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INVARIANCE PRINCIPLES FOR RENEWAL PROCESSES

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We present a general methodology for proving invariance principles for renewal processes, resulting in almost sure and probability inequality approximations. We show that our obtained rates are best possible in the i.i.d. case.

1. Introduction. Let $\{Z(t);\ 0 \le t < \infty\}$ be a real-valued stochastic process and define its inverse, or renewal process, N by

$$N(t) = \inf\{x\colon Z(x) > t\}, \qquad 0 \le t < \infty. \tag{1.1}$$

The question we address in this paper is as follows. Suppose we have an approximation for Z, then what can we say about a similar approximation for N? In particular, assume for example that there exist positive constants μ and σ and a standard Wiener process $\{W(t);\ 0 \le t < \infty\}$ such that, as $T \to \infty$,

$$\sup_{0 \le t < T} \left| \frac{Z(t) - \mu t}{\sigma} - W(t) \right| =_{\text{a.s.}} o(r(T)), \tag{1.2}$$

where $r(T) \uparrow \infty$ (nondecreasing, tends to ∞), and $r(T)/T \downarrow 0$ (nonincreasing, tends to zero). Then we would like to know what we can say about N in the like manner.

Motivated by some problems in sequential analysis, Horváth (1984a, b, c) studied renewals of partial sums. Posing the renewal problem in the above generality, Horváth (1986) proved that if $r(T)$ is regularly varying at infinity and $r(T) = O((T \log\log T)^{1/4}(\log T)^{1/2})$, then (1.2) implies

$$\begin{aligned}\limsup_{T\to\infty}\, (T \log\log T)^{-1/4}(\log T)^{-1/2} \sup_{0 \le t \le T} |t/\mu - N(t) - (\sigma/\mu)W(t/\mu)| \\ =_{\text{a.s.}} 2^{1/4}\sigma^{3/2}\mu^{-7/4},\end{aligned} \tag{1.3}$$

and also

$$\begin{aligned}\limsup_{T\to\infty}\, (T \log\log T)^{-1/4}(\log T)^{-1/2} \sup_{0 \le t \le T} |N(t) + Z(t/\mu) - 2t/\mu| \\ =_{\text{a.s.}} 2^{1/4}\sigma^{3/2}\mu^{-7/4}.\end{aligned} \tag{1.4}$$

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In (1.2) and (1.3) both stochastic processes Z and N are approximated by the *same* Wiener process. By (1.4) we conclude that the best possible rate of any joint approximation of Z and N is of order $(T \log\log T)^{1/4}(\log T)^{1/2}$. This phenomenon for the sup-norm distance of the uniform empirical distribution function and its inverse, the uniform empirical quantile function, was first posed by Bahadur (1966), proved by Kiefer (1970) and was extended by Csörgő and Révész (1978) to the nonuniform case. Applications of the Bahadur–Kiefer phenomenon of (1.4) were given by Horváth (1985a, b) to strong laws for randomly indexed U-statistics and to nonlinear renewal theorems.

The main aim of this paper is to construct best possible approximations for $N(t)$. From our discussions so far it is clear that any improvement of (1.3) can only be accomplished in terms of a Wiener process which must be different from that of (1.2). Steinebach (1986) observed that $N(t)$ can be nearer to a *renewal process based on* $W(t)$ than $N(t)$ itself to $W(t)$. Here we first extend Lemma 5 of Steinebach (1986) to our general setup. Then one can easily check that the renewal process

$$(1.5) \qquad M(t) = \inf\{x\colon \lambda W(x) = t - x\}, \qquad 0 \le t < \infty,$$

where λ is a positive constant, has stationary independent increments whose moment generating function exists. Consequently, we can immediately apply the Komlós, Major and Tusnády (1976) approximation to $M(t)$. The latter then combined with our general approach enables us to prove invariances principles for $N(t)$ which give a better rate of approximation than that of the joint one in (1.3).

Section 2 is on strong approximation of $M(t)$ of (1.5). A general almost sure invariance principle for $N(t)$ of (1.1) is stated and proved in Section 3. When applying the latter to renewals of partial sums of independent identically distributed random variables (i.i.d.r.v.'s), we also demonstrate that the obtained rates are the best possible. Some further examples are also given in Section 3. In Section 4 we establish probability inequalities for the approximation of $N(t)$, which also result in bounds for the Prohorov–Lévy distance of $N(t)$ and its limit process. The latter bounds will be shown to be optimal in the i.i.d. case again.

Without loss of generality we assume that the underlying probability space $(\Omega, \mathscr{A}, P)$ is so rich that it accommodates all r.v.'s and processes introduced so far as well as later on.

2. Approximation of $M(t)$. Here we prove an auxiliary result.

THEOREM 2.1. *Let $\{W(t);\ 0 \le t < \infty\}$ be an arbitrary standard Wiener process, $\lambda > 0$ a constant and $M(t)$ as in* (1.5). *Then we can define another standard Wiener process $\{\tilde{W}(t);\ 0 \le t < \infty\}$ such that for all positive x we have*

$$(2.1) \qquad P\left\{ \sup_{0 \le t \le T} \left| \frac{M(t) - t}{\lambda} - \tilde{W}(t) \right| > A \log T + x \right\} \le B \exp(-Cx),$$

where A, B and C are positive constants.

PROOF. We first show that for $0 < t_1 < t_2 < \infty$ fixed

$$\{M(t_2) - M(t_1) | M(s); 0 \le s \le t_1\} =_{\mathscr{D}} M(t_2 - t_1). \tag{2.2}$$

On replacing x by $x + M(t_1)$ in (1.5), we get

$$\begin{aligned}
&\{M(t_2) - M(t_1) | M(s); 0 \le s \le t_1\} \\
&\quad = \{\inf\{x\colon \lambda W(x + M(t_1)) = t_2 - (x + M(t_1))\} | M(s); 0 \le s \le t_1\} \\
&\quad = \{\inf\{x\colon \lambda(W(x + M(t_1)) - W(M(t_1))) = t_2 - t_1 - x\} | M(s); \\
&\qquad\qquad 0 \le s \le t_1\}.
\end{aligned}$$

By the strong Markov property of the Wiener process we have that $\{W(x + M(t_1)) - W(M(t_1));\ x \ge 0\}$ and $\{W(s);\ 0 \le s \le M(t_1)\}$ are independent. Also, the σ-algebra generated by $\{M(s);\ 0 \le s \le t_1\}$ is contained in that generated by $\{W(s);\ 0 \le s \le M(t_1)\}$. Consequently, we get

$$\begin{aligned}
&\{\inf\{x\colon \lambda(W(x + M(t_1)) - W(M(t_1))) = t_2 - t_1 - x\} | M(s); 0 \le s \le t_1\} \\
&\quad = \{\inf\{x\colon \lambda(W(x + M(t_1)) - W(M(t_1))) = t_2 - t_1 - x\} | W(s); \\
&\qquad\qquad 0 \le s \le M(t_1)\} \\
&\quad =_{\mathscr{D}} \inf\{x\colon \lambda W(x) = t_2 - t_1 - x\} \\
&\quad = M(t_2 - t_1).
\end{aligned}$$

Thus we have (2.2), which in turn implies that $M(t)$ is a stationary independent increment process.

