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BERNOULLI PERCOLATION ABOVE THRESHOLD: AN INVASION PERCOLATION ANALYSIS

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Using the invasion percolation process, we prove the following for Bernoulli percolation on $\mathbb{Z}^d$ ($d > 2$): (1) exponential decay of the truncated connectivity, $\tau'_{xy} \equiv P(x$ and y belong to the same finite cluster$) \leq \exp(-m\|x - y\|)$; (2) infinite differentiability of $P_\infty(p)$, the infinite cluster density, and of $\chi'(p)$, the expected size of finite clusters, as functions of p, the density of occupied bonds; and (3) upper bounds on the cluster size distribution tail, $P_n \equiv P($the cluster of the origin contains exactly n bonds$) \leq \exp(-[c/\log n]n^{(d-1)/d})$. Such results (without the $\log n$ denominator in (3)) were previously known for $d = 2$ and $p > p_c$, the usual percolation threshold, or for $d > 2$ and p close to 1. We establish these results for all $d > 2$ when p is above a limit of "slab thresholds," conjectured to coincide with p_c.

1. Introduction and results. Many of the properties of Bernoulli percolation can be characterized in terms of the behavior of large finite clusters. This behavior is well understood below the threshold at which the expected cluster size diverges. In particular, both the (truncated) connectivity function and the cluster size distribution tail are known to decay exponentially ([10], [13], [3]). On the other hand, previous results about these quantities above the percolation threshold ([15], [2], [14]) are quite incomplete. This note represents a moderately successful attempt to remedy this situation by exploiting the relation between standard percolation and a recently invented dynamic growth model known as invasion percolation ([16], [5], [21], [8]).

In this paper, we will treat standard Bernoulli bond percolation on $\mathbb{Z}^d$ ([4], [14]), in which the nearest neighbor bonds are independently "occupied" or "vacant" with probability p or $1 - p$. All our results can easily be extended to Bernoulli site percolation and to lattices other than $\mathbb{Z}^d$. It is natural to think of the hypercubic bond percolation model as a random graph whose vertex set is $\mathbb{Z}^d$ and whose edges are the occupied nearest neighbor bonds. The cluster $C(x)$ of x in $\mathbb{Z}^d$ is then defined as that connected component of this random graph which contains the vertex x. Its size or volume $|C(x)|$ is the number of occupied bonds in $C(x)$. For x, y in $\mathbb{Z}^d$, we then define the connectivity,

$$\tau_{xy} = P(y \in C(x)) = P(x \in C(y)), \tag{1.1}$$

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and the truncated connectivity,

(1.2) $$\tau'_{xy} = P\big(y \in C(x) \text{ and } |C(x)| < \infty\big).$$

Other basic quantities of interest are the cluster size distribution,

(1.3) $$P_n = P\big(|C(0)| = n\big); \qquad n = 0, 1, 2, \dots,$$

the infinite cluster density,

(1.4) $$P_\infty = P\big(|C(0)| = \infty\big) = 1 - \sum_{n=0}^{\infty} P_n,$$

the expected cluster size,

(1.5) $$\chi = E\big(|C(0)|\big),$$

and the expected size of finite clusters,

(1.6) $$\chi' = E\big(|C(0)| \mathbb{1}_{|C(0)|<\infty}\big) = \sum_{n=0}^{\infty} nP_n.$$

Our main results are stated in terms of the above definitions with $|C(0)|$ as the number of bonds in $C(0)$. They also hold (with suitable changes of constants) when $|C(0)|$ is replaced by the number of sites in $C(0)$.

There are many possible definitions of a threshold or critical point for p. These are nontrivial (i.e., not equal to 1 or 0) for every $d \geq 2$, but are rigorously known to coincide only for $d = 2$ ([12], [14]). Two of these thresholds are

(1.7) $$\pi_c = \sup\{p\colon \chi(p) < \infty\}$$

and

(1.8) $$p_c = \sup\{p\colon P_\infty(p) = 0\}.$$

It is of course clear that $\pi_c \leq p_c$.

For p in $(0, \pi_c)$, there exist constants (depending on p) m_1 and m_2 in $(0, \infty)$, such that

(1.9) $$\exp(-m_1\|y - x\|) \leq \tau_{xy} = \tau'_{xy} \leq \exp(-m_2\|y - x\|) \quad \text{for } x, y \text{ in } \mathbb{Z}^d,$$

where $\|x\|$ denotes (e.g.) the l_1 norm of x in $\mathbb{Z}^d$, and constants c_1 and c_2 in $(0, \infty)$ such that

(1.10) $$\exp(-c_1 n) \leq P_n \leq \exp(-c_2 n) \quad \text{for } n = 1, 2, \dots .$$

The lower bounds of (1.9)–(1.10) are elementary. The upper bounds were originally obtained in [10] and then rederived in [13], [3].

A lower bound for τ' as in (1.9) is easily seen to be valid for any p in $(0, 1)$ and any d. For example, by considering a path from x to y containing exactly $\|y - x\|$ occupied bonds and surrounded by at most $(2d - 2)(\|y - x\| + 1) + 2$ vacant bonds, one obtains

(1.11) $$\tau'_{xy} \geq (1 - p)^{2d}\big[p(1 - p)^{(2d-2)}\big]^{\|y-x\|}.$$

A lower bound for P_n above threshold is less elementary. It is known ([2]) that for p in $(p_c, 1)$, there exists $c_3 = c_3(p) < \infty$ such that

$$(1.12) \qquad P_n \geq \exp\left(-c_3 n^{(d-1)/d}\right) \quad \text{for } n = 1, 2, \ldots,$$

a result which was first derived for p close to 1 in [15]. We remark that this explicitly shows that an upper bound as in (1.10), i.e., an exponential tail for the cluster size distribution, cannot be valid above p_c. Furthermore, the lower bound (1.12) is valid for a large class of nonindependent percolation models ([2]).

