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PROPHET COMPARED TO GAMBLER: AN INEQUALITY FOR TRANSFORMS OF PROCESSES[1]

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A prophet is a player with complete foresight; a gambler knows only the past and the present, but not the future. If each of them bets on differences of consecutive nonnegative random variables X_i such that $E(X_i|X_{i-1}) = EX_i$, the players multiplying their stakes by uniformly bounded variables, then the expected gain of the prophet is at most three times that of the gambler. The constant 3 is optimal.

1. Introduction. A prophet is a player with complete foresight; a gambler knows only the past and the present, but not the future. We show that if each of them bets on differences of consecutive nonnegative random variables such that any two adjacent random variables are independent, then the expected gain of the prophet is at most three times that of the gambler, the constant 3 being optimal. If the random variables are independent and the only permitted action is stopping, then the constant is known to be 2 [5]; see also [6].

The actual setting is somewhat less restrictive. Usually, one assumes that the gambler knows the distribution, but our gambler has to know only the expectations of random variables. Also, less than adjacent independence is required. Let $X_0, X_1, \dots, X_r$ be integrable random variables, with expectations e_n, $e_0 \le X_r$. Let $\mathscr{L}_i$ be sigma-algebras of events such that each X_i is measurable with respect to $\mathscr{L}_i$. At time $i+1$ the gambler wins $U_{i+1}(X_{i+1} - X_i)$, where the factor U_{i+1} is chosen by the gambler on the basis of the information provided by $\mathscr{L}_i$, i.e., U_{i+1} is measurable with respect to $\mathscr{L}_i$. We then say that U_i is *predictable with respect to* $\mathscr{L}_i$. The two main particular cases are $\mathscr{L}_i = \sigma(X_1, \dots, X_i)$, when U_i is simply called *predictable*, and $\mathscr{L}_i = \sigma(X_i)$, when U_i is called *presently predictable*: The player multiplies his stake by the random variable U_{i+1}, which depends only on the present. In any event, the gain of the player up to the time n is

$$\sum_{0 \le i < n} U_{i+1}(X_{i+1} - X_i) = Z_n.$$

The sequence Z_n is called the *transform* of (X_n) by (U_n), a terminology that originated with Burkholder (1966); one writes $Z_n = (X * U)_n$. We will assume that all the U_n's are bounded in absolute value by a fixed constant c. If T is a stopping time, then $X_{T \wedge n} - X_0$ is the sequence X_n transformed by the predictable sequence $1_{\{T \ge n\}}$. In the present article the gambler transforms the process X by a presently predictable U (bounded by c); the collection of all such U's is denoted by Π_s. The set of *all* U's (bounded by c) is Δ_s. The corresponding sets

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limited to nonnegative U's (our main concern) are simply Π and Δ. The maximal gains of the prophet and of the gambler are

$$P_s = \sup_{\Delta_s} E(U * X), \qquad G_s = \sup_{\Pi_s} E(U * X).$$

The corresponding expressions when U is restricted to Δ and Π are denoted by P and G. It is easy to see that the suprema in the expressions for P_s, G_s, P and G are reached. We impose on the X_i's the condition

$$\text{(E)} \qquad E(X_i | X_{i-1}) = e_i, \qquad i = 1, \dots, r.$$

This condition is in particular satisfied if for each i, X_i and X_{i-1} are pairwise independent. The condition (E) also holds if the centered random variables $X_i - e_i$ are martingale differences. The main theorem asserts that $P \leq 3G$; if $X_r = e_0$, then also $P_s \leq 3G_s$.

Clearly, these inequalities remain true if the U_i's are only assumed predictable, instead of presently predictable, since then the suprema in the expressions for G and G_s are taken over larger sets.

2. Proving the prophet inequality. Let $X_0, \dots, X_r$ be integrable random variables with expectations e_i. It will be useful to introduce a functional μ involving moments defined on L_1 by

$$\mu(X) = EX - \tfrac{1}{2}E|X - EX| = EX - E(X - EX)^+.$$

The functional μ is superadditive and homogeneous. We only prove the following simple fact: If $X_r \geq e_0$, then $\mu(X_0) \leq \mu(X_r)$. Indeed, the same constant may be subtracted from X_0 and X_r without changing $\mu(X_r) - \mu(X_0)$; we may therefore assume $e_0 = 0$ and $X_r \geq 0$. Then $\mu(X_0) = -E(X_0)^+ \leq 0$, while $\mu(X_r) = e_r - \frac{1}{2}E|X_r - e_r| \geq e_r - \frac{1}{2}E|X_r| - \frac{1}{2}e_r = e_r - e_r = 0$. Finally, if the random variables $X_1, \dots, X_r$ are nonnegative and both players receive X_1, then the addition in the beginning of the random variable $X_0 = 0$ does not change anything, hence, in this case supposing $X_r \geq e_0$ is not a loss of generality. We observe this to justify the assumption $\mu(X_0) \leq \mu(X_r)$ frequently made in this paper.

