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ON WIENER–HOPF FACTORISATION AND THE DISTRIBUTION OF EXTREMA FOR CERTAIN STABLE PROCESSES

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It is shown that when the index $0 < \alpha < 2$, $\alpha \neq 1$, and the symmetry parameter $-1 \leq \beta \leq 1$ of a stable process $\{X(t);\ t \geq 0\}$ are such that $P\{X(1) > 0\} = l\alpha^{-1} - k$, where l and k are integers, Darling's integral can be evaluated. This leads to explicit formulas for a transform of the Laplace transform of $\sup_{0 \leq t \leq 1} X(t)$ and the Wiener–Hopf factors of $\{X(t),\ t \geq 0\}$.

1. Introduction and results. Wiener–Hopf factorisation is of great importance in the theory of stochastic processes because there are a number of identities which express transforms of various functionals of the process, such as the maximum, first passage time, etc., in terms of the Wiener–Hopf factors. It is therefore unfortunate that explicit expressions for these factors can rarely be found. In the case of stable processes, it follows from the results of Darling (1956) and Heyde (1969) that explicit evaluation of these factors is essentially equivalent to the calculation of a certain definite integral, usually called Darling's integral. This integral was evaluated by Darling (1956) in the case of a symmetric Cauchy process and by Bingham (1973) in the case of a spectrally negative stable process, although in this latter case the Wiener–Hopf factors can be easily found by a probabilistic argument.

In this paper we employ a method similar to that used by Bingham (1973) to evaluate Darling's integral for a large class of stable processes, and hence find explicit forms for the Wiener–Hopf factors.

To state our results, we need some notation. We will assume throughout that $\mathbf{X} = \{X(t),\ t \geq 0\}$ is a stable process whose characteristic exponent $\psi(\theta) = \log\{E(e^{\theta X(1)})\}$ satisfies, for real u,

$$\psi(iu) = -c|u|^{\alpha}\left(1 - i\beta \operatorname{sign} u \tan\frac{\pi\alpha}{2}\right), \tag{1.1}$$

where $1 < \alpha < 2$, $-1 \leq \beta \leq 1$, or $0 < \alpha < 1$, $-1 < \beta < +1$, and for convenience we take $c = \{1 + \beta^2 \tan^2(\pi\alpha/2)\}^{-1/2}$. Notice that the cases $0 < \alpha < 1$, $\beta = \pm 1$, correspond to the situation where $\mathbf{X}$ or $-\mathbf{X}$ is a subordinator, where the factorisation is trivial, the case $\alpha = 1$, $\beta = 0$ (when $\mathbf{X}$ is a symmetric Cauchy process) has been treated by Darling (1956), and the case $\alpha = 1$, $\beta \neq 0$ (when $\mathbf{X}$ is a nonsymmetric Cauchy process) cannot be treated by our methods, because the scaling property fails. This property, which is valid for any $\mathbf{X}$ satisfying (1.1), states that for any $c > 0$, $\mathbf{X} =_D \{c^{-\delta}X(ct),\ t \geq 0\}$, where here, and throughout, $\delta = 1/\alpha$. It is a simple consequence of this property that the Wiener–Hopf

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factorisation for Lévy processes, due originally to Rogozin (1966) [see also Pecherskii and Rogozin (1969) and Gusak and Korolyuk (1969)] reduces in the case of a stable process satisfying (1.1), to

$$\{1-\psi(\theta)\}^{-1}=\psi^{+}(\theta)\psi^{-}(\theta),\qquad \mathrm{Re}(\theta)=0. \tag{1.2}$$

Here $\psi^{+}(\theta)$ [$\psi^{-}(\theta)$] is analytic in the half-plane $\mathrm{Re}(\theta)<0$ [$\mathrm{Re}(\theta)>0$], continuous and nonvanishing on $\mathrm{Re}(\theta)\le 0$ [$\mathrm{Re}(\theta)\ge 0$], and is the Laplace transform of an infinitely divisible distribution on the left (right) half-line. Furthermore, if the supports of one of these infinitely divisible distributions is required to be $(-\infty,0)$ or $(0,\infty)$, respectively, then the factorisation is unique. [Otherwise it is unique modulo multiplication by exponential factors; see Rogozin (1966).]

A fundamental quantity in the study of stable processes is $\rho=\Pr\{X(1)>0\}$; this was first evaluated by Zolotarev (1957), the result being

$$\rho=\frac{1}{2}+\frac{1}{\pi\alpha}\tan^{-1}\left(\beta\tan\frac{\pi\alpha}{2}\right). \tag{1.3}$$

To see the connection between the Wiener–Hopf factors and Darling's integral, observe that if we specialize the fundamental identity of Rogozin (1966) to the case of a stable process with the scaling property, we get

$$\int_0^\infty e^{-x}m^{+}(\theta x^{\delta})\,dx=\psi^{+}(-\theta),\qquad \mathrm{Re}(\theta)\ge 0, \tag{1.4}$$

where $m^{+}(\theta)=E(e^{-\theta M^{+}})$ and $M^{+}=\sup_{0\le t\le 1}X(t)$. On the other hand, Darling (1956) for the special case of a symmetric stable process, and Heyde (1969) have shown that for λ real and ≥ 0,

$$\int_0^\infty e^{-x}m^{+}(\lambda x^{\delta})\,dx=g^{+}(\lambda), \tag{1.5}$$

where

$$g^{+}(\lambda)=\exp\left\{-\frac{\sin\pi\rho}{\pi}\int_0^\infty\frac{\log\{1+(\lambda x)^{\alpha}\}\,dx}{(x^2+2x\cos\pi\rho+1)}\right\}. \tag{1.6}$$

