The Annals of Probability
1987, Vol. 15, No. 4, 1557–1568

THE NUMBER OF PACKETS TRANSMITTED BY COLLISION DETECT RANDOM ACCESS SCHEMES

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We consider infinite source collision detect random access schemes. For such schemes we establish that the number of packets successfully transmitted is infinite with probability 0 or 1 according as the arrival rate is greater than or less than a critical value.

1. Introduction. An infinite number of stations share a single communication channel. Packets for transmission arrive in a Poisson stream of rate $\nu > 0$ from time $t = 0$ onward and no station ever has more than one packet arrive at it. All stations are synchronized and the time axis is slotted so that one packet can be successfully transmitted in the slot $(t, t+1)$, $t = 1, 2, \ldots$. However, if two or more stations transmit in a slot, there is a collision and none of the packets involved is successfully transmitted. When a station transmits a packet, it learns at the end of the slot whether the packet has been successfully transmitted or whether a collision has occurred. Call the collection of packets that have collided and await transmission the backlog. A station with a backlogged packet waits for a random time and then retransmits the packet, repeating this procedure until the packet is successfully transmitted. In this paper we consider the case where the only information a station has concerning other stations or the use of the channel is the history of its own transmission attempts. Retransmission policies which use only this information are called collision detect (or acknowledgment based) random access schemes.

The simplest example of such a retransmission policy is the Aloha scheme (see, for example, [4]). Under this scheme, packets arriving during the interval $(t-1, t)$ are first transmitted in slot $(t, t+1)$, the first complete slot after their arrival. Also, backlogged packets are independently retransmitted with probability $f \in (0,1)$ in slot $(t, t+1)$. Thus, the retransmission delay following an unsuccessful attempt is geometrically distributed with parameter $1 - f$. A more sophisticated retransmission policy is the Ethernet scheme. Under this scheme a station which has attempted unsuccessfully to transmit a packet r times, retransmits after a further delay which has a discrete uniform distribution on $B_r = \{1, 2, 3, \ldots, \lfloor b^r \rfloor\}$. Here $b > 1$ is the backoff factor and the case $b = 2$ is termed binary exponential backoff [11].

In this paper we prove that for a general collision detect random access scheme there exists a critical value $\nu_c \in [0, \infty]$, with the property that the number of packets successfully transmitted is finite with probability 0 or 1

Received October 1985; revised March 1986.
[1]Supported by the Association of Commonwealth Universities and by SERC Grant GR-D-31119.
AMS 1980 *subject classifications*. Primary 60K99; secondary 60F20, 94A99.
Key words and phrases. Random access, exponential backoff.

according as $\nu < \nu_c$ or $\nu > \nu_c$. For example, for the Aloha scheme $\nu_c = 0$. More generally, we show that $\nu_c = 0$ for any scheme with slower than exponential backoff. For the Ethernet scheme with backoff factor b we prove that $\nu_c = \log b$.

The organization of this paper is as follows. In Section 2 we formally define the model and a number of examples, and in Section 3 we obtain our main results. In Section 4 we consider an unslotted version of the model, in which it is not assumed that stations are synchronized, and indicate how our results are altered in this case. For example, we find that an unslotted Ethernet scheme with backoff factor b has a critical value of $\frac{1}{2}\log b$.

Some of the results of this paper are described in [8] as part of a brief essay on probabilistic problems in random access communications. For a much fuller review of the area, the reader is referred to the collection [10]. In particular, the papers of Gallager [4] and Hajek [7] discuss in detail the assumptions underlying the basic model considered in this paper.

2. The model and some examples. Let the arrival times of packets be $0 < t_1 < t_2 < \cdots$, where the sequence $t_1, t_2, \ldots$ is a realization of a Poisson process of rate ν. Label the stations so that station i receives its packet at time t_i. Station i transmits its packet in slots $(\lfloor t_i \rfloor + \tau_{i,r}, \lfloor t_i \rfloor + \tau_{i,r} + 1)$, $r = 1, 2, \ldots$, stopping only when the packet is successfully transmitted. $T_i = (\tau_{i,r}, r \in \mathbb{N})$ is an increasing sequence of positive integers and T_i, $i \in \mathbb{N}$, are independent and identically distributed. Fix a probability space $(\Omega, \mathscr{F}, P)$ that carries the Poisson arrival stream and the sequences T_i, $i \in \mathbb{N}$.

