mathbf{A}x}{|b(y)|r} \\ &+ \frac{1}{r^2}\left[8\cos\eta - 4\mathbf{Q} - \left(\frac{|b(y)|}{b_r(y)}\right)^2\mathbf{Q}\right]\frac{x^{\mathsf{T}}\mathbf{A}x}{r^2} \\ &+ \frac{1}{r^2}\left[\operatorname{trace}\mathbf{A}(\mathbf{Q} - 2\cos\eta) - 6\frac{y^{\mathsf{T}}\mathbf{A}x}{r^2}\right].\end{aligned} \tag{4.36}$$

We define

$$D_t = \int_0^t \mathbf{D}(X_s)\,ds, \tag{4.37}$$

and similarly for E_t.

We apply A1, A3, (4.22), and the fact that $0 < \cos\eta \le 1$ to see that for $x \in C(\hat{r}, \hat{\eta})$,

$$|\mathbf{E}(x)| \le \mathbf{C}_2 r^{-2} \tag{4.38}$$

for $\mathbf{C}_2 = \lambda(6 + 2d + (6 + d)/\rho + 1/\rho^3)$, so by the same calculations as in (4.25) we get, on Ω_1, $0 \le t \le \tau$,

$$|E_t| \le \int_0^t \mathbf{C}_2 R_s^{-2}\, ds \le \frac{C_2}{C_1} h\big(z^{-1}(|X_0|)\big). \tag{4.39}$$

We estimate $\mathbf{D}(x)$ by first noticing that the vector $b'(y)b(y)$ is the acceleration of the curve x_t at $|x_t| = r$. Let s_t denote the distance along this curve from x_0 to x_t; then $\dot{s}_t = |\dot{x}_t| = |b(x_t)| = |b(y)|$, where $t = t(r)$ is given by $r(t) = |x_t|$. We can express this acceleration vector in the following familiar form:

$$\ddot{x}_t = \ddot{s}_t T + \kappa(\dot{s}_t)^2 \mathbf{N}, \tag{4.40}$$

where unit vectors $\mathbf{T}$ and $\mathbf{N}$ are just the Gram–Schmidt orthonormalization of $(\dot{x}_t, \ddot{x}_t)$. We note that $\mathbf{T} = b(y)/|b(y)|$ and decompose x and y in terms of this new basis: $x_{\mathrm{T}} = x \cdot T = x^{\mathrm{T}} b(y)/|b(y)|$, and similarly for y_{T}, $x_{\mathrm{N}} = x \cdot N$, and $y_{\mathrm{N}} = y \cdot N$. We have for $x \in C(\hat{r}, \hat{\eta})$,

$$\begin{aligned} \mathbf{D}(x) &= \frac{\big(x - (x_{\mathrm{T}}/y_{\mathrm{T}})y\big)\big(\ddot{s}_t \mathbf{T} + \kappa(r)\dot{s}_t^2 \mathbf{N}\big)}{\dot{s}_t^2 y_{\mathrm{T}}^2} \frac{x^{\mathrm{T}} \mathbf{A} x}{r^2} \\ &= \frac{y_{\mathrm{T}} x_{\mathrm{N}} - x_{\mathrm{T}} y_{\mathrm{N}}}{y_{\mathrm{T}}^3} \kappa(r) \frac{x^{\mathrm{T}} \mathbf{A} x}{r^2}. \end{aligned} \tag{4.41}$$

Note that

$$y_{\mathrm{T}} = \frac{y^{\mathrm{T}} b(y)}{|b(y)|} = \frac{y^{\mathrm{T}} b(y)/r}{|b(y)|/r} = \frac{b_r(y)}{|b(y)|} r.$$

Using (A3) $b_r(y)/|b(y)| \ge \rho > 0$ we get $y_{\mathrm{T}} \ge \rho r$. Since $|x| = |y| = r$ we have

$$\left| \frac{y_{\mathrm{T}} x_{\mathrm{N}} - x_{\mathrm{T}} y_{\mathrm{N}}}{y_{\mathrm{T}}^3} \right| \le \frac{2}{\rho^3 r}$$

and therefore, for $x \in C(\hat{r}, \hat{\eta})$,

$$|\mathbf{D}(x)| \le \mathbf{C}_3 \frac{|\kappa(r)|}{r}, \tag{4.42}$$

where $\mathbf{C}_3 = 2\lambda/\rho^3$.