Next, by Takács (1967), page 82, for any $y > 0$ and $t > 0$, we have

$$\begin{aligned}
P\{M(t) > y\} &= P\Big\{\sup_{0 \le x \le y} (\lambda W(x) + x) \le t\Big\} \\
&= \Phi\left(\frac{t - y}{\lambda y^{1/2}}\right) - e^{2t/\lambda^2}\Phi\left(\frac{-t - y}{\lambda y^{1/2}}\right).
\end{aligned} \tag{2.3}$$

This immediately implies that the moment generating function of $M(t)$ exists in a neighbourhood of zero. Consequently, $\mu(t) = EM(t)$ and $\sigma^2(t) = E(M(t) - \mu(t))^2$ are finite for any $t > 0$, and they are also continuous in t. Due to $M(t)$ being a stationary independent increment process, we get also that

$$\mu(t + s) = \mu(t) + \mu(s) \quad \text{and} \quad \sigma^2(t + s) = \sigma^2(t) + \sigma^2(s).$$

Hence, there exist two positive constants c_1 and c_2 such that

$$\mu(t) = c_1 t \quad \text{and} \quad \sigma^2(t) = c_2 t.$$

Since

$$\frac{(\lambda W(nt) + nt) - nt}{\lambda n^{1/2}} \to_{\mathscr{D}[0,1]} W(t),$$

and $M(t)$ is the inverse of $\lambda W(x) + x$, by Vervaat (1972) we have also

$$\frac{M(nt) - nt}{\lambda n^{1/2}} \to_{\mathscr{D}[0,1]} W(t). \tag{2.4}$$

On the other hand, $M(t)$ being a stationary independent increment process, it is immediate that

$$\frac{M(nt)-c_1 nt}{(c_2 n)^{1/2}} \to_{\mathscr{D}[0,1]} W(t). \tag{2.5}$$

Hence by (2.4) and (2.5) we must have $c_1 = 1$ and $c_2 = \lambda^2$.

In order to prove now (2.1) we first note that by the Komlós, Major and Tusnády (1975a, 1976) inequality [cf. Theorem 2.6.1 in Csörgő and Révész (1981)], we have

$$P\left\{\max_{1\le k\le [T]+1}\left|\frac{M(k)-k}{\lambda}-\tilde{W}(k)\right| > A_1\log T + x\right\} \le B_1\exp(-C_1x). \tag{2.6}$$

Also,

$$\begin{aligned} &P\left\{\max_{1\le k\le [T]+1}\left|\frac{M(k)-M(k-1)}{\lambda}\right| > A_2\log T + x\right\} \\ &\quad\le ([T]+1)P\{M(1)/\lambda > A_2\log T + x\} \le B_2\exp(-C_2x), \end{aligned} \tag{2.7}$$

due to $M(1)$ having a moment generating function in a neighbourhood of zero. We have also

$$P\left\{\sup_{0\le t\le [T]+1}|\tilde{W}([t])-\tilde{W}(t)| > A_3\log T + x\right\} \le B_3\exp(-C_3x), \tag{2.8}$$

by Lemma 1.2.1 of Csörgő and Révész (1981). Combining now (2.6), (2.7) and (2.8), we obtain (2.1). □

Using a somewhat different method of proof, Mason and van Zwet (1986) and Csörgő, Horváth and Steinebach (1986) also proved Theorem 2.1.

3. Almost sure approximation of $N(t)$ and examples. For any two stochastic processes $\xi(t)$ and $\eta(t)$ the statement $\xi(t) \le_{\text{a.s.}} \eta(t)$ will mean that for almost all $\omega \in \Omega$ there is a $t_0 = t_0(\omega)$ such that $\xi(t) \le \eta(t)$ for $t \ge t_0$. The abbreviations

$$\xi(t) =_{\text{a.s.}} o(\eta(t)) \quad \text{and} \quad \xi(t) =_{\text{a.s.}} O(\eta(t))$$

will stand for

$$\lim_{t\to\infty} \xi(t)/\eta(t) = 0 \quad \text{a.s.} \quad \text{and} \quad P\left\{\limsup_{t\to\infty}|\xi(t)|/|\eta(t)| < \infty\right\} = 1.$$

Let $\{Z(t);\ 0 \le t < \infty\}$ be a real-valued stochastic process, and let $N(t)$ be its inverse as in (1.1).

THEOREM 3.1. *We assume that with some positive constants* μ *and* σ *we have*

$$\sup_{0\le t\le T}\left|\frac{Z(t)-\mu t}{\sigma}-W(t)\right| =_{\text{a.s.}} O(r(T)), \tag{3.1}$$

where $\{W(t);\ 0\le t<\infty\}$ *is a standard Wiener process,* $r(T)\uparrow\infty$ *and* $r(T)/T\downarrow 0$. *Then there exists a standard Wiener process* $\{\hat{W}(t);\ 0\le t<\infty\}$ *such that*

$$\sup_{0\le t\le T}\left|\frac{N(t)-t/\mu}{\sigma/\mu^{3/2}}-\hat{W}(t)\right| =_{\text{a.s.}} O(\log T + r(T)). \tag{3.2}$$

PROOF. By (3.1)

$$Z(t)/t\to\mu \quad \text{a.s.} \quad \text{as} \quad t\to\infty$$

and hence

$$N(t)/t\to 1/\mu \quad \text{a.s.} \quad \text{as} \quad t\to\infty.$$

Thus for every $\varepsilon>0$ we have

$$(1-\varepsilon)/\mu \le_{\text{a.s.}} N(T)/T \le_{\text{a.s.}} (1+\varepsilon)/\mu. \tag{3.3}$$

The statement of (3.1) is equivalent to saying that there are two r.v.'s $A=A(\omega)$ and $T_0=T_0(\omega)$ such that

$$\sup_{0\le t\le T}\left|\frac{Z(t)-\mu t}{\sigma}-W(t)\right| \le_{\text{a.s.}} Ar(T),$$

for all $T\ge T_0$.

