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TWO BOOKS ON RANDOM GRAPHS

BÉLA BOLLOBÁS, *Random Graphs*, Academic, London, 1985, 447 pages, $58.50 (paperback $29.95).

EDGAR PALMER, *Graphical Evolution: An Introduction to the Theory of Random Graphs*, Wiley, New York, 1985, 177 pages, $34.95.

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Problems concerning the structure of random media have challenged mathematicians and physicists for many years, and the last twenty years have witnessed much progress. Probabilists and statistical physicists have developed and refined techniques for approaching quantitative as well as qualitative questions, such as the nature of phase transition in physical systems and the bulk properties thereof. The limited extent of finite-dimensional space (usually $d = 2$ or $d = 3$) has constrained progress. In a parallel development described in these two books, combinatorial theorists have developed an intricate theory of certain random networks not subject to such constraints of dimensionality. Owing to the simplicity of definition of these so-called "random graphs," rich and complex discoveries have been made about their inner structures.

There are various different types of "random graphs," of which the following is perhaps the most basic. Take n vertices labelled $1, 2, 3, \ldots, n$, and from the $\binom{n}{2}$ available unordered pairs draw N at random; join these pairs with edges to obtain a random graph ω_n. What are the properties of ω_n? In a paper which has since received much attention, Erdös and Rényi (1960) began to answer this question. They thought of such random graphs as growing organisms, observing the properties of ω_n as $n \to \infty$ when $N = N(n)$ is a prescribed function of n. Some examples of their findings are as follows:

(a) If $N(n) = o(n)$, then ω_n contains almost surely no cycle.

(b) If $N(n) = cn$ for some constant c, then the number v_n of vertices in the largest component of ω_n is asymptotic (in appropriate senses) to $\alpha(c)\log n$ if $c < \frac{1}{2}$, $n^{2/3}$ if $c = \frac{1}{2}$ and $\beta(c)n$ if $c > \frac{1}{2}$, for constants α and β depending on c.

(c) If $N(n) = \frac{1}{2}n \log n + yn$, then the probability that ω_n is connected converges as $n \to \infty$ to $\exp(-e^{-2y})$.

Note that the expression "ω_n has property π almost surely" is used by random graph theorists in a nonstandard fashion to mean that $P(\omega_n$ has property $\pi) \to 1$ as $n \to \infty$.

In the above model the set of edges is a random set with prescribed cardinality. The following is an alternative model. Fix a number $p = p(n)$ in the interval $[0, 1]$. We examine each distinct pair of vertices from the vertex set $\{1, 2, \ldots, n\}$ in turn, and we join this pair by an edge with probability p independently of all

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other pairs. In this "constant edge density" model, the total number of edges has the binomial distribution with mean $\binom{n}{2}p(n)$, and it is not therefore surprising that the properties of this random graph resemble closely those of the "constant edge number" model with $N(n) = \binom{n}{2}p(n)$. Bollobás has explored the relationship between these two models.

Other "random graphs" include graphs chosen at random from the sets of unlabelled graphs on n vertices, regular graphs on n vertices with degree r, labelled trees on n vertices, as well as various other types of families of trees. Another sort of random graph is the "m-out" model, in which each vertex picks m vertices from the remaining $n - 1$ and joins itself to these, independently of the choices of all other vertices. Thus the term "random graph" is generic rather than specific. The subject can appear to be rather disjointed, using a variety of techniques to approach a ragbag of questions in various contexts.

Here are a few words about techniques. The earliest papers in the area treated the topic as a branch of enumeration theory. For example, in order to determine the chance that ω_n has exactly k components, we may count the number of distinct labelled graphs with n vertices, N edges and k components, and divide the total by $\binom{T}{N}$, where $T = \binom{n}{2}$. It seems to have been Erdös and Rényi who first made real use of genuinely probabilistic techniques such as Chebyshev's inequality, the central limit theorem and the binomial-Poisson limit theorem. The modern theory uses both probabilistic and combinatorial techniques, wound together often in an intricate manner to overcome problems of statistical dependency. On the other hand, we may note that the useful probabilistic techniques are fundamental but usually primitive in terms of their probabilistic content.

Why might we be interested in random graphs? Quite apart from mathematical beauty and sophistication, of which there is no lack, there are two principal types of application. The first is the so-called probabilistic method in combinatorics. Suppose that we wish to show the existence of a labelled graph with n vertices, N edges and property π, but we are unable to construct such a graph. It may be possible to show that $P(\omega_n \text{ has property } \pi) > 0$, using simple methods such as Chebyshev's inequality, thus demonstrating the existence of such a graph. This technique is useful in extremal graph theory for instance [see Bollobás (1978)]; see also the chapter on Ramsey theory in the first book under review.

