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A CENTRAL LIMIT THEOREM UNDER METRIC ENTROPY WITH L_2 BRACKETING[1]

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Let $(\mathbf{S}, \rho)$ be a metric space, $(\mathbf{V}, \mathscr{V}, \mu)$ be a probability space, and $f: \mathbf{S} \times \mathbf{V} \to \mathbb{R}$ be a real-valued function on $\mathbf{S} \times \mathbf{V}$ which has mean zero and is Lipschitz in $L_2(\mu)$ with respect to ρ. Let V be a random variable defined on $(\mathbf{V}, \mathscr{V}, \mu)$, and let $\{V_i: i \geq 1\}$ be a sequence of independent copies of V. The limiting behavior of the process $S_n(s) = n^{-1/2}\sum_{i=1}^{n} f(s, V_i)$ is studied under an integrability condition on the metric entropy with bracketing in $L_2(\mu)$. This metric entropy condition is analogous to one which implies the continuity of the limiting Gaussian process. A tightness result is derived which, in conjunction with the results of Andersen and Dobrić (1987), shows that a central limit theorem holds for the S_n-process. This result generalizes those of Dudley (1978), Dudley (1981) and Jain and Marcus (1975).

1. Introduction. In this paper, a central limit theorem is given for random functions with finite second moments. A metric entropy condition is assumed which is analogous to one which implies the continuity of the limiting Gaussian process. This central limit theorem generalizes those of Dudley (1978), Dudley (1981) and Jain and Marcus (1975).

Let $(\mathbf{S}, \rho)$ be a metric space, $(\mathbf{V}, \mathscr{V}, \mu)$ be a probability space, and $f: \mathbf{S} \times \mathbf{V} \to \mathbb{R}$ be a real-valued function on $\mathbf{S} \times \mathbf{V}$ which has mean zero and is Lipschitz in $L_2(\mu)$ with respect to ρ. Let V be a random variable (r.v.) defined on $(\mathbf{V}, \mathscr{V}, \rho)$ and let $\{V_i: i \geq 1\}$ be a sequence of independent copies of V. We study the limiting behavior of the process

$$S_n(s) = n^{-1/2} \sum_{i=1}^{n} f(s, V_i), \quad \text{for } s \in \mathbf{S}, \tag{1.1}$$

under certain metric entropy integrability conditions. In particular, we derive a tightness result which, in conjunction with the results of Andersen and Dobrić (1987) shows that a central limit theorem holds for the S_n-process. This result, in conjunction with the results of Dudley and Philipp (1983), also gives an invariance principle and a law of the iterated logarithm.

Processes of this type have been studied by numerous authors, frequently under the name empirical processes. [To see the sense in which this is the case, let $P_n(\cdot) = n^{-1}\sum_{i=1}^{n}\delta_{V_i}(\cdot)$ denote the empirical measure and identify $f(s, \cdot)$ with $f_s(\cdot)$. Then $S_n(s) = n^{-1/2}\int f_s \, dP_n$ is an empirical process indexed by the family of functions $\{f_s: s \in S\}$.] For a recent review of limit theorems for empirical

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processes see Giné and Zinn (1984). A key central limit theorem for an important family of uniformly bounded functions is due to Dudley (1978). Under a metric entropy condition analogous to that needed for the continuity of Gaussian measures he shows that a central limit theorem holds for the centered set-indexed empirical measure. In Dudley (1981) a central limit theorem is derived for families of functions which are uniformly bounded in L_p for some $p > 2$. However, the metric entropy condition assumed depends on p and is stronger than that needed for the continuity of the limiting Gaussian process. (See Section 2 below.) This second result of Dudley's parallels those of Pyke (1983) and Bass and Pyke (1984) for set-indexed partial-sum processes composed of random masses attached to fixed lattice points. Jain and Marcus (1975) prove a central limit theorem for families of functions taking values in $C(\mathbf{S})$ under the same metric entropy condition as that needed for the continuity of the limiting Gaussian processes. However (using our notation), they also assume the rather stringent condition

$$|f(s,V) - f(t,V)| \le g(V)\rho(s,t), \quad \text{for } s, t \in \mathbf{S},$$

where $g\colon \mathbf{V} \to \mathbb{R}$ is a measurable function with $Eg^2(V) < \infty$.

A goal of this paper is to unify these seemingly disparate results. [For example, the classical central limit theorem is a corollary of Jain and Marcus (1975), but not of Dudley (1978) or Dudley (1981).] We prove a central limit theorem for the process $\{f(s,V)\colon s \in \mathbf{S}\}$ assuming only finite second moments and the metric entropy integrability condition

$$\int_0^1 (H^B(u,\mathbf{S},\rho))^{1/2}\,du < \infty, \tag{1.2}$$

where $H^B(\cdot,\mathbf{S},\rho)$ denotes the metric entropy with bracketing of f with respect to ρ in L_2. Condition (1.2) is analogous to the metric entropy condition needed for the continuity of the limiting Gaussian process.

The relationship between laws of the iterated logarithm and the central limit theorem for r.v.'s taking values in a real separable Banach space with norm $\|\cdot\|$ has been studied by many authors, cf. Kuelbs and Zinn (1983). Pisier (1975/1976) proves that, in this framework, if X is a mean zero r.v. with $E\|X\|^2 < \infty$ which satisfies the central limit theorem, then X also satisfies the compact law of the iterated logarithm. These results cannot be applied directly in our general framework since we make only the minimal assumption that $f(\cdot,V) \in B(\mathbf{S})$, the bounded real-valued functions on $\mathbf{S}$, which, together with the sup-norm, $\|\cdot\|$, forms a nonseparable Banach space. [If $f(\cdot,V) \in C(\mathbf{S})$ with $\mathbf{S}$ totally bounded, these results immediately give a functional law of the iterated logarithm since $(C(\mathbf{S}), \|\cdot\|)$ form a separable Banach space.] However, under condition (1.2), $\mathbf{S}$ is totally bounded and the limiting Gaussian processes have support in $C(\mathbf{S})$. Thus Pisier's result together with the invariance principles of Dudley and Philipp (1983) can be used to derive a functional law of the iterated logarithm.

In Section 2 the notation and assumptions used throughout are introduced. In particular, metric entropy is defined and related results on the continuity of Gaussian processes are discussed.

The central limit theorem and two important probability bounds are presented in Section 3. These include a tightness criterion and an exponential probability bound. The methods used to prove the central tightness result follow those used by Bass (1985) in the context of partial-sum processes. Throughout, only conditions on the second moment are required.

The final results of this paper are presented in Section 4. We show that the invariance principles of Dudley and Philipp (1983) hold in our general setting. These results are used to derive a Strassen-type law of the iterated logarithm.

2. Notation and assumptions. Let $(\mathbf{S}, \rho)$ denote a separable pseudo-metric space and let $\mathscr{S}$ denote its Borel σ-field. Let $(\mathbf{V}, \mathscr{V}, \mu)$ be a probability space. Let $f: \mathbf{S} \times \mathbf{V} \to \mathbb{R}$ be a function which is measurable-$\mathscr{S} \times \mathscr{V}$ and suppose

$$Ef(s,V) = 0, \quad \text{for all } s \in \mathbf{S}, \tag{2.1}$$

$$Ef^2(s,V) < \infty, \quad \text{for all } s \in \mathbf{S}, \tag{2.2}$$

and

$$\left(E(f(s,V) - f(t,V))^2\right)^{1/2} \leq \rho(s,t), \quad \text{for all } s,t \in \mathbf{S}. \tag{2.3}$$

That is, each $f(s,\cdot)$ is in $L_2(\mu)$ and the L_2-distance between the members of $\mathscr{F} = \{f(s,\cdot): s \in \mathbf{S}\}$ is dominated by ρ, or, in other words, $f(s,\cdot)$ is Lipschitz with respect to ρ in $L_2(\mu)$. We also assume that the process $\{f(s,V): s \in \mathbf{S}\}$ is *separable*. That is, we assume there is a countable dense set $D_{\mathbf{s}} \subset \mathbf{S}$ and a measurable set $N \subset \mathbf{V}$ with $\mu(N) = 0$ such that for any open set $G \subset \mathbf{S}$ and any closed set $F \subset \mathbb{R}$,

$$[f(s,V) \in F \text{ for all } s \in G \cap D_{\mathbf{s}}] \setminus [f(s,V) \in F \text{ for all } s \in G] \subset N.$$

Let $\{V_i: i \geq 1\}$ be a sequence of i.i.d. copies of V, and define the partial-sum process

$$S_n(s) = n^{-1/2} \sum_{i=1}^{n} f(s,V_i), \quad \text{for } s \in \mathbf{S}. \tag{2.4}$$

The space in which f and, consequently, S_n take values is discussed below.

We now assume that $f(s,\cdot)$ can be bounded above and below by simple functions (in $\mathbf{S}$) which are themselves close in $L_2(\mu)$. For $\delta > 0$, let $\mathbf{S}(\delta)$ denote a finite δ-net for $\mathbf{S}$ with respect to the metric ρ. Suppose that for each $\delta > 0$, there exist f_δ^u and f_δ^l, measurable real-valued functions on $\mathbf{S}(\delta) \times \mathbf{V}$, such that for each $s \in \mathbf{S}$ there is some $s^\delta \in \mathbf{S}(\delta)$ satisfying

$$\rho(s, s^\delta) < \delta, \tag{2.5}$$

with

$$f_\delta^l(s^\delta, V) \leq f(s,V) \leq f_\delta^u(s^\delta, V) \quad \text{a.s.} \tag{2.6}$$

and

$$\left(E\left(f_\delta^u(s^\delta,V) - f_\delta^l(s^\delta,V)\right)^2\right)^{1/2} < \delta. \tag{2.7}$$

DEFINITION 2.1. When (2.5), (2.6) and (2.7) above hold we say that $f_\delta^{\,u}$ and $f_\delta^{\,l}$ are, respectively, upper and lower δ-approximations to f in $L_2(\mu)$.

