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CONDITIONAL BOUNDARY CROSSING PROBABILITIES, WITH APPLICATIONS TO CHANGE-POINT PROBLEMS

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For normal random walks $S_1, S_2, \dots,$ formed from independent identically distributed random variables $X_1, X_2, \dots,$ we determine the asymptotic behavior under regularity conditions of

$$P(S_n > mg(n/m) \text{ for some } n < m | S_m = m\xi_0, U_m = m\lambda_0), \qquad \xi_0 < g(1),$$

where $U_m = X_1^2 + \cdots + X_m^2$. The result is applied to a normal change-point problem to approximate null distributions of test statistics and to obtain approximate confidence sets for the change-point.

1. Introduction. A method of developing approximations for boundary crossing probabilities which has received some attention of late is that of writing the probability as an expectation of a conditional boundary crossing probability given an appropriate random variable, and then developing an approximation for the conditional probability. Such a method has been used with some degree of success, as measured by the accuracy of the approximations, by Siegmund (1982, 1985, 1986), Hu (1985) and James, James and Siegmund (1987).

Let $X_1, X_2, \dots$ be independent identically distributed $N(\mu, \sigma^2)$ random variables, with $S_n = X_1 + \cdots + X_n$ and $U_n = X_1^2 + \cdots + X_n^2$. Given a function $g(t)$, $0 < t \le 1$ and $m \ge 1$, let τ be the possibly defective stopping time

$$\tau = \tau_m = \inf\{n \ge 1 : S_n > mg(n/m)\}.$$

Siegmund (1982) studied the asymptotic behavior of the conditional probabilities

$$P(\tau < m | S_m = m\xi_0), \qquad \xi_0 < g(1),$$

and used the results to approximate the tail probability of the Smirnov statistic and the power function of repeated significance tests for a normal mean when σ^2 is known. In this paper, we extend Siegmund's method to study the asymptotic order of the conditional probabilities

$$P(\tau < m | S_m = m\xi_0, U_m = m\lambda_0), \qquad \xi_0 < g(1), \tag{1.1}$$

and apply the result to some change-point problems.

Our main result is stated and proved in the next section. The proof uses a likelihood ratio argument, but we believe the result could also be obtained using

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the method of Woodroofe [(1982), Chapter 8]. On the other hand, the method of mixtures of likelihood ratios [cf. Lai and Siegmund (1977)] and the method of Siegmund [(1985), Theorem 9.54; see also Hu (1985)], which seem particularly simple in certain related problems, appear to be difficult to adapt to the present situation.

Our motivation for studying the conditional probabilities (1.1) comes from our investigation of the following change-point problem: Let $X_1, \dots, X_m$ be independent random variables with $X_i \sim N(\mu_i, \sigma^2)$, and suppose we wish to test the hypothesis of no change in mean H_0: $\mu_1 = \cdots = \mu_m$ versus the alternative of a single change H_1: $\mu_1 = \cdots = \mu_j \neq \mu_{j+1} = \cdots = \mu_m$ for some $j \in \{1, \dots, m-1\}$. We can then use the theorem of the next section to obtain approximations for the significance levels of several tests of H_0, as well as to obtain likelihood-based confidence sets for the change-point j. These applications are discussed in Section 3.

2. Asymptotic conditional boundary crossing probabilities. Throughout this section, the following assumptions and definitions will hold. $X_1, \dots, X_n$ are independent identically distributed normal random variables, without loss of generality assumed to be $N(0,1)$, with $S_n = X_1 + \cdots + X_n$ and $U_n = X_1^2 + \cdots + X_n^2$, $n = 1, 2, \dots, m$. The real-valued function g, defined on $(0,1]$, has two continuous derivatives. For a fixed $\xi_0 < g(1)$, there exists a unique point $t^* \in (0,1)$ which minimizes the function

$$h(t) = \frac{g(t) - \xi_0 t}{\{t(1-t)\}^{1/2}}$$

and further satisfies $h(t^*) > 0$, $\liminf_{t \to 0} h(t) > h(t^*)$ and $h''(t^*) > 0$. The stopping time $\tau = \tau_m$ is defined by

$$\tau = \inf\{n \le m: S_n \ge mg(n/m)\};$$

we let $\tau = +\infty$ if the defining set is empty. Let λ_0 be such that $\lambda_0 > g^2(t^*)(t^*)^{-1} + \{g(t^*) - \xi_0\}^2(1-t^*)^{-1}$, and define μ and σ^2 by $\mu = g(t^*)/t^*$ and $\sigma^2 = \lambda_0 - g^2(t^*)(t^*)^{-1} - \{g(t^*) - \xi_0\}^2(1-t^*)^{-1}$. Let $\xi = m\xi_0$ and $\lambda = m\lambda_0$. Finally, for any $x \in \mathbb{R}$ and $y > 0$, we let

$$P_{x,y}^{(m)}(A) = P(A | S_m = x, U_m = y)$$

for A belonging to the σ-field generated by $X_1, \dots, X_m$.

It can be seen that $\sigma^2 = \lambda_0 - \xi_0^2 - h^2(t^*)$, which in turn implies that $\lambda_0 > \xi_0^2$ and $\sigma^2/(\lambda_0 - \xi_0^2) < 1$. Note also that the condition $h'(t^*) = 0$ implies $\mu - g'(t^*) = (\mu - \xi_0)/\{2(1-t^*)\}$. Since $h(t^*) > 0$, this implies $\mu - g'(t^*) > 0$. It can also be shown that the conditions on h imply that $1 + 2g''(t^*)t^*(1-t^*)\{\mu - g'(t^*)\}^{-1} > 0$. Thus, the terms that appear in (2.1) in the statement of the theorem are all well defined, with the factor $\sigma^2/(\lambda_0 - \xi_0^2)$ taking on a value between 0 and 1.