Second, after a decade of relative quiet, the revival of interest in random graphs in the 1970's reflected a growth of interest in questions involving the design and analysis of algorithms in operations research, and this remains the major field of application of the topic. This growing field treats questions such as "what is the 'typical' run-time of algorithm A, rather than its worst-case run-time?", and "algorithm A may not yield optimal results and may be very bad in the worst case, but how good is it in a 'typical' instance of the problem?" A classical example of the latter question involves the chromatic number problem. The chromatic number $\chi(G)$ of a graph G is the smallest number of colours

required such that each vertex of G may be assigned a colour in such a way that no pair of adjacent vertices have the same colour. The problem of finding the chromatic number of a graph is *NP*-complete. On the other hand, there are some rather simple and fast algorithms which may be used to colour the vertices of G with no adjacent pair of identically coloured vertices, but these algorithms may use more than the minimal number $\chi(G)$ of colours. A standard example is the "greedy" algorithm, which proceeds as follows. Let $1, 2, \ldots, n$ be the vertices of G and $c_1, c_2, \ldots$ an infinite set of distinct colours. We use c_1 to colour vertex 1. If vertices 1 and 2 are adjacent, we use c_2 to colour vertex 2, and otherwise colour c_1. We proceed to colour the vertices of G in numerical order. Having coloured vertices $1, 2, \ldots, i$, we colour $i + 1$ with the earliest colour c_j for which no neighbour of $i + 1$ has already been assigned c_j. Writing $\Gamma(G)$ for the number of colours used by the greedy algorithm, we note that $\Gamma(G) \geq \chi(G)$. Certainly the greedy algorithm is fast, running in $O(n^2)$ time, but it generally uses more colours than are necessary; indeed the ratio $\Gamma(G)/\chi(G)$ has order n in the worst case [see Johnson (1974)]. It turns out that the "typical" discrepancy between $\Gamma(G)$ and $\chi(G)$ is not nearly as bad as the worst-case discrepancy. Suppose that $0 < p < 1$, and that G is a random graph on n labelled vertices with edge-density p. Writing χ_n and Γ_n for $\chi(G)$ and $\Gamma(G)$, we find that Γ_n is almost surely $(1 + o(1))n/\log_d n$, where $d = (1 - p)^{-1}$. On the other hand, it is known that χ_n almost surely exceeds $\frac{1}{2}(1 + o(1))n/\log_d n$, so that the ratio Γ_n/χ_n satisfies $1 \leq \Gamma_n/\chi_n \leq 2 + o(1)$ almost surely. This example is particularly appropriate in the context of this review. First, many combinatorial theorists have been obsessed with chromatic numbers, and the problem of finding chromatic numbers is central to certain aspects of computational complexity and operations research. The second reason lies closer to the two books under review. One of the principal conjectures of the field of random graphs was that χ_n almost surely *equals* $\frac{1}{2}(1 + o(1))n/\log_d n$, but standard methods did not seem sufficient to establish the required upper bound on χ_n. Korshunov (1980) has published a proof of this conjecture. It would appear that his published proof is incomplete, and he has made corrections and additions in private correspondence. Notwithstanding this, this reviewer knows of no one claiming to understand all the details of Korshunov's proof. Perhaps it is valid. In any case, this reviewer and many others had hoped that the matter might be cleared up definitively by one or both of the texts under review. Unfortunately, neither author even refers to the paper in question. (Since the writing of this review, Bollobás has dealt effectively with the matter. He has used martingale inequalities to settle the preceding conjecture affirmatively.)

These books differ greatly. Graph-theorist Palmer has written a brief, largely nontechnical introduction for a course for graduate students in graph theory and combinatorics. Analyst Bollobás has written a long, technical account for "research students and professional mathematicians." Whereas Palmer's sketchy overview emphasizes methods by applying them to "selected highlights" of the subject at the expense of complete proofs or best-possible statements of theorems, Bollobás has placed these methods in the full context of the complex technical morass of mathematical detail.

Palmer restricts himself largely to his models A, B and C, being his rather uncompelling notation for random graphs with given edge-density, given edge-number and the m-out model, respectively. His breezy stroll through the subject begins with the idea of a "threshold function." For a given property π of graphs, we say that $A(n)$ is a *threshold function* (in the fixed edge-number model, say) if $P(\omega_n \text{ has } \pi)$ converges to 0 if $N(n)/A(n) \to 0$ and to 1 if $N(n)/A(n) \to \infty$. For many properties π, a sharper threshold may be found. For each $x \in [0,1]$, there may exist a function $A_x(n)$ such that $P(\omega_n \text{ has } \pi) \to x$ as $n \to \infty$ when $N(n) = A_x(n)$; such a function $A_x(n)$ is called a *sharp threshold function.* Palmer's definition of a threshold function is rather different from that given above. This is perhaps a pity, since the notion is now well established.