Thus, $\psi^{+}(-\theta)$ is the (unique) analytic extension of g^{+} to $\mathrm{Re}(\theta)\ge 0$, and in a sense the problem of determining ψ^{+} is equivalent to that of calculating the integral which appears on the right-hand side of (1.6), i.e., Darling's integral. Furthermore, since the right and left Wiener–Hopf factors for $-\mathbf{X}$ are $\psi^{-}(-\theta)$, $\psi^{+}(-\theta)$, respectively, it follows that $\psi^{-}(\theta)$ is the (unique) analytic extension of g^{-} to $\mathrm{Re}(\theta)\ge 0$, where

$$g^{-}(\lambda)=\exp\left\{\frac{-\sin\pi(1-\rho)}{\pi}\int_0^\infty\frac{\log\{1+(\lambda x)^{\alpha}\}\,dx}{x^2+2x\cos\pi(1-\rho)+1}\right\}. \tag{1.7}$$

Spectrally negative stable processes arise when $1<\alpha<2$ and $\beta=-1$; in this case it follows from (1.3) that $\rho=\delta$ (recall $\delta=\alpha^{-1}$), and this motivates the following definition.

DEFINITION 1. For integers k and l the class $C_{k,l}$ denotes all stable processes satisfying (1.1) such that

$$\rho + k = l\delta. \tag{1.8}$$

We also need

DEFINITION 2. For integers $m \geq 0$, real ε and complex θ,

$$f_m(\varepsilon, \theta) = \prod_{r=0}^{m} (\theta + e^{i\varepsilon(m-2r)\pi}), \tag{1.9}$$

and

$$f_{-1}(\varepsilon, \theta) \equiv 1. \tag{1.10}$$

NOTE. With $f(\varepsilon, \theta) = f_1(\varepsilon, \theta) = 1 + 2\theta \cos \varepsilon\pi + \theta^2$ we have $f_{2n}(\varepsilon, \theta) = (1 + \theta)\prod_{r=1}^{n} f(2r\varepsilon, \theta)$, $f_{2n-1}(\varepsilon, \theta) = \prod_1^n f\{(2r-1)\varepsilon, \theta\}$ for $n \geq 0$.

We can now state our results. (We will see later that taking $k \geq 0$, $l \geq 1$ is no real restriction.)

THEOREM 1. *For any stable process in $C_{k,l}$ with $k \geq 0$, $l \geq 1$, we have, for $\lambda \geq 0$,*

$$g^+(\lambda) = f_{k-1}\big(\alpha, (-1)^l\lambda^\alpha\big)/f_{l-1}\big(\delta, (-1)^k\lambda\big), \tag{1.11}$$

$$g^-(\lambda) = f_{l-1}\big(\delta, (-1)^{k+1}\lambda\big)/f_k\big(\alpha, (-1)^l\lambda^\alpha\big). \tag{1.12}$$

In the next result we adopt the convention that θ^ε stands for $\sigma^\varepsilon e^{i\varepsilon\phi}$ when $\theta = \sigma e^{i\phi}$, $\sigma > 0$, $\pi \geq \phi > -\pi$.

THEOREM 2. *The Wiener–Hopf factors for any stable process in $C_{k,l}$ with $k \geq 0$, $l \geq 1$, are given by*

$$\psi^+(\theta) = f_{k-1}\big(\alpha, (-1)^l(-\theta)^\alpha\big)/f_{l-1}\big(\delta, (-1)^{k+1}\theta\big), \qquad \arg(\theta) \neq 0, \tag{1.13}$$

$$\psi^-(\theta) = f_{l-1}\big(\delta, (-1)^{k+1}\theta\big)/f_k\big(\alpha, (-1)^l\theta^\alpha\big), \qquad \arg(\theta) \neq -\pi. \tag{1.14}$$

It is not difficult to see that the set of points (α, β) such that Theorem 1 applies to the stable process with parameters (α, β) is dense on the set $\{0 < \alpha \leq 2, -1 \leq \beta \leq +1\}$. This suggests the possibility of proving results about the Wiener–Hopf factors of arbitrary stable processes by first establishing them for members of $C_{k,l}$ and then passing to the limit. This technique is illustrated by the proof of the following curious result.

THEOREM 3. *Let $\mathbf{X}$ be any stable process satisfying* (1.1) *with* $1 < \alpha < 2$. *Let $\hat{\mathbf{X}}$ be a stable process satisfying* (1.1) *with parameters $\hat{\alpha}$ and $\hat{\beta}$, where*

$\hat{\alpha} = 1/\alpha$ *and*

$$(1.15)\qquad \frac{2}{\pi}\tan^{-1}\left(-\hat{\beta}\tan\frac{\pi\hat{\alpha}}{2}\right) = \frac{1}{\alpha}\left(1 + \frac{2}{\pi}\tan^{-1}\left(-\beta\tan\frac{\pi\alpha}{2}\right)\right) - 1.$$

Then

$$(1.16)\qquad \hat{g}^{+}(\lambda) = g^{+}(\lambda^{\delta}), \quad \text{for } \lambda \geq 0,$$

where $\hat{g}^{+}(\lambda)$ *denotes the right-hand of* (1.5) *for* $\hat{\mathbf{X}}$.