THEOREM. *Let ν be the function defined for $t > 0$ by*

$$\nu(t) = 2t^{-2}\exp\left\{-2\sum_{n=1}^{\infty} n^{-1}\Phi(-tn^{1/2}/2)\right\},$$

where Φ is the standard normal distribution function. Then, as $m \to \infty$,

$$P_{\xi,\lambda}^{(m)}(\tau < m) \sim \nu\left\{\frac{2(\mu - g'(t^*))}{\sigma}\right\}\left(\frac{\sigma^2}{\lambda_0 - \xi_0^2}\right)^{(m-3)/2} \tag{2.1}$$
$$\times\left\{1 + \frac{2g''(t^*)t^*(1-t^*)}{\mu - g'(t^*)}\right\}^{-1/2}.$$

REMARK 1. The function ν can be evaluated either directly by numerical computation or approximately, at least in the range $0 < t \le 2$, from the local expansion

$$\nu(t) = \exp(-\rho t) + o(t^2), \qquad t \to 0,$$

where ρ is a numerical constant which is approximately equal to 0.583. See Siegmund [(1985), Chapter 10].

The following two lemmas are technical and will be used in the proof of the theorem.

LEMMA 1. *Assume $a_m \to \infty$ with $a_m = o(m^{1/2})$, and let $b_m = m^{1/2}\log m$ and $I_m = (mt^* - a_m m^{1/2}, mt^* + a_m m^{1/2})$. The following bounds all hold as $m \to \infty$:*

(a) $\max_{1\le n\le m-1} P_{\xi,\lambda}^{(m)}(S_n \ge mg(n/m)) = O\{m^{-1/2}(\sigma^2/(\lambda_0 - \xi_0^2))^{(m-3)/2}\}$.
(b) $\sum_{n\in I_m} P_{\xi,\lambda}^{(m)}(S_n \ge mg(n/m)) = O\{(\sigma^2/(\lambda_0 - \xi_0^2))^{(m-3)/2}\}$.
(c) *For $a > 0$,*

$$P_{\xi,\lambda}^{(m)}\{\tau < m, |\tau - mt^*| \ge am^{1/2}\} \le \delta(a)\{\sigma^2/(\lambda_0 - \xi_0^2)\}^{(m-3)/2},$$

where $\delta(a) \to 0$ as $a \to \infty$. In particular,

$$P_{\xi,\lambda}^{(m)}\left(|\tau/m - t^*| \ge a_m m^{-1/2}, \tau < m\right) = o\left\{\left(\sigma^2/(\lambda_0 - \xi_0^2)\right)^{(m-3)/2}\right\}.$$

(d)
$$P_{\xi,\lambda}^{(m)}\left(|\tau/m - t^*| < a_m m^{-1/2},\right.$$
$$\left.S_\tau - mg(\tau/m) \ge b_m\right) = o\left\{\left(\sigma^2/(\lambda_0 - \xi_0^2)\right)^{(m-3)/2}\right\}.$$

(e) *For each fixed $\varepsilon > 0$, uniformly for n and r such that $|n - mt^*| \le a_m m^{1/2}$ and $0 \le r \le b_m$,*

$$P_{\xi,\lambda}^{(m)}\left(|U_n/m - (\sigma^2 + \mu^2)t^*| > \varepsilon | S_n = mg(n/m) + r\right) = o(1).$$

PROOF. Let φ and f_n denote, respectively, the standard normal and the χ_n^2 probability density functions. Then $P\{S_n \in dx, U_n \in dy\} =$

$n^{-1/2}\varphi(n^{-1/2}x)f_{n-1}(y - x^2/n)\,dx\,dy$ for $|x| < (yn)^{1/2}$ and 0 elsewhere. From this one easily obtains the following conditional densities, which are used repeatedly. For $1 < n < m - 1$ and $|\xi| < (m\lambda)^{1/2}$,

$$(2.2)\quad \begin{aligned} &P^{(m)}_{\xi,\lambda}\{S_n \in dx, U_n \in dy\}/dx\,dy \\ &\quad = \pi^{-1/2}\left(\frac{m}{n(m-n)}\right)^{1/2} \\ &\qquad \times\left[\Gamma\left(\frac{m-1}{2}\right)\left(y - \frac{x^2}{n}\right)^{(n-3)/2}\left(\lambda - y - \frac{(\xi - x)^2}{m-n}\right)^{(m-n-3)/2}\right] \\ &\qquad \div\left[\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{m-n-1}{2}\right)\left(\lambda - \frac{\xi^2}{m}\right)^{(m-3)/2}\right] \end{aligned}$$

for $0 < y < \lambda$, $|x| < (yn)^{1/2}$, $|\xi - x| < [(m-n)(\lambda - y)]^{1/2}$ and 0 elsewhere; also for $n \le m - 1$,

$$(2.3)\quad \begin{aligned} P^{(m)}_{\xi,\lambda}\{S_n \in dx\}/dx &= \pi^{-1/2}\left(\frac{m}{n(m-n)}\right)^{1/2}\frac{\Gamma((m-1)/2)}{\Gamma((m-2)/2)} \\ &\quad \times\left(\lambda - \frac{\xi^2}{m}\right)^{-(m-3)/2}\left(\lambda - \frac{(\xi - x)^2}{m-n} - \frac{x^2}{n}\right)^{(m-4)/2} \end{aligned}$$

for $x^2/n + (\xi - x)^2/(m-n) < \lambda$ and 0 elsewhere.

(a) After integrating (2.3) and changing variables, we obtain

$$(2.4)\quad P^{(m)}_{\xi,\lambda}\left(S_n \ge mg\left(\frac{n}{m}\right)\right) = \frac{\Gamma((m-1)/2)}{\pi^{1/2}\Gamma((m-2)/2)}\int_{B_n}(1 - y^2)^{(m-4)/2}\,dy,$$

where $B_n = \{y\colon |y| \le 1,\ y \ge (\lambda_0 - \xi_0^2)^{-1/2}h(n/m)\}$. Now for $0 < a < 1$ we can show, by a change of variables ($x = y^2$) and appropriate bounding of the integrand, that

$$\int_a^1 (1 - y^2)^{(m-4)/2}\,dy \le \frac{(1 - a^2)^{(m-2)/2}}{a(m-2)}.$$

Stirling's formula for the gamma function implies

$$\Gamma((m-1)/2)/\Gamma((m-2)/2) \sim (m/2)^{1/2}.$$

Thus, it follows from (2.4) that

$$(2.5)\quad P^{(m)}_{\xi,\lambda}\left(S_n \ge mg\left(\frac{n}{m}\right)\right) \le Km^{-1/2}\left(\frac{\lambda_0 - \xi_0^2 - h^2(n/m)}{\lambda_0 - \xi_0^2}\right)^{(m-3)/2}$$

for some $K > 0$ and all $m \ge 3$ and n such that $h^2(n/m) < \lambda_0 - \xi_0^2$ (the bound is 0 otherwise). Part (a) now follows from the relations $\sigma^2 = \lambda_0 - \xi_0^2 - h^2(t^*)$ and $h(n/m) \ge h(t^*)$.