Palmer's text continues with a brief account of the evolution of random graphs, following the early conception of Erdös and Rényi. For various choices of the number $N(n)$ of edges, the structure of ω_n is examined for large n. The case $N(n) = cn$ is treated in some detail, particularly with regard to the emergence of the giant component when c is near $\frac{1}{2}$. Next there is an account of the connectedness of ω_n. The next chapter, entitled "Selected highlights" describes vertex degrees, chromatic number, clique number (size of the largest complete subgraph) and the m-out model. There are a few pages devoted to random regular graphs and random trees. Ubiquitous is humour; in a textbook, humour must of course be robust to the ravages of time.

This account is most approachable and relaxed, imparting to the graph theorist much of the flavour of the topic. A probabilist might find the approach rather alien, the level of probability and analysis having been minimized in order to emphasize graph-theoretic arguments.

In contrast, Bollobás presents "the first systematic and extensive account of a substantial body of results." He makes no claim of exhaustiveness for his 450 densely written pages, writing "my main consideration in selecting and presenting the material was to write a book which I would like to read myself." Whilst this approach limits the market somewhat, this book is a remarkable work, containing not only an account of the author's extensive research but also a largely reworked account of much standard material. His notation, $G(n, p)$, $G(n, N)$, $G(n; m\text{-out})$, is as compact as possible whilst preserving meaning. Of additional mathematical value is his espousal of "random graph processes:" starting with n labelled isolated vertices, we add edges one by one, placing them randomly between pairs of (so far) nonadjacent vertices. The result is a random sequence $GP = (G(0), G(1), \ldots, G(T))$ of graphs, where $T = \binom{n}{2}$. Such a coupling device facilitates the refinement of threshold properties, through the use of "stopping times" for properties. As an example of this, consider the classical question of determining how many edges a graph needs before a typical such graph is hamiltonian (that is, there exists a cycle which visits each vertex once exactly). The threshold behaviour of this property is contained in the fact that if $N(n) = \frac{1}{2}n(\log n + \log\log n + \gamma(n))$, then $P(\omega_n \text{ is hamiltonian})$ converges to 0 if $\gamma(n) \to -\infty$ and to 1 if $\gamma(n) \to \infty$. Consider now the property that the minimal vertex degree, δ_n say, of ω_n satisfies $\delta_n \geq 2$. It is clear that this is necessary but not sufficient for ω_n to be hamiltonian. It turns out that the latter property is

"almost surely" sufficient in the following sense. Let τ_n, σ_n be the hitting times of the process GP for the properties of being hamiltonian and satisfying $\delta_n \geq 2$, respectively: that is, $\tau_n = \min\{k\colon G(k) \text{ is hamiltonian}\}$ and $\sigma_n = \min\{k\colon G(k)$ has minimal vertex degree $2\}$. Clearly $\sigma_n \leq \tau_n$, and Komlós and Szemerédi (and Bollobás, independently) have proved that $P(\sigma_n = \tau_n) \to 1$ as $n \to \infty$. The very edge which completes the first hamiltonian cycle is (with large probability) the edge which ultimately ensures that every vertex has degree 2 or more.

After some detailed probabilistic preliminaries (including rather an old-fashioned definition of a distribution function, but so what?) and an introduction to the different types of random graph, Bollobás covers much of the (now) standard theory and more, in great detail, with an eye to analytical detail and (to date) best possible results. There are accounts of sizes of components, connectivity, matchings, long paths, automorphism group, diameter, clique number and chromatic number, together with other chapters on more diverse topics such as Ramsey theory and sorting algorithms. Random trees are omitted, but random regular graphs receive considerable attention. As the author writes "there is little doubt that many readers will use this monograph as a compendium of results," but, given the nature of the material, he should be congratulated and respected for weaving such a coherent symphony onto a framework which to many appeared disjointed.

The reader pays a price for the author's preoccupation with detail. Some of the appeal of beautiful results is occasionally obscured by his way of jumping in at the analytical deep end. This is a pity, and might have been avoided without great cost by more careful introductory sections.

Finally, we note that, in recent works, Ruciński and Vince (1985, 1986) have simplified the approach to nonbalanced subgraphs of random graphs. Karoński (1984) has written a small monograph on balanced subgraphs of random graphs, and he tells me that Theorem 1 on page 57 of Bollobás's book was proved earlier in an obscure Russian journal by Hakimullin (1979). Palmer (but not Bollobás) considers the question of the planarity of random graphs, borrowing heavily from Erdös and Rényi (1960). Unfortunately, there are mistakes in the latter paper at this point, corrected recently by T. Łuczak and J. Wierman.

In conclusion, Palmer's book is a gentle introduction to highlights of the theory, whilst Bollobás's book is a nearly exhaustive technical masterpiece which will be a standard reference text for the foreseeable future.

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