Let $s_i = s$ if station i successfully transmits its packet in slot $(s, s+1)$ and $s_i = \infty$ if it never manages to transmit its packet. Thus, $s_i \geq \lfloor t_i \rfloor$. Write

$$I_t = \{i: t_i < t \leq s_i\}, \qquad t \in \mathbb{N}, \tag{2.1}$$

for the set of stations with packets at time t. The number of transmissions attempted in slot $(t, t+1)$ is

$$Z_t = |\{i \in I_t: t \in \lfloor t_i \rfloor + T_i\}|, \qquad t \in \mathbb{N}. \tag{2.2}$$

A successful transmission occurs only if $Z_t = 1$, and so

$$Z_t = 1 \Rightarrow \begin{matrix} s_i = t, & i \in I_t: t \in \lfloor t_i \rfloor + T_i, \\ s_i > t, & i \in I_t: t \notin \lfloor t_i \rfloor + T_i, \end{matrix} \tag{2.3}$$

while

$$Z_t = 0 \Rightarrow s_i > t, \qquad i \in I_t. \tag{2.4}$$

The recursive relations (2.1)–(2.4) define the stochastic process $(I_t, t \in \mathbb{N})$ on $(\Omega, \mathscr{F}, P)$. Let

$$X_t = \left(|I_t|; \{(\lfloor t_i \rfloor + T_i) \cap \{1, 2, \ldots, t-1\}: i \in I_t\}\right). \tag{2.5}$$

Thus, X_t gives the number of stations with packets at time t, and for each such station it records when, prior to time t, that station attempted transmissions. Observe that $(X_t, t \in \mathbb{N})$ is a Markov chain with stationary transition probabilities. Let $\mathscr{X}$ be its state space and denote the state (0;) by 0. Write

$P(X_t = x | X_0 = 0) = P(X_t = x)$ and let $P_x(X_t = x') = P(X_{t+u} = x' | X_u = x)$ whenever there exists a $u \in \mathbb{N}$ such that $P(X_u = x) > 0$.

Let

$$h(\tau) = P(\tau \in T_i), \qquad \tau \in \mathbb{N}. \tag{2.6}$$

Note that, since T_i is an infinite sequence, $\sum_{\tau=1}^{\infty} h(\tau) = \infty$. Let

$$H(\nu) = \sum_{t=1}^{\infty} \left[\left\{ \sum_{\tau=1}^{t} h(\tau) \right\} \exp\left\{ -\nu \sum_{\tau=1}^{t} h(\tau) \right\} \right]. \tag{2.7}$$

Then $H(\nu)$ is nonincreasing in ν and may be infinite for small enough values of ν. Let

$$\nu_c = \inf\{\nu > 0 \colon H(\nu) < \infty\}, \tag{2.8}$$

where $\nu_c = \infty$ if $H(\nu) = \infty$ for all $\nu \in (0, \infty)$.

REMARK 2.9. Note that the probabilistic structure of the sequence T_i influences ν_c only through the probabilities $h(\tau)$, $\tau \in \mathbb{N}$.

We shall establish in Section 3 that the number of packets successfully transmitted is finite with probability 0 or 1 according as $\nu < \nu_c$ or $\nu > \nu_c$. In the remainder of this section we provide some examples of the general construction. The examples show that ν_c may take any value in the set $[0, \infty]$.

EXAMPLE 2.10. Aloha (see, for example, [4]). For this scheme, described informally in the Introduction, $\tau_{i,1} = 1$ and $\tau_{i,r+1} - \tau_{i,r} - 1$, $r = 1, 2, \dots$, are independent random variables geometrically distributed with parameter $1 - f$, where $f \in (0,1)$. Thus, $h(1) = 1$ and $h(\tau) = f$, $\tau = 2, 3, \dots$, and so, from (2.7) and (2.8), $\nu_c = 0$.