The proof of these results is given in Section 2. In Section 3 we derive some corollaries, discuss some special cases and make some final remarks.

2. Proofs. For our purposes it is convenient to replace the standard representation (1.1) for $\psi(\theta)$ by an alternative form, due originally to Zolotarev (1957). [See also Feller (1971), page 581.] We set

$$(2.1)\qquad \gamma = \frac{2}{\pi}\tan^{-1}\left(-\beta\tan\frac{\pi\alpha}{2}\right),$$

so that (1.1) and (1.3) reduce to

$$(2.2)\qquad \psi(\pm iu) = -u^{\alpha}\exp\left(\pm i\tfrac{1}{2}\pi\gamma\right), \qquad u > 0,$$

and

$$(2.3)\qquad \rho = \tfrac{1}{2}(1 - \gamma\delta).$$

Then $\mathbf{X} \in C_{k,l}$, i.e., (1.8) holds, iff

$$(2.4)\qquad \gamma = (2k+1)\alpha - 2l,$$

and when this happens, (2.2) reduces to

$$(2.5)\qquad \psi(\pm iu) = (-1)^{l+1}u^{\alpha}\exp\left(\pm i\left(k + \tfrac{1}{2}\right)\pi\alpha\right), \qquad u > 0.$$

We start by elucidating the extent of the class $C_k = \bigcup_{l=-\infty}^{\infty} C_{k,l}$.

LEMMA 1. (i) $\mathbf{X} \in C_{k,l}$ *iff* $-\mathbf{X} \in C_{-(k+1),-l}$.
(ii) $\mathbf{X} \in C_0$ *iff* $1 < \alpha < 2$ *and* $\beta = -1$.
(iii) *For* $k \geq 1$, $\mathbf{X} \in C_k$ *iff for some* $l \geq 1$ *either*

$$(2.6)\qquad 0 < \alpha < 1 \quad \text{and} \quad k\alpha < l < (k+1)\alpha,$$

or

$$(2.7)\qquad 1 < \alpha < 2 \quad \text{and} \quad (k+1)\alpha - 1 < l < \alpha k + 1.$$

PROOF. (i) This is immediate from (2.4) since the quantity corresponding to γ for $-\mathbf{X}$ is $-\gamma$.

(ii) For $0 < \alpha < 1$, $\rho = l\delta$ could only occur if $l = 0$, i.e., $\rho = 0$. But this happens only when $\beta = -1$, a case we have excluded. For $1 < \alpha < 2$, $\rho = l\delta$ can only happen if $l = 1$, when $\gamma = \alpha$ by (2.4) and $\beta = -1$ by (2.1).

(iii) Observe from (2.1) that as β increases from -1 to $+1$, when $0 < \alpha < 1$, γ decreases from α to $-\alpha$ and ρ increases from 0 to 1, whereas for $1 < \alpha < 2$, γ

increases from $\alpha - 2$ to $2 - \alpha$, and ρ decreases from δ to $1 - \delta$. (2.6) and (2.7) are then immediate from (2.4). □

For $\mathbf{X} \in C_0$ (i.e., the spectrally negative case), $k = 0$ and $l = 1$ so that, in view of Definition 2, Theorem 1 reduces to the assertion that $g^+(\lambda) = 1/(1 + \lambda)$ and $g^-(\lambda) = (1 - \lambda)/(1 - \lambda^\alpha)$. Both these results are known. As previously remarked, Bingham (1973) contains a direct proof of the first one, and an indirect proof of the second can be found in Bingham (1975), page 760. Thus, we need only deal with the case $\mathbf{X} \in C = \bigcup_1^\infty C_k$, when in view of Lemma 1 $l \geq 1$ when $0 < \alpha < 1$ and $l \geq 2$ when $1 < \alpha < 2$. We write $\Delta = l\delta$ and for any nonnegative integer n define $n^* = [\frac{1}{2}n]$, where $[x]$ denotes the integer part of x, and put $n = 2n^* + \eta(n)$, so that $\eta(n) = 1$ or 0 according as n is odd or even. Also, for $t \geq 0$ let

$$I(t) = \frac{\Delta \sin \pi\rho}{\pi} \int_0^\infty \frac{x^{l+\Delta-1}\,dx}{(1 + tx^l)(x^{2\Delta} + 2x^\Delta \cos \pi\rho + 1)}. \tag{2.8}$$

LEMMA 2. *If $\mathbf{X} \in C$, then for $t > 0$,*

$$\begin{aligned} I(t) = \Delta t^{-1} - \operatorname{Re}\Bigg\{ & 2\sum_{n=1}^{l^*} \delta t^{-(\delta+1)}\left\{t^{-\delta} + e^{i\pi(\rho-(2n-1)\delta)}\right\}^{-1} \\ & + \eta(l)\delta t^{-(\delta+1)}\left\{t^{-\delta} + (-1)^k\right\}^{-1} \\ & + 2\sum_{m=1}^{k^*} \left\{t + e^{-i\pi\alpha(\rho+2m-1)}\right\}^{-1} + \eta(k)\left\{t + (-1)^l\right\}^{-1}\Bigg\}. \end{aligned} \tag{2.9}$$