The definition of metric entropy is as follows.

DEFINITION 2.2. For $\delta > 0$, let

$$\nu(\delta, \mathbf{S}, \rho) = \min\{\text{card}\,\mathbf{S}(\delta)\colon \mathbf{S}(\delta) \text{ is a } \delta\text{-net for } \mathbf{S} \text{ with respect to the metric } \rho\} \tag{2.8}$$

and define the metric entropy of $\mathbf{S}$ with respect to ρ,

$$H(\delta, \mathbf{S}, \rho) = \ln \nu(\delta, \mathbf{S}, \rho). \tag{2.9}$$

Notice that from equation (2.3), for all $\delta > 0$ and each $s \in \mathbf{S}$, there exists some $s^\delta \in \mathbf{S}(\delta)$ such that

$$\left(E\left(f(s,V) - f(s^\delta, V)\right)^2\right)^{1/2} < \delta.$$

Let σ represent the pseudo-metric defined on $\mathbf{S}$ by the L_2-distance,

$$\sigma(s,t) = \left(E\left(f(s,V) - f(t,V)\right)^2\right)^{1/2}, \tag{2.10}$$

and consider the mean zero Gaussian process $\{Z(s)\colon s \in \mathbf{S}\}$ with covariance function

$$\text{Cov}(Z(s), Z(t)) = \text{Cov}(f(s,V), f(t,V)), \quad \text{for } s, t \in \mathbf{S}. \tag{2.11}$$

A criterion for the a.s. continuity of Z in terms of metric entropy is given by Dudley (1967), namely that if

$$\int_0^1 (H(u, \mathbf{S}, \sigma))^{1/2}\, du < \infty, \tag{2.12}$$

then there exists a continuous version of Z.

In order to calculate the necessary probability bounds a definition of a stronger form of metric entropy proves useful.

DEFINITION 2.3. Let

$$\nu^B(\delta, S, \rho) = \min\{\text{card}\,\mathbf{S}(\delta)\colon (2.5), (2.6) \text{ and } (2.7) \text{ are satisfied}\} \tag{2.13}$$

and define the metric entropy with bracketing of f with respect to ρ to be

$$H^B(\delta, \mathbf{S}, \rho) = \ln \nu^B(\delta, \mathbf{S}, \rho). \tag{2.14}$$

The term "bracketing" stems from the condition imposed by (2.6); f must now be "bracketed" between the "step" functions $f_\delta^{\,u}$ and $f_\delta^{\,l}$ defined on $\mathbf{S}(\delta) \times \mathbf{V}$. For the special case $\mathbf{S} = \mathscr{A}$, a family of sets, when the indicator function of the set s is identified with s, the metric entropy with bracketing of the family of indicator functions can be seen to be the same as the metric entropy with inclusion of $\mathscr{A}$. [Henceforth $\mathbf{S}(\delta)$ and $\mathbf{S}^B(\delta)$ will be understood to represent those δ-nets for $\mathbf{S}$ which have cardinality $\nu(\delta, \mathbf{S}, \rho)$ and $\nu^B(\delta, \mathbf{S}, \rho)$, respectively.]

Taking σ to be the metric defined by (2.10), we will see in Section 3 that the condition analogous to (2.12),

$$\int_0^1 \big(H^B(u,\mathbf{S},\sigma)\big)^{1/2}\,du < \infty, \tag{2.15}$$

plays a central role in the estimation of probability bounds on the partial-sum process $\{S_n(s)\colon s \in \mathbf{S}\}$ defined by (2.4) above.

It is frequently more difficult to calculate the metric entropy with bracketing, H^B, of a function f than to calculate the ordinary metric entropy, H, of f. Below we show that, when the local oscillation of f is bounded in L_2, H^B can be bounded in terms of H.

From separability, we may assume that, for each $\delta > 0$, $\mathbf{S}(\delta) \subset \mathbf{S}$. Let $\{B_\delta(s)\colon s \in \mathbf{S}(\delta)\}$ be a collection of δ-neighborhoods of the members of $\mathbf{S}(\delta)$ which covers $\mathbf{S}$. Suppose that for each $s \in \mathbf{S}(\delta)$

$$\Big(E^* \sup_{t \in B_\delta(s)} |f(s,V) - f(t,V)|^2\Big)^{1/2} \le g(\delta), \tag{2.16}$$

for some strictly increasing continuous function $g\colon [0,\infty) \to [0,\infty)$. ($E^*$ denotes upper expectation.)

Lemma 2.1. *If* (2.16) *holds, then for* $u > 0$,

$$H^B(u,\mathbf{S},\rho) \le H\big(g^{-1}(u/2),\mathbf{S},\rho\big). \tag{2.17}$$

Proof. Fix $\delta > 0$, let $\{B_\delta(s)\colon s \in \mathbf{S}\}$ be as above, and define

$$f_\delta^u = \inf\Big\{\psi \text{ meas: } \mathbf{S}(\delta)\times\mathbf{V} \to \mathbb{R}\colon \psi(s,\cdot) \ge \sup_{t\in B_\delta(s)} f(t,\cdot) \text{ a.s. for all } s \in \mathbf{S}(\delta)\Big\},$$

$$f_\delta^l = \sup\Big\{\psi \text{ meas: } \mathbf{S}(\delta)\times\mathbf{V} \to \mathbb{R}\colon \psi(s,\cdot) \le \inf_{t\in B_\delta(s)} f(t,\cdot) \text{ a.s. for all } s \in \mathbf{S}(\delta)\Big\}.$$

Since $\mathbf{S}(\delta) \subset \mathbf{S}$, for each $s \in \mathbf{S}(\delta)$

$$f_\delta^l(s,\cdot) \le f(t,\cdot) \le f_\delta^u(s,\cdot), \quad \text{for all } t \in B_\delta(s).$$

By Minkowski's inequality, the definition of upper expectation, and (2.16) for $s \in \mathbf{S}(\delta)$

$$\begin{aligned}
&\Big(E\big(f_\delta^u(s,V) - f_\delta^l(s,V)\big)^2\Big)^{1/2}\\
&\quad\le \Big(E\big(f_\delta^u(s,V) - f(s,V)\big)^2\Big)^{1/2} + \Big(E\big(f(s,V) - f_\delta^l(s,V)\big)^2\Big)^{1/2}\\
&\quad\le 2\Big(E^* \sup_{t\in B_\delta(s)} |f(s,V) - f(t,V)|^2\Big)^{1/2}\\
&\quad\le 2g(\delta).
\end{aligned}$$

Setting $\delta = g^{-1}(u/2)$, the result follows. □

COROLLARY 2.1. *If* (2.16) *holds, then for* $\delta > 0$

$$\int_0^{\delta}\left(H^B(u,\mathbf{S},\rho)\right)^{1/2}du \le 2\int_0^{g^{-1}(\delta/2)}\left(H(u,\mathbf{S},\rho)\right)^{1/2}dg(u). \tag{2.18}$$

PROOF. By Lemma 2.1,

$$\int_0^{\delta}\left(\left(H^B(u,\mathbf{S},\rho)\right)^{1/2}du \le \int_0^{\delta}\left(H\left(g^{-1}(u/2),\mathbf{S},\rho\right)\right)^{1/2}du.$$

Using the change of variables $v = g^{-1}(u/2)$, the result follows. □

Occasionally in the definition of metric entropy with bracketing, the stipulation that the upper and lower δ-approximations to f be within δ in L_2 is replaced with the stipulation that they be close in L_α for some $0 < \alpha < 2$; that is, condition (2.7) is replaced with

$$\left(E\left|f_\delta^u(s^\delta,V) - f_\delta^l(s^\delta,V)\right|^{\alpha}\right)^{1/\alpha} < \delta. \tag{2.19}$$

DEFINITION 2.4. For $0 < \alpha < 2$, let

$$\nu_\alpha^B(\delta,\mathbf{S},\rho) = \min\{\operatorname{card}\mathbf{S}(\delta)\colon (2.5), (2.6) \text{ and } (2.19) \text{ are satisfied}\}$$

and define the metric entropy with bracketing of f in L_α to be

$$H_\alpha^B(\delta,\mathbf{S},\rho) = \ln \nu_\alpha^B(\delta,\mathbf{S},\rho). \tag{2.20}$$

Henceforth let $\mathbf{S}_\alpha(\delta)$ represent the δ-net with cardinality $\nu_\alpha^B(\delta,\mathbf{S},\rho)$. From the Cauchy–Schwarz inequality we see that $H_\alpha^B(\delta,\mathbf{S},\rho) \le H^B(\delta,\mathbf{S},\rho)$ for each $\alpha \in (0,2)$. In certain cases, conditions on the integrability of H_α^B imply that the integrability condition (2.15) holds.