The focus of this paper is on upper bounds complementary to (1.11)–(1.12) above threshold. Previously such results were known only for $d = 2$ or for p close to 1. The upper bound for $d = 2$ and $p > \frac{1}{2}$ $(= p_c = \pi_c)$,

$$(1.13) \qquad P_n \leq \exp\left(-c_4 n^{1/2}\right),$$

was first obtained by Kesten ([14]). For general d, it is known ([15]) that

$$(1.14) \qquad P_n \leq \exp\left(-c_4 n^{(d-1)/d}\right)$$

for p close to 1. In fact, it is not hard to show ([8]) that (1.14) holds for $p > 1 - \pi_c^*$, where π_c^* is the analogue of π_c for the dual percolation model (bonds for $d = 2$, plaquettes for $d = 3$, etc.). An argument similar to that of [8], but for τ' rather than P_n, yields an exponential upper bound for τ', as in (1.9) for $p > 1 - \pi_c^*$. Since $1 - \pi_c^* = 1 - \pi_c = \frac{1}{2} = \pi_c = p_c$ for $d = 2$ ([18], [19], [12]), the two-dimensional noncritical behavior of τ, τ', and P_n are all well characterized.

In this paper, we treat dimension $d \geq 3$ and derive upper bounds complementary to (1.11)–(1.12) and some related results above one or the other of two thresholds introduced in [1], $p^{\infty}_{c,d-1}$ and $\hat{p}^{\infty}_{c,2}$, which are conjectured to coincide with p_c (and π_c). To define $p^{\infty}_{c,d-1}$, first denote by $p^{K}_{c,d-1}$ for $K = 0, 1, 2, \ldots$, the percolation threshold defined as in (1.8) but with $\mathbb{Z}^d$ replaced by $\mathbb{Z}^{d-1} \times \{0, \ldots, K\}$; $p^{\infty}_{c,d-1}$ is then the limit of $p^{K}_{c,d-1}$ as $K \to \infty$. The threshold $\hat{p}^{\infty}_{c,2}$ is defined similarly as the limit of $\hat{p}^{K}_{c,2}$, the percolation threshold in $(\mathbb{Z}^+)^2 \times \{0, \ldots, K\}^{d-2}$, where $(\mathbb{Z}^+)^2$ is the positive quadrant of $\mathbb{Z}^2$. Our main results are as follows.

THEOREM 1. *For $d > 2$ and $p > p^{\infty}_{c,d-1}$, there exists $m = m(p) > 0$ such that*

$$(1.15) \qquad \tau'_{xy} \leq \exp(-m\|y - x\|) \quad \textit{for all } x, y \textit{ in } \mathbb{Z}^d.$$

Our proof of Theorem 1 leads to an inequality relating the critical behavior of $m(p)$, $P_\infty(p)$, and a quantity characterizing the approach of $p^{K}_{c,d-1}$ to $p^{\infty}_{c,d-1}$. See the remark following the proof of Theorem 5 in Section 2.

As a fairly direct consequence of Theorem 1, we obtain the following result concerning the smoothness of P_∞ and χ'.

THEOREM 2. *For $d > 2$, both P_∞ and χ' are infinitely differentiable functions of p on $(p^{\infty}_{c,d-1}, 1)$.*

We note that infinite differentiability of $P_\infty(p)$ on $(p_c, 1)$ was known previously only for $d = 2$ ([18]), although continuity of $P_\infty(p)$ on $(\hat{p}_{c,2}^\infty, 1)$ for $d > 2$ follows from uniqueness of the infinite cluster ([1]) and the results of [20].

Finally, we obtain upper bounds on P_n.

THEOREM 3. *For $d > 2$ and $p > \hat{p}_{c,2}^\infty$, there exists $c = c(p) > 0$ such that*

$$P_n \le \exp(-[c/(\log n)]\, n^{(d-1)/d}) \quad \text{for } n = 2, 3, \dots. \tag{1.16}$$

The extra ingredients needed beyond Theorems 1 and 3 to complete (together with (1.11)–(1.12)) the characterization of the asymptotic behavior of τ' and P_n above threshold would be a proof that $\pi_c = p_c = p_{c,\,d-1}^\infty = \hat{p}_{c,2}^\infty$ and a replacement of $c/(\log n)$ by a somewhat larger "constant."

The proofs of Theorems 1 and 2 are presented in Section 2 of this paper; that of Theorem 3 in Section 3. Theorem 2 is a consequence of Theorem 1 and conventional percolation theory arguments. On the other hand, our proofs of Theorems 1 and 3 are based on the use of invasion percolation as a tool for analyzing standard Bernoulli percolation. We complete this section with some equivalent definitions of invasion percolation and a proposition which gives two simple but crucial inequalities relating the invasion and standard percolation models. For a more detailed discussion of invasion percolation, see [8] or [6].

Invasion percolation was introduced and studied numerically as a model of transport in a random medium in [16], [5], [21]. There are at least three equivalent definitions of the model ([8]), all of which are useful in obtaining results about invasion percolation itself and in relating it to standard percolation. These definitions are as follows.

(i) *Deterministic invasion.* Denote by $\mathbb{B}_d$ the set of nearest neighbor bonds on $\mathbb{Z}^d$. Let $\{W_b\colon b \in \mathbb{B}_d\}$ be i.i.d. random variables uniformly distributed on $[0,1]$. The invasion percolation process is specified by a sequence $\varnothing = C_0 \subset C_1 \subset C_2 \cdots$ of random subsets of $\mathbb{B}_d$, defined as follows. C_1 is that bond b touching the origin with the minimum value of W_b. For $n \ge 1$, C_{n+1} is the union of C_n with the single additional bond b in ∂C_n with the minimum value of W_b, where

$$\partial C_n \equiv \{b\colon b \text{ shares an endpoint with some } b' \text{ in } C_n\} \setminus C_n. \tag{1.17}$$

Denote the sequence of bonds absorbed into the growing invaded region by b_n, so that $C_n = \{b_1, \dots, b_n\}$ for each n. Denote the sequence of W values of these absorbed bonds by

$$X_n = W_{b_n}. \tag{1.18}$$

(ii) *Dynamic growth.* This is an algorithmic procedure in which bond values are assigned to bonds in $\mathbb{B}_d$ as needed while the invaded region is growing. Choose some deterministic ordering on $\mathbb{B}_d$. Let $Y_1, Y_2, \dots$ be i.i.d. and uniformly distributed on $[0,1]$. For each of the $2d$ bonds touching the origin, assign one of $Y_1, \dots, Y_{2d}$ according to the chosen bond order. The Y_i assigned to the bond b is then defined to be W_b. Define C_1 as above. Note that there are now $2d - 1$

bonds in ∂C_1 which do not touch the origin and therefore have not been assigned values; to each of these bonds b, assign in order one of $Y_{2d+1}, \dots, Y_{4d-1}$ and define that to be W_b. For $n \geq 2$ define C_n as above and then define W_b for each $b \in \partial C_n \setminus \partial C_{n-1}$ by assigning in order one of the next unassigned Y_i's. Define b_n and X_n as above.