The term C_t given by (4.15) is the controlling term that drives η_t to zero. We use the toe-in function $T(x)$ defined in A5 (1.8) which can be calculated in terms of x, y, $b(x)$, and $b(y)$,

$$
\begin{aligned}
T(x) &= \frac{b_\eta(x, y)}{b_r(x)} + \frac{b_\eta(y, x)}{b_r(y)} \\
&= \frac{y^{\mathsf T} b(x) - (\cos\eta) x^{\mathsf T} b(x)}{(\sin\eta) x^{\mathsf T} b(x)} + \frac{x^{\mathsf T} b(y) - (\cos\eta) y^{\mathsf T} b(y)}{(\sin\eta) y^{\mathsf T} b(y)} \qquad (4.43)\\
&= \frac{1}{\sin\eta}\left(\frac{y^{\mathsf T} b(x)}{x^{\mathsf T} b(x)} + \frac{x^{\mathsf T} b(y)}{y^{\mathsf T} b(y)} - 2\cos\eta\right).
\end{aligned}
$$

For $x \in C(\hat{r}, \hat{\eta})$,

$$
\begin{aligned}
(\nabla \cos\eta)^{\mathsf T} b(x) &= \frac{1}{r^2}\left[y^{\mathsf T} b(x) + (\mathbf{Q} - 2\cos\eta) x^{\mathsf T} b(x)\right] \\
&= \frac{x^{\mathsf T} b(x)}{r^2}\left[\frac{y^{\mathsf T} b(x)}{x^{\mathsf T} b(x)} + \frac{x^{\mathsf T} b(y)}{y^{\mathsf T} b(y)} - 2\cos\eta\right] \qquad (4.44)\\
&= \sin\eta \frac{b_r(x)}{r} T(x).
\end{aligned}
$$

PROOF OF (i) AND (iii). Define, for some $0 < c \le \gamma < \hat{\eta}$, the (non-Markov) time

$$S_t = \max\{0 \le s \le t \colon \eta_s \le c\} \qquad (4.45)$$

(we take $S_t = 0$ if $\eta_s > c$ for all $0 \le s \le t$) and write

$$\cos\eta_t = \cos\eta_{S_t} + M_t - M_{S_t} + E_t - E_{S_t} + C_t - C_{S_t} + D_t - D_{S_t}. \qquad (4.46)$$

For each path in $\Omega_1 \cap \Omega_2$ with $|X_0|$ sufficiently large we show $\eta_t < \hat{\eta}$ for $0 \le t \le \tau$. If $S_t = t$ then we must have $\eta_t \le c < \hat{\eta}$; otherwise $0 \le S_t < t$ and $\eta_{S_t} = \eta_0$ or c so

$$\eta_{S_t} \le \gamma < \frac{\gamma + \hat{\eta}}{2} < \hat{\eta}. \qquad (4.47)$$

By (4.31), (4.39), and (4.29) we see, for $|X_0|$ large enough, on $\Omega_1 \cap \Omega_2$, $0 \le t \le \tau$,

$$
\begin{aligned}
|M_t - M_{S_t}| + |E_t - E_{S_t}| &\le 4h\left(z^{-1}(|X_0|)\right)^{1/3} + 2\frac{\mathbf{C}_2}{\mathbf{C}_1} h\left(z^{-1}(|X_0|)\right) \qquad (4.48)\\
&\le \cos\frac{\gamma + \hat{\eta}}{2} - \cos\hat{\eta}.
\end{aligned}
$$

