The Annals of Probability
1987, Vol. 15, No. 4, 1491–1500

THE RADIAL PART OF BROWNIAN MOTION ON A MANIFOLD: A SEMIMARTINGALE PROPERTY[1]

BY WILFRID S. KENDALL

University of Strathclyde

The usual Itô formula fails to apply for $r(X)$ when r is a distance function and X a Brownian motion on a general manifold, since r fails to be differentiable on the cut-locus. It is shown that the discrepancy between the two sides of Itô's formula forms a monotonic random process (and hence is of locally bounded variation). In particular, $r(X)$ is a semimartingale.

1. Introduction. A recurrent theme in stochastic differential geometry concerns the use of comparison theorems to analyze the behaviour of Brownian motion on a manifold. We refer for example to the papers of Malliavin (1974), Debiard, Gaveau and Mazet (1976), Prat (1975), Vauthier (1972) and the survey by Pinsky (1978). The basic procedure is as follows. Let $\mathbf{M}$ be a complete Riemannian manifold. Given a fixed point p in $\mathbf{M}$ we can define the radius function

$$r(x) = \operatorname{dist}(x, p), \quad \text{for } x \in \mathbf{M},$$

as the distance in $\mathbf{M}$ of x from p. If X is Brownian motion on $\mathbf{M}$ begun at X_0, then by Itô's lemma

$$r(X_t) - r(X_0) = W_t + \tfrac{1}{2}\int_0^t \Delta r(X_s)\, ds, \tag{1.1}$$

up to the time that X leaves the domain of smoothness of r. Here Δ is the Laplace–Beltrami operator for $\mathbf{M}$ and W is a real-valued Brownian motion. Geometrical comparison theorems can be applied to bound Δr by some function $f(r)$, given suitable bounds on the sectional curvatures of $\mathbf{M}$. Then a comparison theorem for stochastic differential equations can be applied to deduce a bound for $r(X)$ in terms of Y, the solution to

$$Y_t - Y_0 = W_t + \tfrac{1}{2}\int_0^t f(Y_s)\, ds. \tag{1.2}$$

Greene and Wu (1979) give useful geometric comparison theorems, while Ikeda and Watanabe (1981) provide an exposition of comparison theorems for stochastic differential equations.

This procedure enables probabilistic proofs of theorems which give conditions for X to explode, to be transient or to possess a 0–1 law at infinity. Examples can be found in the papers quoted above, the book by Ikeda and Watanabe, and

Received September 1985; revised April 1986.

[1] This work was carried out while visiting the Mathematical Institute at Warwick University as part of the Stochastic Analysis Year funded by the SERC.

AMS 1980 *subject classifications*. Primary 60J65; secondary 58G32.

Key words and phrases. Brownian motion, Laplace–Beltrami operator, cut-locus, comparison theorem.

Elworthy (1982). Kendall (1981, 1983, 1986) and Goldberg and Mueller (1983) discuss applications to geometric function theory. Kendall (1984) discusses a similar result in which the point p is replaced by a geodesic.

The method suffers one irritating difficulty. For many manifolds $\mathbf{M}$, r is not smooth away from p. Of course, r is never smooth at p, but except in the one-dimensional case the singleton $\{p\}$ is polar.

If $\mathbf{M}$ is the unit sphere S^2, then r is also not smooth at the point antipodal to p; again single points are polar and so can be disregarded. If $\mathbf{M}$ is the real projective plane $\mathbf{R}P^2$ (obtained from the unit sphere by identifying antipodal points), then r fails to be smooth on "the circle at infinity" (the locus derived from the great circle forming the equator if p is the north pole). This set is not polar and in this case $r(X)$ satisfies (1.2) in differential form only while X does not belong to the circle at infinity. The full specification for $r(X)$ is obtained by requiring $r(X)$ to undergo reflection whenever $r(X) = \mathrm{dist}(p, \text{circle at infinity}) = \pi/2$. More complicated examples can be constructed by smoothly pasting portions of $\mathbf{R}P^m$ together with other parts of a Riemannian manifold.