(b) and (c) Note that

$$P^{(m)}_{\xi,\lambda}\left(|\tau/m - t^*| \ge a_m m^{-1/2},\ \tau < m\right) \le \sum_{\substack{n \notin I_m \\ n < m}} P^{(m)}_{\xi,\lambda}(S_n \ge mg(n/m))$$

and that (2.5) implies

$$P_{\xi,\lambda}^{(m)}\left(S_n \geq mg\left(\frac{n}{m}\right)\right) \leq Km^{-1/2}\left(\frac{\sigma^2}{\lambda_0 - \xi_0^2}\right)^{(m-3)/2} \times\left(\frac{\lambda_0 - \xi_0^2 - h^2(n/m)}{\lambda_0 - \xi_0^2 - h^2(t^*)}\right)^{(m-3)/2}. \tag{2.6}$$

Both (b) and (c) will follow by developing bounds for appropriate sums of the last factor of (2.6). By the assumptions on h, this factor will be of exponentially small order in m if n/m lies outside any fixed neighborhood of t^*; thus, for any fixed $\delta > 0$, we may restrict attention to n such that $|n/m - t^*| < \delta$. But Taylor's series expansions on $\log(1 + x)$, to one derivative, and $h(t)$ around t^*, to two derivatives, yield the existence of $K_0 > 0$ and $\delta > 0$ such that

$$\left(\frac{\lambda_0 - \xi_0^2 - h^2(n/m)}{\lambda_0 - \xi_0^2 - h^2(t^*)}\right)^{(m-3)/2} < \exp\left(-mK_0\left(\frac{n}{m} - t^*\right)^2\right)$$

for n such that $|n/m - t^*| < \delta$. Part (b) and the second statement in (c) following by summing these bounds over n in I_m and I_m^c and bounding the sums appropriately by integrals. The first statement in (c) follows by essentially the same argument.

(d) By a process similar to that used to obtain (2.6), we have that

$$\begin{aligned}P_{\xi,\lambda}^{(m)}&(|\tau/m - t^*| \leq a_m m^{-1/2},\ S_\tau - mg(\tau/m) \geq b_m)\\ &\leq \sum_{n\in I_m} P_{\xi,\lambda}^{(m)}(S_n \geq mg(n/m) + b_m)\end{aligned}$$

and for some $K' > 0$, all $m \geq 3$ and all n such that $[h(n/m) + b_m/\{n(m - n)\}^{1/2}]^2 < \lambda_0 - \xi_0^2$,

$$\begin{aligned}P_{\xi,\lambda}^{(m)}&\left(S_n \geq mg\left(\frac{n}{m}\right) + b_m\right)\\ &\leq Km^{-1/2}\left(\frac{\lambda_0 - \xi_0^2 - \left[h(n/m) + b_m/\{n(m-n)\}^{1/2}\right]^2}{\lambda_0 - \xi_0^2}\right)^{(m-3)/2}\\ &\leq K'm^{-1/2}\left(\frac{\sigma^2}{\lambda_0 - \xi_0^2}\right)^{(m-3)/2}\\ &\quad\times\left(\frac{\lambda_0 - \xi_0^2 - h^2(n/m) - b_m^2/\{n(m-n)\}}{\lambda_0 - \xi_0^2 - h^2(t^*)}\right)^{(m-3)/2}\\ &\leq K'm^{-1/2}\left(\frac{\sigma^2}{\lambda_0 - \xi_0^2}\right)^{(m-3)/2}\left(1 - \frac{4b_m^2}{\sigma^2 m^2}\right)^{(m-3)/2},\end{aligned}$$

where the last inequality uses the fact that $h(n/m) \geq h(t^*)$. Part (d) now follows by using the relation $1 - a \leq e^{-a}$ for $0 \leq a \leq 1$.

(e) By Markov's inequality

$$
\begin{aligned}
&P_{\xi,\lambda}^{(m)}\big(|U_n/m - (\sigma^2+\mu^2)t^2| > \varepsilon | S_n = x\big)\\
&\quad \le 1/\varepsilon^2\Big[\operatorname{Var}(U_n/m | S_n = x,\, S_m = \xi,\, U_m = \lambda)\\
&\qquad + \big\{E(U_n/m | S_n = x,\, S_m = \xi,\, U_m = \lambda)\\
&\qquad\qquad - (\sigma^2+\mu^2)t^*\big\}^2\Big].
\end{aligned} \tag{2.7}
$$

Conditionally, U_n is a linear function of a beta-distributed random variable. In fact it can be shown that the random variable

$$V = \frac{U_n - S_n^2/n}{U_m - S_n^2/n - (S_m - S_n)^2/(m-n)},$$

whose numerator is one of two independent chi-squares making up the denominator, has a beta distribution with parameters $(n-1)/2$ and $(m-n-1)/2$ and is independent of the vector (S_n, S_m, U_m). Therefore,

$$E(U_n | S_n = x,\, S_m = \xi,\, U_m = \lambda) = \frac{x^2}{n} + \left(\lambda - \frac{(\xi - x)^2}{m-n} - \frac{x^2}{n}\right)\frac{(n-1)}{(m-2)}$$

and

$$
\begin{aligned}
&\operatorname{Var}(U_n | S_n = x,\, S_m = \xi,\, U_m = \lambda)\\
&\quad = 2\left(\lambda - \frac{(\xi - x)^2}{m-n} - \frac{x^2}{n}\right)^2 \frac{(n-1)(m-n-1)}{m(m-2)^2}.
\end{aligned}
$$

Part (e) now follows from (2.7) and the preceding by algebra. □

LEMMA 2. *For each* $\varepsilon > 0$,

(a) $P_{\xi,\lambda}^{(m)}(|S_\tau/m - \mu t^*| > \varepsilon,\, \tau < m) = o\{(\sigma^2/(\lambda_0 - \xi_0^2))^{(m-3)/2}\}$ *and*
(b) $P_{\xi,\lambda}^{(m)}(|U_\tau/m - (\sigma^2+\mu^2)t^*| > \varepsilon,\, \tau < m) = o\{(\sigma^2/(\lambda_0 - \xi_0^2))^{(m-3)/2}\}$.