Let

$$M(t)=\inf\left\{x:\ \frac{\sigma}{\mu}W(x)=t-x\right\}. \tag{3.4}$$

Next we show that for every $\varepsilon>0$

$$M\left(t/\mu-\sigma Ar\left(\frac{1+\varepsilon}{\mu}t\right)/\mu\right) \le_{\text{a.s.}} N(t) \le_{\text{a.s.}} M\left(t/\mu+\sigma Ar\left(\frac{1+\varepsilon}{\mu}t\right)/\mu\right). \tag{3.5}$$

This is so, for on using (3.1), (3.3) and the definition of $N(t)$, we get

$$\begin{aligned} N(t) &\le_{\text{a.s.}} \inf\{x:\ x\le(1+\varepsilon)t/\mu,\ Z(x)>t\}\\ &\le_{\text{a.s.}} \inf\{x:\ \sigma W(x)+\mu x>t+\sigma Ar((1+\varepsilon)t/\mu)\}\\ &= M(t/\mu+\sigma Ar((1+\varepsilon)t/\mu)/\mu), \end{aligned}$$

and the left-hand side inequality of (3.5) is proved similarly.

Using now Theorem 2.1 for $M(t)$, we have a Wiener process $\tilde{W}$ such that with some constant A_4,

$$\sup_{0\le t\le T}\left|\frac{M(t)-t}{\sigma/\mu}-\tilde{W}(t)\right| \le_{\text{a.s.}} A_4\log T. \tag{3.6}$$

Also, from (3.5) we get

$$
\begin{aligned}
&\left|N(t) - t/\mu - \frac{\sigma}{\mu}\tilde{W}(t/\mu)\right| \\
&\le_{\text{a.s.}} \left|M\left(t/\mu + \sigma A r\left(\frac{1+\varepsilon}{\mu}t\right)/\mu\right) - \left(t/\mu + \sigma A r\left(\frac{1+\varepsilon}{\mu}t\right)/\mu\right) - \frac{\sigma}{\mu}\tilde{W}\left(t/\mu + \sigma A r\left(\frac{1+\varepsilon}{\mu}t\right)/\mu\right)\right| \\
&\quad + \left|M\left(t/\mu - \sigma A r\left(\frac{1+\varepsilon}{\mu}t\right)/\mu\right) - \left(t/\mu - \sigma A r\left(\frac{1+\varepsilon}{\mu}t\right)/\mu\right) - \frac{\sigma}{\mu}\tilde{W}\left(t/\mu - \sigma A r\left(\frac{1+\varepsilon}{\mu}t\right)/\mu\right)\right| \\
&\quad + \left|\frac{\sigma}{\mu}\left(\tilde{W}(t/\mu) + \tilde{W}\left(t/\mu + \sigma A r\left(\frac{1+\varepsilon}{\mu}t\right)/\mu\right)\right)\right| \\
&\quad + \left|\frac{\sigma}{\mu}\left(\tilde{W}(t/\mu) - \tilde{W}\left(t/\mu - \sigma A r\left(\frac{1+\varepsilon}{\mu}t\right)/\mu\right)\right)\right| \\
&\quad + (2\sigma A/\mu) r\left(\frac{1+\varepsilon}{\mu}t\right) \\
&=_{\text{a.s.}} O(\log t) + O\left(\left(r\left(\frac{1+\varepsilon}{\mu}t\right)\log t\right)^{1/2}\right) + O\left(r\left(\frac{1+\varepsilon}{\mu}t\right)\right) \\
&= O\left(\log t + r\left(\frac{1+\varepsilon}{\mu}t\right)\right),
\end{aligned}
\tag{3.7}
$$

where we applied (3.6) and Theorem 1.2.1 of Csörgő and Révész (1981). Hence by monotonicity of the rates in (3.7) we arrive at

$$
\sup_{0 \le t \le T}\left|N(t) - t/\mu - \frac{\sigma}{\mu}\tilde{W}(t/\mu)\right| =_{\text{a.s.}} O\left(\log T + r\left(\frac{1+\varepsilon}{\mu}T\right)\right). \tag{3.8}
$$

Now observing that, on account of $r(t)\uparrow$, we have

$$
r\left(\frac{1+\varepsilon}{\mu}T\right) \le r(T), \quad \text{if } \frac{1+\varepsilon}{\mu} \le 1,
$$

and, due to $r(t)/t\downarrow$, we have

$$
r\left(\frac{1+\varepsilon}{\mu}T\right)\Big/\left(\frac{1+\varepsilon}{\mu}T\right) \le r(T)/T, \quad \text{if } \frac{1+\varepsilon}{\mu} > 1,
$$

we conclude

$$
r\left(\frac{1+\varepsilon}{\mu}T\right) \le \max\left(1, \frac{1+\varepsilon}{\mu}\right) r(T).
$$

Since $\hat{W}(t) = \mu^{1/2}\tilde{W}(t/\mu)$ is a standard Wiener process, by (3.8) the proof of (3.2) is now complete. □

Next we consider some examples.

EXAMPLE 3.1. Let X, $\{X_i;\ i \geq 1\}$ be i.i.d.r.v.'s with

$$EX = \mu > 0 \quad \text{and} \quad 0 < E(X - \mu)^2 = \sigma^2 < \infty. \tag{3.9}$$

Define $Z(t) = \sum_{i=1}^{t} X_i$. Then $N(t)$, defined in (1.1), becomes the usual generalized renewal process considered in the literature [cf., e.g., Feller (1966)].

THEOREM 3.2. *Assume* (3.9) *and* $EH(|X|) < \infty$, *where* $H(x) \geq 0$, $x \geq 0$, *is a nondecreasing continuous function such that* $x^{-2-\gamma}H(x)$ *is nondecreasing for some* $\gamma > 0$ *and* $x^{-1}\log H(x)$ *is nonincreasing. Then we can define a standard Wiener process* $\{\hat{W}(t);\ 0 \leq t < \infty\}$ *so that*

$$\sup_{0 \leq t \leq T} \left| \frac{N(t) - t/\mu}{\sigma/\mu^{3/2}} - \hat{W}(t) \right| =_{\text{a.s.}} O(G(T)), \tag{3.10}$$

for every $\varepsilon > 0$, *where* G *is the inverse function of* H. *If we also assume*

$$\liminf_{x \to \infty} H(\delta x)/H(x) > 0, \quad \text{for all } \delta > 0, \tag{3.11}$$

then

$$\sup_{0 \leq t \leq T} \left| \frac{N(t) - t/\mu}{\sigma/\mu^{3/2}} - \hat{W}(t) \right| =_{\text{a.s.}} o(G(T)). \tag{3.12}$$

PROOF. Komlós, Major and Tusnády (1975, 1976) and Major (1976) [cf. Theorem 2.6.6 in Csörgő and Révész (1981)] constructed a Wiener process $\{W(t);\ 0 \leq t < \infty\}$ such that

$$\sup_{0 \leq t \leq T} \left| \frac{Z(t) - \mu t}{\sigma} - W(t) \right| =_{\text{a.s.}} O(G(T)). \tag{3.13}$$

On account of $x^{-2-\gamma}H(x)$ being a nondecreasing function for some $\gamma > 0$, we obtain that $H(x)/x \uparrow \infty$, $x \to \infty$, and so $G(T)/T \downarrow 0$. Hence Theorem 3.1 immediately implies (3.10). Now under condition (3.11) we have that (3.13) holds true with $O(\delta G(T))$ for every positive constant δ [cf. Lemmas 2.6.1 and 2.6.2 of Csörgő and Révész (1981)]. Therefore Theorem 3.1 implies also (3.12). □

REMARK 3.1. Due to $Z(t)$ being defined as a partial sum of i.i.d.r.v.'s the Blumenthal 0–1 law [cf. Itô and McKean (1965)] implies that the constant of $O(G(T))$ in (3.13) is nonrandom. Therefore the statement of (3.10) is equivalent to

$$\sup_{0 \leq t \leq T} \left| \frac{N(t) - t/\mu}{\sigma/\mu^{3/2}} - \hat{W}(t) \right| \leq_{\text{a.s.}} A_5 G(T), \tag{3.14}$$

where A_5 is a constant.