2.1. THEOREM. *Let $\mathscr{L}_i$ be sigma-algebras such that for each i, X_i and U_{i+1} are $\mathscr{L}_i$-measurable. Assume that $\mu(X_0) \leq \mu(X_r)$ and*

$$\text{(E)} \qquad E(X_i | X_{i-1}) = e_i, \qquad i = 1, \dots, r.$$

Then $P \leq 3G$. This inequality is strict unless all the random variables are identically equal to the same constant.

PROOF. Without loss of generality, we can and do assume that $\mathscr{L}_i$ is the sigma-algebra generated by X_i because the gain of the gambler using optimal strategies increases if the sigma-algebra is enlarged.

For the gambler, since U_i is X_{i-1}-measurable, we have

$$E[U_i(X_i - X_{i-1})] = E[E(U_i(X_i - X_{i-1}) | \mathscr{L}_{i-1})] = E[U_i(e_i - X_{i-1})].$$

Assuming $0 \le U_i \le 1$, this is maximal for $U_i = 1_{\{X_{i-1} < e_i\}}$. Hence,

$$G = \sum_{i=1}^{r} E(e_i - X_{i-1})^+.$$

On the other hand,

$$\begin{aligned} P &= \sum_{i=1}^{r} E(X_i - X_{i-1})^+ \\ &\le \sum_{i=1}^{r} \left[E(X_i - e_i)^+ + E(e_i - X_{i-1})^+ \right] \\ &= \sum_{i=1}^{r} E(X_i - e_i)^+ + G. \end{aligned}$$

It is therefore sufficient to show that

$$\sum_{i=1}^{r} E(X_i - e_i)^+ \le 2G. \tag{2.2}$$

The identity $E(e_i - X_{i-1})^+ = E(X_{i-1} - e_i)^+ + e_i - e_{i-1}$ yields

$$G = \sum_{i=0}^{r-1} E(X_i - e_{i+1})^+ + e_r - e_0. \tag{2.3}$$

For the comparison of terms $a_i = E(X_i - e_i)^+$ and $b_i = E(X_i - e_{i+1})^+$, we prove a lemma.

2.4. Lemma. *If X is an integrable random variable and a and b are constants, $a \le b$, then*

$$(b - a)P(X \ge b) \le E(X - a)^+ - E(X - b)^+ \le (b - a)P(X > a).$$

Proof. Let $F(x) = P(X \le x)$. Then

$$E(X - a)^+ = \int_a^\infty (x - a)F(dx).$$

Hence,

$$\begin{aligned} E(X - a)^+ - E(X - b)^+ &= \int_{b-0}^{\infty} (x - a - x + b)F(dx) + \int_a^{b-0} (x - a)F(dx) \\ &\ge \int_{b-0}^{\infty} (b - a)F(dx) = (b - a)P(x \ge b), \end{aligned}$$

$$E(X - a)^+ - E(X - b)^+ \le \int_a^\infty (x - a - x + b)F(dx) = (b - a)P(X > a).$$

□

2.5. Lemma. *Let X be an integrable random variable with $EX = e$. Then for any constant e',*

$$E(X - e)^+ \le E(X - e')^+ [1 + P(X < e)] + e' - e.$$

If $e \le e'$, the term $P(X < e)$ may be omitted.

PROOF. If $e \le e'$, then by the previous lemma,

$$E(X-e)^+ \le E(X-e')^+ + e' - e.$$

Now assume $e > e'$. Then the first inequality in Lemma 2.4 gives

$$\begin{aligned} E(X-e)^+ &\le E(X-e')^+ + (e'-e)P(X \ge e) \\ &= E(X-e')^+ + (e'-e) - (e'-e)P(X<e) \\ &\le E(X-e')^+\,[1+P(X<e)] + e' - e \end{aligned}$$

because $E(X-e')^+ \ge E(X-e') = e - e'$. □

Applying the previous lemma with $X = X_i$, $e = e_i$, $e' = e_{i+1}$ and summing, we find

$$\sum_{i=1}^{r-1} a_i \le \sum_{i=0}^{r-1} [1 + P(X_i < e_i)]\, b_i + e_r - e_0. \tag{2.6}$$