In general, it is not clear even whether $r(X)$ is a semimartingale (actually this is the case and shall be proved so in the course of this note). However, it is a well-known principle among geometers that, "In order to compute an upper bound of the Laplacian of the distance function, it is sufficient to perform the computation within the cut locus of the point under consideration" (Yau, 1976), since the same bound is then satisfied by the Laplacian as an operator on distributions. This encourages us to suppose that $r(X_t) - \int_0^t U(X_s)\,ds$ will actually be a supermartingale for a suitable positive function U.

The purpose of this paper is to give a relatively self-contained account of the semimartingale property of $r(X)$, proceeding along these lines.

We conclude this introduction by fixing some notation and stating the theorem to be proved. As above, X will denote a Brownian motion on $\mathbf{M}$ and Δ will denote the Laplace–Beltrami operator. The function r: $\mathbf{M} \to [0, \infty)$ will always be the distance from a specified point p. It will be convenient to regard X as arising from a stochastic system of Stratonovich differentials

$$
\begin{aligned}
d_s X &= \Xi\, d_s B, \\
d_s \Xi &= H_\Xi\, d_s X, \qquad (1.3)\\
X_0 = x_0, &\qquad \Xi_0 = \xi_0,
\end{aligned}
$$

where Ξ is the lift of X to the frame-bundle $O(\mathbf{M})$ using the Levi–Civita connection and B is a Brownian motion on $\mathbf{R}^m$ the model space for $\mathbf{M}$. The map H_Ξ is the horizontal lift of $T_X\mathbf{M}$ to $T_\Xi O(\mathbf{M})$. This construction is used in the proof only to identify the martingale part of $r(X)$ as being $\int \mathrm{grad}\, r(X)\Xi\, d_I B$ (Itô integral).

The *cut-locus* $C(p)$ is a geometrically defined subset of $\mathbf{M}$ with the properties that r is smooth on $\mathbf{M} - \{p\} - C(p)$, that $\mathbf{M} - C(p)$ is a dense open subset of $\mathbf{M}$ and that $\mathrm{dist}(p, C(p))$ is positive. The definition and properties of $C(p)$ are discussed in Section 2.

It is only necessary to discuss the case of compact **M**, in which case the diffusion X does not explode to infinity and all the curvatures of **M** are bounded above and below.

THEOREM 1.1. *If* **M** *is compact, then we have*

$$r(X_t) - r(X_0) = \int_0^t \operatorname{grad} r(X)\Xi\, d_I B + \tfrac{1}{2}\int_0^t \Delta r(X_s)\, ds - L_t^{(p)}, \tag{1.4}$$

where grad r *and* Δr *are defined to be zero where* r *fails to be differentiable; and* $L^{(p)}$ *is an increasing process, locally constant when* X *does not belong to the cut-locus* $C(p)$. *In particular,* $r(X)$ *is a semimartingale.* [*N.B.* $\int F\, d_I B$ *is the Itô integral of* F *with respect to* B.]

If **M** is noncompact, then a localization argument shows that (1.4) is satisfied at least up to the first time that $r(X)$ exceeds a given value. This is immediate because we may alter **M** to be a compact manifold, altering **M** only outside a suitably large ball. Consequently,

COROLLARY 1.2. *If* **M** *is noncompact, then* (1.4) *holds up to the explosion time*

$$\zeta = \sup\{t > 0\colon r(X) \text{ bounded on } [0, t]\}.$$

Of course, even if **M** is noncompact we may still have $\zeta = \infty$ almost surely. This follows from a comparison argument [applied to (1.4) but otherwise as indicated above] when the Ricci curvatures of **M** are bounded below.

Theorem 1.1 is proved first by establishing basic geometrical properties of the cut-locus in Section 2 and then by exploiting these to bound the difference between the two sides of (1.1). The idea here originated in a geometrical setting in the paper of Calabi (1958). Alternatively, it would be possible to prove the result by using a generalized Itô formula as discussed by Brosamler (1970) and Meyer (1978). However, we prefer a direct and relatively self-contained approach.