(iii) *Percolation cluster method.* This is an intermediate scheme which is dynamic in the sense of method (ii) but provides partial advance global information in the sense of method (i). Let y be in $(0,1)$ and choose a random configuration ω of occupied bonds from the standard Bernoulli bond percolation model with density y. Then, as in method (ii), values are assigned dynamically to bonds as they join the boundary of the invaded region. Here, however, if b is occupied (in ω), its value is assigned uniformly from $[0, y]$, while unoccupied bonds are assigned values uniformly from $[y, 1]$.

Within any of the above formulations, we define for $0 \neq x \in \mathbb{Z}^d$, the time T_x at which x is invaded as

$$(1.19) \qquad T_x = \begin{cases} \min\{n: C_n \text{ touches } x\}, & \text{if } C_\infty \equiv \bigcup_{n=1}^{\infty} C_n \text{ touches } x, \\ +\infty, & \text{otherwise.} \end{cases}$$

Here, C touches x means that x is an endpoint of some $b \in C$. Thus $T_x < \infty$ if and only if x is eventually invaded. T_0 is defined to be 0.

The following proposition states two simple inequalities which explain why an analysis of invasion percolation should be useful in obtaining Theorems 1 and 3. In the proposition, we specifically display the p dependence of τ' and P_n.

PROPOSITION 4. *For $d \geq 1$ and any p in* $(0,1)$,

$$(1.20) \quad \tau'_{0x}(p) \leq P(T_x < \infty \text{ and } X_n > p \text{ for some } n > T_x) \quad \text{for all } x \text{ in } \mathbb{Z}^d,$$

$$(1.21) \qquad P_n(p) \leq B_{n+1}(p) \equiv P(X_{n+1} > p) \quad \text{for } n = 0,1,2,\dots .$$

PROOF. These inequalities are essentially immediate in the context of the percolation cluster method (method (iii) above) with $y = p$. Indeed, to prove (1.20) it suffices to show that if the invasion process is constructed on any configuration ω which contributes to $\tau'_{0x}(p)$, then $T_x < \infty$ and $X_n > p$ for some $n > T_x$. To this end, suppose that, in the configuration ω, $C(0)$ is finite and contains x. Then the invasion process on ω will have $\{b_1, \dots, b_{|C(0)|}\}$ coinciding with the bonds of $C(0)$, so that $T_x \leq |C(0)| < \infty$. Furthermore, $b_{|C(0)|+1}$ must be a vacant bond of ω and hence $X_{|C(0)|+1} > p$. Inequality (1.21) follows in the same fashion by considering Bernoulli configurations at density p in which $|C(0)| = n$. □

2. Decay of the truncated connectivity.

PROOF OF THEOREM 1. Let $z \in \mathbb{Z}^d$ and assume, without loss of generality, that $z_d \geq |z_1|, \ldots, |z_{d-1}|$. Defining T_N to be the time at which the invaded region first reaches the hyperplane $x_d = N$, i.e.,

$$T_N = \min\{T_x \colon x_d = N\}, \tag{2.1}$$

we may easily bound the right-hand side of (1.20) by

$$\tau'_{0z}(p) \leq P\big(T_{z_d} < \infty \text{ and } X_n > p \text{ for some } n > T_{z_d}\big). \tag{2.2}$$

Since

$$z_d = \max_i |z_i| \geq \|z\|/d, \tag{2.3}$$

Theorem 1 is an immediate consequence of the following:

THEOREM 5. *Suppose $d > 2$ and $p > p^K_{c,d-1}$ for some $K \geq 0$. Then for $N \geq 0$,*

$$P(T_N < \infty \text{ and } X_n > p \text{ for some } n > T_N) < (1 - \rho(K,p))^{N/(K+1)}, \tag{2.4}$$

where $\rho(K,p) > 0$ is the probability that $|C(0)| = \infty$ (i.e., P_∞) for standard Bernoulli bond percolation on $\mathbb{Z}^{d-1} \times \{0, \ldots, K\}$.

PROOF. This argument uses a combination of the dynamic growth method and a variation of the percolation cluster method fitted to percolation in slabs. Let L_j, for $j \in \mathbb{Z}$, denote the slab which is a translate by $(j-1)(K+1)$ of $\mathbb{Z}^{d-1} \times \{0, \ldots, K\}$,

$$L_j = \mathbb{Z}^{d-1} \times \{(j-1)(K+1), (j-1)(K+1)+1, \ldots, j(K+1)-1\}. \tag{2.5}$$

Let us denote by ψ_j the time at which L_j is first invaded and by S_j the site at which it is first invaded; $\psi_j = \inf\{T_x \colon x \in L_j\}$ and S_j is the y such that $\psi_j = T_y$ (defined only if $\psi_j < \infty$). Invasion percolation may be constructed by using the percolation cluster method (with $y = p$) *within* each slab L_j by choosing a configuration ω_j (for Bernoulli percolation on L_j) *at the time* ψ_j when it is first needed. The bond variables for bonds between L_j and L_{j+1} may be assigned according to the dynamic growth method, as they are needed. Furthermore, ω_j may be chosen by first choosing a configuration ω'_j from a Bernoulli percolation model on $\mathbb{Z}^{d-1} \times \{0, \ldots, K\}$ which is independent of all previously chosen ω_j's and all previously assigned bond variables, and then translating in $\mathbb{Z}^d$ so that the origin in $\mathbb{Z}^{d-1} \times \{0, \ldots, K\}$ is shifted to S_j.