Hence, by (2.3)

$$\begin{aligned} \sum_{i=1}^{r} a_i &\le a_r - a_0 + 2\left(\sum_{i=0}^{r-1} b_i + e_r - e_0\right) - 2(e_r - e_0) + e_r - e_0 \\ &= 2G - (e_r - a_r - e_0 + a_0) \\ &= 2G - [\mu(X_r) - \mu(X_0)]. \end{aligned}$$

Thus (2.2) holds because we assume $\mu(X_r) \ge \mu(X_0)$. Hence $P \le 3G$.

Finally, we show that $P < 3G$, unless all $X_i = e_0$. The computation following (2.6) shows that the inequality is strict unless $\mu(X_0) = \mu(X_r)$. Therefore, we can and do assume $\mu(X_0) = \mu(X_r)$. Now, if one of the b_i's is strictly positive, then $P < 3G$ follows from the inequality $1 + P(X_i < e_i) < 2$. We can therefore assume that all $b_i = 0$ and $X_i \le e_{i+1}$. Hence, $a_i \le e_{i+1} - e_i$ and this inequality is strict unless $e_{i+1} = e_i$ and $X_i = e_i$. We may therefore assume $\sum_{i=0}^{r-1} a_i < e_r - e_0$. Now (2.3) shows that $G = e_r - e_0$, hence, $G = a_r - a_0$ because $\mu(X_0) = \mu(X_r)$. The estimate preceding (2.2) now implies that

$$P \le G + \sum_{i=1}^{r} a_i = 2G + \sum_{i=0}^{r-1} a_i < 2G + e_r - e_0 = 3G.$$

Thus $P < 3G$ unless all $X_i = e_0$. This completes the proof of the theorem. □

Actually, the ratio P/G is sometimes ≤ 2. Such is the case if all the e_i's are equal.

2.7. COROLLARY. *Suppose that* $e_0 \le \cdots \le e_r$. *Then, under the assumptions of Theorem* 2.1, $P \le 2G + e_r - e_0$.

PROOF. If $e_i \le e_{i+1}$, then, by Lemma 2.5, the term $P(X_i < e_i)$ may be omitted in (2.6). Hence,

$$\sum_{i=1}^{r} a_i \le a_r - a_0 + \sum_{i=0}^{r-1} b_i + e_r - e_0 = a_r - a_0 + G.$$

Using this instead of (2.2), one obtains that $P \le 2G + a_r - a_0$. [This is true also without the assumption $\mu(X_r) \ge \mu(X_0)$.] Since $\mu(X_0) \le \mu(X_r)$, $a_r - a_0 \le e_r - e_0$, hence, $P \le 2G + e_r - e_0$. □

3. The case of signed U_i's. We now allow U_i's with $-c \le U_i \le +c$, $c > 0$. Without loss of generality, we may assume $c = 1$. In the notation of the introduction, the optimal gain of the gambler is G_s and the gain of the prophet is P_s.

3.1. THEOREM. *Let $\mathscr{L}_i$ be sigma-algebras such that for i, X_i and U_{i+1} are $\mathscr{L}_i$-measurable. Assume*

(E) $$E(X_i | X_{i-1}) = e_i, \qquad i = 1, \dots, r.$$

Assume, also, $\mu(X_0) \le \mu(X_r)$ and $e_0 \ge e_r$ (this is in particular true if $X_r = e_0$). Then $P_s \le 3G_s$. If $e_r = e_0$, then $P_s = 2P$ and $G_s = 2G$.

PROOF. With the U_i's signed, the optimal gambler receivers $X_i - X_{i-1}$ on the set $\{X_{i-1} < e_i\}$ and $-(X_i - X_{i-1})$ on the set $\{X_{i-1} \ge e_i\}$. Hence

$$G_s = \sum_{i=1}^{r} \left[E(e_i - X_{i-1})^+ + E(e_i - X_{i-1})^- \right].$$

The difference of the summands in the brackets is $e_i - e_{i-1}$, hence

$$G_s = 2 \sum_{i=1}^{r} E(e_i - X_{i-1}) - \sum_{i=1}^{r} (e_i - e_{i-1})$$
$$= 2G - e_r + e_0.$$

Similarly,

$$P_s = 2P - e_r + e_0.$$

Now, $P \le 3G$ and $e_0 \ge e_r$ implies $P_s \le 3G_s$. □

4. Optimality of the constant 3. We now show that the constant 3 cannot be replaced by a lower constant in Theorems 2.1 and 3.1. Given $\varepsilon > 0$, we construct independent integrable nonnegative random variables $X_0', X_1', \dots, X_s'$, with $P' \ge (3 - \varepsilon)G'$. Here P' and G' are the expected gains of the prophet and of the gambler for the primed process. $\mathscr{L}_i'$ is the σ-algebra generated by $X_0', \dots, X_i'$. It will follow that 3 is optimal also for a smaller $\mathscr{L}_i'$.