PROOF. Let $a_m = \log m$. Applying first Lemma 1(c) and then the triangle inequality, we have

$$
\begin{aligned}
\text{LHS(a)} &= P_{\xi,\lambda}^{(m)}\left(\left|\frac{S_\tau}{m} - \mu t^*\right| > \varepsilon, \left|\frac{\tau}{m} - t^*\right| < a_m m^{-1/2}\right) + o\left\{\left(\frac{\sigma^2}{\lambda_0 - \xi_0^2}\right)^{(m-3)/2}\right\}\\
&\le P_{\xi,\lambda}^{(m)}\left(S_\tau - mg\left(\frac{\tau}{m}\right) > \frac{m\varepsilon}{2}, \left|\frac{\tau}{m} - t^*\right| < a_m m^{-1/2}\right)\\
&\quad + P_{\xi,\lambda}^{(m)}\left(\left|g\left(\frac{\tau}{m}\right) - \mu t^*\right| > \frac{\varepsilon}{2}, \left|\frac{\tau}{m} - t^*\right| < a_m m^{-1/2}\right)\\
&\quad + o\left\{\left(\frac{\sigma^2}{\lambda_0 - \xi_0^2}\right)^{(m-3)/2}\right\}.
\end{aligned}
$$

The second summand on the right-hand side is null for n sufficiently large, since g is continuous and $g(t^*) = \mu t^*$. The first summand can be handled by Lemma 1(d), thus completing the proof of (a).

Apply parts (c) and (d) of Lemma 1 to obtain

$$\text{LHS(b)} = P_{\xi,\lambda}^{(m)}\left(\left|\frac{U_\tau}{m} - (\sigma^2 + \mu^2)t^*\right| > \varepsilon,\ S_\tau - mg\left(\frac{\tau}{m}\right) \le b_m,\right.$$
$$\left.\left|\frac{\tau}{m} - t^*\right| < a_m m^{-1/2}\right) + o\left\{\left(\frac{\sigma}{\lambda_0 - \xi_0^2}\right)^{(m-3)/2}\right\}.$$

Now decompose the preceding event according to the value of τ, letting I_m be the interval $(mt^* - a_m m^{1/2},\ mt^* + a_m m^{1/2})$:

$$\text{LHS(b)} \le \sum_{n \in I_m} P_{\xi,\lambda}^{(m)}\left(\left|\frac{U_n}{m} - (\sigma^2 + \mu^2)t^*\right| > \varepsilon,\ 0 \le S_n - mg\left(\frac{n}{m}\right) \le b_m\right)$$
$$+ o\left\{\left(\frac{\sigma^2}{\lambda_0 - \xi_0^2}\right)^{(m-3)/2}\right\}$$
$$\le \sum_{n \in I_m} P_{\xi,\lambda}^{(m)}\left(S_n \ge mg\left(\frac{n}{m}\right)\right)$$
$$\times P_{\xi,\lambda}^{(m)}\left(\left|\frac{U_n}{m} - (\sigma^2 + \mu^2)t^*\right| > \varepsilon \,\middle|\, 0 \le S_n - mg\left(\frac{n}{m}\right) \le b_m\right)$$
$$+ o\left\{\left(\frac{\sigma^2}{\lambda_0 - \xi_0^2}\right)^{(m-3)/2}\right\}.$$

Part (b) now follows by applying first Lemma 1(e) and then Lemma 1(b). □

REMARK 2. We will show in the course of the proof of the main theorem that $P_{\xi,\lambda}^{(m)}(\tau < m) > K\{\sigma^2/(\lambda_0 - \xi_0^2)\}^{(m-3)/2}$ for some $K > 0$. Then Lemmas 1(c) and 2 will give us convergence of τ/m, S_τ/m and U_τ/m to t^*, μt^* and $(\sigma^2 + \mu^2)t^*$ in conditional $P_{\xi,\lambda}^{(m)}$-probability given $\{\tau < m\}$, i.e., for each $\varepsilon > 0$

$$P_{\xi,\lambda}^{(m)}(|\tau/m - t^*| > \varepsilon | \tau < m) \to 0,$$
$$P_{\xi,\lambda}^{(m)}(|S_\tau/m - \mu t^*| > \varepsilon | \tau < m) \to 0$$

and

$$P_{\xi,\lambda}^{(m)}\left(|U_\tau/m - (\sigma^2 + \mu^2)t^*| > \varepsilon | \tau < m\right) \to 0.$$

REMARK 3. Formula (2.4) shows that the marginal probability $P_{\xi,\lambda}^{(m)}(S_n \ge mg(n/m))$ is maximized by that n which minimizes $h(n/m)$, i.e., by some n not far from mt^*. When m is large, then it would seem reasonable that if the partial sum process were to cross the curve at all, it would do so for n near mt^*. We see from Remark 2 that this holds.

PROOF OF THE THEOREM. Let $P_{x,y,n}^{(m)}$ denote the restriction of $P_{x,y}^{(m)}$ to the σ-field generated by $X_1, \dots, X_n$. Let $\xi_1 = m\mu$ and $\lambda_1 = m(\sigma^2 + \mu^2)$. The idea of the proof is to use a likelihood ratio argument, based on the likelihood ratio of $P_{\xi,\lambda}^{(m)}$ with respect to $P_{\xi_1,\lambda_1}^{(m)}$. The values of ξ_1 and λ_1 are chosen because of the approximately equivalent local behavior of the pre-τ process under $P_{\xi_1,\lambda_1}^{(m)}$ and, conditionally, under $P_{\xi,\lambda}^{(m)}$ given $\{\tau < m\}$. In fact, given $\{\tau < m\}$, Remark 2 tells us that $S_\tau/\tau \to \mu$ and $U_\tau/\tau \to \sigma^2 + \mu^2$ in $P_{\xi,\lambda}^{(m)}$-probability.