Define $D_j(p)$ as the event that in ω_j, the cluster $C(S_j)$ is infinite. If $T_N < \infty$ for some $N \geq 0$, then $\psi_{N'} < \infty$ for every integer $N' \geq 0$ such that $(N'-1) \times (K+1) \leq N$. The largest such integer $\overline{N}$ satisfies

$$\overline{N} > N/(K+1). \tag{2.6}$$

If $\psi_{\overline{N}} < \infty$ and $D_j(p)$ occurs for some $j = 1, \ldots, \overline{N}$ (say j'), then clearly $X_n \leq p$

for every $n \geq \psi_{j'}$ and hence

$$P(T_N < \infty \text{ and } X_n > p \text{ for some } n > T_N) \tag{2.7}$$
$$\leq P\left(\psi_{\bar{N}} < \infty \text{ and } \bigcap_{j=1}^{\bar{N}} (D_j(p))^c\right).$$

By the above construction, conditional on $\psi_{\bar{N}} < \infty$, $D_{\bar{N}}(p)$ occurs with probability $\rho(K, p)$ independently of the occurrence of $D_1(p), \ldots, D_{\bar{N}-1}(p)$. Thus

$$P\left(\psi_{\bar{N}} < \infty \text{ and } \bigcap_{j=1}^{\bar{N}} (D_j(p))^c\right)$$
$$= (1 - \rho(K, p))P\left(\psi_{\bar{N}} < \infty \text{ and } \bigcap_{j=1}^{\bar{N}-1} (D_j(p))^c\right) \tag{2.8}$$
$$\leq (1 - \rho(K, p))P\left(\psi_{\bar{N}-1} < \infty \text{ and } \bigcap_{j=1}^{\bar{N}-1} (D_j(p))^c\right).$$

Since

$$P(\psi_1 < \infty \text{ and } (D_1(p))^c) = P((D_1(p))^c) = 1 - \rho(K, p), \tag{2.9}$$

(2.4) follows inductively from (2.6), (2.7), and (2.8). This completes the proof of Theorem 5 and hence of Theorem 1. □

REMARK. (2.2), (2.3), and (2.4) combine to give the explicit bound

$$\tau'_{0x} \leq (1 - \rho(K, p))^{\|x\|/(dK+d)} \leq \exp\left(-\frac{\rho(K, p)}{d(K+1)}\|x\|\right) \quad \text{for all } x \text{ in } \mathbb{Z}^d. \tag{2.10}$$

Let us define the inverse correlation length $(\xi(p))^{-1}$ as the supremum of those $m \geq 0$ such that

$$\tau'_{0x} \leq \exp(-m\|x\|) \quad \text{for all sufficiently large } x. \tag{2.11}$$

Then (2.11) implies that

$$\xi(p) \leq \frac{d(K+1)}{\rho(K, p)}. \tag{2.12}$$

Let us next define a length scale $\bar{K}(p)$, associated with the convergence of $p_{c,d-1}^K$ to $p_{c,d-1}^{\infty}$, by

$$\bar{K}(p) = \min\{K: \rho(K, p) \geq (\tfrac{1}{2})\bar{P}_{\infty}(p)\}, \tag{2.13}$$

where $\bar{P}_{\infty}(p)$ is the percolation density in the half-space, $\mathbb{Z}^{d-1} \times \{0, 1, 2, \ldots\}$. Then (2.12) yields

$$\xi(p) \leq \frac{d}{2} \frac{(\bar{K}(p) + 1)}{\bar{P}_{\infty}(p)}. \tag{2.14}$$

The inequality (2.14) is nonvacuous only if the length scale $\bar{K}(p)$ is finite. This can be verified whenever the infinite cluster in the half-space is unique and contains an infinite path within some finite slab, $\mathbb{Z}^{d-1} \times \{0, \dots, K_0\}$, a condition which is known to hold whenever $p > \hat{p}^{\infty}_{d,2}$ ([1]). Indeed, when this is the case, the probability that the origin is connected to an infinite path within the slab $\mathbb{Z}^{d-1} \times \{0, \dots, K\}$ converges, as $K \to \infty$, to $\overline{P}_{\infty}$.

Assuming that $p_c = p^{\infty}_{c,d-1} = \hat{p}^{\infty}_{c,2}$ and that as p approaches p_c from above, ξ, $\overline{P}_{\infty}$, and $\overline{K}$ exhibit power law behavior in some appropriate sense,

$$\xi(p) \sim (p - p_c)^{-\nu'}, \qquad \overline{P}_{\infty}(p) \sim (p - p_c)^{\overline{\beta}}, \quad \text{and} \qquad \overline{K}(p) \sim (p - p_c)^{-\kappa}, \tag{2.15}$$

one may obtain from (2.14) the critical exponent inequality

$$\nu' \le \overline{\beta} + \kappa. \tag{2.16}$$

Proof of Theorem 2. We first obtain an estimate on the tail of the finite cluster distribution as a corollary of Theorem 1. There is a positive constant h, depending only on d, such that $|C(0)| = n$ implies that some x in $\mathbb{Z}^d$ with $\|x\| \ge hn^{1/d}$ belongs to $C(0)$. Thus, by elementary arguments and (1.15),

$$\begin{aligned} P_n &\le P\big(|C(0)| < \infty \text{ and } x \in C(0) \text{ for some } x \text{ with } \|x\| \ge hn^{1/d}\big) \\ &\le \sum_{\|x\| \ge hn^{1/d}} \tau'_{0x} \le \sum_{\|x\| \ge hn^{1/d}} \exp(-m\|x\|) \\ &\le \exp\big(-m_1 n^{1/d}\big) \quad \text{for } n = 0, 1, 2, \dots, \end{aligned} \tag{2.17}$$

where $m_1(p) > 0$ whenever $m(p) > 0$. As an immediate consequence of (2.17) one has that all moments of the finite cluster size distribution are finite:

$$E\Big(|C(0)|^k \mathbb{1}_{|C(0)| < \infty}\Big) = \sum_{n=0}^{\infty} n^k P_n < \infty \quad \text{for } k = 1, 2, \dots .$$

Furthermore, it follows from (2.10), (2.17) and the monotonicity in p of $\rho(K, p)$ that $\Sigma n^k P_n$ is uniformly convergent away from $p^{\infty}_{c,d-1}$; i.e., that for any $\varepsilon > 0$,

$$\lim_{N \to \infty} \sup_{p \ge p^{\infty}_{c,d-1} + \varepsilon} \left[\sum_{n=N}^{\infty} n^k P_n(p) \right] = 0, \quad \text{for } k = 1, 2, \dots . \tag{2.18}$$