PROOF. Consider $\int_\Gamma h(z)\,dz$, where $h(z) = l^{l+\Delta-1}(1 + tz^l)^{-1}(z^\Delta + e^{i\pi\rho})^{-1}$ and Γ consists of Γ_0, the semicircle of radius R in the upper half plane centered at the origin, Γ_1, a similar semicircle of radius ε, Γ_2, a semicircle of radius ε centered at the point -1, and Γ_3, another semicircle of radius ε centered at the point $-t^{-1/l}$, all suitably orientated and connected by portions of the real axis. It is easily seen that

(i) as $R \to \infty$, $\int_{\Gamma_0} h(z)\,dz \to i\pi t^{-1}$;
(ii) as $\varepsilon \to 0$, $\int_{\Gamma_1} h(z)\,dz \to 0$;
(iii) as $\varepsilon \to 0$, $\int_{\Gamma_2} h(z)\,dz \to -\eta(l) i\pi (lt)^{-1} t^{-\delta}\{t^{-\delta} + (-1)^k\}^{-1}$;
(iv) as $\varepsilon \to 0$, $\int_{\Gamma_3} h(z)\,dz \to -\eta(k) i\pi \Delta^{-1}\{t + (-1)^l\}^{-1}$.

Since $\operatorname{Im}(h(z)) = -\sin \pi\rho x^{\Delta+l-1}(1 + tx^l)^{-1}(x^{2\Delta} + 2x^\Delta \cos \pi\rho + 1)^{-1}$ when $\operatorname{Im}(z) = 0$, $\operatorname{Re}(z) = x > 0$, and $\operatorname{Im}(h(z)) = 0$ when $\operatorname{Im}(z) = 0$, $\operatorname{Re}(z) < 0$, $\neq -1$, $\neq -t^{-1/l}$, the result follows from the residue theorem and a careful evaluation of the residues of $h(z)$ at the points $t^{-1/l}e^{i\pi(2n-1)l^{-1}}$, $1 \leq n \leq l^*$, and the points $e^{i\pi(\rho+2m-1)\Delta^{-1}}$, $1 \leq m \leq k^*$, where $h(z)$ has simple poles within Γ. □

PROOF OF (1.11). Referring to (1.6) and setting $J(t) = -\log g^+(t^\delta)$ we see, after making the substitution $x = y^l$, that $J'(t) = I(t)$. Observing that for any real ε, $\mathrm{Re}\{\log(1 + t^{\delta}e^{i\pi\varepsilon})\} = \delta\int_0^t[y^{-1} - \mathrm{Re}\{(y^{-\delta} + e^{i\pi\varepsilon})^{-1}y^{-(\delta+1)}\}]\,dy$ and writing $\Delta t^{-1} = (2l^* + \eta(l))\delta t^{-1}$ we may integrate (2.9) and use $J(0) = 0$ to get

$$J(t) = \mathrm{Re}\Bigg\{2\sum_{n=1}^{l^*}\log\{1 + t^{\delta}e^{i\pi(\rho-(2n-1)\delta)}\} + \eta(l)\log\big(1 + (-1)^k t^{\delta}\big) \tag{2.10}$$
$$- 2\sum_{m=1}^{k^*}\log\big(t + e^{-i\pi\alpha(\rho+2m-1)}\big) - \eta(k)\log\big(t + (-1)^l\big)\Bigg\}.$$

Since $2\,\mathrm{Re}(\log(1 + \lambda e^{i\pi\varepsilon})) = 2\,\mathrm{Re}(\log(\lambda + e^{-i\pi\varepsilon})) = \log f(\varepsilon, \lambda)$ when $\lambda \geq 0$, we may rewrite (2.10), using the facts that $\rho - (2n-1)\delta = \delta(l + 1 - 2n) - k$, $-\alpha(\rho + 2m - 1) = \alpha(k + 1 - 2m) - l$ to get

$$J(t) = \sum_{n=1}^{l^*}\log f\big(\delta(l + 1 - 2n), (-1)^k t^{\delta}\big) \tag{2.11}$$
$$- \sum_{m=1}^{k^*}\log f\big(\alpha(k + 1 - 2m), (-1)^l t\big)$$
$$+ \eta(l)\log\big|1 + (-1)^k t^{\delta}\big| - \eta(k)\log\big|1 + (-1)^l t\big|.$$

It is easy to check, by considering separately the cases when none, one or both of k and l are even, that the last two terms in (2.11) may be rewritten as $\eta(l)\log(1 + (-1)^k t^{\delta}) - \eta(k)\log(1 + (-1)^l t)$, and then (1.11) follows immediately. □

It is possible to establish (1.12) by a similar calculation; however, the value of $g^-(\lambda)$ follows immediately once we know $\psi^-(\theta)$, and we now show that Theorem 2 follows from (1.11). We break the proof into a sequence of lemmas. (Note that again we need only treat the case $k \geq 1$.)