By (4.44) and the toe-in condition, A5,

$$(\nabla \cos\eta)^{\mathsf T} b(x) \ge \sin\eta \frac{b_r(x)}{r} \varepsilon(\eta) > 0, \qquad (4.49)$$

so by (4.37) and (4.42), since $\eta_s \geq c$ for $S_t \leq s \leq t$,

$$C_t - C_{S_t} + D_t - D_{S_t} \geq \int_{S_t}^{t} \left[\varepsilon(c) \sin c - \mathbf{C}_3 \frac{|\kappa(R_s)|}{b_r(X_s)} \right] \frac{b_r(X_s)}{R_s} \, ds. \tag{4.50}$$

By A2 and A4, since $R_s \geq |X_0|/2$ for $0 \leq s \leq \tau$, for $|X_0|$ large enough the integrand in (4.50) is positive.

Putting (4.46), (4.47), (4.48), and (4.50) together we get, for some $\varepsilon_1, \varepsilon_2 > 0$,

$$\begin{aligned} \cos \eta_t &\geq \cos \eta_{S_t} - \cos \frac{\gamma + \hat{\eta}}{2} + \cos \hat{\eta} \\ &\geq \cos \gamma - \cos \frac{\gamma + \hat{\eta}}{2} + \cos \hat{\eta} \\ &> \cos \hat{\eta} + \varepsilon_1, \end{aligned} \tag{4.51}$$

and hence $\eta_t < \hat{\eta} - \varepsilon_2$, $0 \leq t \leq \tau$. Since $R_t \geq |X_0|/2 > \hat{r}$, X_t never exits the cone $C(\hat{r}, \hat{\eta})$ on $\Omega_1 \cap \Omega_2$ which with (4.33) proves (i).

By conclusion (ii) and (4.26)–(4.28) we know that R_t^{-2} is integrable along (almost) any path in $\{\tau = \infty\}$ since $z(t)^{-2}$ is. This and (4.25) imply that $\langle M \rangle_t \to \langle M \rangle_\infty$, and hence $M_t \to M_\infty$, a finite random variable on the set $\{\tau = \infty\}$; similarly, $E_t \to E_\infty$ by (4.39). We now bolster the previous argument a little to get (iii).

The key fact is that (4.45) $S_t = t$ infinitely often, for otherwise the integrand in (4.50) will eventually be greater than const. $b_r(X_s)/R_s$ and hence the term $D_t - D_{S_t} + C_t - C_{S_t}$ of (4.46) will be unbounded, which is impossible since the other terms of $\cos \eta_t$ (4.48) are bounded. We see this by calculating $\log R_t$, using Itô's formula and (4.1),

$$\log R_t - \log R_0 = \int_0^t \frac{b_r(X_s)}{R_s} \, ds + \int_0^t \frac{dN_s}{R_s} - \frac{1}{2} \int_0^t \frac{d\langle N \rangle_s}{R_s^2}. \tag{4.52}$$

The latter two integrals converge since R_t^{-2} is integrable; note that the second term is a martingale whose variance process is $-2 \times$ the third term, which is convergent by A1 and (4.3); also (4.9), (4.10). Since $\log R_t \to \infty$, the first integral must diverge.

Given $0 < c < \theta < \hat{\eta}$, and given a sample path X_t in $\{\tau = \infty\}$, we can choose $\mathbf{T}$ so large that for any $t \geq \mathbf{T}$, $|M_t - M_\infty| + |E_t - E_\infty| < \cos c - \cos \theta$; also $S_t > 0$ and hence $\cos \eta_{S_t} = \cos c$, and (see (4.50)) $D_t - D_{S_t} + C_t - C_{S_t} > 0$. Now (4.46) becomes

$$\cos \eta_t \geq \cos c - (\cos c - \cos \theta) + 0 = \cos \theta, \tag{4.53}$$

and hence $\eta_t \leq \theta$, $t \geq \mathbf{T}$. Since $0 < c < \theta < \hat{\eta}$ were arbitrary, this proves $\eta_t \to 0$. □

Acknowledgment. The work presented here was done in completion of the dissertation requirements for the Ph.D. in Mathematics at the University of California at Los Angeles. I want to take this opportunity to thank my advisor, Professor Richard T. Durrett, for his patient guidance and support.

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