In conclusion we should note that the probabilistic consequences of the geometrical principle noted above have already been investigated by several authors, for example, Dodziuk (1983), Ichihara (1984) and Yau (1978). However, the probabilistic interpretation given here is new.

2. Definition and properties of the cut-locus. Consider a compact manifold **M**.

The analogue of the stochastic differential system (1.3) when the Brownian motion B is replaced by a straight-line trajectory is the geodesic equation

$$\begin{aligned} d\gamma(t)/dt &= \xi_{(t)}v, \\ d\xi(t)/dt &= H_{\xi(t)}(d\gamma/dt), \\ \gamma(0) = x, &\qquad \xi_0 \text{ is a frame in } O_x(\mathbf{M}). \end{aligned} \tag{2.1}$$

The solution $\gamma(1)$ depends smoothly on the initial conditions x, ξ_0, v and is in fact a well-defined function of $\xi_0(v)$ an element of the tangent bundle $T\mathbf{M}$. This defines the *exponential map*, a smooth function

$$\text{Exp}: T\mathbf{M} \to \mathbf{M}.$$

We write Exp_x for Exp restricted to $T_x\mathbf{M}$.

Given $u \in S_x\mathbf{M}$ (the sphere bundle or set of unit vectors in $T_x\mathbf{M}$) consider the geodesic

$$\gamma_u(t) = \text{Exp}_x(tu), \quad \text{for } t \geq 0.$$

The *distance to the cut-point* for $u \in S_x\mathbf{M}$ is

$$f(u) = \sup\{t > 0: \text{dist}(\gamma_u(t), x) = t\}$$

and the *cut-locus* at x is

$$C(x) = \{\gamma_u(f(u)) = \text{Exp}_x(f(u)u): u \in S_x(\mathbf{M})\}.$$

Since $\mathbf{M}$ is compact we know $f(u) < \infty$.

We require some basic facts about $C(x)$ as expounded for example in Cheeger and Ebin (1975). We refer to this book as C & E in the sequel.

A point y is *conjugate to* x *along* γ_u if it is a singular value of Exp_x at some point along the ray $\{ut: 0 < t < f(u)\}$ (C & E, page 18). The cut-locus is made up of conjugate points and points where injectivity fails:

(C & E, Lemma 5.2) The *cut-point* $\gamma_u(f(u))$ is the first point y along γ_u for which either

(a) y is conjugate to x along γ_u, or
(b) there is more than one geodesic of shortest length connecting y to x.

Since Exp_x is injective on sufficiently small balls (C & E, page 8) we know f is always positive.

Note that $C(x)$ cannot intersect the half-open segment $\{\gamma_u(\lambda f(u)): \lambda \in [0,1)\}$. For otherwise a point $\tilde{y} = \gamma_u(\lambda f(u))$ is conjugate to x along another geodesic γ_v.

However, $\text{dist}(x, \tilde{y}) = \lambda$ must equal the distance from $\tilde{y}$ to x along γ_v (by definition of the cut-locus) so this would contradict property (b) above.

LEMMA 2.1. *The function r is smooth on $\mathbf{M} - \{p\} - C(p)$.*

PROOF. An application of the inverse function theorem to Exp_p shows that if r fails to be smooth at y, then either there is more than one shortest geodesic from p to y or y is conjugate to p along the shortest geodesic. □

The argument about $r(X)$ depends on an interesting "monotonicity" property of cut-loci.