LEMMA 2.2. *If* $\sup_{s\in\mathbf{S}}|f(s,V)| \le Y$ *a.s. with* $Y \in L_p(\mu)$ *for some* $p > 2$ *and*

$$\int_0^1\left(H_\alpha^B(u,\mathbf{S},\rho)\right)^{1/2}u^{-p(2-\alpha)/2(p-\alpha)} < \infty, \tag{2.21}$$

then (2.15) *holds.*

PROOF. This result is an easy consequence of Hölder's inequality. Fix $\alpha \in (0,2)$ and for $\delta > 0$, let u_δ and l_δ represent, respectively, upper and lower δ-approximations to f in L_α. We can assume $|u_\delta|$ and $|l_\delta|$ are both bounded by Y. For $s \in \mathbf{S}_\alpha(\delta)$,

$$\begin{aligned}
&E\left(u_\delta(s,V) - l_\delta(s,V)\right)^2\\
&\quad\le E\left((2Y)^{p(2-\alpha)/(p-\alpha)}\left(u_\delta(s,V) - l_\delta(s,V)\right)^{\alpha(p-2)/(p-\alpha)}\right)\\
&\quad\le 2^{p(2-\alpha)/(p-\alpha)}\left(EY^p\right)^{(2-\alpha)/(p-\alpha)}\left(E\left(u_\delta(s,V) - l_\delta(s,V)\right)^{\alpha}\right)^{(p-2)/(p-\alpha)}\\
&\quad\le 2^{p(2-\alpha)/(p-\alpha)}\left(EY^p\right)^{(2-\alpha)/(p-\alpha)}\delta^{\alpha(p-2)/(p-\alpha)}.
\end{aligned}$$

Thus, letting $c_p = 2^{p(2-\alpha)/2(p-\alpha)}(EY^p)^{(2-\alpha)/2(p-\alpha)}$,

$$H^B\big(c_p\delta^{\alpha(p-2)/2(p-\alpha)}, \mathbf{S}, \rho\big) \le H_\alpha^B(\delta, \mathbf{S}, \rho),$$

so

$$\begin{aligned}
&\int_0^\delta \big(H^B(u, \mathbf{S}, \rho)\big)^{1/2}\,du \\
&\quad= \frac{c_p\alpha(p-2)}{2(p-\alpha)} \int_0^{(\delta/c_p)^{2(p-\alpha)/\alpha(p-2)}} \Big(H^B\big(c_p x^{\alpha(p-2)/2(p-\alpha)}, \mathbf{S}, \rho\big)\Big)^{1/2} \\
&\qquad\qquad \times x^{-p(2-\alpha)/2(p-\alpha)}\,dx \\
&\quad\le \frac{c_p\alpha(p-2)}{2(p-1)} \int_0^{(\delta/c_p)^{2(p-\alpha)/\alpha(p-2)}} \big(H_\alpha^B(x, \mathbf{S}, \rho)\big)^{1/2} x^{-p(2-\alpha)/2(p-\alpha)}\,dx. \qquad \square
\end{aligned}$$

In particular when f is bounded a.s. by a constant (so $f \in L_\infty$), this demonstrates that $H_\alpha^B(u, \mathbf{S}, \rho)/u^{2-\alpha}$ can replace $H^B(u, \mathbf{S}, \rho)$ in condition (2.15). Although this result has been stated for metric entropy with bracketing, it is clear the analogous result holds for the usual metric entropy as well.

For a function φ: $\mathbf{S} \to \mathbb{R}$, let

$$\|\varphi\|_{\mathbf{S}} = \sup_{s \in \mathbf{S}} |\varphi(s)| \tag{2.22}$$

denote the sup of $|\varphi|$ on $\mathbf{S}$. Whenever unambiguous we write $\|\cdot\|$ instead of $\|\cdot\|_{\mathbf{S}}$. Also let

$$\|\varphi\|_\delta = \sup_{s,\, t \in \mathbf{S}:\, \rho(s,t) < \delta} |\varphi(s) - \varphi(t)| \tag{2.23}$$

denote the sup of $|\varphi(s) - \varphi(t)|$ over the pairs (s, t) in the set of diameter δ about the diagonal of the space $\mathbf{S} \times \mathbf{S}$. It is well known that $B(\mathbf{S})$ is complete in the sup-norm, so that $(B(\mathbf{S}), \|\cdot\|)$ form a Banach space. Throughout this paper we make the natural assumption that, with probability one, $f(\cdot, V) \in B(\mathbf{S})$.

3. Weak convergence and probability bounds. In this section we give a central limit theorem for the S_n-process as well as the statement and proof of two important probability bounds. The method of proof of the probability bounds utilizes a chaining argument adapted from that of Bass (1985), who studied set-indexed partial-sum processes made up of random masses attached to fixed lattice points. This method allows us to prove a tightness result and an exponential probability bound using only the first and second moments of the random process $\{f(s, \cdot): s \in \mathbf{S}\}$.

We begin by defining convergence in law. Let $L(\mathbf{S}) \subset B(\mathbf{S})$.

DEFINITION 3.1. A sequence of $L(\mathbf{S})$-valued r.v.'s $\{Y_n: n \ge 1\}$ converges in law to a $L(\mathbf{S})$-valued r.v. Y ($Y_n \to_L Y$) if

$$Eg(Y) = \lim_{n\to\infty} E^*g(Y_n) \quad \forall\, g \in C\big(L(\mathbf{S}), \|\cdot\|_{\mathbf{S}}\big),$$

where $C(L(\mathbf{S}), \|\cdot\|_{\mathbf{S}})$ is the set of all bounded continuous functions from $(L(\mathbf{S}), \|\cdot\|_{\mathbf{S}})$ into $\mathbb{R}$. Here E^* denotes upper expectation.

This definition is due to Hoffman-Jørgensen (1985).

Again let $\{Z(s)\colon s \in \mathbf{S}\}$ be the mean zero Gaussian process with

$$\mathrm{Cov}(Z(s), Z(t)) = \mathrm{Cov}(f(s, V), f(t, V)).$$

THEOREM 3.1 (Central limit theorem). *If the metric entropy integrability condition*

$$\int_0^1 (H^B(u, \mathbf{S}, \rho))^{1/2}\, du < \infty \tag{3.1}$$

holds, then $S_n \to_L Z$.

PROOF. The proof is immediate, using the following finite-dimensional convergence theorem (Theorem 3.2), tightness theorem (Theorem 3.3) and Theorem 5.5 of Andersen and Dobrić (1987). □

THEOREM 3.2 (Finite-dimensional convergence). *The finite dimensional distributions of $\{S_n(s)\colon s \in \mathbf{S}\}$ converge weakly to those of $\{Z(s)\colon s \in \mathbf{S}\}$.*

PROOF. The finite-dimensional convergence of the S_n-process is an immediate consequence of the classical central limit theorem. □

THEOREM 3.3 (Tightness). *If the metric entropy integrability condition* (3.1) *holds, then for all $\eta, \varepsilon > 0$, there exists $\delta > 0$ such that for all n sufficiently large,*

$$P^*(\|S_n\|_\delta > \eta) < \varepsilon, \tag{3.2}$$

where P^* denotes outer probability.

Dudley (1981) shows the above tightness condition holds under more stringent conditions. He assumes that the family of functions $\mathscr{F} = \{f(s, \cdot)\colon s \in \mathbf{S}\}$ is dominated by $Y \in L_p(\mu)$ for some $p > 2$ and that

$$H_1^B(\delta, \mathbf{S}, \rho) \le c\delta^{-\gamma}$$

for some $\gamma \in (0, 1 - 2/p)$. Setting $\alpha = 1$ in Lemma 2.2 we see that Theorem 3.1 implies the result of Dudley (1981).

We begin with a well-known symmetrization lemma. See, for example, Pollard (1982) for a proof.

LEMMA 3.1 (Symmetrization). *Let $\{V_i'\colon i \ge 1\}$ be an independent copy of $\{V_i\colon i \ge 1\}$, and let $\{S_n'\colon n \ge 1\}$ denote the partial-sum processes constructed from it. Then for all $0 < \delta < \varepsilon < \eta$,*

$$P^*(\|S_n\|_\delta > \eta) \le \varepsilon^2(\varepsilon^2 - \delta^2)^{-1} P^*(\|S_n - S_n'\|_\delta > \eta - \varepsilon). \tag{3.3}$$

This result allows us to bound the tail probabilities of the S_n-process with those of the *symmetric* $(S_n - S_n')$-process. After renormalizing (by dividing by $\sqrt{2}$) we see the covariance function of the symmetrized process is exactly that of the original partial-sum process. Thus we can replace, throughout the remainder of the paper, $f(\cdot, V_i)$ with $\varepsilon_i f(\cdot, V_i)$ for $i \geq 1$, where $\{\varepsilon_i: i \geq 1\}$ is a sequence of i.i.d. symmetric Bernoulli r.v.'s ($P(\varepsilon_1 = 1) = P(\varepsilon_1 = -1) = \frac{1}{2}$), independent of the sequence $\{V_i: i \geq 1\}$.

The initial step in the proof of a tightness result for a partial-sum or empirical process usually involves a truncation of the original process. Thus we begin the proof of our tightness theorem with the proof of a probability bound for a truncated form of the S_n-process. Using the same methods an exponential probability bound can be derived for a (different) truncated form of the S_n-process.