We now turn to the question of optimality of the obtained rate in Theorem 3.2. Let $X^- = -(0 \wedge X)$ and $X^+ = 0 \vee X$.

THEOREM 3.3. *Assume* (3.9) *and that*

$$E\exp(sX^-) < \infty, \quad \text{in a neighbourhood of zero}. \tag{3.15}$$

Let $G(T) \uparrow \infty$ be a positive continuous function, $G(T)/T \downarrow 0$, $\log T/G(T)$ bounded and H be the inverse function of G. If there exists a standard Wiener process $\{\hat{W}(t);\ 0 \le t < \infty\}$ such that

$$\sup_{0 \le t \le T}\left|\frac{N(t) - t/\mu}{\sigma/\mu^{3/2}} - \hat{W}(t)\right| =_{\text{a.s.}} O(G(T)), \tag{3.16}$$

then

$$EH(K|X|) < \infty, \quad \text{for some } K > 0. \tag{3.17}$$

PROOF. Let

$$Y(t) = \inf\{x\colon N(x) > t\}, \qquad 0 \le t < \infty.$$

Then Theorem 3.1 implies the existence of a standard Wiener process $\{\hat{\hat{W}}(t);\ 0 \le t < \infty\}$ so that

$$\sup_{0 \le t \le T}\left|\frac{Y(t) - \mu t}{\sigma} - \hat{\hat{W}}(t)\right| =_{\text{a.s.}} O(G(T)). \tag{3.18}$$

Let $R_1, R_2, \dots$ be the ascending ladder points of the process $\{Z(t);\ 0 \le t < \infty\}$, and put

$$U_i = R_i - R_{i-1} \quad \text{and} \quad V_i = Z(R_i) - Z(R_{i-1}), \qquad i = 1, 2, \dots,$$

with $R_0 = 0$. Then by Feller (1966), Chapter 12, we have that

$$\{(U_i, V_i);\ i \ge 1\} \text{ are i.i.d.r. vectors,} \tag{3.19}$$

and that, as $n \to \infty$,

$$R_n/n \to_{\text{a.s.}} c > 0. \tag{3.20}$$

We observe that

$$Y(t) = Z(R_i), \quad \text{if } R_{i-1} \le t < R_i.$$

Hence by (3.18) and (3.20) we get

$$\begin{aligned}\sup_{1 \le i \le T}\left|\frac{Z(R_i) - \mu R_i}{\sigma} - \hat{\hat{W}}(R_i)\right| &=_{\text{a.s.}} O\big(G(R_{[T]})\big) \\ &=_{\text{a.s.}} O\big(G(T)\big).\end{aligned} \tag{3.21}$$

Next we show that

$$E\exp(sU_1) < \infty, \quad \text{in a neighbourhood of zero.} \tag{3.22}$$

In order to see this, we recall that the $\{X_i;\ i \ge 1\}$ are i.i.d.r.v.'s. Also, by definition

$$\begin{aligned}P\{U_1 > n\} &= P\{Z(1) \le 0, \dots, Z(n) \le 0\} \\ &\le P\{Z(n) \le 0\} \le E\exp(sZ(n)) = \big(E\exp(sX)\big)^n, \qquad s < 0.\end{aligned}$$

Hence by (3.15) and the assumption that $EX = \mu > 0$,

$$P\{U_1 > n\} \le \rho^n, \quad \text{for some } 0 < \rho < 1,$$

we get (3.22). The latter in turn implies

$$\sup_{1\le i\le T} U_i =_{\text{a.s.}} O(\log T). \tag{3.23}$$

Consequently, by Theorem 1.2.1 of Csörgő and Révész (1981) we get

$$\sup_{1\le i\le T} \left|\hat{W}(R_i) - \hat{W}(R_{i-1})\right| =_{\text{a.s.}} O(\log T). \tag{3.24}$$

On combining now (3.21), (3.23) and (3.24) we obtain

$$\sup_{1\le i\le T} V_i =_{\text{a.s.}} O(G(T)). \tag{3.25}$$

Hence, on account of (3.25) and recalling that the $\{V_i;\ i \ge 1\}$ are i.i.d.r.v.'s by the Kolmogorov 0–1 law with some constant $K_1 > 0$, we have

$$P\{V_i > K_1 G(i) \text{ i.o.}\} = 0.$$

Consequently, by the Borel–Cantelli lemma,

$$\sum_{i=1}^{\infty} P\{V_i > K_1 G(i)\} < \infty,$$

which in turn implies [cf., e.g., Corollary 3 on page 89 of Chow and Teicher (1978)]

$$EH(KV_1) < \infty, \quad \text{for some } K > 0. \tag{3.26}$$

We have also

$$\begin{aligned} EH(KV_1) &\ge \int_{\{R_1=1\}} H(KZ(1))\, dP = \int_{\{X>0\}} H(KX)\, dP \\ &\ge EH(KX^+) - H(0). \end{aligned} \tag{3.27}$$

Thus by (3.15), (3.26) and (3.27) we conclude that (3.17) holds true. □

THEOREM 3.4. *Assume* (3.9) *and*

$$E(X^-)^r < \infty, \quad \textit{for some } r > 2. \tag{3.28}$$

Let $G(T)\uparrow\infty$ *be a positive continuous function,* $G(T)/T \downarrow 0$ *and* H *be the inverse function of* G*, satisfying the condition that* $H(x)/x^r$ *is bounded. If there exists a standard Wiener process* $\{\hat{W}(t);\ 0 \le t < \infty\}$ *such that*

$$\sup_{0\le t\le T} \left|\frac{N(t) - t/\mu}{\sigma/\mu^{3/2}} - \hat{W}(t)\right| =_{\text{a.s.}} O(G(T)), \tag{3.29}$$

then

$$EH(K|X|) < \infty, \quad \textit{for some } K > 0. \tag{3.30}$$

PROOF. Following the lines of proof of Theorem 3.3 we arrive at

$$\sup_{1\le i\le T} \left|\frac{Z(R_i) - \mu R_i}{\sigma} - \hat{W}(R_i)\right| =_{\text{a.s.}} O(G(T)). \tag{3.31}$$