Let L_n denote the likelihood ratio of the absolutely continuous part of $P_{\xi,\lambda,n}^{(m)}$ relative to $P_{\xi_1,\lambda_1,n}^{(m)}$. It follows from sufficiency and (2.2) that for $n \le m - 2$,

$$L_n = \left(\frac{\lambda - U_n - (\xi - S_n)^2/(m-n)}{\lambda_1 - U_n - (\xi_1 - S_n)^2/(m-n)}\right)^{(m-n-3)/2} \left(\frac{\lambda_1 - \xi_1^2/m}{\lambda - \xi^2/m}\right)^{(m-3)/2}$$

if $\lambda_1 - U_n - (\xi_1 - S_n)^2/(m-n) > 0$ and $\lambda - U_n - (\xi - S_n)^2/(m-n) > 0$, and $L_n = 0$ if $\lambda - U_n - (\xi - S_n)^2/(m-n) \le 0 < \lambda_1 - U_n - (\xi_1 - S_n)^2/(m-n)$.

Let I_m and b_m be as in Lemma 1, and put $B = \{\tau \in I_m,\ S_\tau - mg(\tau/m) < b_m,\ |m^{-1}S_\tau - \mu t^*| + |m^{-1}U_\tau - (\sigma^2 + \mu^2)t^*| < \varepsilon\}$. By Lemmas 1 and 2 it suffices to show

$$P_{\xi,\lambda}^{(m)}(B) \sim \text{RHS}(2.1).$$

It is easily shown that $\lambda_1 - U_\tau - (\xi_1 - S_\tau)^2/(m-\tau) > 0$ on B and hence by Wald's likelihood ratio identity [see, e.g., Siegmund (1985), page 13]

$$\begin{aligned} P_{\xi,\lambda}^{(m)}(B) &= \int_B L_\tau \, dP_{\xi_1,\lambda_1}^{(m)} \\ &= \left(\frac{\sigma^2}{\lambda_0 - \xi_0^2}\right)^{(m-3)/2} \\ &\quad \times \int_B \left(\frac{\lambda - U_\tau - (\xi - S_\tau)^2/(m-\tau)}{\lambda_1 - U_\tau - (\xi_1 - S_\tau)^2/(m-\tau)}\right)^{(m-\tau-3)/2} dP_{\xi_1,\lambda_1}^{(m)}. \end{aligned} \tag{2.8}$$

Law of large numbers arguments indicate that under $P_{\xi_1,\lambda_1}^{(m)}$, as $m \to \infty$,

$$\frac{\tau}{m} \to_P t^*, \qquad \frac{S_\tau}{m} \to_P \mu t^* \quad \text{and} \quad \frac{U_\tau}{m} \to_P (\sigma^2 + \mu^2)t^*,$$

so that

$$\begin{aligned} m^{-1}\left(\lambda - U_\tau - \frac{(\xi - S_\tau)^2}{m-\tau}\right) &\to_P \sigma^2(1 - t^*), \\ m^{-1}\left(\lambda_1 - U_\tau - \frac{(\xi_1 - S_\tau)^2}{m-\tau}\right) &\to_P \sigma^2(1 - t^*). \end{aligned} \tag{2.9}$$

Since $\log(1 + x) = x + O(x^2)$ as $x \to 0$, we have

$$\frac{\lambda - U_\tau - (\xi - S_\tau)^2/(m-\tau)}{\lambda_1 - U_\tau - (\xi_1 - S_\tau)^2/(m-\tau)}$$

$$(2.10) \qquad = \exp\left[\frac{\lambda - \lambda_1 - \{\xi^2 - \xi_1^2 - 2S_\tau(\xi - \xi_1)\}/(m-\tau)}{\lambda_1 - U_\tau - (\xi_1 - S_\tau)^2/(m-\tau)} + O_p\left\{\left(\lambda_0 - \sigma^2 - \mu^2 - \frac{(\xi_0^2 - \mu^2 - 2(\xi_0 - \mu)S_\tau/m)}{1 - \tau/m}\right)^2\right\}\right].$$

Letting R_m be the excess over the boundary, i.e., $R_m = S_\tau - mg(\tau/m)$, and using a Taylor expansion on g at t^*, we get

$$(2.11) \qquad S_\tau = R_m + m\left\{g(t^*) + g'(t^*)\left(\frac{\tau}{m} - t^*\right) + \frac{g''(t^*)}{2}\left(\frac{\tau}{m} - t^*\right)^2 + \varepsilon\left(\frac{\tau}{m}\right)\left(\frac{\tau}{m} - t^*\right)^2\right\},$$

where $\varepsilon(t) \to 0$ as $t \to t^*$. To obtain the limiting joint distribution of R_m and $m^{1/2}(\tau/m - t^*)$, we must appeal to an appropriate nonlinear renewal theorem for the conditional process governed by $P^{(m)}_{\xi_1, \lambda_1}$. For an intuitive discussion of nonlinear renewal theory which leads one to the correct limiting joint distribution, see Siegmund [(1986), Appendix 2 and Lemma 2.16]. Hu [(1985), Chapter 4, Theorem 10] has proved a general result which provides a rigorous justification. The upshot is that R_m and $m^{1/2}(\tau/m - t^*)$ converge in distribution and are asymptotically independent under $P^{(m)}_{\xi_1, \lambda_1}$; the limiting distributions will be seen later. This, together with some algebra, means that the right-hand side of (2.10) can be written as

$$\exp\left\{\frac{2(\xi_0 - \mu)\left(R_m + mg''(t^*)(\tau/m - t^*)^2/2 + m\varepsilon(\tau/m)(\tau/m - t^*)^2\right)}{(1 - \tau/m)\left(\lambda_1 - U_\tau - (\xi_1 - S_\tau)^2/(m-\tau)\right)} + O_p(m^{-2})\right\}.$$

If we insert this in the integrand, (2.8) becomes

$$\left(\frac{\sigma^2}{\lambda_0 - \xi_0^2}\right)^{(m-3)/2} \times \int_B \exp\left\{\frac{(m - \tau - 3)(\xi_0 - \mu)\left(R_m + mg''(t^*)(\tau/m - t^*)^2/2 + o_p(1)\right)}{(1 - \tau/m)\left(\lambda_1 - U_\tau - (\xi_1 - S_\tau)^2/(m-\tau)\right)} + O_p(m^{-1})\right\} dP^{(m)}_{\xi_1, \lambda_1}.$$

Application of (2.9) then yields

$$(2.8) = \left(\frac{\sigma^2}{\lambda_0 - \xi_0^2}\right)^{(m-3)/2}$$

(2.12)

$$\times \int_B \exp\left\{\frac{(\xi_0 - \mu)\left(R_m + mg''(t^*)(\tau/m - t^*)^2/2\right)}{\sigma^2(1 - t^*)} + o_p(1)\right\} dP^{(m)}_{\xi_1, \lambda_1}.$$