EXAMPLE 2.11. Ethernet [11]. Set $\tau_{i,1} = 1$ and let $\tau_{i,r+1} - \tau_{i,r}$, $r = 1, 2, \dots$, be independent random variables with $\tau_{i,r+1} - \tau_{i,r}$ uniformly distributed on the set $B_r = \{\tau \in \mathbb{N} \colon 1 \le \tau \le \lfloor b^r \rfloor\}$. Here, $b > 1$ is the backoff factor. Let

$$R(t) = \sum_{\tau=1}^{t} I[\tau \in T_1]$$

be the random number of transmission attempts a packet will make before it has been delayed by a time t. Then, certainly,

$$\sum_{r=0}^{R(t)} b^r \ge t;$$

equivalently,

$$(b^{R(t)+1} - 1)/(b - 1) \ge t$$

and, hence,

$$P(R(t) \ge \log_b((b-1)t + 1) - 1) = 1. \tag{2.12}$$

But for $k \geq 1$,

$$\begin{aligned} P(R(t) \geq k+1) &= P(\tau_{i,k+1} \leq t) \\ &\leq P\left(\sum_{r=1}^{k} (\tau_{i,r+1} - \tau_{i,r}) \leq t \right) \\ &\leq P(\tau_{i,r+1} - \tau_{i,r} \leq t,\ r = 1,2,\dots,k) \\ &= \prod_{r=1}^{k} \min\left\{1, \frac{t}{\lfloor b^r \rfloor}\right\} \\ &= \prod_{r=\lfloor \log_b t \rfloor + 1}^{k} \frac{t}{\lfloor b^r \rfloor} \\ &\leq \prod_{r=0}^{k-1-\lfloor \log_b t \rfloor} \frac{1}{\lfloor b^r \rfloor}. \end{aligned}$$

Thus, for $j \geq 1$,

$$P(R(t) \geq \lfloor \log_b t \rfloor + j + 1) \leq \prod_{r=0}^{j-1} \frac{1}{\lfloor b^r \rfloor}. \tag{2.13}$$

From (2.12) and (2.13),

$$\begin{aligned} \sum_{\tau=1}^{t} h(\tau) &= E(R(t)) \\ &= \log_b t + O(1) \quad \text{as } t \to \infty. \end{aligned}$$

Hence,

$$\begin{aligned} \exp\left\{ -\nu \sum_{\tau=1}^{t} h(\tau) \right\} &= \exp\{-\nu \log_b t + O(1)\} \\ &= t^{-\nu/\log b} \exp\{O(1)\} \quad \text{as } t \to \infty \end{aligned}$$

(logarithms are natural unless otherwise indicated) and so, from its definition (2.7),

$$\begin{aligned} H(\nu) &= \infty, \qquad \nu \in (0, \log b], \\ &< \infty, \qquad \nu \in (\log b, \infty). \end{aligned}$$

Thus, from (2.8), $\nu_c = \log b$.

Hajek [6], Rosenkrantz [12], Fayolle [3], Goodman, Greenberg, Madras and March [5], Szpankowski and Rego [13] and Aldous [1] have considered a variant of this example, where $\tau_{i,r+1} - \tau_{i,r}$, $r = 1, 2, \dots,$ are again independent random variables, but where $\tau_{i,r+1} - \tau_{i,r} - 1$ is geometrically distributed with mean b^r. These authors were primarily interested in the recurrence or transience of the stochastic process $(I_t, t \in \mathbb{N})$, an issue to which we return in Remark 3.14, and the geometric assumption permits a substantial reduction of the Markov description $(X_t, t \in \mathbb{N})$. The geometric assumption complicates the preceding calculations, but does not alter the critical value $\nu_c = \log b$ [9].

EXAMPLE 2.14. Let the events $\{\tau \in T_i\}$, $\tau \in \mathbb{N}$, be independent, with $P(\tau \in T_i) = a/\tau$, where $a > 0$. Thus, a packet which arrived in the interval $(t-1, t)$ and has not been successfully transmitted by time $t + \tau - 1$ is retransmitted in slot $(t + \tau - 1, t + \tau)$ with probability a/τ, and the retransmission rate of a packet depends only upon the delay it has incurred. This scheme is more tractable mathematically than the previous scheme, although its general behaviour is very similar. Note that

$$\begin{aligned}\sum_{\tau=1}^{t} h(\tau) &= \sum_{\tau=1}^{t} \frac{a}{\tau} \\ &= a \log t + O(1) \quad \text{as } t \to \infty\end{aligned}$$

and, hence, $\nu_c = a^{-1}$. Observe that in this example and the next, the sets T_i are infinite with probability 1 by the second Borel–Cantelli lemma.