LEMMA 2.2. *Consider $u \in S_x(\mathbf{M})$, $\lambda \in (0,1)$ and $y = \gamma_u(f(u))$ while $\tilde{y} = \gamma_u(\lambda f(u))$. Then*

$$y \notin C(\tilde{y}).$$

PROOF. By the definition of $f(u)$ we have

$$\operatorname{dist}(y, x) = f(u),$$
$$\operatorname{dist}(\tilde{y}, x) = \lambda f(u)$$

and hence by the triangle inequality

$$\operatorname{dist}(y, \tilde{y}) \geq (1-\lambda) f(u).$$

But $\tilde{y}$ is at distance $(1-\lambda)f(u)$ from y along the geodesic γ_u. Since x is at distance $f(u)$ from y along γ_u it follows that $\tilde{y}$ cannot be the cut-point for y along γ_u. The characterization of $C(y)$ given above (C & E, Lemma 5.2) assures us that $\tilde{y}$ cannot be the cut-point for y along any other geodesic, and so $\tilde{y} \notin C(y)$. But the symmetry property for cut-loci (C & E, Lemma 5.3) then implies $y \notin C(\tilde{y})$. □

Because $\mathbf{M}$ is compact the function f is everywhere finite. So from C & E (Proposition 5.4) we know f: $S(\mathbf{M}) \to (0, \infty)$ is continuous.

Let $E_{(x)} = \{v \in T_x(\mathbf{M})\colon v = 0 \text{ or } |v| < f(v/|v|)\}$ and $E = \bigcup_{x \in \mathbf{M}} E(x) \subset T\mathbf{M}$.

THEOREM 2.3. Exp *provides a diffeomorphism of* E *onto a dense open subset of* $\mathbf{M} \times \mathbf{M}$ *via*

$$(x, v) \to (x, \operatorname{Exp}_x v).$$

PROOF. The complement of the image is given by

$$C = \{(x, y)\colon y \in C(x)\}.$$

Moreover, by the remarks after the characterization of $C(x)$ we know that the minimal geodesic segment from x up to (but not including) y lies in E_x. Therefore $C \subset \overline{\operatorname{Exp} E}$.

From the characterization of the cut-locus it can be shown that Exp: $E \to \mathbf{M} \times \mathbf{M}$ is of full rank everywhere and injective. Hence it is a diffeomorphism onto its image. Note that E is open by the continuity of f.

It follows from Sard's theorem that $C(x)$ is of measure zero in $\mathbf{M}$. □

3. Analysis of $r(X)$. As observed in the introduction it is enough to consider compact $\mathbf{M}$. Consequently, X is defined for all time and we can impose bounds

$$-\beta^2 \leq \operatorname{Sect}(\mathbf{M}) \leq \alpha^2,$$

on the sectional curvatures of $\mathbf{M}$ and

$$\rho = \inf\{\operatorname{dist}(x, C(x))\colon x \in \mathbf{M}\} > 0,$$

on the injectivity radius.

Consider the function $g(u)$ defined for $u \in S(\mathbf{M})$ as the distance in $\mathbf{M} \times \mathbf{M}$ (using the product-manifold metric) between the set C and the point-pair

$$(\gamma_u(f(u)u), \gamma_u(\rho u/3)).$$

By Lemma 2.2 we know g is everywhere positive. Since f is continuous the point-pair is a continuous function of u. By Theorem 2.3 the set C is compact. Hence we can choose a positive δ with

$$g(u) \geq \delta, \quad \text{for all } u \text{ in } S(\mathbf{M}).$$

In addition, we may suppose $\delta < \rho/3$.

The function U: $\mathbf{M} \to (0, \infty)$ is defined by

$$U(x) = \begin{cases} \dfrac{m-1}{2}\beta \coth \beta r(x), & \text{if } r(x) \leq \rho/3, \\ \dfrac{m-1}{2}\beta \coth \beta\rho/3, & \text{otherwise.} \end{cases}$$

By the second-variation formula and comparison with hyperbolic space $\mathbf{H}^{m-1}(-\beta^2)$ [see, for example, Greene and Wu (1979)], we see

$$\tfrac{1}{2}\Delta r(x) \leq U(x), \quad \text{if } x \notin C(p).$$

That $r(X)$ is a semimartingale follows if

$$r(X_t) - r(X_0) - \int_0^t U(X_s)\, ds$$

defines a supermartingale. This in turn follows if

$$E\left(r(X_t) - r(X_0) - \int_0^t U(X_s)\, ds\right) \leq 0,$$

for all X_0 in $\mathbf{M}$, since we may then apply the strong Markov property for X.