Next we show that

$$EU_1^{r-1} < \infty. \tag{3.32}$$

By Theorem 1 on page 362 of Chow and Teicher (1978) we have

$$\begin{aligned} EU_1^{r-1} &= \sum_{n=1}^{\infty} n^{r-1} P\{U_1 = n\} \\ &\le 1 + \sum_{n=1}^{\infty} \left((n+1)^{r-1} - n^{r-1}\right) P\Big\{\max_{1\le i\le n} Z(i) \le 0\Big\} \\ &\le 1 + (r-1) \sum_{n=1}^{\infty} n^{r-2} P\Big\{\max_{1\le i\le n} Z(i) \le 0\Big\} \\ &= 1 + (r-1) \sum_{n=1}^{\infty} n^{r-2} P\Big\{n - \max_{1\le i\le n} Z(i)/\mu \ge n\Big\} \\ &\le 1 + (r-1) \sum_{n=1}^{\infty} n^{r-2} P\Big\{\max_{1\le i\le n} (i - Z(i)/\mu) \ge n\Big\} < \infty, \end{aligned}$$

provided $E((-(X-\mu))^+)^r < \infty$. The latter condition in turn holds true by (3.28). Now (3.32) and (3.19) give

$$\sup_{1\le i\le T} U_i =_{\text{a.s.}} O(T^{1/(r-1)}). \tag{3.33}$$

Hence by Theorem 1.2.1 of Csörgő and Révész (1981)

$$\sup_{1\le i\le T} \left|\hat{W}(R_1) - \hat{W}(R_{i-1})\right| =_{\text{a.s.}} o(T^{1/r}). \tag{3.34}$$

Therefore by (3.31) and (3.34) we get

$$\sup_{1\le i\le T} |V_i - U_i\mu| =_{\text{a.s.}} O(G(T)). \tag{3.35}$$

Using now (3.19) and (3.35), by the Borel–Cantelli lemma we obtain

$$EH(K|V_1 - U_1\mu|) < \infty, \quad \text{for some } K > 0. \tag{3.36}$$

An estimation, similar to that of (3.27), now gives $EH(KX^+) < \infty$. This also completes the proof of (3.30). □

COROLLARY 3.1. *Assume* (3.9) *and* $P\{X \ge 0\} = 1$.

(i) *There exists a standard Wiener process* $\{\hat{W}(t);\ 0 \le t < \infty\}$ *such that*

$$\sup_{0\le t\le T} \left| \frac{N(t) - t/\mu}{\sigma/\mu^{3/2}} - \hat{W}(t) \right| =_{\text{a.s.}} O(\log T),$$

if and only if $E\exp(sX) < \infty$ *in a neighbourhood of zero.*

(ii) *There exists a standard Wiener process* $\{\hat{W}(t);\ 0 \le t < \infty\}$ *such that*

$$\sup_{0 \le t \le T} \left| \frac{N(t) - t/\mu}{\sigma/\mu^{3/2}} - \hat{W}(t) \right| =_{\text{a.s.}} o(T^{1/r}), \quad \textit{for some } r > 2,$$

if and only if $EX^r < \infty$.

EXAMPLE 3.2. Let $\mathbf{X} = (X^{(1)}, \dots, X^{(d)})$, $\{\mathbf{X}_i;\ i \ge 1\}$ be a sequence of i.i.d.r. vectors in $\mathbb{R}^d$, $d \ge 1$, with mean $\boldsymbol{\mu}$. We assume

(3.37) $$E|X^{(j)}|^r < \infty, \quad 1 \le j \le d, \quad \text{for some } r > 2.$$

Let $h\colon \mathbb{R}^d \to \mathbb{R}$ be a function which satisfies the following regularity conditions:

(3.38) h is homogeneous of degree one, i.e., for all $\mathbf{x} \in \mathbb{R}^d$ and $\lambda \ge 0$, $h(\lambda\mathbf{x}) = \lambda h(\mathbf{x})$,

(3.39) $h(\boldsymbol{\mu}) > 0$,

(3.40) h has continuous partial derivatives of the second order in a neighbourhood of $\boldsymbol{\mu}$.

The partial sums of the random vectors are denoted by $\mathbf{S}(t) = \sum_{i=1}^{t} \mathbf{X}_i$. We define the extended renewal process $N^*(t)$ by

$$N^*(t) = \inf\{x\colon h(\mathbf{S}(x)) > tx^p\}, \qquad 0 \le p < 1.$$

We note that (3.38) and (3.39) imply in general that $\boldsymbol{\mu} \ne \mathbf{0}$. If h is any norm inducing Euclidean topology in $\mathbb{R}^d$, then (3.38) and (3.39) are automatically satisfied with $\boldsymbol{\mu} \ne \mathbf{0}$ and (3.40) usually places a condition on the expectation vector $\boldsymbol{\mu} = E\mathbf{X}$. In particular, if

$$h(\mathbf{x}) = \left(\sum_{j=1}^{d} x_j^2 \right)^{1/2} \quad \text{or} \quad h(\mathbf{x}) = \sum_{j=1}^{d} |x_j|,$$

then (3.40) is satisfied if and only if the components of $\boldsymbol{\mu}$ are all different from 0. If $h(\mathbf{x}) = \max_{1 \le i \le d} |x_i|$, the L_∞ norm, then we find that (3.40) is satisfied if and only if the components of $\boldsymbol{\mu}$ are different from each other and from 0. In these cases when $p = 0$, $N^*(t)$ denotes the instant when d-dimensional random walk leaves for the first time the sphere of radius t about the point $\mathbf{0}$ for the norm h.

The case of $h(x) = x$, $0 \le p < 1$, was considered by Gut (1973, 1975) who proved functional central limit theorems for extended renewal processes of this special case. Horváth (1984c) developed a strong approximation approach and Bahadur–Kiefer type representations for extended multidimensional renewal theory in the above-described general setup. For further references we refer to Gut (1973, 1975) and Horváth (1984c).

By (3.40) the function h has continuous derivatives in $\boldsymbol{\mu}$, and on setting

$$\nabla h(\boldsymbol{\mu}) = (\partial h/\partial x_1, \dots, \partial h/\partial x_d)|_{\mathbf{x}=\boldsymbol{\mu}},$$

we define σ^2 to be the second moment of $\nabla h(\boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})'$ (where $'$ denotes transpose) and assume

(3.41) $$\sigma^2 > 0.$$

THEOREM 3.5. *Assume* (3.37)–(3.41) *hold true. Then we can define a standard Wiener process* $\{\hat{W}(t);\ 0 \le t < \infty\}$ *such that*

$$\sup_{0\le t\le T}\left|N^*(t) - \left(\frac{t}{h(\mu)}\right)^{1/q} - \frac{\sigma}{qh(\mu)}\hat{W}\left(\left(\frac{t}{h(\mu)}\right)^{1/q}\right)\right| =_{\text{a.s.}} o(T^{1/rq}),$$

where $q = 1 - p$.