The proof will be completed by using standard arguments ([18]) to show that (2.18) implies infinite differentiability of P_{∞} and χ, in fact of all moments of the finite cluster size distribution. Since

$$P_{\infty}(p) = 1 - \sum_{n=0}^{\infty} P_n(p) \quad \text{and} \quad \chi(p) = \sum_{n=0}^{\infty} nP_n(p), \tag{2.19}$$

it suffices by (2.18) to show that $P_n(p)$ is infinitely differentiable and that for each k, there is some H_k and k' so that

$$\left| \left(\frac{d}{dp} \right)^k P_n(p) \right| \le H_k n^{k'} P_n(p) \quad \text{for } n = 0, 1, \dots, \tag{2.20}$$

where k' is independent of p, and H_k is uniformly bounded on compact subsets of (0, 1). Now if A denotes a (bond) lattice animal, i.e., a possible configuration of a finite $C(0)$, $|A|$ denotes the number of bonds in A and $|\partial A|$ denotes the number of boundary bonds of A, then

$$P_n(p) = \sum_{A:|A|=n} p^{|A|}(1-p)^{|\partial A|}, \tag{2.21}$$

and hence P_n is smooth. It is straightforward to show that

$$\left|\left(\frac{d}{dp}\right)^k p^m(1-p)^n\right| \le H_k'(p)\,[\max(m,n)]^k p^m(1-p)^n, \tag{2.22}$$

where H_k' is a polynomial in p^{-1} and $(1-p)^{-1}$ independent of m and n. Since for any $|A|$, $|\partial A| \le 2d|A|$, it follows that (2.20) is valid with $H_k = (2d)^k H_k'$ and $k' = k$. The proof is now complete. □

3. Upper bounds on the finite cluster size distribution. Note that as a consequence of the truncated connectivity bound (1.15) of Theorem 1, we have already obtained in (2.17) an upper bound on the cluster size distribution. This upper bound, in view of the lower bound (1.12) and the high density upper bound (1.14), has the wrong power of n (for $d > 2$). Our object in this section is to obtain the upper bound (1.16) of Theorem 3 which has the correct power of n [but also an unfortunate logarithmic factor which, as explained in Section 1, is known not to be needed for $d = 2$ ([14]) or for p close to 1 ([15], [8])].

The proof of Theorem 3 uses an extension of the invasion percolation technique introduced above in the proof of Theorem 1 together with renormalization methods such as those of [1] and [7]. It also exploits an elegant inequality of Loomis and Whitney [17] concerning the volumes of $(d-1)$-dimensional projections of subsets of $\mathbb{Z}^d$. Since this inequality is perfectly suited for use in percolation and other models of aggregation, we state it as a separate theorem in order to bring it to the attention of other researchers in the field. We omit the proof which involves nothing more complicated than Hölder's inequality and applies equally well to subsets of $\mathbb{R}^d$.

THEOREM 6 ([17]). *Suppose* $\Lambda \subset \mathbb{Z}^d$, $d \ge 2$. *Denote by* Λ_i *for* $i = 1, \dots, d$, *the* $(d-1)$-*dimensional projection of* Λ *perpendicular to coordinate* i:

$$\Lambda_i = \left\{(x_1,\dots,x_{d-1}) \in \mathbb{Z}^{d-1}: (x_1,\dots,x_{i-1},x_0,x_i,\dots,x_{d-1}) \in \Lambda \text{ for some } x_0 \in \mathbb{Z}\right\}. \tag{3.1}$$

Denote by $|\Lambda'|$ *the number of sites in* Λ'. *Then*

$$\prod_{i=1}^{d} |\Lambda_i| \ge |\Lambda|^{d-1}. \tag{3.2}$$

We remark that the proof of Theorem 6 shows that equality holds between the two sides of (3.2) (for finite $|\Lambda|$) if and only if Λ is a generalized rectangle, i.e., if Λ is the product of its d one-dimensional projections. We further note that (3.2) implies that

$$\max_i |\Lambda_i| \geq |\Lambda|^{(d-1)/d}, \tag{3.3}$$

which is the only consequence of Theorem 6 that we will use below.

PROOF OF THEOREM 3. By (1.21), it suffices to bound $P(X_{n+1} > p)$ by the right-hand side of (1.16) for $p > \hat{p}_{c,2}^{\infty}$. By adjusting the constant c, it clearly suffices to obtain the bound only for large n. At time n, the invaded region C_n contains exactly n bonds and hence touches at least n/d sites in $\mathbb{Z}^d$. Let $C_{n,i}$ denote the $(d-1)$-dimensional projection perpendicular to coordinate i of the set of sites touched by C_n. By (3.3), at least one of these projections contains $(n/d)^{(d-1)/d}$ or more sites. Thus

$$\begin{aligned} P(X_{n+1} > p) &= P\left(X_{n+1} > p \text{ and } \bigcup_{i=1}^{d} \left\{|C_{n,i}| \geq (n/d)^{(d-1)/d}\right\}\right) \\ &\leq dP\left(X_{n+1} > p \text{ and } |C_{n,d}| \geq (n/d)^{(d-1)/d}\right). \end{aligned} \tag{3.4}$$

The remainder of the proof parallels that of Theorem 5 but with two major differences. First we replace the slabs, L_j $(j \in \mathbb{Z})$ of size $(\infty)^{d-1} \times K$ used previously, by pillars, U_j $(j \in \mathbb{Z}^{d-1})$ of size $(K_n)^{d-1} \times \infty$:

$$U_j = U_{n,j} = \left(\{0,\ldots,K_n\}^{d-1} \times \mathbb{Z}\right) + j(K_n + 1,\ldots,K_n + 1, 0). \tag{3.5}$$

Second, the K_n's, unlike the previous K, will be chosen to increase with n. This will be done in such a way as to insure that the following quantity is bounded away from zero as $n \to \infty$:

$$\rho_n(p) \equiv \inf_{x \in U_{n,0}} P(\text{the cluster } C(x) \text{ in standard Bernoulli bond percolation on } U_{n,0} \text{ contains more than } n \text{ bonds}). \tag{3.6}$$

As in the proof of Theorem 5, it is most convenient to consider the invasion percolation process as constructed by using the percolation cluster method within each pillar, with the Bernoulli configuration assigned at the time the pillar is first invaded, and using the dynamic growth method for the bonds not within a pillar. For each n, we define θ_j $(= \theta_{n,j})$, for $j = 1, 2, \ldots,$ as the time that the jth pillar is invaded (the slabs are ordered (randomly) in the order of invasion) and σ_j $(= \sigma_{n,j})$ as the "beachhead" point (on the surface of that jth pillar) which is invaded at time θ_j. Thus $\theta_1 = 0$, $\sigma_1 = (0,\ldots,0)$, and the event $\theta_j \leq n$ is identical to the event that C_n touches at least j distinct pillars. If fewer than j pillars are eventually invaded, then $\theta_j = \infty$ and σ_j is undefined.