LEMMA 3. *For $k \geq 1$,*

$$f_k\big(\alpha, (-1)^l\theta^{\alpha}\big) = \big(1 + (-1)^l\theta^{\alpha}e^{\pm ik\alpha\pi}\big)f_{k-1}\big(\alpha, (-1)^l(-\theta)^{\alpha}\big)$$

according as $\arg(\theta) \in (0, \pi]$ *or* $\arg(\theta) \in (-\pi, 0]$.

PROOF. This follows from the identities

$$f_k(\alpha, z) = (1 + ze^{\mp ik\alpha\pi})f_{k-1}(\alpha, ze^{\pm i\alpha\pi}) \quad \text{and} \quad (-\theta)^{\alpha} = e^{\mp i\alpha\pi}\theta^{\alpha},$$

each according as $\arg(\theta) \in (0, \pi]$ or $\arg(\theta) \in (-\pi, 0]$. □

COROLLARY. *With ψ^+ and ψ^- given by* (1.13) *and* (1.14), (1.2) *holds.*

PROOF. Just put $\theta = \pm iu$ in Lemma 3 and use (2.5). □

LEMMA 4. *When either* (2.6) *or* (2.7) *holds the zeros of $f_{k-1}(\alpha, (-1)^l(-\theta)^{\alpha})$ coincide with the zeros of $f_k(\alpha, (-1)^l\theta^{\alpha})$.*

PROOF. By Lemma 3, it suffices to show that $1 + (-1)^l\theta^\alpha e^{\pm ik\alpha\pi}$ does not vanish. But if l is even, for $0 < \arg(\theta) \le \pi$ we have

$$\arg(\theta^\alpha e^{ik\alpha\pi}) \in (k\alpha\pi, (k+1)\alpha\pi]$$

and for $-\pi < \arg(\theta) \le 0$ we have $\arg(\theta^\alpha e^{-ik\alpha\pi}) \in (-(k+1)\alpha\pi, -k\alpha\pi]$. If (2.6) holds we have

$$\begin{aligned}(k\alpha\pi, (k+1)\alpha\pi] &= ((k+1)\alpha\pi - \alpha\pi, k\alpha\pi + \alpha\pi] \subset (l\pi - \alpha\pi, l\pi + \alpha\pi]\\ &\subset ((l-1)\pi, (l+1)\pi],\end{aligned}$$

and similarly $(-(k+1)\alpha\pi, -k\alpha\pi] \subset (-(l+1)\pi, -(l-1)\pi)$ so that $\arg(\theta^\alpha e^{\pm ik\alpha\pi}) \ne -\pi(\text{mod}\, 2\pi)$. If (2.7) holds it is immediate that $(k\alpha\pi, (k+1)\alpha\pi] \subset ((l-1)\pi, (l+1)\pi)$ and $(-(k+1)\alpha\pi, -k\alpha\pi] \subset (-(l+1)\pi, -(l-1)\pi)$, so again $\arg(\theta^\alpha e^{\pm ik\alpha\pi}) \ne -\pi(\text{mod}\, 2\pi)$. Finally, if l is odd, a similar calculation establishes the result. □

LEMMA 5. *For $k \ge 1$ and $0 < \alpha < 2$, $\alpha \ne 1$, let $Z_k(\alpha) = \{z_n, 1 - p \le n \le p\}$, $Y_k(\alpha) = \{y_n, -q \le n \le q\}$, where $z_n = (-1)^k e^{i(2n-1)\delta\pi}$, $y_n = (-1)^k e^{i2n\delta\pi}$, $p = [\frac{1}{2}(k\alpha + \alpha + 1)]$, $q = [\frac{1}{2}\alpha(k+1)]$. Then*

(i) *if $\frac{1}{2}(k\alpha + \alpha + 1)$ is not an integer, $f_k(\alpha, \theta^\alpha)$ has simple zeros at the points of $Z_k(\alpha)$ and has no other zeros;*

(ii) *if $\frac{1}{2}\alpha(k+1)$ is not an integer, $f_k(\alpha, -\theta^\alpha)$ has simple zeros at the points of $Y_k(\alpha)$ and has no other zeros.*

PROOF. We prove (i) only, the proof of (ii) being similar. First observe that $z_n^\alpha = e^{i\alpha((2n-1)\delta + k - 2m_n)\pi} = -e^{i\alpha(k-2m_n)\pi}$, where m_n is the unique integer with $-1 < (2n-1)\delta + k - 2m_n \le 1$, or $(n - \frac{1}{2})\delta + \frac{1}{2}(k-1) \le m_n < (n - \frac{1}{2})\delta + \frac{1}{2}(k+1)$. Clearly, $1 - p \le n \le p$ implies $m_{1-p} \le m_n \le m_p$, and from the definition of p it follows easily that $m_{1-p} \ge 0$, $m_p \le k$ [unless $p = \frac{1}{2}(k\alpha + \alpha + 1)$ when $m_{1-p} = -1$]. Thus $\theta \in Z_k(\alpha) \Rightarrow f_k(\alpha, \theta^\alpha) = 0$. On the other hand, $\theta^\alpha = -e^{i\alpha\pi(k-2m)} \Leftrightarrow \theta = e^{i\phi}$, where $-\pi < \phi \le \pi$ and $\alpha\phi = \pi(\alpha(k-2m) + 2n - 1)$ for some integer n. Thus, $\phi = \pi(k - 2m + (2n-1)\delta)$, $\theta = z_n$ and $-\alpha < \alpha(k-2m) + 2n - 1 \le \alpha$. For $0 \le m \le k$ this implies $1 - \alpha(k+1) < 2n \le 1 + \alpha(k+1)$ and hence $f_k(\alpha, \theta^\alpha) = 0 \Rightarrow \theta \in Z_k(\alpha)$. □