EXAMPLE 2.15. Let the events $\{\tau \in T_i\}$, $\tau \in \mathbb{N}$, be independent, with $P(\tau \in T_i) \sim (\tau \log \tau)^{-1}$ as $\tau \to \infty$. Then

$$\sum_{\tau=1}^{t} h(\tau) \sim \log \log t \quad \text{as } t \to \infty$$

and from this it follows that $H(\nu) = \infty$ for any $\nu \in [0, \infty)$. Thus, $\nu_c = \infty$.

REMARK 2.16. We have assumed that the sets T_i are infinite. Our model and our subsequent results generalize to allow $P(|T_i| < \infty) > 0$, with the interpretation that station i discards its packet after $|T_i|$ unsuccessful transmission attempts. Under this generalization $E|T_i| < \infty$ implies that $\sum_{\tau=1}^{\infty} h(\tau) < \infty$ and, hence, that $\nu_c = \infty$.

(Example 2.11 is, of course, a mathematical idealization. The authors note that the implementation of Ethernet [2] discards a packet after 16 attempts; it also permits at most 1024 stations.)

REMARK 2.17. The Ethernet scheme is often termed "exponential backoff," since the size of the set B_r grows exponentially with r. Part (a) of the following proposition makes precise the statement "$\nu_c = 0$ for any scheme with slower than exponential backoff." Parts (a) and (b) can be regarded as generalizations of Examples 2.10 and 2.15, respectively.

PROPOSITION 2.18. (a) *If*

$$\lim_{t \to \infty} \frac{\sum_{\tau=1}^{t} h(\tau)}{\log t} = \infty,$$

then $\nu_c = 0$.

(b) *If*

$$\lim_{t \to \infty} \frac{\sum_{\tau=1}^{t} h(\tau)}{\log t} = 0,$$

then $\nu_c = \infty$.

PROOF. Let

$$g(t) = \frac{\sum_{\tau=1}^{t} h(\tau)}{\log t}.$$

From the definition (2.7)

$$H(\nu) = \sum_{t=1}^{\infty} \frac{g(t)\log t}{t^{\nu g(t)}}.$$

Recalling (2.6), $g(t)\log t \le t$ and for any $K < \infty$ there exists $t(K) < \infty$ such that $g(t)\log t \ge K$ for $t > t(K)$. Hence

$$K \sum_{t=t(K)}^{\infty} t^{-\nu g(t)} \le H(\nu) \le \sum_{t=1}^{\infty} t^{1-\nu g(t)}.$$

The first inequality shows that

$$\lim_{t\to\infty} g(t) = 0 \Rightarrow H(\nu) = \infty, \qquad \forall\, \nu \in (0,\infty),$$

while the second inequality shows that

$$\lim_{t\to\infty} g(t) = \infty \Rightarrow H(\nu) < \infty, \qquad \forall\, \nu \in (0,\infty).$$

Results (a) and (b) now follow from the definition (2.8) of ν_c. □

3. Results. We start by introducing a partial ordering on the state space $\mathscr{X}$. For $x, x' \in \mathscr{X}$, define $x \gtrsim x'$ to mean $|I| \ge |I'|$ and

$$\{(x_{ij};\ j = 1,2,\dots, j(i)), i \in I\} \supset \{(x'_{ij};\ j = 1,2,\dots, j'(i)); i \in I'\}.$$

Recall from relation (2.2), that Z_t is the number of transmissions attempted in slot $(t, t+1)$. Let $x_1, x_2 \in \mathscr{X}$ and suppose $x_1 \gtrsim x_2$. Write X_t^k, Z_t^k, $k = 1,2$, for copies of X_t, Z_t started from x_1, x_2, respectively. There is a natural coupling of the processes X_t^1 and X_t^2 such that for every t, $X_t^1 \gtrsim X_t^2$ and $Z_t^1 \ge Z_t^2$. Without essential loss of generality, assume $\min\{\tau\colon P(\tau_{i,1} = \tau) > 0\} = 1$.