It also follows from the preceding arguments that $P^{(m)}_{\xi_1, \lambda_1}(B) \to 1$, so if we may interchange expectation and limit in (2.12), we will be able to evaluate the order of (2.8) by using Hu's result. This result states that as $m \to \infty$

$$P^{(m)}_{\xi_1, \lambda_1}\left(\frac{(\tau - mt^*)(\mu - g'(t^*))}{(mt^*(1 - t^*))^{1/2}\sigma} \le x,\ R_m \le y\right) \tag{2.13}$$
$$\to \Phi(x) \lim_{c \to \infty} P(R_c^* \le y),$$

where Φ is the standard normal distribution function and R_c^* is the excess over the constant boundary c of a random walk, generated by independent identically distributed $N(\mu - g'(t^*), \sigma^2)$ random variables, which is stopped the first time it exceeds c. If R is a random variable such that $R_c^* \to_{\mathscr{D}} R$ as $c \to \infty$, then renewal theory [see Siegmund (1985), Chapter 8] allows us to calculate

$$E \exp\left\{\frac{(\xi_0 - \mu)R}{(1 - t^*)\sigma^2}\right\} = \nu\left\{\frac{2(\mu - g'(t^*))}{\sigma}\right\},$$

where we use the fact, noted earlier in this section, that $\mu - g'(t^*) = (\mu - \xi_0)/\{2(1 - t^*)\}$. If X has a chi-squared distribution with 1 degree of freedom, the remaining factor will have the form

$$E \exp\left\{-\frac{g''(t^*)t^*(1 - t^*)}{\mu - g'(t^*)} X\right\} = \left\{1 + \frac{2g''(t^*)t^*(1 - t^*)}{\mu - g'(t^*)}\right\}^{-1/2}.$$

Therefore, we will be able to conclude that the right-hand side of (2.1) is asymptotically equivalent to (2.8) if we can make the exchange of expectation and limit alluded to previously.

Fatou's lemma for convergence in law implies that the right-hand side of (2.1) is an asymptotic lower bound for (2.8). To obtain an upper bound let $A = \{|\tau - mt^*| < am^{1/2}\}$ and write

$$\int_B L_\tau \, dP^{(m)}_{\xi_1, \lambda_1} = \int_{BA} L_\tau \, dP^{(m)}_{\xi_1, \lambda_1} + \int_{BA^c} L_\tau \, dP^{(m)}_{\xi_1, \lambda_1}. \tag{2.14}$$

By Wald's likelihood ratio identity and Lemma 1(c)

$$\int_{BA^c} L_\tau \, dP^{(m)}_{\xi_1, \lambda_1} \le \delta(a)\left\{\sigma^2/\left(\lambda_0 - \xi_0^2\right)\right\}^{(m-3)/2}, \tag{2.15}$$

where $\delta(a) \to 0$ as $a \to \infty$. For the first integral on the right-hand side of (2.14), note that for ε sufficiently small $(\lambda_1 - U_\tau - (\xi_1 - S_\tau)^2/(m - \tau)) > m\sigma^2(1 - t^*)/2$ on B. Then we may use the fact that $\log(1 + x) < x$ for

$x > 0$ to dominate the integrand in the region of integration:

$$\left(\frac{\lambda - U_\tau - (\xi - S_\tau)^2/(m-\tau)}{\lambda_1 - U_\tau - (\xi_1 - S_\tau)^2/(m-\tau)}\right)^{(m-\tau-3)/2}$$

$$\le 1 \vee \exp\left[\frac{(m-\tau-3)}{2}\left(\frac{\lambda - \lambda_1 - \{\xi^2 - \xi_1^2 - 2S_\tau(\xi - \xi_1)\}/(m-\tau)}{\lambda_1 - U_\tau - (\xi_1 - S_\tau)^2/(m-\tau)}\right)\right]$$

$$\le 1 \vee \exp\big[2(m-\tau-3)(\xi_0 - \mu)\big\{R_m + mg''(t^*)(\tau/m - t^*)^2/2 + m\varepsilon(\tau/m)(\tau/m - t^*)^2\big\}/\big\{\sigma^2(1-t^*)(m-\tau)\big\}\big] \quad \text{[by (2.11)]}$$

$$\le 1 \vee \max\left[\frac{m(\tau/m - t^*)^2|\xi_0 - \mu|\{|g''(t^*)| + 2|\varepsilon(\tau/m)|\}}{\sigma^2(1-t^*)}\right] \quad (m \text{ large})$$

$$\le \exp(Ka^2)$$

for some $K > 0$ and m large enough; the last inequality uses the fact that $R_m > 0$ and $\xi_0 - \mu < 0$. Letting $m \to \infty$ and appealing to the weak convergence indicated in (2.13), then letting $a \to \infty$ and using (2.15), we can justify the desired interchange of integral and limit, thus completeing the proof. □

As an aid in applying the theorem, we note that if g has two continuous derivatives, then the other conditions on g and h will be satisfied if (i) t^* is the only point at which $h' = 0$, (ii) $h(t^*) > 0$ and (iii) $h''(t^*) > 0$. This will be the case in the examples considered in the next section.

3. Application: Tests and confidence sets for a change-point. Let X_i, $i = 1, \dots, m$, be independent normal random variables with unknown means μ_i and constant unknown variance $\sigma^2 > 0$. We consider testing the hypothesis of a constant mean against the alternative of a single change-point, i.e., testing H_0: $\mu_1 = \cdots = \mu_m$ versus H_1: for some $j \in \{1, \dots, m-1\}$, $\mu_1 = \cdots = \mu_j \ne \mu_{j+1} = \cdots = \mu_m$.

In a comparative study of different tests James, James and Siegmund (1987) gave without justification approximations for the significance levels of three different tests. The approximations may all be obtained as applications of the theorem in Section 2. We illustrate those results by deriving the suggested approximation for the significance level of (a slight generalization of) the likelihood ratio test. We also give the results of a Monte Carlo experiment to assess the accuracy of the approximation and discuss briefly probability approximations to obtain likelihood ratio based confidence sets for the change-point.