Let $\{\varphi_n: n \geq 1\}$ represent some increasing sequence with $\varphi_1 \geq 1$ and $\varphi_n = o(n^{1/2})$ as $n \to \infty$. For $a > 0$, let

$$\psi(a, x) = \begin{cases} a & \text{if } a < x, \\ x & \text{if } -a \leq x \leq a, \\ -a & \text{if } x < -a. \end{cases}$$

For each $\theta > 0$, $n \geq 1$ and $s \in \mathbf{S}$, let

$$f_n^{(\theta)}(s, \cdot) = \psi\big(\theta n^{1/2}\varphi_n^{-1}, f(s, \cdot)\big) \tag{3.4}$$

and

$$S_n^{(\theta)}(s) = n^{-1/2} \sum_{i=1}^{n} \varepsilon_i f_n^{(\theta)}(s, V_i). \tag{3.5}$$

PROPOSITION 3.1. *Take $\varphi_n = 1$ for all $n \geq 1$. If condition* (3.1) *above holds, then for all $\eta, \delta > 0$ and each $\theta \leq \delta(3/32(H^B(\delta, \mathbf{S}, \rho) + \eta^2))^{1/2}$*

$$P^*\left(\|S_n^{(\theta)}\|_\delta > K\left(\int_0^\delta \big(H^B(u, \mathbf{S}, \rho)\big)^{1/2} du + \eta\delta\right)\right) \leq 3 \sum_{k \geq 0} \exp\{-\eta^2 Lk\}, \tag{3.6}$$

where K is a fixed constant and $Lx = \ln x$ for $x \geq e$, $Lx = 1$ for $x < e$.

THEOREM 3.4 (Exponential probability bound). *Take $\{\varphi_n: n \geq 1\}$ to be any nondecreasing sequence with $\varphi_1 \geq 1$ and $\varphi_n = o(n^{1/2})$ as $n \to \infty$. If condition* (3.1) *above holds, then for all $\eta, \varepsilon > 0$, and $\delta \in (0, 1]$, there exist constants $c, \theta > 0$, depending on η, ε and δ, such that*

$$P^*\big(\|S_n^{(\theta)}\|_\delta > \eta\varphi_n\big) \leq c\exp\big\{-\eta^2\varphi_n^2/\delta(2 + \varepsilon)\big\}. \tag{3.7}$$

PROOF OF THEOREM 3.3. Fix $\eta, \varepsilon > 0$ and $\varphi_n = 1$ for all n. Notice that, for any $\delta > 0$,

$$0 \leq \|f(\cdot, V)\| \leq \sum_{s \in \mathbf{S}(\delta)} \big(|f_\delta^u(s, V)| + |f_\delta^l(s, V)|\big)$$

with probability one. Thus the family of r.v.'s $\{f(s, V): s \in \mathbf{S}\}$ is uniformly bounded by a positive r.v. $Y \in L_2(\mu)$. Thus for $\theta > 0$,

$$\begin{aligned}\|S_n - S_n^{(\theta)}\| &\le n^{-1/2} \sum_{i=1}^{n} \| f(\cdot, V_i) - f_n^{(\theta)}(\cdot, V_i)\| \\ &\le n^{-1/2} \sum_{i=1}^{n} \| f(\cdot, V_i)\| \mathbf{1}_{[\|f(\cdot, V_i)\| > \theta n^{1/2}]} \\ &\le n^{1/2} \sum_{i=1}^{n} Y_i \mathbf{1}_{[Y_i > \theta n^{1/2}]},\end{aligned}$$

where $\{Y_i\colon i \ge 1\}$ is a sequence of independent copies of Y. Thus

$$\begin{aligned}P\big(\|S_n - S_n^{(\theta)}\| > \eta/4\big) &\le P\left(n^{-1/2} \sum_{i=1}^{n} Y_i \mathbf{1}_{[Y_i > \theta n^{1/2}]} > \eta/4\right) \\ &\le P\big(Y_i > \theta n^{1/2} \text{ for some } 1 \le i \le n\big) \\ &\le nP(Y > \theta n^{1/2}) = o(1) \quad \text{as } n \to \infty.\end{aligned}$$

Since

$$\|S_n\|_\delta \le \|S_n^{(\theta)}\|_\delta + 2\|S_n - S_n^{(\theta)}\|,$$

it remains to show

$$P^*\big(\|S_n^{(\theta)}\|_\delta > \eta/2\big) < \varepsilon/2. \tag{3.8}$$

Choose η' so that $3\Sigma_{k \ge 0}\exp\{-(\eta')^2Lk\} < \varepsilon/2$. Now choose δ sufficiently small to have

$$K\left(\int_0^\delta \big(H^B(u, \mathbf{S}, \rho)\big)^{1/2}\, du + \delta\eta'\right) \le \eta/2.$$

Thus (3.8) holds for $\theta < \delta(3/32(H^B(\delta, \mathbf{S}, \rho) + (\eta')^2))^{1/2}$. □

PROOF OF PROPOSITION 3.1. The following setup is common to the proof of both Proposition 3.1 and Theorem 3.3, so temporarily let φ_n represent any nondecreasing sequence with $\varphi_1 = 1$ and $\varphi_n = o(n^{1/2})$.

Fix $\eta, \delta > 0$. For $k \ge 0$, let $\delta_k = \lambda\delta\beta^k$ and $\gamma_k = \Sigma_{j \le k} H^B(\delta_j, \mathbf{S}, \rho)$, where $\lambda \in (0,1]$ and $\beta \in (0,1)$ are constants with values to be specified later. Let $\{a_k\colon k \ge 0\}$ be a strictly decreasing sequence with $\lim_{k \to \infty} a_k = 0$. The values of a_k will be specified later. The intervals $I_k := [a_{k+1}, a_k)$ partition the interval $(0, a_0)$. Also define $\bar{I}_k = [a_{k+1}, \infty)$.

This proof relies heavily upon a special stratification of the truncated process $S_n^{(\theta)}$. The strata will be determined by the magnitude of the difference of the upper and lower δ_k approximations to the individual $f_n^{(\theta)}$'s. Fix $\theta \le a_0/2$. Construct a *nested* sequence of upper and lower δ_k-approximations to $f_n^{(\theta)}$ in L_2 in the following manner. For $s \in \mathbf{S}$, let

$$\begin{aligned}u_k(s, \cdot) &= \bigwedge_{j \le k} f_{\delta_j}^u(s^{\delta_j}, \cdot), \\ l_k(s, \cdot) &= \bigvee_{j \le k} f_{\delta_j}^l(s^{\delta_j}, \cdot), \\ u_{n,k}(s, \cdot) &= \psi\big(\theta n^{1/2}\varphi_n^{-1}, u_k(s, \cdot)\big),\end{aligned} \tag{3.9}$$

and

$$l_{n,k}(s,\cdot) = \psi(\theta n^{1/2}\varphi_n^{-1}, l_k(s,\cdot)). \tag{3.10}$$

Notice that $l_{n,k}$ and $u_{n,k}$ only depend on $\mathbf{S}$ through the $k+1$ values $s^{\delta_0},\dots,s^{\delta_k}$. Since $\sup_{s\in\mathbf{S}}|f_n^{(\theta)}(s,\cdot)| \le a_0 n^{1/2}/2\varphi_n$ a.s., we can assume

$$\sup_{s\in\mathbf{S}} |u_{n,k}(s,\cdot)| \vee |l_{n,k}(s,\cdot)| \le a_0 n^{1/2}/2\varphi_n \quad \text{a.s.},$$

as well as

$$\sup_{s\in\mathbf{S}} |u_{n,k}(s,\cdot) - l_{n,k}(s,\cdot)| \le a_0 n^{1/2}/\varphi_n \quad \text{a.s.} \tag{3.11}$$

Notice that for each $k \ge 0$,

$$l_{n,k}(s,\cdot) \le f_n^{(\theta)}(s,\cdot) \le u_{n,k}(s,\cdot) \quad \text{a.s.}, \tag{3.12}$$

$$0 \le u_{n,k+1}(s,\cdot) - l_{n,k+1}(s,\cdot) \le u_{n,k}(s,\cdot) - l_{n,k}(s,\cdot) \tag{3.13}$$

and

$$\begin{aligned}\left(E(u_{n,k}(s,V) - l_{n,k}(s,V))^2\right)^{1/2} &\le \left(E\left(f_{\delta_k}^u(s^{\delta_k},V) - f_{\delta_k}^l(s^{\delta_k},V)\right)^2\right)^{1/2} \\ &\le \delta_k.\end{aligned} \tag{3.14}$$

We can now construct the sets with which we partition $S_n^{(\theta)}$. Choose k_n so that

$$n\varphi_n^{-2}a_{k_n+1} < \int_0^{\delta_0}(H^B(u,\mathbf{S},\rho))^{1/2}\,du + \eta\delta_0 \le n\varphi_n^{-2}a_{k_n}. \tag{3.15}$$

For $0 \le k \le k_n$, let

$$A_{n,k}(s) = \left[(u_{n,k}(s,\cdot) - l_{n,k}(s,\cdot))\varphi_n/n^{1/2} \in I_k\right] \tag{3.16}$$

and

$$\tilde{A}_{n,k}(s) = \left[(u_{n,k}(s,\cdot) - l_{n,k}(s,\cdot))\varphi_n/n^{1/2} \in \bar{I}_k\right]. \tag{3.17}$$