LEMMA 3.1. *Suppose* $X_0 = x_0 \in C(p)$ *and* $T = \inf\{t > 0\colon \operatorname{dist}(X_t, x_0) = \delta\}$. *Then*

$$E\left\{r(X_{t\wedge T}) - r(X_0) - \int_0^{t\wedge T} U(X_s)\, ds\right\} \leq 0.$$

PROOF. Let $\gamma(t) = \operatorname{Exp}_p(tu)$ define a distance-minimizing geodesic from p to x_0. If

$$r^+(x) = \operatorname{dist}(\gamma(\rho/3), x) + \rho/3,$$

then

$$r^+(x_0) = r(x_0),$$
$$r^+(x) \geq r(x), \quad \text{by the triangle inequality.}$$

By choice of δ we know r^+ is smooth at X_t for $t < T$ and so $r^+(X)$ satisfies (1.1) with obvious changes. If U^+ is constructed using $\operatorname{dist}(\gamma(\rho/3), x)$ rather than $r(x)$, then (since $\delta < \rho/3$) we know

$$U^+(X_t) = U(X_t), \quad \text{up to time } T,$$

and moreover

$$\tfrac{1}{2}\Delta r^+(X_t) \leq U^+(X_t), \quad \text{up to time } T.$$

Consequently,

$$r(X_{t\wedge T}) - r(X_0) - \int_0^{t\wedge T} U(X_s)\,ds$$
$$\le r^+(X_{t\wedge T}) - r^+(X_0) - \int_0^{t\wedge T} U^+(X_s)\,ds$$

and this last is a supermartingale of initial value zero and consequently has nonpositive expectation. □

Lemma 3.2.

$$E\left(r(X_t) - \int_0^t U(X_s)\,ds\right) \le r(X_0), \quad \textit{for all } t, \textit{ all } X_0.$$

Proof. Consider the stopping times $T_0 = 0$,

$$S_n = \inf\{t \ge T_{n-1}: X_t \in C(p)\},$$
$$T_n = \inf\{t \ge S_n: \operatorname{dist}(X_t, X_{S_n}) = \delta\}.$$

Let $\mathscr{F}_{S_n}$, $\mathscr{F}_{T_n}$ be the σ-fields of events occurring before S_n, T_n, respectively. Then

$$E\left(r\left(X_{S_n\wedge t}\right) - r\left(X_{T_{n-1}\wedge t}\right) - \int_{T_{n-1}\wedge t}^{S_n\wedge t} U(X_s)\,ds \mid \mathscr{F}_{T_{n-1}}\right) \le 0,$$

by a straightforward comparison argument, while

$$E\left(r\left(X_{T_n\wedge t}\right) - r\left(X_{S_n\wedge t}\right) - \int_{S_n\wedge t}^{T_n\wedge t} U(X_s)\,ds \mid \mathscr{F}_{S_n}\right) \le 0,$$

by Lemma 3.1. Consequently, the result follows if we can show $T_n \to \infty$ as $n \to \infty$.

But the process

$$t \to \operatorname{dist}\left(X_{S_n+t}, X_{S_n}\right)$$

(conditional on $\mathscr{F}_{S_n}$) can be compared with the radial part of BM($\mathbf{H}^{m-1}(-\beta^2)$) begun at 0, as indicated in the introduction and using the bounds on Laplacians of distance function discussed at the beginning of this section. Consequently, for some $\varepsilon > 0$ we have

$$P\{T_n - S_n > \varepsilon \mid \mathscr{F}_{S_n}\}$$
$$\ge P\left\{\mathrm{BM}_0\left(\mathbf{H}^{m-1}(-\beta^2)\right) \text{ remains in the ball of radius } \delta \text{ until time } \varepsilon\right\}$$
$$= \tfrac{1}{2}.$$

It follows that $T_n \to \infty$ almost surely. □

We now know $r(X)$ is the sum of an increasing process and a supermartingale, and hence is a semimartingale. To establish Theorem 1.1 we need to discuss the decomposition of $r(X)$ into the sum of a local martingale and a process of locally bounded variation.