PROOF. Let

$$Z(t) = \left(\frac{h(\mathbf{S}(t))}{t^p h(\mu)}\right)^{1/q}.$$

Horváth (1984c), Theorem 2.2, constructed a Wiener process $\{W(t);\ 0 \le t < \infty\}$ such that

$$\sup_{0\le t\le T}\left|\frac{Z(t)-t}{\sigma/(qh(\mu))} - W(t)\right| =_{\text{a.s.}} o(T^{1/r}). \tag{3.42}$$

Let $N(t)$ be the inverse of $Z(t)$ as defined in (1.1). Then Theorem 3.1 gives

$$\sup_{0\le t\le T}\left|\frac{N(t)-t}{\sigma/(qh(\mu))} - \hat{W}(t)\right| =_{\text{a.s.}} o(T^{1/r}). \tag{3.43}$$

Now observing that $N(t) = N^*(t^q h(\mu))$, we obtain Theorem 3.5 from (3.43). □

EXAMPLE 3.3. The result of Theorem 3.2 can be extended to also cover sequences $\{X_i;\ i \ge 1\}$ of non-i.i.d.r.v.'s. Let $Z(t) = \sum_{i=1}^{t} X_i$. Many authors [cf., e.g., Philipp and Stout (1975), Berkes (1975), Berkes and Philipp (1977, 1979), Kuelbs and Philipp (1980), Berkes and Morrow (1981), Sahanenko (1982) and Sun (1984)] proved strong invariance principles for partial sums of r.v.'s under various dependency structures. A general formulation of these types of results can be stated as follows. There exist constants μ and $\sigma > 0$ and a standard Wiener process $\{W(t);\ 0 \le t < \infty\}$ such that

$$\sup_{0\le t\le T}\left|\frac{Z(t)-t\mu}{\sigma} - W(t)\right| =_{\text{a.s.}} o(T^{\rho}), \tag{3.44}$$

for some $0 < \rho < 1/2$. If $\mu > 0$, then (3.44) immediately implies a strong approximation theorem for the renewal process of $\{X_i;\ i \ge 1\}$ via Theorem 3.1.

Recently Janson (1983) considered renewals of stationary m-dependent r.v.'s, obtaining a central limit theorem. We extend the latter result into a strong approximation theorem. We assume

$$X, \{X_i;\ i \ge 1\} \text{ are stationary } m\text{-dependent r.v.'s,} \tag{3.45}$$

$$\mu = EX > 0, \qquad \sigma^2 = E(X-\mu)^2 + 2\sum_{i=1}^{m-1}\operatorname{cov}(X_1, X_{i+1}) > 0 \tag{3.46}$$

and

$$E|X|^r < \infty, \quad \text{for some } r > 2. \tag{3.47}$$

Let $N(t)$ be the renewal process of $Z(t)$, defined as in (1.1).

THEOREM 3.6. *Assume* (3.45), (3.46) *and* (3.47) *hold true. Then we can define a standard Wiener process* $\{\hat{W}(t);\ 0 \le t < \infty\}$ *such that*

$$\sup_{0 \le t \le T} \left| \frac{N(t) - t/\mu}{\sigma/\mu^{3/2}} - \hat{W}(t) \right| =_{\text{a.s.}} o(T^{\rho}),$$

for all $\rho > 5/12 + 1/(6r)$.

PROOF. Philipp and Stout (1975) proved that (3.44) is satisfied with $\rho > 5/12 + 1/(6r)$. Consequently, Theorem 3.1 implies the result. □

EXAMPLE 3.4. Let $\{\xi_i;\ i \ge 1\}$ be i.i.d.r.v.'s with continuous distribution function. For integer $p \ge 1$ we say that $\xi_k, \dots, \xi_{k+p}$ is a run down of length p if $\xi_{k-1} \le \xi_k$, $\xi_k > \xi_{k+1} > \cdots > \xi_{k+p}$, $\xi_{k+p} \le \xi_{k+p+1}$. Let $N^{(p)}(n)$ be the smallest such integer for which the sequence $\xi_1, \dots, \xi_{N^{(p)}(n)}$ contains exactly n run downs of length p.

We define

$$X_k = I\{\xi_k \le \xi_{k+1}, \xi_{k+1} > \xi_{k+2} > \cdots > \xi_{k+p+1}, \xi_{k+p+1} \le \xi_{k+p+2}\},$$

where $I\{A\}$ is indicator function of event A. Then the sequence $\{X_i;\ i \ge 1\}$ is stationary and $\{p + 3)$-dependent. Levene and Wolfowitz (1944) and Wolfowitz (1944) computed

$$\mu = EX_1 = (p^3 + 3p + 1)/(p + 3)!$$

and

$$\sigma^2 = E(X_1 - \mu)^2 + 2\sum_{i=1}^{p+2} \operatorname{cov}(X_1, X_{i+1})$$

$$= (p^3 + 3p + 1)(p^3 + 2p^2 + 2p - 4)/(p!(p + 3)!).$$

Introduce now $Z(t) = \sum_{i=1}^{t} X_i$ and $N(t)$, the inverse of $Z(t)$ defined as in (1.1). It is easy to see that

$$N^{(p)}(n) = N(n - 1) + p + 2. \tag{3.48}$$

THEOREM 3.7. *Assume that the conditions in Example* 3.4 *are satisfied. Then we can define a standard Wiener process* $\{\hat{W}(t);\ 0 \le t < \infty\}$ *such that*

$$\max_{1 \le i \le n} \left| \frac{N^{(p)}(i) - i/\mu}{\sigma/\mu^{3/2}} - \hat{W}(i) \right| =_{\text{a.s.}} o(n^{\rho}),$$

for all $\rho > 5/12$.

PROOF. Immediate by Theorem 3.6 and (3.48). □

Other processes based on runs were considered by Pittel (1980), Révész (1983) and Horváth (1986).

4. Probability inequalities for $N(t)$ and examples. So far we have concentrated on almost sure approximations of the renewal process $N(t)$. Frequently there are probability inequalities available for $Z(t)$, and then it is only natural to ask whether these inequalities could also be inherited by its inverse $N(t)$. One of the aims of this section is to answer this question.

THEOREM 4.1. *We assume that with some positive constants μ and σ we have*

$$P\left\{\sup_{0\le t\le T}\left|\frac{Z(t)-\mu t}{\sigma}-W_T(t)\right|>x(T)\right\}\le y(T), \tag{4.1}$$

where $\{W_T(t);\ 0\le t<\infty\}$ is a standard Wiener process for each T, $x(T)>0$, $y(T)>0$, and for some $c>0$,

$$c\log T\le x^{(1)}(T)\le x(T)\le x^{(2)}(T),\qquad x^{(2)}(T)/T\to 0,\qquad T\to\infty. \tag{4.2}$$

Then there exist standard Wiener processes $\{\hat W_T(t);\ 0\le t<\infty\}$ such that for each positive α we have

$$P\left\{\sup_{0\le t\le T}\left|\frac{N(t)-t/\mu}{\sigma/\mu^{3/2}}-\hat W_T(t)\right|>C_1x(2T/\mu)\right\}\le C_2(y(2T/\mu)+T^{-\alpha}), \tag{4.3}$$

where $C_1=C_1(\alpha)$ and C_2 are constants.