LEMMA 6. *When either* (2.6) *or* (2.7) *holds the zeros of $f_k(\alpha, (-1)^l\theta^\alpha)$ coincide with the zeros of $f_{l-1}(\delta, (-1)^{k+1}\theta)$.*

PROOF. Note first that the zeros of $f_{l-1}(\delta, (-1)^{k+1}\theta)$ are located at the points $x_r = (-1)^k e^{i\delta(l-1-2r)\pi}$, $0 \le r \le l-1$. Suppose first that l is even, equal to 2λ, so that $\{x_r, 0 \le r \le l-1\} = \{z_n, 1 - \lambda \le n \le \lambda\}$. If (2.6) holds, $2\lambda + 1 = l + 1 < (k+1)\alpha + 1 < l + \alpha + 1 < 2\lambda + 2$, and if (2.7) holds, $\alpha > 1 \Rightarrow l \ge k + 1 \Rightarrow (l-1)k^{-1} \ge l(k+1)^{-1}$ so that $2\lambda + 1 = l + 1 < (k+1)\alpha + 1 < l + 2 = 2\lambda + 2$. Thus, in both cases, $[\frac{1}{2}(k\alpha + \alpha + 1)] = \lambda$ and the result follows from (i) of Lemma 5. If l is odd, equal to $2\lambda + 1$, we have $\{x_r, 0 \le r \le l-1\} = \{y_n, -\lambda \le n \le \lambda\}$ and the result follows from (ii) of Lemma 5, since it is easy to check that $[\frac{1}{2}\alpha(k+1)] = \lambda$. □

Proof of Theorem 2. It follows from (1.5) that for $\lambda \geq 0$, $g^+(-\lambda)$ is the Laplace transform of an infinitely divisible distribution with support $(-\infty, 0)$, and since Lemmas 4, 5 and 6 show that the right-hand side of (1.13) is the unique analytic extension of the right-hand side of (1.11) to the region $\arg(\theta) \neq 0$, (1.13) is established. The Wiener–Hopf factorisation (1.2) then fixes the value of $\psi^-(\theta)$ for $\mathrm{Re}(\theta) = 0$, using the corollary to Lemma 2, and since we have shown that the right-hand side of (1.14) is analytic for $\arg(\theta) \neq -\pi$, (1.14) in turn is established. □

Proof of Theorem 1. We have already established (1.11), and now (1.12) follows from (1.14) by the statements analogous to (1.4) and (1.5) for g^- and ψ^-. □

Proof of Theorem 3. Notice first that when $\beta = -1$, (1.15) gives $\hat{\beta} = +1$, and since Theorem 1 applies with $k = 0$ and $l = 1$ to give $g^+(\lambda) = (1 + \lambda)^{-1}$, (1.16) states that $\hat{g}^+(\lambda) = (1 + \lambda^\delta)^{-1}$. Since the process $\hat{\mathbf{X}}$ is a subordinator, $\hat{m}^+(\lambda) = e^{-\lambda^\delta}$, and this follows from (1.5). Suppose now that $\mathbf{X} \in C_{k,l}$, where in view of Lemma 1 we take $k \geq 1$ and $l \geq 2$. Since (1.15) can be rewritten as $\hat{\gamma} = \delta(\gamma + 1) - 1$, (2.4) then gives $-\hat{\gamma} = (2l - 1)\hat{\alpha} - 2k$ so that $-\hat{\mathbf{X}} \in C_{\hat{k}, \hat{l}}$, where $\hat{k} = l - 1 \geq 0$, $\hat{l} = k \geq 1$. Thus, Theorem 1 applies to both $\mathbf{X}$ and $-\hat{\mathbf{X}}$ and since $\hat{g}^+$ coincides with g^- evaluated for $-\hat{\mathbf{X}}$ we get

$$\hat{g}^+(\lambda) = f_{\hat{l}-1}\big(\hat{\delta}, (-1)^{\hat{k}+1}\lambda\big)/f_{\hat{k}}\big(\hat{\alpha}, (-1)^{\hat{l}}\lambda^{\hat{\alpha}}\big) = f_{k-1}\big(\alpha, (-1)^{l}\lambda\big)/f_{l-1}\big(\delta, (-1)^{k}\lambda^{\delta}\big)$$
$$= g^+(\lambda^\delta).$$