Denote by $C(\sigma_j)$ the Bernoulli cluster of σ_j within the jth pillar. A (lengthy) moment's thought yields the following inequality, analogous to (2.7):

$$P(X_{n+1} > p \text{ and } \theta_M \leq n) \leq P\left(\theta_M \leq n \text{ and } \bigcap_{j=1}^{M} \left(|C(\sigma_j)| \leq n\right)\right). \tag{3.7}$$

We may use (3.4) together with (3.7) to bound $P(X_{n+1} > p)$, since

$|C_{n,d}| \geq (n/d)^{(d-1)}$ implies that C_n touches at least $(n/d)^{(d-1)/d}/(K_n+1)^{d-1}$ pillars. Let us define

$$M_n = \left[\text{greatest integer } \leq (n/d)^{(d-1)/d}/(K_n+1)^{d-1}\right]. \tag{3.8}$$

Then

$$\begin{aligned} P(X_{n+1} > p) &\leq dP\left(\theta_{M_n} \leq n \text{ and } \bigcap_{j=1}^{M_n} \left(|C(\sigma_j)| \leq n\right)\right) \\ &\leq dP\left(\theta_{M_n} < \infty \text{ and } \bigcap_{j=1}^{M_n} \left(|C(\sigma_j)| \leq n\right)\right). \end{aligned} \tag{3.9}$$

The conditional probability that $|C(\sigma_M)| \leq n$ is *not* independent of the history of the percolation process up to time θ_M (as was the analogous slab probability which led to the first equality of (2.8)) since it depends on the position of the beachhead point σ_M, but this conditional probability *is* bounded above by $1 - \rho_n(p)$. Thus,

$$\begin{aligned} &P\left(\theta_M < \infty \text{ and } \bigcap_{j=1}^{M} \left(|C(\sigma_j)| \leq n\right)\right) \\ &\quad\leq (1-\rho_n(p))P\left(\theta_M < \infty \text{ and } \bigcap_{j=1}^{M-1} \left(|C(\sigma_j)| \leq n\right)\right) \\ &\quad\leq (1-\rho_n(p))P\left(\theta_{M-1} < \infty \text{ and } \bigcap_{j=1}^{M-1} \left(|C(\sigma_j)| \leq n\right)\right) \cdots \\ &\quad\leq (1-\rho_n(p))^M. \end{aligned} \tag{3.10}$$

Hence by (1.21) and (3.9), we have the inequality

$$P_n(p) \leq B_{n+1}(p) \equiv P(X_{n+1} > p) \leq d(1-\rho_n(p))^{M_n}, \tag{3.11}$$

where $\rho_n(p)$ is defined in (3.5)–(3.6) and M_n is defined in (3.8). If K_n is chosen so that for some $\overline{K} < \infty$,

$$K_n \leq \overline{K}(\log n)^{1/(d-1)} \quad \text{for large } n, \tag{3.12}$$

then M_n will be eventually larger than $c'n^{(d-1)/d}/(\log n)$ for some $c' > 0$ and consequently (3.11) will imply (1.16) for some $c > 0$, providing that

$$\liminf_{n\to\infty} \rho_n(p) > 0 \tag{3.13}$$

for such a choice of K_n. The following theorem implies that K_n can be chosen so that (3.12) and (3.13) will be simultaneously satisfied, providing $p > \hat{p}_{c,2}^{\infty}$, and hence completes the proof of Theorem 3. □

THEOREM 7. *Define*

$$U_n(K) = \{0,\dots,K\}^{d-1} \times \{0,\dots,n+1\} \tag{3.14}$$

and $\rho_{n,p}(K,x)$ *to be the probability, in standard Bernoulli bond percolation on*

$U_n(K)$ with bond density p, that $|C(x)| > n$. If $d \geq 3$ and $p > \hat{p}_{c,2}^{\infty}$, then there exists a sequence of positive integers, K_n, such that

$$\limsup_{n\to\infty} K_n/(\log n)^{1/(d-1)} < \infty \tag{3.15}$$

and

$$\inf_{n,x}\left\{\rho_{n,p}(K_n, x) \colon n \geq 1, x \in \partial_d U_n(K_n)\right\} > 0, \tag{3.16}$$

where $\partial_d U_n(K_n)$ denotes the $x_d = 0$ portion of the boundary of $U_n(K_n)$ perpendicular to the x_d axis.

PROOF. If $p > \hat{p}_{c,2}^{\infty}$, then there is some $N < \infty$ such that $p > \hat{p}_{c,2}^{N}$, where $\hat{p}_{c,2}^{N}$ is the percolation threshold in a quadrant layer $\hat{L}_N = (\mathbb{Z}^+)^2 \times \{0, \dots, N\}^{d-2}$. Define $\Delta = \frac{1}{2}(p - \hat{p}_{c,2}^{N})$ so that, at bond density $p - \Delta$ there is also percolation in the quadrant layer $\hat{L}_N$.

First we will show that there exist constants $c_1 < \infty$ and $c_2 > 0$ (independent of n) such that if $K_n \geq c_1(\log n)^{1/(d-1)}$, then for $n \geq 1$,

$$\max_{x}\left\{\rho_{n,p-\Delta}(K_n, x) \colon x \in \partial_d U_n(K_n)\right\} \geq c_2. \tag{3.17}$$

In other words, we will show that, at density $p - \Delta$, *some* point x in $\partial_d U_n(K_n)$ has $|C(x)| \geq n$ with uniformly positive probability (in contrast to (3.16) which says that, at density p, *every* point x in $\partial_d U_n(K_n)$ has $|C(x)| \geq n$ with uniformly positive probability). By adjusting c_2, it clearly suffices to prove (3.17) for n large.