PROOF. We have

$$\begin{aligned}P\{N(T)>2T/\mu\}&\le P\{Z(2T/\mu)\le T\}\\&=P\{Z(2T/\mu)-2T\le -T\}\\&\le y(2T/\mu)+P\left\{\frac{\sigma W_T(2T/\mu)}{T^{1/2}}\le\frac{-T+\sigma x(2T/\mu)}{T^{1/2}}\right\}\\&\le y(2T/\mu)+K_1\exp(-K_2T),\end{aligned}\tag{4.4}$$

where K_1 and K_2 are positive constants. Let

$$M_T(t)=\begin{cases}\inf\left\{x:\ \dfrac{\sigma}{\mu}W_T(x)=t-x\right\}, & \text{if } t\ge 0,\\ 0, & \text{otherwise.}\end{cases}$$

We prove

$$\begin{aligned}P\Bigg\{M_{2T/\mu}\left(\frac{t}{\mu}-\frac{1}{\mu}x(2T/\mu)\right)\le N(t)\le M_{2T/\mu}\left(\frac{t}{\mu}+\frac{1}{\mu}x(2T/\mu)\right),&\\ 0\le t\le T\Bigg\}&\\ \ge 1-(4y(2T/\mu)+2K_1\exp(-K_2T)).&\end{aligned}\tag{4.5}$$

The proof of (4.5) is like that of (3.5), only we use (4.4) here instead of (3.3), and

(4.1) instead of (3.1). By Theorem 2.1 we can define standard Wiener processes $\{\tilde{W}_T(t);\ 0 \le t < \infty\}$ such that for each positive K and λ,

$$(4.6)\quad P\left\{\sup_{0\le t\le K}\left|\frac{M_{2T/\mu}(t)-t}{\sigma/\mu}-\tilde{W}_T(t)\right|>A\log K+\lambda\right\}\le B\exp(-C\lambda),$$

where a, B and C are constants of Theorem 2.1. Next by Lemma 1.2.1 of Csörgő and Révész (1981) we have

$$(4.7)\quad \begin{aligned} P\left\{\sup_{0\le t\le T}\left|\tilde{W}_T\left(t+\frac{1}{\mu}x(2T/\mu)\right)-\tilde{W}_T(t)\right|\ge\left(K_3\frac{1}{\mu}x(2T/\mu)\log T\right)^{1/2}\right\}\\ \le K_4T^{1-K_3/3}/x(2T/\mu).\end{aligned}$$

Given $\alpha > 0$, we choose K_3 so that

$$K_4T^{1-K_3/3}/x(2T/\mu)\le T^{-\alpha}.$$

By condition (4.2) there is a constant K_5 such that

$$\left(K_3\frac{1}{\mu}x(2T/\mu)\log T\right)^{1/2}\le K_5x(2T/\mu).$$

Consequently, by (4.7) we obtain

$$(4.8)\quad P\left\{\sup_{0\le t\le T}\left|\tilde{W}_T\left(t+\frac{1}{\mu}x(2T/\mu)\right)-\tilde{W}_T(t)\right|\ge K_5x(2T/\mu)\right\}\le T^{-\alpha},$$

and a similar argument yields also

$$(4.9)\quad P\left\{\sup_{0\le t\le T}\left|\tilde{W}_T(t)-\tilde{W}_T\left(t-\frac{1}{\mu}x(2T/\mu)\right)\right|\ge K_5x(2T/\mu)\right\}\le T^{-\alpha}.$$

On letting $\hat{W}_T(t) = \mu^{1/2}\tilde{W}_T(t/\mu)$, we get a standard Wiener process for each T. Using now (4.6) with $K = T/\mu = (1/\mu)x(2T/\mu)$ and $\lambda = (\alpha/C)\log T$, (4.5), (4.8) and (4.9) result in (4.3). □

Now we illustrate the usefulness of Theorem 4.1 by a few examples.

EXAMPLE 4.1. Let X, $\{X_i;\ i \ge 1\}$ be i.i.d.r.v.'s as in Example 3.1. We give a probability inequality version of Theorem 3.2.

THEOREM 4.2. *Assume* (3.9) *and* $EH(|X|) < \infty$, *where* $H(x) \ge 0$, $x \ge 0$, *is a nondecreasing continuous function such that* $x^{-2-\gamma}H(x)$ *is nondecreasing for some* $\gamma > 0$ *and* $x^{-1}\log H(x)$ *is nonincreasing. Then we can define a standard Wiener process* $\{\hat{W}(t);\ 0 \le t < \infty\}$ *so that for* $x \in (D_3G(T), D_4(T\log T)^{1/2})$ *and every* $\alpha > 0$ *we have*

$$(4.10)\quad P\left\{\sup_{0\le t\le T}\left|\frac{N(t)-t/\mu}{\sigma/\mu^{3/2}}-\hat{W}(t)\right|>x\right\}\le D_1(T/H(D_2x)+T^{-\alpha}),$$

where G *is the inverse function of* H *and* D_i, $i = 1,\dots,4$, *are constants. If we* *also assume* (3.11), *then*

$$(4.11)\quad P\left\{\sup_{0\le t\le T}\left|\frac{N(t)-t/\mu}{\sigma/\mu^{3/2}}-\hat{W}(t)\right|>x\right\}\le D_T\left(T/H(D_2x)+T^{-\alpha}\right),$$

where $D_T \to 0$ *as* $T \to \infty$.

PROOF. Komlós, Major and Tusnády (1975a, 1976) [cf. Theorem 2.6.7 in Csörgő and Révész (1981)] proved

$$(4.12)\qquad P\left\{\sup_{0\le t\le T}\left|\frac{Z(t)-\mu t}{\sigma}-W(t)\right|>x\right\}\le D_5\left(T/H(D_6x)\right),$$

for every $x \in (G(T), D_7(T\log T)^{1/2})$, where $\{W(t);\ 0 \le t < \infty\}$ is a standard Wiener process. Using (4.12) and Theorem 4.1, (4.10) is proven. Now by Lemmas 2.6.1 and 2.6.2 of Csörgő and Révész (1981), (4.10) implies (4.11). □

A usual application of inequalities like those of Theorem 4.2 is the estimation of the Prohorov–Lévy distance d of measures generated by N and its limit process. Let

$$L_T(s)=\frac{N(sT)-sT/\mu}{T^{1/2}\sigma/\mu^{3/2}},\qquad 0\le s\le 1,$$

and let L_T be the probability measure generated by the latter stochastic process. Write W for the Wiener measure.