LEMMA 3.3. *If* grad r *is taken to be zero on the cut-locus* $C(p)$, *then*

$$\int \operatorname{grad} r(X) \Xi \, d_I B$$

is the martingale part of $r(X)$.

PROOF. By the martingale representation theorem [see, for example, Dellacherie and Meyer (1982), Chapter 8, Theorem 62], we can find a vector-valued process η such that

$$\int \eta \Xi \, d_I B$$

is the martingale part of $r(X)$. By Itô calculus we can show that the local martingale

$$N = \int \eta \Xi \, d_I B - \int \operatorname{grad} r(X) \Xi \, d_I B$$

must be constant on any stochastic interval during which X does not visit $C(p)$. Since the complement of $C(p)$ is a dense open set, the set $\{t\colon X_t \notin C(p)\}$ is a countable union of stochastic intervals. Since $C(p)$ is of measure zero and X has a transition density it follows that $\{t\colon X_t \in C(p)\}$ is almost surely a null set. It follows that the increasing process $[N, N]$ can increase only on the null set $\{t\colon X_t \in C(p)\}$. Since $[N, N]$ is absolutely continuous it must therefore be constant, and, consequently,

$$\int \eta \Xi \, d_I B = \int \operatorname{grad} r(X) \Xi \, d_I B. \qquad \square$$

The proof of Theorem 1.1 is concluded by the following lemma.

LEMMA 3.4. *If* Δr *is taken to be zero on the cut-locus* $C(p)$, *then the process*

$$L_t^{(p)} = \tfrac{1}{2} \int_0^t \Delta r(X_s) \, ds + \int_0^t \operatorname{grad} r(X) \Xi \, d_I B - (r(X_t) - r(X_0))$$

is an increasing process, increasing only when $X \in C(p)$.

PROOF. Let $I_\delta = \bigcup_{n>0} [S_n, T_n]$, where S_n, T_n are as in Lemma 3.2 above. Then it follows from the work of this section that

$$r(X_t) - r(X_0) - \int_0^t \operatorname{grad} r(X) \Xi \, d_I B - \tfrac{1}{2} \int_{[0,t] - I_\delta} \Delta r(X_s) \, ds - \int_{I_\delta} U(X_s) \, ds$$

is a decreasing process. The same holds when I_δ is replaced by $I_\delta \cap I_{\delta/n}$: Let the resulting process be $-L^{(p,n)}$. From comparison of $\frac{1}{2}\Delta r$ with U we know $L_t^{(p,n)} \geq L_t^{(p,n+1)}$. Consequently, $L^{(p,\infty)} = \lim_{n\to\infty} L^{(p,n)}$ is a decreasing process given by

$$-L_t^{(p,\infty)} = r(X_t) - r(X_0) - \int_0^t \operatorname{grad} r(X) \Xi\, d_I B$$

$$-\tfrac{1}{2}\int_{[0,t]-I_*} \Delta r(X_s)\, ds - \int_{I_*} U(X_s)\, ds,$$

where $I_* = \lim_{n\to\infty} I_\delta \cap I_{\delta/n}$. Analysis of I_* shows that it equals $\{t\colon X_t \in C(p)\}$ which is the time set when X belongs to a set of measure zero. Since X has a transition density it follows that I_* has measure zero and so $L^{(p,\infty)} = L^{(p)}$. (That X has a transition density is a consequence of the regularity theory for the strongly elliptic operator Δ.)

It is a consequence of Itô's lemma that $L^{(p)}$ changes only when $X \in C(p)$. □

From the construction $L^{(p)}$ is clearly related to some kind of local time on the cut-locus $C(p)$. This is exactly the case in our special example of $\mathbf{R}P^2$. It would be interesting if such a relationship could be made precise in general.