The sequence (X_i') will be obtained from a sequence $X_0, \dots, X_r$ by inserting constant random variables between the X_i's, namely, the expectation of the X_i

to the right. We put $s = 2r$ and $X_0' = X_0$, $X_1' = e_1$, $X_2' = X_1, \dots, X_{2i-1}' = e_i$ and $X_{2i}' = X_i, \dots, X_{2r=s}' = X_r$. For $0 < j = 2i$, we have $e_j' = e_i$ and $X_{j-1}' = e_i$. For $j = 2i - 1$, we have $e_j = e_i$ and $X_{j-1}' = X_{i-1}$. Hence,

$$G' = \sum_{j=1}^{2r} E(e_j' - X_{j-1}')^+ = \sum_1^r E(e_i - X_{i-1})^+ = G.$$

Similarly,

$$P' = \sum_{j=1}^{2r} E(X_j' - X_{j-1}')^+ = \sum_{i=1}^{r} E(e_i - X_{i-1})^+ + \sum_{i=1}^{r} E(X_i - e_i)^+$$
$$= G + \sum_{i=1}^{r} E(X_i - e_i)^+.$$

Recall the definitions of a_i and b_i from Section 2:

$$a_i = E(X_i - e_i)^+, \qquad b_i = E(X_i - e_{i+1})^+.$$

We will construct $X_0, \dots, X_r$ so that $\sum_{i=1}^r a_i \geq (2 - \varepsilon)G$ and $X_0 = X_r = 0$. Then (2.3) gives $G = \sum_{i=0}^{r-1} b_i$. Let $X_1 = \eta = e_1$ and $e_{i+1} = e_i/\rho$ $(1 \leq i \leq r - 2)$, where η and ρ are constants, $\rho < 1$. r will be a larger integer to be chosen later. For $2 \leq i \leq r - 2$, let X_i be random variables such that

$$P(X_i = e_{i+1}) = \rho, \qquad P(X_i = 0) = 1 - \rho.$$

Then $b_i = 0$ and $a_i = (e_{i+1} - e_i)\rho$ for $2 \leq i \leq r - 2$. Choosing $\eta = e_1$ small and ρ close to 1, we can have the sum

$$\sum_{i=1}^{r-2} a_i = \sum_{i=2}^{r-2} a_i = (e_{r-1} - e_2)\rho,$$

arbitrarily close to e_{r-1}. Choosing r big, we may assume that $e_{r-1} = e_1\rho^{2-r} \geq 1$. Now define X_{r-1} so that $P(X_{r-1} = e_{r-1}/\alpha) = \alpha$ and $P(X_{r-1} = 0) = 1 - \alpha$, where α is a constant. Then, $a_{r-1} = ((e_{r-1}/\alpha) - e_{r-1})\alpha = e_{r-1} - e_{r-1}\alpha$ will be close to e_{r-1} if α is small. Now, $a_r = 0$ and $b_{r-1} = e_{r-1}$. Thus, $\sum_{i=1}^r a_i$ is arbitrarily close to $2e_{r-1}$, while $\sum_{i=0}^{r-1} b_i = e_{r-1} = G$. Hence, the inequality in (2.2) is arbitrarily close to an equality, which implies that $3G$ is arbitrarily close to P. Thus, the constant 3 is optimal in Theorem 2.1.

Since $X_0' = X_s'$, Theorem 3.1 implies that $P_s = 2P$ and $G_s = 2G$, which shows that the constant 3 is also optimal in Theorem 3.1.

REMARK. It is known that a prophet inequality is true for averages X_n of positive random variables if transforms are restricted to stopping (see [5] and [4]). It is, therefore, natural to consider the problem of such averages in our present setting. It is easy to show that the answer is negative, even if $X_n = (1/n)(Y_1 + \cdots + Y_n)$, where Y_i are independent random variables, with $P(Y_i = 0) = P(Y_i = 2) = \frac{1}{2}$: There is no bound for P/G.

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