COROLLARY 4.1. *Assume* (3.9) *holds true.*

(i) *If* $E(\exp(tX)) < \infty$ *in a neighbourhood of zero, then*

$$d(L_T, W) = O\left(\log T/T^{1/2}\right).$$

(ii) *If* $E|X|^r < \infty$ *for some* $r > 2$, *then*

$$d(L_T, W) = o\left(T^{-(r-2)/2(r+1)}\right).$$

PROOF. It is well known [cf. Komlós, Major and Tusnády (1975b)] that

$$(4.13)\quad d(L_T,W)\le \inf_{0\le\varepsilon<1}\left(\varepsilon+P\left\{\sup_{0\le s\le 1}|L_T(s)-\hat{W}(sT)/T^{1/2}|>\varepsilon\right\}\right).$$

In case of (i) we let $\varepsilon = D_8\log T/T^{1/2}$, where D_8 is large and use (4.10) with $x = \varepsilon T^{1/2}$ and $\alpha = 1$. In the case of (ii) let $\varepsilon = D_T^{1/(r+1)}T^{-(r-2)/(2r+2)}$ and use (4.11) with $x = \varepsilon T^{1/2}$ and $\alpha = 1$. □

We note that, using a different method, Corollary 4.1 was proved by Borovkov (1982) in the special case of $P\{X \ge 0\} = 1$. For related results we refer to Csörgő, Deheuvels and Horváth (1987).

REMARK 4.1. If $P\{X > 0\} = 1$, then the inverse of the renewal $N(t)$ is the original partial sum. Hence the optimality of the Prohorov–Lévy distance for partial sums is inherited by L_T. In particular, if $\{X_i;\ i \geq 1\}$ are Poisson r.v.'s with mean one, then there is a constant $D_9 > 0$ such that

$$d(L_T, W) > D_9 \log T / T^{1/2}.$$

Also, for any function $\omega_T \to \infty$, $T \to \infty$, there exists a positive r.v. X with $EX^r < \infty$ for some $r > 2$, such that

$$\limsup_{T \to \infty} \omega_T T^{-(r-2)/(2r+2)} d(L_T, W) = \infty.$$

These two statements follow immediately by Theorems 4 and 5 of Komlós, Major and Tusnády (1975b) and our Theorem 4.1.

When the moment generating function of X exists, then Komlós, Major and Tusnády (1976) proved a more precise inequality than the one used in (4.12). Using the latter inequality we can slightly improve upon (4.10) as follows, in this special case.

COROLLARY 4.2. *Assume* (3.9) *and that* $E(\exp(tX)) < \infty$ *in a neighbourhood of zero. Then we can define a standard Wiener process* $\{\hat{W}(t);\ 0 \leq t < \infty\}$ *such that for every* $x > 0$,

$$P\left\{\sup_{0 \leq t \leq T}\left|\frac{N(t) - t/\mu}{\sigma/\mu^{3/2}} - \hat{W}(t)\right| > \hat{A}\log T + x\right\} \leq \hat{B}\exp(-\hat{C}x),$$

where $\hat{A}$, $\hat{B}$ *and* $\hat{C}$ *are positive constants.*

PROOF. We follow the proof of Theorem 4.1. However, instead of (4.1) we use the Komlós, Major and Tusnády (1976) inequality

$$(4.14)\quad P\left\{\sup_{0 \leq t \leq T}\left|\frac{Z(t) - \mu t}{\sigma} - W(t)\right| > \hat{A}_1 \log T + x\right\} \leq \hat{B}_1 \exp(-\hat{C}_1 x),$$

where $\hat{A}_1$, $\hat{B}_1$ and $\hat{C}_1$ are positive constants and $\{W(t);\ 0 \leq t < \infty\}$ is a standard Wiener process. First, similar to the proof of (4.4), one gets

$$P\{N(T) > 2T/\mu + c_1 x\} \leq c_2 \exp(-c_3 x),$$

and then we obtain

$$P\{M(t/\mu - (c_4 \log T + c_5 x)) \leq N(t) \leq M(t/\mu + (c_4 \log T + c_5 x)), 0 \leq t \leq T\}$$
$$\geq 1 - c_6 \exp(-c_7 x).$$

Now the latter combined with Theorem 2.1, and Lemma 1.2.1 of Csörgő and Révész (1981) completes the proof. □

We note that Corollary 4.2 was also proved by Mason and van Zwet (1986) and by Csörgő, Horváth and Steinebach (1986).

EXAMPLE 4.2. Many authors worked on establishing Prohorov–Lévy distance rates for partial sums of non-i.i.d.r.v.'s. These results can be inverted into

Prohorov–Lévy distance rates for the renewals of the said sums via Theorem 4.1. Here we will only consider such results for m-dependent r.v.'s. We will again use the process $\{L_T(s);\ 0 \le s \le 1\}$ of Example 4.1, where now N is the renewal of stationary m-dependent r.v.'s.

THEOREM 4.3. *Assume* (3.45), (3.46) *and* (3.47) *hold true with* $r > 4$. *Then*

$$d(L_T, W) = o(T^{-\rho}), \tag{4.15}$$

where $\rho < r/(6r+6)$.

PROOF. Kanagawa (1982) proved that there are standard Wiener processes $\{W_T(t);\ 0 \le t < \infty\}$ satisfying

$$P\left\{\sup_{0 \le t \le T}\left|\frac{Z(t) - \mu t}{\sigma} - W_T(t)\right| > \delta T^{(1/2)-\rho}\right\} \le \delta T^{-\rho}, \tag{4.16}$$

for every $\delta > 0$. Using Theorem 4.1 we get

$$P\left\{\sup_{0 \le t \le T}\left|\frac{N(t) - t/\mu}{\sigma/\mu^{3/2}} - \hat{W}_T(t)\right| > \delta T^{(1/2)-\rho}\right\} \le \delta T^{-\rho}, \tag{4.17}$$

for every $\delta > 0$, where the Wiener processes $\hat{W}_T$ are those of Theorem 4.1. By (4.13) we get that (4.17) implies (4.15). □

EXAMPLE 4.3. This example is on runs down of Example 3.4. Let

$$L_T^{(p)}(s) = \frac{N^{(p)}(sT) - sT/\mu}{T^{1/2}\sigma/\mu^{3/2}}, \qquad 0 \le s \le 1.$$

THEOREM 4.4. *Assume that the conditions in Example* 3.4 *are satisfied. Then*

$$d\left(L_T^{(p)}, W\right) = o(T^{-\rho}),$$

where $\rho < 1/6$.

PROOF. We noted in Example 3.4 that $N^{(p)}$ is a renewal of bounded stationary $(p+3)$-dependent r.v.'s. Therefore Theorem 4.3 implies Theorem 4.4. □

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