Denote the related disjoint events by

$$\begin{aligned}B_{n,k}(s) &= \tilde{A}_{n,k}(s) \Big\backslash \bigcup_{j=0}^{k-1} \tilde{A}_{n,j}(s) \\ &= A_{n,k}(s) \Big\backslash \bigcup_{j=0}^{k-1} \tilde{A}_{n,j}(s),\end{aligned} \tag{3.18}$$

where $\bigcup_{j=0}^{-1}\tilde{A}_{n,j}(s)$ is understood to be the null set. Let

$$\begin{aligned}B_{n,k_n+1}(s) &= \left(\bigcup_{k=0}^{k_n} B_{n,k}(s)\right)^c \\ &= \left(\bigcup_{k=0}^{k_n} \tilde{A}_{n,k}(s)\right)^c \\ &= \bigcap_{k=0}^{k_n} \tilde{A}_{n,k}^c(s).\end{aligned}$$

The collection of sets $\{B_{n,k}(s): 0 \le k \le k_n + 1\}$ partitions the sample space. For $k \ge 1$, let

$$
\begin{aligned}
C_{n,k}(s) &= \left(\bigcup_{j=0}^{k-1} \tilde{A}_{n,j}(s)\right)^c \\
&= \left(\bigcup_{j=0}^{k-1} B_{n,j}(s)\right)^c \qquad (3.19)\\
&= \bigcup_{j=k}^{k_n+1} B_{n,j}(s).
\end{aligned}
$$

Since $C_{n,k}(s) \subset \tilde{A}^c_{n,k-1}(s)$, we have, on the set $C_{n,k}(s)$,

$$
\begin{aligned}
|l_{n,k}(s,\cdot) - l_{n,k-1}(s,\cdot)| &\le u_{n,k-1}(s,\cdot) - l_{n,k-1}(s,\cdot) \qquad (3.20)\\
&\le a_k \varphi_n^{-1} n^{1/2} \quad \text{a.s.}
\end{aligned}
$$

Notice that, using Chebyshev's inequality,

$$
\begin{aligned}
P(\tilde{A}_{n,k}(s)) &\le E(u_{n,k}(s,V) - l_{n,k}(s,V))^2 \varphi_n^2 / a_{k+1}^2 n \qquad (3.21)\\
&\le \delta_k^2 \varphi_n^2 / a_{k+1}^2 n.
\end{aligned}
$$

We now stratify $S_n^{(\theta)}(s)$ using the partition $\{B_{n,k}(s): 0 \le k \le k_n + 1\}$ constructed above. Within each stratum, we compare each $f_n^{(\theta)}$ to its lower δ_k-approximation, $l_{n,k}$. For $0 \le k \le k_n + 1$, let

$$S_{n,k}(s) = n^{-1/2} \sum_{i=1}^{n} \varepsilon_i f_n^{(\theta)}(s, V_i) \mathbf{1}_{B_{n,k}(s)}(V_i)$$

and

$$L^{(1)}_{n,k}(s) = n^{-1/2} \sum_{i=1}^{n} \varepsilon_i l_{n,k}(s, V_i) \mathbf{1}_{B_{n,k}(s)}(V_i).$$

Then, since $\theta \le a_0/2$,

$$S_n^{(\theta)}(s) = \sum_{k \le k_n+1} S_{n,k}(s). \qquad (3.22)$$

For $k \le k_n$,

$$
\begin{aligned}
|S_{n,k}(s) - L^{(1)}_{n,k}(s)| &\le n^{-1/2} \sum_{i=1}^{n} \left(f_n^{(\theta)}(s, V_i) - l_{n,k}(s, V_i)\right) \mathbf{1}_{B_{n,k}(s)}(V_i) \\
&\le n^{-1/2} \sum_{i=1}^{n} (u_{n,k}(s, V_i) - l_{n,k}(s, V_i)) \mathbf{1}_{A_{n,k}(s)}(V_i) \qquad (3.23)\\
&:= R^{(1)}_{n,k}(s).
\end{aligned}
$$

Likewise, we have

$$\begin{aligned}
&\left|S_{n,k_n+1}(s) - L^{(1)}_{n,k_n+1}(s)\right| \\
(3.24)\qquad &\leq n^{-1/2}\sum_{i=1}^{n}\left(u_{n,k_n+1}(s,V_i) - l_{n,k_n+1}(s,V_i)\right)\mathbf{1}_{B_{n,k_n+1}(s)}(V_i) \\
&:= R^{(1)}_{n,k_n+1}(s).
\end{aligned}$$

On the set $B_{n,k_n+1}(s)$, we have by (3.18)

$$\begin{aligned}
\left(u_{n,k_n+1}(s,\cdot) - l_{n,k_n+1}(s,\cdot)\right)/n^{1/2} &\leq a_{k_n+1}\varphi_n^{-1} \\
&\leq \left(\int_0^{\delta_0}\left(H^B(u,\mathbf{S},\rho)\right)^{1/2}du + \eta\delta_0\right)\varphi_n/n.
\end{aligned}$$

Thus

$$R^{(1)}_{n,k_n+1}(s) \leq \left(\int_0^{\delta_0}\left(H^B(u,\mathbf{S},\rho)\right)^{1/2}du + \eta\delta_0\right)\varphi_n. \tag{3.25}$$

Now, on the individual $B_{n,k}(s)$'s, we compare each lower δ_k-approximation, $l_{n,k}$, to the lower δ_0-approximation, $l_{n,0}$. Let

$$L^{(0)}_n(s) = n^{-1/2}\sum_{i=1}^{n}\varepsilon_i l_{n,0}(s,V_i).$$

For $k \leq k_n + 1$, let

$$L^{(0)}_{n,k}(s) = n^{-1/2}\sum_{i=1}^{n}\varepsilon_i l_{n,0}(s,V_i)\mathbf{1}_{B_{n,k}(s)}(V_i)$$

so that

$$L^{(0)}_n(s) = \sum_{k\leq k_n+1} L^{(0)}_{n,k}(s). \tag{3.26}$$

Notice that

$$L^{(0)}_{n,0}(s) - L^{(1)}_{n,0}(s) = 0,$$

and for $1 \leq k \leq k_n + 1$,

$$l_{n,k}(s,\cdot) - l_{n,0}(s,\cdot) = \sum_{j=1}^{k}\left(l_{n,j}(s,\cdot) - l_{n,j-1}(s,\cdot)\right).$$

Thus, recalling (3.19),

$$\sum_{k=0}^{k_n+1} \left(L_{n,k}^{(1)}(s) - L_{n,k}^{(0)}(s)\right)$$

$$(3.27)\quad \begin{aligned} &= \sum_{k=1}^{k_n+1} n^{-1/2} \sum_{i=1}^{n} \varepsilon_i \sum_{j=1}^{k} \left(l_{n,j}(s, V_i) - l_{n,j-1}(s, V_i)\right) \mathbf{1}_{B_{n,k(s)}}(V_i) \\ &= \sum_{j=1}^{k_n+1} n^{-1/2} \sum_{i=1}^{n} \varepsilon_i \left(l_{n,j}(s, V_i) - l_{n,j-1}(s, V_i)\right) \mathbf{1}_{C_{n,j(s)}}(V_i) \\ &:= \sum_{j=1}^{k_n+1} R_{n,j}^{(2)}(s). \end{aligned}$$

We now compare $S_n^{(\theta)}$ to $L_n^{(0)}$ defined above. Combining (3.23), (3.24) and (3.27), we see that, for each $s \in \mathbf{S}$, when $\theta \le a_0$,

$$\begin{aligned} |S_n^{(\theta)}(s) - L_n^{(0)}(s)| &= \left| \sum_{k \le k_n+1} \left(S_{n,k}(s) - L_{n,k}^{(0)}(s)\right) \right| \\ &\le \sum_{k \le k_n+1} |S_{n,k}(s) - L_{n,k}^{(1)}(s)| + \left| \sum_{k \le k_n+1} \left(L_{n,k}^{(1)}(s) - L_{n,k}^{(0)}(s)\right) \right| \\ &\le \sum_{k \le k_n+1} R_{n,k}^{(1)}(s) + \sum_{1 \le k \le k_n+1} |R_{n,k}^{(2)}(s)|. \end{aligned}$$

Thus,

$$(3.28)\quad \begin{aligned} \|S_n^{(\theta)}\|_\delta &\le \|L_n^{(0)}\|_\delta + 2\|S_n^{(\theta)} - L_n^{(0)}\| \\ &\le \|L_n^{(0)}\|_\delta + 2 \sum_{k \le k_n+1} \|R_{n,k}^{(1)}\| + 2 \sum_{1 \le k \le k_n+1} \|R_{n,k}^{(2)}\|. \end{aligned}$$

Then, recalling (3.25), when η_0, $\{\eta_k^{(1)}: 0 \le k \le k_n\}$ and $\{\eta_k^{(2)}: 1 \le k \le k_n + 1\}$ are constants which satisfy

$$(3.29)\quad \eta_0 + \sum_{k \le k_n} \eta_k^{(1)} + \sum_{1 \le k \le k_n+1} \eta_k^{(2)} \le c \int_0^{\delta_0} \left(H^B(u, \mathbf{S}, \rho)\right)^{1/2} du + \eta\delta_0,$$

for c some positive constant, we have

$$(3.30)\quad \begin{aligned} &P^*\left(\|S_n^{(\theta)}\|_\delta > (c+1)\left(\int_0^{\delta_0} \left(H^B(u, \mathbf{S}, \rho)\right)^{1/2} du + \eta\delta_0\right)\right) \\ &\quad \le P\left(\|L_n^{(0)}\|_\delta > \eta_0 \varphi_n\right) + \sum_{k \le k_n} P\left(\|R_{n,k}^{(1)}\| > \eta_k^{(1)} \varphi_n\right) \\ &\qquad + \sum_{1 \le k \le k_n+1} P\left(\|R_{n,k}^{(2)}\| > \eta_k^{(2)} \varphi_n\right). \end{aligned}$$

[The values of the constants η_0, $\{\eta_k^{(1)}: 0 \le k \le k_n\}$, and $\{\eta_k^{(2)}: 1 \le k \le k_n + 1\}$ will be specified later.]