REFERENCES

BROSAMLER, G. A. (1970). Quadratic variation of potentials and harmonic functions. *Trans. Amer. Math. Soc.* **149** 243–257.

CALABI, E. (1958). An extension of E. Hopf's maximum principle with an application to geometry. *Duke Math. J.* **25** 45–56.

CHEEGER, J. and EBIN, D. (1975). *Comparison Theorems in Riemannian Geometry.* North-Holland, Amsterdam.

DEBIARD, A., GAVEAU, B. and MAZET, E. (1976). Théoremes de comparaison en géometrie Riémannienne. *Publ. Res. Inst. Math. Sci.* **12** 391–425.

DELLACHERIE, C. and MEYER, P.-A. (1982). *Probabilities and Potential B.* North-Holland, Amsterdam.

DODZIUK, J. (1983). Maximum principle for parabolic inequalities and the heat flow on open manifolds. *Indiana Univ. Math. J.* **32** 703–716.

ELWORTHY, K. D. (1982). *Stochastic Differential Equations on Manifolds.* Cambridge Univ. Press, New York.

GOLDBERG, S. I. and MUELLER, C. (1983). Brownian motion, geometry, and generalizations of Picard's little theorem. *Ann. Probab.* **11** 833–846.

GREENE, R. E. and WU, H. (1979). *Function Theory on Manifolds which Possess a Pole.* Springer, Berlin.

ICHIHARA, K. (1984). Comparison theorems for Brownian motions on Riemannian manifolds and their applications. Nagoya Dept. Math. preprint.

IKEDA, N. and WATANABE, S. (1981). *Stochastic Differential Equations and Diffusion Processes.* North-Holland, Amsterdam.

KENDALL, W. S. (1981). Brownian motion, negative curvature, and harmonic maps. In *Stochastic Integrals* (D. Williams, ed.). *Lecture Notes in Math.* **851** 479–491. Springer, Berlin.

KENDALL, W. S. (1983). Brownian motion and a generalized little Picard's theorem. *Trans. Amer. Math. Soc.* **275** 751–760.

KENDALL, W. S. (1984). Brownian motion on a surface of negative curvature. In *Séminaire de Probabilités XVIII 1982/83* (J. Azéma and M. Yor, eds.). *Lecture Notes in Math.* **1059** 70–76. Springer, Berlin.

KENDALL, W. S. (1986). Stochastic differential geometry, a coupling property, and harmonic maps. *J. London Math. Soc.* **33** 554–566.

MALLIAVIN, P. (1974). Asymptotic of the Green's function of a Riemannian manifold and Itô's stochastic integral. *Proc. Nat. Acad. Sci. U.S.A.* **71** 381–383.

MEYER, P.-A. (1978). La formule d'Itô pour le mouvement Brownian d'apres G. Brosamler. In *Séminaire de Probabilités XII* (C. Dellacherie, P.-A. Meyer, and M. Weil, eds.). *Lecture Notes in Math.* **649** 763–769. Springer, Berlin.

PINSKY, M. (1978). Stochastic Riemannian geometry. In *Probabilistic Analysis and Related Topics* (A. T. Bharucha-Reid, ed.) 199–236. Academic, New York.

PRAT, J. J. (1975). Etude asymptotique et convergence angulaire du mouvement brownian sur une variété a courbure négative. *C. R. Acad. Sci. Paris Ser. A–B* **280** A1539–A1542.

VAUTHIER, J. (1972). Diffusion sur une variété riemannienne complète à courbure négative. *C. R. Acad. Sci. Paris Ser. A–B* **275** A925–A926.

YAU, S.-T. (1976). Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry. *Indiana Univ. Math. J.* **25** 659–670.

YAU, S.-T. (1978). On the heat kernel of a complete Riemannian manifold. *J. Math. Pures Appl.* **57** 191–201.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF STRATHCLYDE
LIVINGSTONE TOWER
26 RICHMOND STREET
GLASGOW G1 1XH
UNITED KINGDOM