LEMMA 3.1. *Suppose $x_1 \in \mathscr{X}$ is such that $P_x(X_t = x_1) > 0$ for some $x \in \mathscr{X}$, $t \in \mathbb{N}$. Then there exists $x_2 \in \mathscr{X}$ such that $x_2 \gtrsim x_1$ and*

$$P_x(X_t = x_2) = P_x(X_t = x_2, Z_s \ge 2,\ s = 1,2,\dots,t) > 0.$$

PROOF. Construct a process $(X'_\tau, \tau \in \mathbb{N})$ from $(X_\tau, \tau \in \mathbb{N})$ by attaching to each Poisson arrival an independent Bernoulli trial with success probability $\frac{1}{2}$, say. Admit only arrivals with successful trials to the system. Then $(X'_\tau, \tau \in \mathbb{N})$ is a version of the process defined by (2.1)–(2.5) on an arrival process of rate $\nu/2$, but if $P_x(X_t = x_1) > 0$ then $P_x(X'_t = x_1) > 0$, as all the Bernoulli trials up to time t may be successes. Again, since t is finite, $P(Z'_s \ge 2,\ s = 1,2,\dots,t) > 0$ and, thinking of the rate ν arrival process as a superposition of two independent rate $\frac{1}{2}\nu$ processes, the existence of an x_2 is assured. □

Next, let $\Phi = \{t \in \mathbb{N}\colon Z_t < 2\}$ denote those slots where transmission is possible but which are not the occasion of a collision.

PROPOSITION 3.2. *If* $\nu > \nu_c$, *then* $P(\Phi = \phi) > 0$.

PROOF. Define

$$\tilde{Z}_t = |\{i \in \mathbb{N}\colon t - \lfloor t_i \rfloor \in T_i\}|, \qquad t \in \mathbb{N}. \tag{3.3}$$

This random variable has the following interpretation. If the channel is externally jammed from time $t = 1$ onward so that no packets are ever successfully transmitted, then $\tilde{Z}_t$ is the number of transmissions attempted in slot $(t, t+1)$. Define, for fixed t, $u = 0, 1, \dots, t-1$,

$$\tilde{Y}_u = |\{i \in \mathbb{N}\colon u < t_i < u + 1, t - \lfloor t_i \rfloor \in T_i\}|$$

and observe that $\tilde{Y}_u$, $u = 0, 1, \dots, t-1$, are independent random variables and that $\tilde{Y}_u$ has a Poisson distribution with mean $E(\tilde{Y}_u) = \nu h(t - u)$. But

$$\tilde{Z}_t = \sum_{u=0}^{t-1} \tilde{Y}_u$$

and, hence, $\tilde{Z}_t$ has a Poisson distribution with mean $\nu \sum_{\tau=1}^{t} h(\tau)$. Thus,

$$P(\tilde{Z}_t < 2) = \left\{1 + \nu \sum_{\tau=1}^{t} h(\tau)\right\} \exp\left\{-\nu \sum_{\tau=1}^{t} h(\tau)\right\}$$

and so, since t was arbitrary,

$$\tilde{\Phi} = \{t \in \mathbb{N}\colon \tilde{Z}_t < 2\} \tag{3.4}$$

has expected cardinality

$$E(|\tilde{\Phi}|) = \sum_{t=1}^{\infty} \left\{1 + \nu \sum_{\tau=1}^{t} h(\tau)\right\} \exp\left\{-\nu \sum_{\tau=1}^{t} h(\tau)\right\}. \tag{3.5}$$

Suppose now that $\nu > \nu_c$. Then, from (2.8) and (2.7), $H(\nu) < \infty$ and so, from (3.5), $E(|\tilde{\Phi}|) < \infty$. Thus, $\tilde{\Phi}$ is finite with probability 1.

Let F be a finite subset of $\mathbb{N}$ such that $P(\tilde{\Phi} = F) > 0$ and let $t = \max F$. Choose $x_1 \in \mathscr{X}$ so that $P(X_{t+1} = x_1, \Phi \cap \{1, 2, \dots, t\} = F) > 0$. Use Lemma 3.1 and the coupling argument preceding it to conclude that

$$\begin{aligned} P(\Phi = \phi) &\geq P(X_{t+1} = x_2, \Phi = \phi) \\ &= P(X_{t+1} = x_2) P_{x_2}(Z_t \geq 2, t \in \mathbb{N}) \\ &\geq P(X_{t+1} = x_2) P_{x_1}(Z_t \geq 2, t \in \mathbb{N}) \\ &= P(X_{t+1} = x_2) P_{x_1}(\tilde{Z}_t \geq 2, t \in \mathbb{N}) > 0. \end{aligned}$$