To establish (3.17), we will rely on the construction of [1] which shows that whenever the bond density exceeds $\hat{p}_{c,2}^{N}$, it is possible to construct renormalized "block" bond events $B_{(i,j)}^{N,L,J}$ between nearest neighbor "sites" i, j of a rescaled lattice $V^{N,L,J}$. Here $V^{N,L,J}$ is a rotation of $\hat{L}_N$ through 45° parallel to the x_1, x_2 plane, with squares of side J as "sites" and a nearest neighbor Euclidean distance $|i - j| = \sqrt{2}L$. (See [1] for precise definitions.) The relevant properties of the renormalized bond events are as follows: (1) The density of block bonds may be made arbitrarily large in the sense that $P(B_{(i,j)}^{N,L,J}) \to 1$ as L and J tend to infinity in an appropriate manner. (2) The bond events are transitive in the sense that if both $B_{(i,j)}^{N,L,J}$ and $B_{(j,k)}^{N,L,J}$ occur, then the sites i and k are connected by a path of occupied bonds in $\hat{L}_N$. (3) Although the bond events are not entirely independent, the only dependence is due to bond events that share a vertex. The relevance of this construction for our purposes is that whenever the bond density exceeds $\hat{p}_{c,2}^{N}$, it is possible to estimate the probability of certain events by choosing the scales L and J of the lattice $V^{N,L,J}$ so that $P(B_{(i,j)}^{N,L,J})$ is in a Peierls regime and doing estimates on (say) $2d$ independent sublattices.

Let $\delta > 0$ and consider the system at density $p - \Delta$. Choose L and J so that

$$1 - P\left(B_{(i,j)}^{N,L,J}\right) < e^{-2d(\lambda_d+\delta)}, \tag{3.18}$$

where λ_d is the Peierls constant for the numbers of contours composed of

$(d-1)$ cells passing through a given $(d-2)$ cell. Take n much larger than L. Let $\mathscr{C}_x$ denote the event of a crossing of the box $U_n(K_n)$ by occupied bonds from the point $x \in \partial_d U_n(K_n)$ to some point on the opposite (i.e., $x_d = n+1$) boundary, so that $\cap_x \mathscr{C}_x^c$ is the event that there is no left–right crossing of $U_n(K_n)$. A straightforward counting argument shows that

$$P\Big(\bigcap_x \mathscr{C}_x^c\Big) \le (\text{const})\big[2(d-1)nK_n^{d-2}\big]e^{-\delta K_n^{d-1}}. \tag{3.19}$$

(See [7], Theorem 4.1b and use the additional factor of $2d$ in the exponent in (3.18) to guarantee an independent sublattice estimate.) Thus if

$$n \le e^{\delta K_n^{d-1}/2}, \tag{3.20}$$

then $P(\cap_x \mathscr{C}_x^c) \le e^{-\delta' K_n^{d-1}}$ for some $\delta' > 0$. However, by the Harris–FKG inequality ([11], [9]), $P(\cap_x \mathscr{C}_x^c) \ge \prod_x P(\mathscr{C}_x^c)$, which implies that

$$\max_x P_{p-\Delta}(\mathscr{C}_x) \ge 1 - e^{-\delta'} > 0. \tag{3.21}$$

Since the event $\mathscr{C}_x$ insures that $|C(x)| \ge n$, this proves the claim (3.17).

Next, we use (3.17) to obtain a lower bound on $\min_x P_p(\mathscr{C}_x)$. Denote by $\bar{x}_\Delta$ the maximizing point of (3.17) (or (3.21)) and consider any $x \in \partial_d U_n(K_n)$. Let $G_{x,\bar{x}_\Delta}$ denote the event that x and $\bar{x}_\Delta$ are connected by a path of occupied bonds within $U_n(K_n)$. Clearly, if the event $G_{x,\bar{x}_\Delta} \cap \mathscr{C}_{\bar{x}_\Delta}$ occurs, then $\mathscr{C}_x$ occurs; hence, it suffices to bound $P_p(G_{x,\bar{x}_\Delta} \cap \mathscr{C}_{\bar{x}_\Delta})$ below. In order to do this, we will bound the probability that $G_{x,\bar{x}_\Delta} \cap \mathscr{C}_{\bar{x}_\Delta}$ "almost occurs" at density $p - \Delta$, and then use a lemma of [1] which allows us to relate this to the probability that $G_{x,\bar{x}_\Delta} \cap \mathscr{C}_{\bar{x}_\Delta}$ actually occurs at density p. The lemma is as follows:

Let A be an event which is nondecreasing in the sense of FKG, and let $E^M(A) \supset A$ denote that set of configurations in which the event A occurs if the configuration is altered on no more than M bonds. Then

$$P_p(A) \ge \varepsilon^M P_{p-\varepsilon}\big(E^M(A)\big). \tag{3.22}$$

We will also use the easily verified fact that for any nondecreasing events, A and B,

$$E^{M_1+M_2}(A \cap B) \subset E^{M_1}(A) \cap E^{M_2}(B). \tag{3.23}$$

We have

$$\begin{aligned} P_p(\mathscr{C}_x) &\ge P_p\big(G_{x,\bar{x}_\Delta} \cap \mathscr{C}_{\bar{x}_\Delta}\big) \\ &\ge \Delta^{2(d-2)^2N} P_{p-\Delta}\Big(E^{2(d-2)^2N}\big(G_{x,\bar{x}_\Delta} \cap \mathscr{C}_{\bar{x}_\Delta}\big)\Big) \\ &\ge \Delta^{2(d-2)^2N} P_{p-\Delta}\Big(E^{2(d-2)^2N}\big(G_{x,\bar{x}_\Delta}\big)\Big)\max_x P_{p-\Delta}(\mathscr{C}_x). \end{aligned} \tag{3.24}$$

Hence it suffices to bound

$$P_{p-\Delta}\Big(E^{2(d-2)^2N}\big(G_{x,\bar{x}_\Delta}\big)\Big).$$

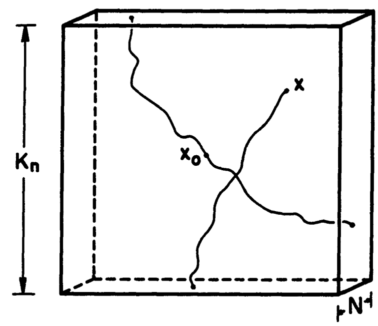

FIG. 1. *Connection of x to x_0 within $T_N(K_n)$.*

To do this, we first note that if each of the points x and $\bar{x}_\Delta$ is connected to the center x_0 of $\partial_d U_n(K_n)$ within $U_n(K_n)$, then $G_{x,\bar{x}_\Delta}$ occurs. Thus