The individual terms of equation (3.28) above are now bounded in probability using both Bernstein's inequality [cf. Bennett (1962)] and the upper bound, $\exp\{\gamma_k\}$, on the cardinality of $\bigcup_{j=0}^{k}\mathbf{S}(\delta_j)$.

For $k \le k_n$, $R_{n,k}^{(1)}(s)$ is a sum of i.i.d. nonnegative r.v.'s, each bounded by $a_k\varphi_n^{-1}$. Hence

$$\begin{aligned} ER_{n,k}^{(1)}(s) &= n^{1/2}E(u_{n,k}(s,V) - l_{n,k}(s,V))\mathbf{1}_{A_{n,k}(s)}(V) \\ &\le n^{1/2}\Big(E(u_{n,k}(s,V) - l_{n,k}(s,V))^2 P(\tilde{A}_{n,k}(s))\Big)^{1/2} \\ &\le \delta_k^2\varphi_n/a_{k+1} \end{aligned}$$

using (3.14), (3.20) and Hölder's inequality. Likewise,

$$\operatorname{Var} R_{n,k}^{(1)}(s_k) \le a_k\varphi_n^{-1}ER_{n,k}^{(1)}(s) < \delta_k^2 a_k/a_{k+1}.$$

Thus by Bernstein's inequality, for each $s \in \mathbf{S}$, when $\eta_k^{(1)} \ge 2\delta_k^2/a_{k+1}$,

$$\begin{aligned} P\big(R_{n,k}^{(1)}(s) > \eta_k^{(1)}\varphi_n\big) &\le P\big(R_{n,k}^{(1)}(s) - ER_{n,k}^{(1)}(s) > \eta_k^{(1)}\varphi_n/2\big) \\ &\le \exp\big\{-\eta_k^{(1)2}\varphi_n^2/8\big(\delta_k^2 a_k a_{k+1}^{-1} + a_k\eta_k^{(1)}/6\big)\big\} \\ &\le \exp\big\{-3\eta_k^{(1)}\varphi_n^2/16a_k\big\}. \end{aligned}$$

Hence, since $R_{n,k}^{(1)}$ depends on $\mathbf{S}$ only through the (at most) $\exp\{\gamma_k\}$ members of $\bigcup_{j=0}^{k}\mathbf{S}(\delta_j)$,

$$\begin{aligned} \sum_{k \le k_n} P\big(\|R_{n,k}^{(1)}\| > \eta_k^{(1)}\varphi_n\big) &\le \sum_{k \le k_n} \exp\{\gamma_k\}\big\|P\big(R_{n,k}^{(1)}(\cdot) > \eta_k^{(1)}\varphi_n\big)\big\| \\ &\le \sum_{k \le k_n} \exp\big\{\gamma_k - 3\eta_k^{(1)}\varphi_n^2/16a_k\big\}. \end{aligned} \tag{3.31}$$

For $1 \le k \le k_n + 1$, $R_{n,k}^{(2)}(s)$ is a sum of i.i.d. mean zero r.v.'s, each bounded by $a_k\varphi_n^{-1}$ by equation (3.20). Also

$$\begin{aligned} \operatorname{Var} R_{n,k}^{(2)}(s) &\le E(l_{n,k}(s,V) - l_{n,k-1}(s,V))^2 \\ &\le E(u_{n,k-1}(s,V) - l_{n,k-1}(s,V))^2 \\ &\le \delta_{k-1}^2. \end{aligned}$$

Again by Bernstein's inequality, for each $s \in \mathbf{S}$,

$$P\big(|R_{n,k}^{(2)}(s)| > \eta_k^{(2)}\varphi_n\big) \le 2\exp\big\{-\eta_k^{(2)2}\varphi_n^2/2\big(\delta_{k-1}^2 + a_k\eta_k^{(2)}/3\big)\big\}.$$

Then, as for $R_{n,k}^{(1)}$,

$$\begin{aligned} &\sum_{1 \le k \le k_n+1} P\big(\|R_{n,k}^{(2)}\| > \eta_k^{(2)}\varphi_n\big) \\ &\quad \le 2\sum_{i \le k \le k_n+1} \exp\big\{\gamma_k - \eta_k^{(2)2}\varphi_n^2/2\big(\delta_{k-1}^2 + a_k\eta_k^{(2)}/3\big)\big\}. \end{aligned} \tag{3.32}$$

Lastly, for $s, t \in \mathbf{S}$, $L_n^{(0)}(s) - L_n^{(0)}(t)$ is a sum of i.i.d. mean zero r.v.'s, each bounded by $a_0\varphi_n^{-1}$, and, when $\rho(s,t) < \delta$,

$$\begin{aligned}\operatorname{Var}\big(L_n^{(0)}(s) - L_n^{(0)}(t)\big) &= E\big(l_{n,0}(s,V) - l_{n,0}(t,V)\big)^2\\ &\le \Big(\big(E(f(s,V) - f(t,V))^2\big)^{1/2}\\ &\quad + \big(E(u_{n,0}(s,V) - l_{n,0}(s,V))^2\big)^{1/2}\\ &\quad + \big(E(u_{n,0}(t,V) - l_{n,0}(t,V))^2\big)^{1/2}\Big)^2\\ &\le \big(\rho(s,t) + 2\delta_0\big)^2\\ &< \delta^2(1+2\lambda)^2.\end{aligned}$$

Again using Bernstein's inequality, for all $s, t \in \mathbf{S}$ with $\rho(s,t) < \delta$,

$$P\big(|L_n^{(0)}(s) - L_n^{(0)}(t)| > \eta_0\eta\varphi_n\big) \le 2\exp\Big\{-\eta_0^2\varphi_n^2/2\big(\delta^2(1+2\lambda)^2 + a_0\eta_0/3\big)\Big\}.$$

Then

$$(3.33)\quad P\big(\|L_n^{(0)}\|_\delta > \eta_0\varphi_n\big) \le 2\exp\{2\gamma_0\}\exp\Big\{-\eta_0^2\varphi_n^2/2\big(\delta^2(1+2\lambda)^2 + a_0\eta_0/3\big)\Big\}.$$

We now specialize to the proof of Proposition 3.1. Set $\varphi_n = 1$ for all n and fix $\lambda = 1$. Recall $Lx = \ln x$ for $x \ge e$, $Lx = 1$ for $x < e$. For $k \ge 0$, let $a_k = \delta_k(3/8(\gamma_k + \eta^2 Lk))^{1/2}$, $\eta_k^{(1)} = 2\delta_k^2/a_{k+1}$ and $\eta_k^{(2)} = \delta_{k-1}^2/a_k$. Thus, since $\gamma_k - 3\eta_k^{(1)}/16a_k \le -\eta^2 Lk$ and $\gamma_k - 3\eta_k^{(2)^2}/2(3\delta_{k-1}^2 + a_k\eta_k^{(2)}) \le -\eta^2 Lk$, we have from (3.31) and (3.32)

$$(3.34)\qquad \sum_{k\le k_n} P\big(\|R_{n,k}^{(1)}\| > \eta_k^{(1)}\big) \le \sum_{k\le k_n}\exp\{-\eta^2 Lk\}$$

and

$$(3.35)\qquad \sum_{1\le k\le k_n+1} P\big(\|R_{n,k}^{(2)}\| > \eta_k^{(2)}\big) \le 2\sum_{1\le k\le k_n}\exp\{-\eta^2 Lk\}.$$

Also set $\eta_0 = 7\delta^2/a_0$, so that $2\gamma_0 - \eta_0^2/2(27\delta^2 + a_0\eta_0/3) \le -\eta^2$, and thus, by (3.33),

$$(3.36)\qquad P\big(\|L_n^{(0)}\|_\delta > \eta_0\big) \le 2\exp\{-\eta^2\}.$$

By (3.30), it remains to show that (3.29) holds for some fixed constant c. First notice that

$$\begin{aligned}\sum_{0\le k\le k_n}\eta_k^{(1)} &\le (32/3)^{1/2}\sum_{k\ge 0}\big(\delta_k^2/\delta_{k+1}\big)\big(\gamma_{k+1} + \eta^2 L(k+1)\big)^{1/2}\\ &\le (32/3)^{1/2}\beta^{-2}\sum_{k\ge 1}\delta_k\big(\gamma_k^{1/2} + \eta L^{1/2}k\big).\end{aligned}$$