□

LEMMA 3.6. *Suppose* E_n, $n \in \mathbb{N}$, *is a collection of disjoint sets and* E, e_n, $n \in \mathbb{N}$, *are such that*

$$P(E | E_n) \geq e_n, \qquad n \in \mathbb{N}.$$

Then

$$P\left(E \middle| \bigcup_n E_n\right) \geq \inf_n e_n.$$

PROOF. Direct. □

PROPOSITION 3.7. *If* $P(\Phi = \phi) = p$, *then*

$$P(|\Phi| \geq n) \leq (1-p)^n.$$

PROOF. Define, for $s = 0, 1, 2, \dots,$

$$\Gamma_s = \{\omega \in \Omega\colon Z_t \geq 2,\ t = s+1, s+2, \dots\} \tag{3.8}$$

and, further, $\theta_0 \equiv 0$,

$$\theta_i = \inf\{t > \theta_{i-1}\colon Z_t < 2\}, \qquad i = 1, 2, \dots, \tag{3.9}$$

where $\inf \phi = \infty$. Thus, if finite, θ_i is the ith slot which is not the occasion of a collision. Let θ be any stopping time with respect to $\{Z_t,\ t \in \mathbb{N}\}$. Observe that $\{\theta = t\} \in \sigma(X_0, X_1, \dots, X_t)$ and, hence, using the Markov property,

$$\begin{aligned} &P(\Gamma_t \cap \{\theta = t\} \cap \{X_i = x_i,\ i = 0, 1, \dots, t\}) \\ &\quad = P_{x_t}(\Gamma_0) P(\{\theta = t\} \cap \{X_i = x_i,\ i = 0, 1, \dots, t\}). \end{aligned}$$

Since $P_{x_t}(\Gamma_0) \geq P(\Gamma_0) = P(\Phi = \phi)$, applying Lemma 3.6, we see that

$$P(\Gamma_t | \theta = t) \geq p, \qquad t = 1, 2, \dots.$$

Writing $\Gamma_\theta = \bigcup_{t=1}^{\infty}\{\theta = t\} \cap \Gamma_t$, we can apply Lemma 3.6 again to obtain $P(\Gamma_\theta | \theta < \infty) \geq p$. Next, set $\theta = \theta_i$. Then, from this inequality and definitions (3.8) and (3.9),

$$P(\theta_{i+1} < \infty | \theta_i < \infty) \leq 1 - p, \qquad i = 1, 2, \dots.$$

Evidently,

$$\{|\Phi| \geq n\} = \{\theta_n < \infty\} \quad \text{and} \quad \{\theta_1 = \infty\} = \{\Phi = \phi\},$$

which establishes the proposition. □

THEOREM 3.10. *If* $\nu \in (\nu_c, \infty)$, *then* $P(|\Phi| = \infty) = 0$ *and so, with probability* 1, *the scheme transmits successfully only finitely many packets.*

PROOF. This follows from Propositions 3.2 and 3.7. □

THEOREM 3.11. *If* $\nu \in (0, \nu_c)$, *then* $P(|\Phi| = \infty) = 1$.

PROOF. Recall definition (3.8) and observe that

$$P(\Gamma_t) = \sum_x P(X_t = x) P_x(\Gamma_0), \qquad t = 1, 2, \dots.$$

Suppose that for some $t \in \mathbb{N}$, $P(\Gamma_t) > 0$. Then $P_x(\Gamma_0) > 0$ for some x such that

$P(X_t = x) > 0$. Apply Lemma 3.1 with $x_1 = x$ to obtain x_2 such that

$$P(\Gamma_0) \geq P(X_t = x_2, Z_s \geq 2, s = 1, 2, \dots, t) P_{x_2}(\Gamma_0) > 0.$$