The test is to reject the hypothesis of no change-point for large values of

$$T = \max_{m_0 \le n \le m_1} \frac{|S_n - n\overline{X}_m|}{\{n(1 - n/m)\}^{1/2} S},$$

where $S_n = X_1 + \cdots + X_n$, $\overline{X}_n = n^{-1}S_n$, $S^2 = m^{-1}\Sigma_1^m(X_i - \overline{X}_m)^2$ and $1 \le m_0 < m_1 \le m - 1$.

Under H_0 the distribution of the process $(S_n - n\bar{X}_m)/S$, $n = 0, 1, \dots, m$, does not depend on μ, σ^2 and hence by Basu's theorem [Lehmann (1959), Theorem 5.2] the process is independent of the complete sufficient statistic (S_m, U_m).

Therefore,

$$P(T \geq b) = P_{0,m}^{(m)}(T \geq b)$$
$$= P_{0,m}^{(m)}\left(\max_{m_0 \leq n \leq m_1} \frac{|S_n|}{\{n(1 - n/m)\}^{1/2}} \geq b \right).$$

Conditioning with respect to the values of S_{m_1} and U_{m_1} and using the Markov property, we have

$$\begin{aligned} P(T \geq b) &= P_{0,m}^{(m)}\left(|S_{m_1}| \geq b\{m_1(1 - m_1/m)\}^{1/2}\right) \\ &\quad + \iint_{A_m} P_{x,y}^{(m_1)}(\tau < m_1) P_{0,m}^{(m)}\left(S_{m_1} \in dx, U_{m_1} \in dy\right), \end{aligned} \tag{3.1}$$

where $\tau = \inf[n \geq m_0\colon |S_n| \geq b\{n(1 - n/m)\}^{1/2}]$ and A_m is the set of (x, y) such that $|x| < b\{m_1(1 - m_1/m)\}^{1/2}$ and the $P_{0,m}^{(m)}$-joint density of S_{m_1} and U_{m_1}, as a function of x and y, is positive.

The first summand in the right-hand side of (3.1) can be calculated exactly, as in (2.4). The theorem can be used to approximate the integrand in the second summand. For this, assume that $b = cm^{1/2}$, $m_0 = mt_0$, $m_1 = mt_1$, $x = m_1 x_0$ and $y = m_1 y_0$, where $0 < c < 1$, $0 < t_0 < t_1 < 1$, $|x_0| < ct_1^{-1/2}(1 - t_1)^{1/2}$ and $m_1 y_0$ is a $P_{0,m}^{(m)}$-possible value of U_{m_1}, given $S_{m_1} = m_1 x_0$. For $g(t) = c\{tt_1^{-1}(1 - tt_1)\}^{1/2}$, $\xi_0 = x_0$ and $\lambda_0 = y_0$, we obtain

$$t^* = \frac{x_0^2 t_1}{c^2(1 - t_1)^2 + x_0^2 t_1^2}.$$

The only hitch in applying the theorem is that $\tau \geq m_0$; a direct application requires $m_0 = 1$. However, this is no problem if $t^* > t_0/t_1$, because in this case it follows from (2.6) that the $P_{x,y}^{(m_1)}$-probability of the process crossing the boundary before $n = m_0$ is of exponentially smaller order than that of crossing before $n = m_1$, so that we may replace m_0 by 1. On the other hand, if $t^* < t_0/t_1$, which corresponds to $|x_0| < ct_1^{-1}(1 - t_1)\{t_0(1 - t_0)^{-1}\}^{1/2}$, we can approximate the integrand by 0. This follows again from (2.6), which implies that the integrand will be of exponentially smaller order than other values of the integrand corresponding to $t^* > t_0/t_1$. Therefore, we are led to the approximation

$$\begin{aligned} P(T \geq b) &\cong \frac{2\Gamma((m-1)/2)}{\pi^{1/2}\Gamma((m-2)/2)} \int_c^1 (1 - x^2)^{(m-4)/2}\,dx \\ &\quad + 2c\left(\frac{1 - t_1}{t_1}\right)^{1/2} \iint_B \frac{1}{x_0} \nu\left(\frac{x_0}{t^*(1 - t_1)\sigma}\right)\left(\frac{\sigma^2}{y_0 - x_0^2}\right)^{(m_1 - 3)/2} \\ &\qquad \times P_{0,m}^{(m)}\left(\frac{S_{m_1}}{m_1} \in dx_0, \frac{U_{m_1}}{m_1} \in dy_0\right), \end{aligned}$$

where $\sigma^2 = \sigma^2(x_0, y_0) = y_0 - c^2 t_1^{-1} + x_0^2 t_1(1 - t_1)^{-1}$ and

$$B = \left\{(x_0, y_0): \frac{c^2}{t_1} - \frac{x_0^2 t_1}{1 - t_1} \le y_0 \le \frac{1}{t_1} - \frac{x_0^2 t_1}{1 - t_1}, \frac{c(1 - t_1)}{t_1}\left(\frac{t_0}{1 - t_0}\right)^{1/2} < x_0 < c\left(\frac{1 - t_1}{t_1}\right)^{1/2}\right\}.$$

The factor 2 is due to restriction to positive values of x, by symmetry.

A further approximation can be made upon insertion of the conditional density into the integral, with subsequent utilization of the fact that U_{m_1} is conditionally, given $S_m = 0$, $U_m = m$ and $S_{m_1} = m_1 x_0$, a linear function of random variable with a beta distribution with parameters $(m_1 - 1)/2$ and $(m - m_1 - 1)/2$, which as $m \to \infty$ with $m_1/m \to t_1$ collapses to a point mass at t_1. Following this procedure, we can insert the $P_{0,m}^{(m)}$-density of $(S_{m_1}/m_1, U_{m_1}/m_1)$, to wit

$$\frac{\{t_1^{m_1}/(1 - t_1)\}^{1/2}\Gamma((m - 1)/2)}{\pi^{1/2}\Gamma((m_1 - 1)/2)\Gamma((m - m_1 - 1)/2)}(y_0 - x_0^2)^{(m_1 - 3)/2} \times\left[1 - t_1\left\{y_0 + \frac{x_0^2 t_1}{1 - t_1}\right\}\right]^{(m - m_1 - 3)/2},$$

make the change of variable (in y_0)

$$z = t_1(1 - c^2)^{-1}\left\{y_0 + x_0^2 t_1(1 - t_1)^{-1} - c^2 t_1^{-1}\right\},$$

integrate with respect to z and use Stirling's formula to approximate the remaining gamma functions to show that the preceding integral reduces asymptotically to a single integral in x_0. Thus we are led to the approximation

$$\begin{aligned} P(T \ge b) \cong \left(\frac{2m}{\pi}\right)^{1/2} \int_c^1 (1 - x^2)^{(m-4)/2}\, dx + c\left(\frac{2m}{\pi}\right)^{1/2}(1 - c^2)^{(m-4)/2} \\ \times \int_{c\{(t_1^{-1} - 1)/(1 - c^2)\}^{1/2}}^{c\{(t_0^{-1} - 1)/(1 - c^2)\}^{1/2}} \frac{1}{x}\nu\left\{x + \frac{c^2}{(1 - c^2)x}\right\} dx. \end{aligned} \tag{3.2}$$