But for arbitrary $\mathbf{X}$ satisfying (1.1) with $1 < \alpha < 2$ there exists a sequence of stable processes $\{\mathbf{X}_k,\ k \geq 1\}$ with $\mathbf{X}_k$ having parameters α_k and γ, where $\alpha_k = (\gamma + 2l_k)/(2k + 1)$ [so that (2.4) holds and $\mathbf{X}_k \in C_{k, l_k}$] and the integers l_k are chosen with $1 < \alpha_k < 2$ and $\alpha_k \to \alpha$ as $k \to +\infty$. It is clear that if $\rho_k = \Pr\{X_k(1) > 0\}$, then $\rho_k = \frac{1}{2}(1 - \gamma\delta_k) \to \frac{1}{2}(1 - \gamma\delta) = \rho$, and it follows easily from (1.6) that $g_k^+(\lambda) \to g^+(\lambda)$, and hence $g_k^+(\lambda^{\delta_k}) \to g^+(\lambda^\delta)$ as $k \to \infty$. In the same way we see that the sequence $\{\hat{\mathbf{X}}_k,\ k \geq 1\}$ is such that $\hat{\alpha}_k \to \hat{\alpha}$, $\hat{\rho}_k \to \hat{\rho}$ and $\hat{g}_k^+(\lambda) \to \hat{g}^+(\lambda)$, so that (1.16) is immediate. □

3. Remarks. (i) As previously mentioned, if $\mathbf{X} \in C_{k,l}$ with k (and hence l) negative, then by Lemma 1, Theorems 1 and 2 apply to $-\mathbf{X}$.

(ii) The most interesting special cases of our results occur when $k = 0$, $l = 1$ (when $1 < \alpha < 2$, $\rho = \delta$ and $\mathbf{X}$ is spectrally negative), when $k = 1$, $l = 1$ (when $0 < \alpha < 1$ and $\rho = \delta - 1$) and when $k = 1$, $l = 2$ (when $1 < \alpha < 2$ and $\rho = 2\delta - 1$). Referring to these situations as case I, case II and case III, respectively, we record the corresponding statements of Theorem 1 in

Corollary 1. *In cases* I, II *and* III, *respectively*,

(3.1) $g^+(\lambda) = 1/(1 + \lambda), \qquad g^-(\lambda) = (1 - \lambda)/(1 - \lambda^\alpha),$

(3.2) $g^+(\lambda) = (1 - \lambda^\alpha)/(1 - \lambda), \qquad g^-(\lambda) = (1 + \lambda)/(1 - 2\lambda^\alpha \cos \pi\alpha + \lambda^{2\alpha}),$

(3.3) $g^+(\lambda) = (1 + \lambda^\alpha)/(1 - 2\lambda \cos \pi\delta + \lambda^2),$
$g^-(\lambda) = (1 + 2\lambda \cos \pi\delta + \lambda^2)/(1 - 2\lambda^\alpha \cos \pi\alpha + \lambda^{2\alpha}).$

(iii) In the cases where Theorem 1 applies we know explicitly the right-hand side of the integral equation (1.5); this raises the possibility of finding $m^+(\lambda)$ explicitly, or even $\mu^+(x)$, the probability density function of M^+, which is known to exist. As has been pointed out by Darling (1956), a theoretical way to achieve this is by the use of Mellin transforms, since it follows from (1.5) that $\mathscr{G}^+(s) = \int_0^\infty x^{s-1}g^+(x)\,dx$ and $\mathscr{M}^+(s) = \int_0^\infty x^{s-1}\mu^+(x)\,dx$ are connected by $\mathscr{G}^+(1-s) = \mathscr{M}^+(s)\Gamma(1-s)\Gamma(1-\delta+\delta s)$. However, only in cases I, II and III can $\mathscr{G}^+$ and $\mathscr{G}^-$ (the corresponding quantity for M^-) be found explicitly. We recorded these explicit results for $\mathscr{M}^+$ and $\mathscr{M}^-$ in

COROLLARY 2. *In cases* I, II *and* III, *respectively*,

$$\begin{aligned} &\mathscr{M}^+(s) = \pi/\sin\pi s, \\ &\mathscr{M}^-(s) = \pi\delta\sin\pi\delta/\{\Gamma(1-s)\Gamma(1-\delta+\delta s)\sin\pi\delta s\sin\pi\delta(1+s)\}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\mathscr{M}^+(s) = \pi/\{\Gamma(1-s)\Gamma(1-s+\delta s)\sin\pi(\alpha+1-s)\}, \\ &\mathscr{M}^-(s) = \pi\left\{\frac{\sin\{\pi(1-\alpha)(1-\delta s)\}}{\sin\pi\delta s} + \frac{\sin\{\pi(1-\alpha)(1-\delta(1+s))\}}{\sin\pi\delta(1+s)}\right\} \\ &\qquad\times\{\alpha\sin\pi\alpha\Gamma(1-s)\Gamma(1-\delta+\delta s)\}^{-1}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\mathscr{M}^+(s) = \pi\left\{\frac{\sin\{\pi(1-\delta)(1-s)\}}{\sin\pi s} + \frac{\sin\{\pi(1-\delta)(1-\alpha-s)\}}{\sin\pi(s+\alpha)}\right\} \\ &\qquad\times\{\sin\pi\delta\Gamma(1-s)\Gamma(1-\delta+\delta s)\}^{-1}, \\ &\mathscr{M}^-(s) = -\pi\left\{\frac{\sin\{\pi(\alpha-1)(1-\delta s)\}}{\sin\pi\delta s}\right. \\ &\qquad + \frac{2\cos\pi\delta\sin\{\pi(\alpha-1)(1-\delta(1+s))\}}{\sin\pi\delta(1+s)} \\ &\qquad \left. + \frac{\sin\{\pi(\alpha-1)(1-\delta(s+2))\}}{\sin\pi\delta(2+s)}\right\} \\ &\qquad\times\{\sin\pi\alpha\Gamma(1-s)\Gamma(1-\delta+\delta s)\}^{-1}. \end{aligned} \tag{3.6}$$