Clearly

$$\sum_{k\ge 1}\delta_k L^{1/2}k = \delta\sum_{k\ge 1}\beta^k L^{1/2}k \le c\delta$$

for some constant c. From the definition of γ_k,

$$\begin{aligned}\sum_{k\geq 1}\delta_k\gamma_k^{1/2} &\leq \sum_{k\geq 1}\delta_k\sum_{j\leq k}\left(H^B(\delta_j,\mathbf{S},\rho)\right)^{1/2}\\ &\leq \sum_{j\geq 0}\left(H^B(\delta_j,\mathbf{S},\rho)\right)^{1/2}\sum_{k\geq j}\delta_k\\ &= (1-\beta)^{-1}\sum_{j\geq 0}\delta_j\left(H^B(\delta_j,\mathbf{S},\rho)\right)^{1/2}\\ &\leq (1-\beta)^{-2}\sum_{j\geq 0}\int_{\delta_{j+1}}^{\delta_j}\left(H^B(u,\mathbf{S},\rho)\right)^{1/2}du\\ &= (1-\beta)^{-2}\int_0^{\delta}\left(H^B(u,\mathbf{S},\rho)\right)^{1/2}du.\end{aligned}\tag{3.37}$$

Since $\eta_k^{(2)} = \eta_{k-1}^{(1)}/2$ and

$$\begin{aligned}\eta_0 &= 7(8/3)^{1/2}\delta\left(\gamma_0+\eta^2\right)^{1/2}\\ &\leq 7(8/3)^{1/2}\left(\delta\left(H^B(\delta,\mathbf{S},\rho)\right)^{1/2}+\delta\eta\right)\\ &\leq 7(8/3)^{1/2}\left(\int_0^{\delta}\left(H^B(u,\mathbf{S},\rho)\right)^{1/2}du+\delta\eta\right),\end{aligned}$$

the result follows. □

PROOF OF THEOREM 3.3. Pick up the proof of Proposition 3.1 above at (3.33). For $k\geq 0$, let $a_k = \delta_k(3/8(\gamma_k+2Lk+\eta^2/2\delta))^{1/2}$, $\eta_k^{(1)} = 2\delta_k^2/a_{k+1}$ and $\eta_k^{(2)} = \delta_{k-1}^2/a_k$. Then

$$\gamma_k - 3\eta_k^{(1)}\varphi_n^2/16a_k \leq -2\varphi_n^2Lk - \eta^2\varphi_n^2/2\delta$$

and

$$\gamma_k - 3\eta_k^{(2)^2}\varphi_n^2/2\left(3\delta_{k-1}^2+a_k\eta_k^{(2)}\right) \leq -2\varphi_n^2Lk - \eta^2\varphi_n^2/2\delta,$$

so that by (3.31) and (3.32)

$$\sum_{k\leq k_n}P\left(\|R_{n,k}^{(1)}\| > \eta_k^{(1)}\varphi_n\right) \leq \exp\left\{-\eta^2\varphi_n^2/2\delta\right\}\sum_{k\leq k_n}\exp\left\{-2\varphi_n^2Lk\right\}\tag{3.38}$$

and

$$\begin{aligned}&\sum_{1\leq k\leq k_n+1}P\left(\|R_{n,k}^{(2)}\| > \eta_k^{(2)}\varphi_n\right)\\ &\quad\leq \exp\left\{-\eta^2\varphi_n^2/2\delta\right\}\sum_{1\leq k\leq k_n+1}\exp\left\{-2\varphi_n^2Lk\right\}.\end{aligned}\tag{3.39}$$

Now fix $\varepsilon>0$ and set $\eta_0 = (2\eta^2a_0/3\delta(2+\varepsilon)) + \eta(1+2\lambda)(2\delta/(2+\varepsilon))^{1/2}$, so that $-\eta_0^2/2(\delta^2(1+2\lambda)^2+a_0\eta_0/3) \leq -\eta^2/\delta(2+\varepsilon)$. Then, by (3.36),

$$P\left(\|L_n^{(0)}\|_\delta > \eta_0\varphi_n\right) \leq 2\exp\left\{\gamma_0\right\}\exp\left\{-\eta^2\varphi_n^2/\delta(2+\varepsilon)\right\}.\tag{3.40}$$

As in the proof of Proposition 3.1, it is straightforward to show that $\sum_{0\leq k\leq k_n}\eta_k^{(1)} + \sum_{1\leq k\leq k_n+1}\eta_k^{(2)} \leq c(\int_0^{\delta_0}H^{1/2}(u)\,du + \eta\delta_0)$ for some fixed constant *c.* Also notice that

$$\eta_0 = 6^{-1/2}\eta^2\lambda\left(\gamma_0 + 2 + \eta^2/2\delta\right)^{-1/2}(2+\varepsilon)^{-1} + \eta(1+2\lambda)(2\delta/(2+\varepsilon))^{1/2}$$
$$\leq \eta\lambda/2 + \eta(1+2\lambda)(1-\varepsilon/3)^{1/2}.$$

Thus, recalling that $\delta_0 = \delta\lambda$, and choosing λ small enough to satisfy

$$(c+1)\left(\int_0^{\delta_0}(H^B(u,\mathbf{S},\rho))^{1/2}\,du + \eta\delta_0\right) \leq \eta\varepsilon/12,$$

$\lambda \leq \varepsilon/6$ and

$$(1+2\lambda)(1-\varepsilon/3)^{1/2} \leq (1-\varepsilon/6),$$

we have

$$\eta_0 + \sum_{1\leq k\leq k_n+1}\eta_k^{(1)} + \sum_{1\leq k\leq k_n+1}\eta_k^{(2)} + \int_0^{\delta_0}(H^B(u,\mathbf{S},\rho))^{1/2}\,du + \eta\delta_0 \leq \eta.$$

Combining (3.29), (3.30), (3.38), (3.39) and (3.40) the result follows. □

4. Invariance principles and a law of the iterated logarithm. Recall the definition of the S_n-process,

$$S_n(s) = n^{-1/2}\sum_{i=1}^{n}\varepsilon_i f(s,V_i), \quad \text{for } s \in \mathbf{S},$$

and let $\{Z(s)\colon s \in \mathbf{S}\}$ be the mean zero Gaussian process with covariance function given by

$$\operatorname{Cov}(Z(s),Z(t)) = \operatorname{Cov}(f(s,V),f(t,V)), \quad \text{for } s,t \in \mathbf{S}. \tag{4.1}$$

In the following, we show an invariance principle holds for the S_n-process by utilizing a strong invariance formulation due to Dudley and Philipp (1983). The ensuing results also yield a functional law of the iterated logarithm.

Let $(\mathbf{V}^\infty, \mathscr{V}^\infty, \mu^\infty)$ denote the countable product of copies of $(\mathbf{V}, \mathscr{V}, \mu)$, and (Ω, Σ, P) denote the product of $(\mathbf{V}^\infty, \mathscr{V}^\infty, \mu^\infty)$ with $(\mathbf{I}, \mathscr{B}(\mathbf{I}), \lambda)$ where $\mathscr{B}(\mathbf{I})$ is the collection of Borel sets in $\mathbf{I} = [0,1]$ and λ is Lebesgue measure on $\mathbf{I}$. Assume for now that $f(\cdot, V) \in B(\mathbf{S})$ and recall that $(B(\mathbf{S}), \|\cdot\|)$ is a Banach space. The conditions imposed on f by equations (2.5), (2.6) and (2.7) together with the metric entropy integral condition,

$$\int_0^1 (H^B(u,\mathbf{S},\rho))^{1/2}\,du < \infty, \tag{4.2}$$

insure that the family of functions $\{f(s,\cdot)\colon s \in \mathbf{S}\}$ is totally bounded as well as uniformly bounded in $L_2(\mu)$. Condition (4.2) also implies that $\{Z(s)\colon s \in \mathbf{S}\}$, the mean zero Gaussian process with covariance defined by (4.1), has versions which are uniformly continuous. The following theorem is a restatement of Theorem 1.3 of Dudley and Philipp (1983).

THEOREM 4.1. *If $\mathscr{F} = \{f(s,V)\colon s \in \mathbf{S}\}$ is totally bounded in $L_2(\mu)$ and for all η, $\varepsilon > 0$ there exists $\delta > 0$ such that, for n sufficiently large,*

$$P^*(\|S_n\|_\delta > \eta) < \varepsilon, \tag{4.3}$$

then there exists a sequence of i.i.d. copies of $\{Z(s)\colon s \in \mathbf{S}\}$, defined on

(Ω, Σ, P), *almost surely uniformly continuous on* $\mathbf{S}$, *and in* $L_2(P)$, *such that*

$$n^{-1/2} \max_{k \le n} \left\| \sum_{i \le k} (\varepsilon_i f(\cdot, V_i) - Z_i(\cdot)) \right\| \to 0 \quad \text{as } n \to \infty \tag{4.4}$$

in probability as well as $L_p(P)$ *for any* $p \le 2$. *The* Z_i*'s can also be chosen so that*

$$\left\| \sum_{i \le n} (\varepsilon_i f(\cdot, V_i) - Z_i(\cdot)) \right\| \le U_n = o\big((nLLn)^{1/2}\big), \tag{4.5}$$

with probability one for some sequence of measurable U_n*'s.*

The following corollary essentially states that, when the metric entropy integral condition holds, our S_n-processes can be approximated by continuous Gaussian processes defined on the same probability space and having the same covariance structure as the S_n's.