$$(3.25)\qquad \begin{aligned} P_{p-\Delta}\Big(E^{2(d-2)^2N}\big(G_{x,\bar{x}_\Delta}\big)\Big) &\geq P_{p-\Delta}\Big(E^{2(d-2)^2N}\big(G_{x,x_0}\cap G_{x_0,\bar{x}_\Delta}\big)\Big)\\ &\geq \Big[\min_x P_{p-\Delta}\Big(E^{(d-2)^2N}\big(G_{x,x_0}\big)\Big)\Big]^2, \end{aligned}$$

where $x \in \partial_d U_n(K_n)$. Thus we must show that, with uniformly positive probability, any point in $\partial_d U_n(K_n)$ can be connected to x_0 within $U_n(K_n)$ (by altering at most $(d-2)^2N$ bonds). In fact, we will show that this occurs in the "leftmost" N layer (along the x_d axis) of $U_n(K_n)$:

$$(3.26)\qquad T_n(K_n) = \{0,\dots,K_n\}^{d-1} \times \{0,\dots,N\}.$$

For simplicity, let us first consider $d = 3$, for which $T_N(K_n)$ is simply a $K_n \times K_n \times N$ rectangle, the $x_d = 0$ plane of which is the square boundary $\partial_d U_n(K_n)$ of size $K_n \times K_n$. Take $x \in \partial_d U_n(K_n)$ and assume, without loss of generality, that x lies in the "upper right" quadrant of $U_n(K_n)$ (see Figure 1).

For the remainder of this argument, it is convenient to consider $T_N(K_n)$ as being embedded in the N layer $L_N = \mathbb{Z}^2 \times \{0,\dots,N\}^{d-2}$. Let D^+ (D^-) denote the event that x_0 is connected to infinity within the upper left (lower right) quadrant of the N layer centered at x_0. Let D_x denote the event that x is connected to infinity within the lower left quadrant of an N layer with x_0 translated to x. If $D^+ \cap D^- \cap D_x$ occurs, the two-dimensional projection of the path from x must intersect the projection of one of the paths from x_0, so that the actual paths must be within a distance N of each other. Thus, by the Harris–FKG inequality

$$(3.27)\qquad P_{p-\Delta}\big(E^N(G_{x,x_0})\big) \geq P_{p-\Delta}(D^+ \cap D^- \cap D_x) \geq \big[\hat{P}_\infty^{N,3}(p-\Delta)\big]^3 \quad \text{in } d = 3,$$

where $\hat{P}_\infty^{N,d}(p-\Delta)$ is the percolation probability for the quadrant N layer $\hat{L}_N$ in dimension d.

The construction is somewhat more intricate for $d > 3$. Now we can regard $T_N(K_n)$ as consisting of $N+1$ $(d-1)$-hypercubes, each of size K_n^{d-1}, the "leftmost" of which (along the x_d axis) is the boundary $\partial_d U_n(K_n)$. The object is to show that any point within $\partial_d U_n(K_n)$ misses being connected to the center point x_0 by at most $(d-2)^2 N$ bonds. Thus consider the point $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{d-1}, 0) \in \partial_d U_n(K_n)$. Let us take a "slice" of $T_N(K_n)$ of size $K_n^2 \times N^{d-2}$, which contains the point $\tilde{x}$,

$$S_N^{1,2}(K_n) = \{x: 0 \le x_1; x_2 \le K_n; \tilde{x}_3 - N \le x_3 \le \tilde{x}_3; \ldots; \tilde{x}_{d-1} - N \le x_{d-1} \le \tilde{x}_{d-1}; 0 \le x_d \le N\}. \tag{3.28}$$

In this expression, we have assumed, without loss of generality, that $\tilde{x}_k \ge N$ for $k = 3, \ldots, d-1$. We can repeat the previous layer construction (see Figure 1) in an attempt to connect $\tilde{x}$ to the central point of $\partial_d S_N^{1,2}(K_n)$. It again costs no more than $[\hat{P}_\infty^{N,d}(p-\Delta)]^3$ to insure that the two-dimensional projections of the paths intersect. In this case, however, the actual paths must be within a distance $(d-2)N$ (= maximum distance between two points in a cube of size N^{d-2}).

In order to "connect" the central point of $\partial_d S_N^{1,2}(K_n)$ to x_0, we must repeat this argument on $d-3$ additional slices $S_N^{i,j}(K_n)$, orthogonal to the slice $\partial_d S_N^{1,2}(K_n)$, each time "connecting" the central point of the boundary of one slice to that of the next slice. Each such "connection" costs $[\hat{P}_\infty^{N,d}(p-\Delta)]^3$ and is guaranteed to miss by no more than $(d-2)N$ bonds. Thus, at a total cost of no more than $[\hat{P}_\infty^{N,d}(p-\Delta)]^{3(d-2)}$, we can insure a connection between x_0 and $\tilde{x}$ which is missing a maximum of $(d-2)[(d-2)N]$ bonds. We have

$$\min_x P_{p-\Delta}\Big(E^{(d-2)^2 N}\big(G_{x,x_0}\big)\Big) \ge \big[\hat{P}_\infty^{N,d}(p-\Delta)\big]^{3(d-2)}, \tag{3.29}$$

which, along with (3.21), (3.24), and (3.25), provides a uniform lower bound on $\min_x P_p(\mathscr{C}_x)$ and thus completes the theorem. □

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Note added in proof. Although it does not directly affect our results, in this paper we distinguished between the critical points p_c and π_c [cf. (1.7)–(1.8)]. It has now been proved that $p_c = \pi_c$ in all dimensions [see M. V. Menshikov, *Dokl. Akad. Nauk. SSSR* **288** 1308–1311 (1986) (in Russian); M. V. Menshikov, S. A. Molchanov and A. F. Sidorenko, *Itogi Nauki i Tekhniki* (*Series of Probability Theory, Mathematical Statistics, Theoretical Cybernetics*) **24** 53–110 (1986) (in Russian); M. Aizenman and D. J. Barsky, *Comm. Math. Phys.* **108** 489–526 (1987)]. Of more direct relevance is a recent proof that the quantity τ_{0n}' [cf. (1.2)] has a well-defined rate of exponential decay: $-1/\xi'(p) = \lim_{n\to\infty} n^{-1}\log \tau'_{0n}(p)$, for which our $m(p)$ in Theorem 1 provides a nontrivial bound (see J. T. Chayes, L. Chayes, G. R. Grimmett, H. Kesten and R. H. Schonmann, in preparation).

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