PROOF. Just apply (3), (12) and (16) of Erdélyi (1954), pages 308–309, to (3.1), (3.2) and (3.3). □

As has been pointed out by Bingham (1973), in case I (3.4) implies that M^+ has a Mittag–Leffler distribution, a result also obtained by Heyde (1969), but in the other cases inversion to obtain μ^+ and μ^- seems impossible. Also in case I, Doney (1987) contains a probabilistic derivation of an integral formula for $m^-(\lambda)$ which is equivalent to the statement that if T is nonnegative, independent of M^- and has a stable distribution of order δ then $T(M^-)^\alpha$ has a Pareto distribution. It is easily seen that this statement is equivalent to (3.1).

(iv) Another interesting special case arises when $\mathbf{X}$ is symmetric, i.e., $\beta = 0$, $\rho = \frac{1}{2}$ and $\gamma = 0$. It follows from (2.4) that in this situation, $\mathbf{X} \in C_{k,l}$ iff $\alpha = 2l/(2k+1)$. Of course, $-\mathbf{X}$ has the same distribution as $\mathbf{X}$, so in this case the apparently different forms for $g^+(\lambda)$ and $g^-(\lambda)$ given by Theorem 1 must coincide. A simple example of this is when $\alpha = \frac{2}{3}$ (and $\beta = 0$): Then we have case II and Corollary 1 gives $g^+(\lambda) = (1 - \lambda^{2/3})(1 - \lambda)^{-1}$, $g^-(\lambda) = (1 + \lambda)(1 - \lambda^{2/3} + \lambda^{4/3})^{-1} = (1 + \lambda)(1 - \lambda^{2/3})(1 - \lambda^2)^{-1} = (1 - \lambda^{2/3})(1 - \lambda)^{-1}$.

(v) If α and β are fixed, then for irrational values of α it is clear that there is at most one pair (k, l) such that (1.8) holds. However, if α is rational, equal to m/n and (1.3) holds, then it will also hold with k and l replaced by k' and l', where $k' = k + rn$, $l' = l + rm$ and r is any integer ≥ 1. In this situation Theorem 1 gives many apparently different forms for g^+ and g^-, but, of course, the difference is only apparent, and cancellation of appropriate terms reduces all the forms to the one corresponding to the smallest values of k and l. For example, when $\alpha = \frac{2}{3}$ and $\beta = 0$ we have $\mathbf{X} \in C_{4,3}$ as well as $\mathbf{X} \in C_{1,1}$, and Theorem 1 gives the alternative expression $g^+(\lambda) = f_3(\frac{2}{3}, -\lambda^{2/3})/f_2(\frac{3}{2}, \lambda)$. However, it is easy to check that $f_3(\frac{2}{3}, -\lambda^{2/3}) = (1 - \lambda^2)(1 - \lambda^{2/3})$ and $f_2(\frac{3}{2}, \lambda) = (1 - \lambda^2)(1 - \lambda)$, so that $g^+(\lambda) = (1 - \lambda^{2/3})/(1 - \lambda)$, in accordance with (3.2).

(vi) Since it is a simple consequence of (1.5) that $g^+(\lambda)$ [and similarly $g^-(\lambda)$] is the Laplace transform of an infinitely divisible distribution concentrated on $(0, \infty)$, Theorem 1 exhibits a large class of such Laplace transforms. Even in the cases I, II and III it does not seem to be easy to check this fact directly.

(vii) An alternative approach to calculating g^+ is to use the fact, established by Bingham (1973), that $-\log g^+(\lambda)$ has a known Mellin transform. This approach works in cases I, II and III, but apparently not for general k and l.

(viii) Theorem 3 is an interesting complement to the well-known result [see Feller (1971), page 583] which links the density $\hat{p}$ of $\hat{X}(1)$ with the density p of $X(1)$, viz.

$$\hat{p}(x) = x^{-(1+\alpha)} p(x^{-\alpha}), \qquad x > 0. \tag{3.7}$$

Notice, of course, that, by applying Theorem 3 to $-\mathbf{X}$ we get a relationship of the form $g^-(\lambda^\delta) = \tilde{g}^-(\lambda)$, where $\tilde{\mathbf{X}}$ is a process with parameters $\tilde{\alpha} = \delta$, $\tilde{\gamma} = 2 - \delta(2 - \gamma)$, but, of course, $\tilde{\gamma} \neq \hat{\gamma}$, unless $\alpha = 1$, $\gamma = 0$, when (1.16) does actually hold, but it tautologous.

Although we have not considered the case of Brownian motion explicitly, since the results corresponding to Theorems 1 and 2 are then well known, it is worth remarking that for Brownian motion the result corresponding to Theorem 3 is just a disguised version of the relationship between the one-sided stable density of order $\frac{1}{2}$ and the normal density function, which of course is the special case $\alpha = 2$ of (3.7).

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