Now, $P(\Gamma_0) = P(\Phi = \phi) = P(\tilde{\Phi} = \phi)$, where $\tilde{\Phi}$ is as defined by (3.4). The proof of Proposition 3.7 can be applied to the process $\tilde{Z}$, defined by (3.3), to establish that $P(|\tilde{\Phi}| \geq n) \leq (1 - p)^n$, where $p = P(\tilde{\Phi} = \phi)$ and, hence, $E(|\tilde{\Phi}|) < \infty$. But this contradicts the assumption $\nu \in (0, \nu_c)$. Hence, $P(\Gamma_t) = 0 \ \forall \ t$. But then

$$P\left(\bigcup_{t=1}^{\infty} \Gamma_t \right) = 0$$

and

$$\{|\Phi| < \infty\} = \bigcup_{t=1}^{\infty} \Gamma_t. \qquad \square$$

THEOREM 3.12. *If $\nu \in (0, \nu_c)$, then with probability 1 the scheme transmits successfully an infinite number of packets.*

PROOF. Let

$$W_t = \left| \{ i \in \mathbb{N} \colon \lfloor t_i \rfloor + \tau_{i,1} = t \} \right|,$$

the number of first transmission attempts in slot $(t, t+1)$. For $z = 0, 1$, define

$$\Psi_z = \{ t \geq d \colon Z_t - W_t = z \},$$

so Ψ_z is the set of slots where z retransmission attempts are made. Observe that

$$|\Phi| = \sum_{t=1}^{\infty} \left(I[t \in \Psi_1, W_t = 0] + I[t \in \Psi_0, W_t = 1] + I[t \in \Psi_0, W_t = 0] \right).$$

But W_t is independent of $(X_t, I[t \in \Psi_0])$ and so with probability 1,

$$\sum_{t=1}^{\infty} \left(I[t \in \Psi_0, W_t = 1] + I[t \in \Psi_0, W_t = 0] \right) = \infty$$

$$\Rightarrow \sum_{t=1}^{\infty} I[t \in \Psi_0, W_t = 1] = \infty.$$

From Theorem 3.11 we have that $|\Phi| = \infty$ with probability 1 and, hence,

$$\sum_{t=1}^{\infty} \left(I[t \in \Psi_1, W_t = 0] + I[t \in \Psi_0, W_t = 1] \right) = \infty$$

with probability 1. $\square$

REMARK 3.13. It has been convenient to arrange our results according as $\nu < \nu_c$ or $\nu > \nu_c$, but observe that the proof of Proposition 3.2 used only the fact that $H(\nu) < \infty$, while the proof of Theorem 3.11 used just that $H(\nu) = \infty$. These observations allow the critical case $\nu = \nu_c$ to be decided. They show that if $\nu = \nu_c \in (0, \infty)$, then P(infinitely many successful transmissions) $= 0$ or 1 according as $H(\nu_c) < \infty$ or $H(\nu_c) = \infty$. Example 2.11 illustrates the case $H(\nu_c) = \infty$. If

Example 2.14 is amended so that

$$P(\tau \in T_i) = a\left(\frac{1}{\tau} + \frac{3}{\tau \log \tau}\right),$$

then $\nu_c = a^{-1}$ and $H(\nu_c) < \infty$.

REMARK 3.14. The Markov process $(X_t, t \in \mathbb{N})$ is clearly transient if $\nu > \nu_c$. It is known that $(X_t, t \in \mathbb{N})$ may be transient even if $\nu < \nu_c$ ([8], or the following example).

Consider the variant of Example 2.11 in which $\tau_{i,r+1} - \tau_{i,r} - 1$ is geometrically distributed with mean 2^r and let N_t be the number of successful transmissions by time t. For this scheme, Aldous [1] has shown that $(X_t, t \in \mathbb{N})$ is transient and

$$\lim_{t \to \infty} \frac{N_t}{t} = 0 \quad \text{a.s.}$$

for any arrival rate $\nu > 0$. Hence, for $\nu \in (0, \log 2]$ this scheme transmits successfully an infinite number of packets, but has a limiting throughput of 0.

It is an open question whether there exists a collision detect random access scheme and a value $\nu > 0$ such that $(X_t, t \in \mathbb{N})$ is recurrent.

REMARK 3.15. The methods described in this section can be extended to the case where messages are m slots in length and arrive at rate ν/m. For example, in the case of exponential backoff it can then be shown that $\nu_c = m(2m-1)^{-1}\log b$; see [9].