COROLLARY 4.1. *If equation* (4.2) *holds, then there exists a sequence* $\{\tilde{Z}_n\colon n \ge 1\}$ *of copies of* $\{Z(s)\colon s \in \mathbf{S}\}$, *defined on* (Ω, Σ, P), *almost surely uniformly continuous on* $\mathbf{S}$, *and in* $L_2(P)$ *such that*

$$\|S_n - \tilde{Z}_n\| \to 0 \quad \text{as } n \to \infty \tag{4.6}$$

in probability as well as $L_p(P)$ *for any* $p \le 2$. *The* $\tilde{Z}_n$*'s can also be chosen so that*

$$\|S_n - \tilde{Z}_n\| \le U_n = o\big((LLn)^{1/2}\big), \tag{4.7}$$

with probability one for some sequence of measurable U_n.

PROOF. This corollary follows from Theorem 4.1 and the probability bound of Theorem 3.3 after setting $\tilde{Z}_n = n^{-1/2}\sum_{i=1}^n Z_i$ and observing that $\tilde{Z}_n$ has the same distribution as Z for all n. □

Since we assume $Ef(s, V) = 0$ and $Ef^2(s, V) < \infty$, the classical law of the iterated logarithm, Hartman and Wintner (1941), implies that, for each $s \in \mathbf{S}$, the set of limit points of $\{S_n(s)/\varphi_n\colon n \ge 1\}$ is, with probability one, the closed interval

$$\Big[-\big(Ef^2(s, V)\big)^{1/2}, \big(Ef^2(s, V)\big)^{1/2}\Big].$$

[Here we set $\varphi_n = (2LLn)^{1/2}$, where $Lx = \ln x \vee 1$.] We are, however, interested in a functional law of the iterated logarithm of the type first proved by Strassen (1964).

Let

$$\mathscr{G} = \left\{g \in L_2(\mu)\colon \int g^2(v)\, d\mu(v) \le 1\right\} \tag{4.8}$$

and

$$\mathscr{G}(\mathbf{S}) = \left\{G\colon \mathbf{S} \to \mathbb{R}\colon G(s) = \int f(s, v) g(v)\, d\mu(v) \text{ for some } g \in \mathscr{G}\right\}. \tag{4.9}$$

We prove that, with probability one, the sequence $\{S_n(s)/\varphi_n\colon s \in \mathbf{S}\}$ is relatively compact and its set of limit points coincides with $\mathscr{G}(\mathbf{S})$ in the topology generated by $\|\cdot\|$.

THEOREM 4.2 (Functional law of the iterated logarithm). *If the metric entropy integral condition* (4.2) *holds, then with probability one,* $\{S_n/\varphi_n\colon n \geq 1\}$ *is relatively compact with respect to* $\|\cdot\|$, *and the set of its limit points coincides with* $\mathscr{G}(\mathbf{S})$.

The proof of this theorem relies heavily upon the invariance results of Theorem 4.1 and the relationship between the central limit theorem and the law of the iterated logarithm in separable Banach spaces. [See, for example, Heinkel (1979).] An alternate proof is available using the exponential probability bound of Theorem 3.4; see Ossiander (1985).

PROOF. By (4.4) of Theorem 4.1 and Theorem 3.3, there exists a sequence $\{Z_i\colon i \geq 1\}$ of i.i.d. copies of $\{Z(s)\colon s \in \mathbf{S}\}$, the mean zero Gaussian process with covariance given by (4.1), such that

$$\left\|\left(S_n - n^{-1/2}\sum_{i=1}^{n} Z_i\right)/\varphi_n\right\| \leq Y_n = o(1), \tag{4.10}$$

with probability one for some sequence of measurable Y_n's. The r.v. Z obviously satisfies the central limit theorem. Since $\|Z\|$ is square integrable and Z takes values in $C(\mathbf{S})$, which together with the sup-norm, $\|\cdot\|$, forms a separable Banach space, by Theorem 4.3 of Pisier (1975/1976) Z satisfies the law of the iterated logarithm as well. That is, for each $\varepsilon > 0$, $P(n^{-1/2}\varphi_n^{-1}\sum_{i=1}^{n} Z_i \notin \mathscr{G}^{\varepsilon}(\mathbf{S})$ i.o.$) = 0$, and with probability one each $G \in \mathscr{G}(\mathbf{S})$ is a limit point of $n^{-1/2}\varphi_n^{-1}\sum_{i=1}^{n} Z_i$. From (4.10) the result follows. □

5. Remarks. Jain and Marcus (1975) present, in addition to the theorem mentioned above in Section 1, a central limit theorem for continuous processes with sub-Gaussian increments. They require that $(\mathbf{S}, \rho)$ be a compact metric space, with $f(\cdot, V) \in C(\mathbf{S})$,

$$\int_0^1 (H(u, \mathbf{S}, \sigma))^{1/2}\, du < \infty,$$

where σ represents the (pseudo-) metric on $\mathbf{S}$ given by the L_2 norm, and that the process $\{f(s, \cdot)\colon s \in \mathbf{S}\}$ has sub-Gaussian increments. This last assumption imposes a rather strong condition on the tails of $f(s, \cdot) - f(t, \cdot)$ for each pair $s, t \in \mathbf{S}$. Giné and Zinn (1984), Section 8, and Pollard (1982) study weak convergence of function-indexed processes which are uniformly bounded in L_2. The theorem of Giné and Zinn requires certain integrability conditions on a *random* measure of metric entropy. The theorem of Pollard (1982) requires that a combinatoric entropy condition be satisfied. Both of the above conditions are related to the combinatoric notions of Vapnik and Červonenkis (1971). The connection between all of these conditions and the integrability condition (4.2) needs study.

The notation used to represent the random process $\{f(s, V): s \in \mathbf{S}\}$ is subject to choice. Let $\{X(s): s \in \mathbf{S}\}$ represent a $B(S)$-valued random variable defined on some probability space (Ω, Σ, P). A realization of this process at the point s is generally written $X(s, \omega)$ for $\omega \in \Omega$. We can equate $f(s, V)$ with $X(s, \omega)$ by equating f with X and V with ω. Thus, suppressing the V, we can write $f(s, V)$ as $f(s)$ and, when $\{f_i: i \geq 1\}$ is a sequence of independent copies of f,

$$(5.1) \qquad S_n(s) = n^{-1/2} \sum_{i=1}^{n} f_i(s), \quad \text{for } s \in \mathbf{S}.$$

The above representation of the S_n-process suggests generalizations of the results of Sections 3 and 4. The probability bounds on S_n derived in Section 3 all depend, ultimately, on bounds on the tail of the distribution of $f(s) - f(t)$ for pairs $s, t \in \mathbf{S}$. These bounds in turn are calculated from the second moment conditions imposed on f. This suggests that the results could readily be extended to processes of the form (5.1) where the f_i's are independent, but not necessarily identically distributed. When it is stipulated that each f_i satisfy conditions (2.5), (2.6), (2.7) for the *same* δ-net $\mathbf{S}^B(\delta)$ as well as conditions (2.1), (2.2) and (2.3), then, when (4.2) holds, the crucial bounds in the proof of Theorems 3.3 and 3.4 [equations (3.31), (3.32) and (3.33)] easily follow.

The reader familiar with the study of partial-sum processes composed of random masses attached to fixed lattice points has perhaps noticed that the methods used in deriving the probability bounds of Section 3 have been previously seen in the context of partial sum processes, see for example Bass (1985). Perhaps because of this, many of the conditions which arise in the study of the convergence of the process $\{S_n(s): s \in \mathbf{S}\}$ where S_n is as defined in (5.1), are analogous to those which arise in the study of partial-sum processes. A unified approach to these two areas of research could be considered; this could be done by studying generalizations of the S_n-process.

A tightness result for set-indexed partial sum processes (analogous to Theorem 3.3) is due to Alexander and Pyke (1986). A different method of proof, which also allows the derivation of an exponential probability bound (analogous to Theorem 3.4), was subsequently developed by Bass (1985). Both proofs begin by symmetrizing and stratifying the partial-sum process, whereas Alexander and Pyke complete the proof by using Gaussian domination, Bass goes on to use an extension of a classic chaining argument. The latter techniques are more closely related to those seen in the proofs of Theorems 3.3 and 3.4. For set-indexed processes made up of random masses attached to random locations a slightly different method of proof can be seen in Ossiander (1985). There the use of a sub-Gaussian inequality [cf. Jain and Marcus (1978)] replaces the Gaussian domination argument of Alexander and Pyke. Indeed a (longer) proof of Theorem 3.3 is available using either Gaussian domination or a sub-Gaussian inequality; however, these methods do not give the exponential probability bound of Theorem 3.4.

Another generalization is the study of the S_n-process, as defined in (5.1), with the V_i's no longer assumed to be independent. This would involve the assumption of a mixing condition on the f_i's. Goldie and Greenwood (1986a, b) have considered problems of this nature for partial-sum processes composed of random masses attached to fixed lattice points.

Due to the second moment conditions imposed upon the process $\{f(s, \cdot): s \in \mathbf{S}\}$, all S_n-processes considered have limiting Gaussian distributions. It would be of interest to consider

$$S_n(s) = n^{-1/\alpha} \sum_{i=1}^{n} f(s, V_i), \quad \text{for } s \in \mathbf{S},$$

where each $f(s, \cdot)$ is in the domain of attraction of a stable law of exponent α. In the context of set-indexed partial-sum processes consisting of random masses attached to fixed lattice points, Bass and Pyke (1985) prove a uniform central limit theorem for a problem of this nature. Their work suggests that the metric entropy condition imposed on $\mathbf{S}$ depends upon the value of α.

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