REMARK 4. It is easy to see that for each $i = 0, 1, \dots,$

$$\int_c^1 (1 - x^2)^{(m-i)/2}\, dx = (cm)^{-1}(1 - c^2)^{(m-i+2)/2}\{1 + m^{-1}(i - 1 - c^{-2}) + o(m^{-1})\}$$

as $m \to \infty$. Use of this approximation simplifies slightly the computational burden associated with application of (3.2). From this expansion it is evident that the first term on the right-hand side of (3.2) is asymptotically of smaller order than the second and mathematically could be neglected. In a number of related problems, Siegmund (1985) shows numerically that including this term

TABLE 1

$$P\left\{\max_{m_0 \le n \le m_1} \frac{|S_n - n\bar{X}_m|}{\{n(1-n/m)\}^{1/2} S} \ge a\right\}$$

a	m	m_0	m_1	Monte Carlo	(3.2)
2.75	20	1	19	0.0458	0.0483
2.45	20	3	17	0.0936	0.0969
2.65	20	3	17	0.0526	0.0510
3.05	20	3	17	0.0104	0.0096
3.05	80	1	79	0.0448	0.0473
2.65	80	8	72	0.0940	0.0994
2.90	80	8	72	0.0478	0.0496
3.40	80	8	72	0.0112	0.0094

typically improves the approximation, and hence we have included it for numerical purposes.

REMARK 5. It is natural to ask what precise mathematical meaning can be attached to (3.2). As noted in Remark 4, the first integral on the right-hand side of (3.2) is asymptotically negligible. With some additional work it can be shown that $P(T \ge b)$ is asymptotically equivalent to the second integral on the right-hand side of (3.2). It suffices to show that for each $x_0 \in (ct_1^{-1}(1 - t_1)\{t_0/(1 - t_0)\}^{1/2}, c\{(1 - t_1)/t_1\}^{1/2})$ the asymptotic behavior of the conditional probability indicated in the theorem holds uniformly for y_0 in a neighborhood of $1 + c^2 t_1^{-1}(1 - t_1) - x_0^2 t_1(1 - t_1)^{-1}$ of width $a_m/m^{1/2}$, where $a_m \to \infty$, together with appropriate uniformity in Lemma 1. The details are tedious and have been omitted.

Table 1 gives results of a 9999 repetition Monte Carlo experiment to assess the accuracy of (3.2). These figures along with similar ones not reported here show that (3.2), although usually giving answers which are slightly too large, is sufficiently accurate for use in practice without fear of being misled.

In the approximation (3.2) the conditional probabilities of the theorem appear indirectly, via the integral in (3.1). A problem in which they enter directly is that of giving confidence sets for a change-point based on the likelihood ratio statistic. The case of known σ is discussed by Siegmund [(1986), Section 3.5]. If σ is unknown, essentially the same argument leads to the probabilities

$$p_1 = P_{\xi,\lambda_1}^{(m_1)}\left\{|S_k| \ge mc_0\left\{km^{-1}(1 - km^{-1})\right\}^{1/2} \text{ for some } k < m_1\right\}$$

and

$$p_2 = P_{\xi,\lambda_2}^{(m-m_1)}\left\{|S_k| \ge mc_0\left\{km^{-1}(1 - km^{-1})\right\}^{1/2} \text{ for some } k < m - m_1\right\},$$

where for some constant $0 < c < 1$,

$$c_0 = \left\{c(\lambda_1 + \lambda_2)/m + (1 - c)\xi^2/\left[m_1(m - m_1)\right]\right\}^{1/2}.$$

Assuming $m_1 = mp$, $\xi = m_1\xi_0$, $\lambda_1 = m_1\lambda_{10}$ and $\lambda_2 = (m - m_1)\lambda_{20}$, where $0 < p < 1$, ξ_0, λ_{10} and λ_{20} are fixed and applying the theorem with $g_1(t) = c_0\{t(1-pt)/p\}^{1/2}$ and $g_2(t) = c_0\{t[1-(1-p)t]/(1-p)\}^{1/2}$, we obtain

$$p_1 \sim \nu\{\xi_0/[(1-p)t_1^*\sigma_1]\}\left[\sigma_1^2/(\lambda_{10} - \xi_0^2)\right]^{(m_1-3)/2} c_0\xi_0^{-1}[(1-p)/p]^{1/2}$$

and

$$p_2 \sim \nu[\xi_0/(pt_2^*\sigma_2)]\left[\sigma_2^2/(\lambda_{20} - \xi_0^2)\right]^{(m-m_1-3)/2} c_0\xi_0^{-1}[p/(1-p)]^{1/2},$$

where $t_1^* = \xi_0^2 p/\{c_0^2(1-p)^2 + \xi_0^2 p^2\}$, $t_2^* = \xi_0^2(1-p)/\{c_0^2p^2 + \xi_0^2(1-p)^2\}$, $\sigma_1^2 = \lambda_{10} - c_0^2/p + \xi_0^2 p/(1-p)$ and $\sigma_2^2 = \lambda_{20} - c_0^2/(1-p) + \xi_0^2(1-p)/p$.

These approximations can be used directly or after integration with respect to the conditional distribution of U_{m_1} given $S_m = 0$, $S_{m_1} = \xi$, $U_m = \lambda_1 + \lambda_2 \cdots$ as in (3.17) and (3.18) of Siegmund (1986) (where incidentally the event A should be A^c) to obtain approximate confidence sets for the change-point.

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