4. Unslotted schemes. The model of Section 2 assumed that stations are synchronized and, hence, that the time axis can be slotted. In view of the very limited amount of information available to a station under a collision detect random access scheme, it is worth looking briefly at the case where stations are unable to maintain synchronization and, hence, the time axis must be considered unslotted. In this section we indicate how our results are altered in this case.

Amend the model of Section 2 as follows. Suppose now that $T_i = (\tau_{i,r}, r \in \mathbb{N})$ is a sequence of real numbers satisfying $\tau_{i,r+1} - \tau_{i,r} \geq 1$, $r \in \mathbb{N}$, with the interpretation that the packet which arrives at station i at time t_i is transmitted in the intervals $(t_i + \tau_{i,r}, t_i + 1 + \tau_{i,r})$, $r = 1, 2, \ldots$, stopping only when the packet is successfully transmitted. A transmission in the interval $(t, t+1)$, $t \in \mathbb{R}_+$, is unsuccessful if any other station transmits for any part of the interval; otherwise, the transmission is successful. As before, assume that the arrival times form a Poisson process and that the sequences T_i, $i \in \mathbb{N}$, are independent and identically distributed. Let A be the event that infinitely many successful transmissions take place. Let

$$L(t) = E|T_i \cap [0, t]|$$

and let

$$M(t) = P\{T_i \cap (t-1, t+1) \neq \phi\}.$$

THEOREM 4.1. *(a) If*

$$\int_0^\infty L(t)\exp\left\{-\nu\int_0^t M(\tau)\,d\tau\right\}dt < \infty, \tag{4.2}$$

then $P(A) = 0$.

(b) *If*

$$\int_0^\infty \exp\left\{-\nu\int_0^t M(\tau)\,d\tau\right\}dt = \infty, \tag{4.3}$$

then $P(A) = 1$.

SKETCH OF PROOF. For $t \in \mathbb{R}_+$, define

$$\begin{aligned}\tilde{Z}_t &= 0, \quad \text{if} \quad |\{i \in \mathbb{N}: (t_i + T_i) \cap (t-1, t) \neq \phi\}| = 0\\ &= 1, \quad \text{if} \quad \exists\, j, r \in \mathbb{N} \text{ such that } (t_j + \tau_{j,r}) \in (t-1, t)\\ &\qquad \text{and} \quad \{i \in \mathbb{N}: (t_i + T_i) \cap (t_j + \tau_{j,r} - 1, t_j + \tau_{j,r} + 1) \neq \phi\} = \{j\}\\ &= c, \quad \text{otherwise.}\end{aligned}$$

This random variable has a simple interpretation when the channel is externally jammed, so that no packets are ever successfully transmitted; then $\tilde{Z}_t = 0$ if no transmission attempts are in progress at time t and $\tilde{Z}_t = 1$ if one transmission attempt is in progress at time t and this attempt does not overlap with any other transmission attempts. Calculations similar to those leading to (3.5) give that

$$\int_0^\infty I[\tilde{Z}_t = 1]\,dt = \int_0^\infty \nu L(t)\exp\left\{-\nu\int_0^t M(\tau)\,d\tau\right\}dt.$$

Condition (4.2) implies that this integral is finite and from this it can be deduced that $P(A) = 0$.

Conversely,

$$\int_0^\infty I[\tilde{Z}_\tau = 0, \tau \in (t, t+1)]\,dt = \int_0^\infty \exp\left\{-\nu\int_0^t M(\tau)\,d\tau\right\}dt.$$

If this integral is infinite, then an argument parallel to the proof of Theorem 3.11 shows that $P(A) = 1$. □

EXAMPLE 4.4. Consider the following unslotted version of Example 2.11. Set $\tau_{i,1} = 0$ and let $\tau_{i,r+1} - \tau_{i,r}$ be independent random variables with $\tau_{i,r+1} - \tau_{i,r}$ uniformly distributed over the interval $[1, b^r]$, where $b > 1$. Then,

$$L(t) = \log_b t + O(1) \quad \text{as } t \to \infty,$$

$$\int_0^t M(\tau)\,d\tau = 2\log_b t + O(1) \quad \text{as } t \to \infty.$$

Hence, $P(A) = 1$ if $\nu \leq \frac{1}{2}\log b$, while $P(A) = 0$ if $\nu > \frac{1}{2}\log b$.

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