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A ZERO OR ONE LAW FOR ONE DIMENSIONAL RANDOM WALKS IN RANDOM ENVIRONMENTS

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We prove a zero or one law for one dimensional random walks in random environments for which the probability of making jumps of size n decays exponentially. As an application we conclude that these random walks are recurrent if the distribution of the random environment is symmetric.

1. Introduction and statement of results. Let Γ be the set of elements of the form $\{s_x\}_{x \in \mathbb{Z}^d}$, where for each $x \in \mathbb{Z}^d$, s_x is a probability measure on $\mathbb{Z}^d$. We consider the product topology on Γ, derived from the topology of weak convergence on each of its coordinates; Γ is now endowed with its Borel σ-algebra. We will call elements of Γ environments and probability measures on Γ random environments. We also introduce the product topology on $(\mathbb{Z}^d)^{\mathbb{Z}_+}$ and the topology of weak convergence on the set of Borel probability measures on $(\mathbb{Z}^d)^{\mathbb{Z}_+}$. Given an environment s and $x \in \mathbb{Z}^d$, consider the Markov chain on $\mathbb{Z}^d$ whose transition probabilities are given by $p_s(y, z) = s_y(z - y)$. Let this Markov chain start at x and denote by P_x^s the probability measure it induces on $(\mathbb{Z}^d)^{\mathbb{Z}_+}$. Now define the random walk in the random environment M starting at x as the stochastic process $\{X_n\}_{n \in \mathbb{Z}_+}$ corresponding to the probability $P_x = \int P_x^s \, dM(s)$ on $(\mathbb{Z}^d)^{\mathbb{Z}_+}$. (It can be easily verified that the function $s \to P_x^s$ is measurable; thus P_x is well defined.) The expectation operators corresponding to P_x^s and P_x will be denoted by E_x^s and E_x, respectively. The subindex x will be deleted when $x = 0$.

One dimensional nearest neighbor random walks in random environments have been extensively studied and their behavior has been well characterized [see Solomon (1975), Kesten, Kozlov and Spitzer (1975) and Sinai (1982)]. Much less is known about these processes in higher dimensions or when jumps of size larger than 1 are allowed. A sufficient condition for transience in a rather general context has been given by Kalikow (1981) and a necessary and sufficient condition for recurrence for one dimensional random walks in random environments has been proved by Key (1984). To be applicable this second condition requires the size of the possible jumps to be bounded. In that same paper Key proves a zero or one law under the following hypothesis: The size of all possible jumps is bounded and M is a translation invariant product measure. Ledrappier (1984) generalized this to the case in which M is translation invariant and ergodic. In this paper we give a further generalization which applies to a larger class of events and includes cases in which jumps are unbounded.

From now on we assume that $d = 1$ and use the following notation: Θ denotes the shift operator on $(\mathbb{Z})^{\mathbb{Z}_+}$, i.e., $\Theta(X_0, X_1, X_2, \dots) = (X_1, X_2, \dots)$, and T de-

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notes the operator defined by $T(X_0, X_1, \dots) = (1 + X_0, 1 + X_1, \dots)$ on the same space. Borel subsets of $(\mathbb{Z})^{\mathbb{Z}_+}$ will be called events. We will say that an event A is shift invariant if $\Theta^{-1}A = A$ and that it is translation invariant if $T^{-1}A = A$.

For any shift invariant event A and any environment s, we define the function

$$\alpha_s(x) = P_x^s(\{X_n\}_{n \in \mathbb{Z}_+} \in A), \tag{1.1}$$

which is harmonic for $p_s(x, y)$.

Throughout this paper we will assume that the following three conditions are satisfied:

(i) M is stationary and ergodic.

(ii) For almost all $s(dM)$ the Markov chain associated to $p_s(x, y)$ is irreducible:

$$\forall\, x, y \in \mathbb{Z}, \qquad \sum_{n=0}^{\infty} p_s^n(x, y) > 0. \tag{1.2}$$

(iii) There exist $c > 0$ and $r \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, all $x \in \mathbb{Z}$ and almost all $s(dM)$, we have

$$\sum_{j=r+n}^{\infty} p_s(x, x+j) \le (1 - c) \sum_{j=n}^{\infty} p_s(x, x+j) \tag{1.3}$$

and

$$\sum_{j=-r-n}^{-\infty} p_s(x, x+j) \le (1 - c) \sum_{j=-n}^{-\infty} p_s(x, x+j). \tag{1.4}$$

Condition (iii) means that the decay at infinity of $p_s(0, j)$ is controlled by an exponential whose parameter is independent of s. It will be used rather heavily in this paper, although it is likely that most of the results proved here are true under much weaker assumptions.

We can now state a zero or one law.

Theorem 1.5. *If M satisfies conditions* (i)–(iii) *and A is a shift invariant and translation invariant event, then $P(A) = 0$ or* 1.

Since (1.2) implies that

$$P^s(\limsup X_n = +\infty) + P^s(\limsup X_n = -\infty) = 1$$

and

$$P^s(\liminf X_n = +\infty) + P^s(\liminf X_n = -\infty) = 1,$$

Corollary 1.6 is an immediate consequence of Theorem 1.5.

Corollary 1.6. *Under conditions* (i)–(iii) *we have*

$$P(\lim X_n = +\infty) = 1$$

or

$$P(\lim X_n = -\infty) = 1$$

or

$$P(\limsup X_n = +\infty \text{ and } \liminf X_n = -\infty) = 1.$$

Our next corollary also follows immediately from Theorem 1.5. It gives a zero or one law for a class of events not covered by previous results.

COROLLARY 1.7. *Let a_n be a sequence of real numbers such that* $\lim a_n = +\infty$ *and* $\lim(a_{n+1}/a_n) = 1$. *If* (i)–(iii) *hold, then*

$$P\left(\limsup \frac{X_n}{a_n} > c\right) = 0 \text{ or } 1$$

and

$$P\left(\liminf \frac{X_n}{a_n} > c\right) = 0 \text{ or } 1$$

for any $c \in \mathbb{R}$.

It follows from the Hewitt–Savage zero or one law that random walks on $\mathbb{Z}^d$ satisfy $P(X_n \in B \text{ i.o.}) = 0$ or 1, whenever B is a subset of $\mathbb{Z}^d$. It is not known whether the same result holds for random walks in random environments. Our next result gives an affirmative answer to this question for one dimensional cases satisfying (i)–(iii).

THEOREM 1.8. *Suppose M satisfies conditions* (i)–(iii). *Then* $P(X_n \in B \text{ i.o.}) = 1$ *in three cases*:

(a) $P(\lim X_n = +\infty) = 1$ *and* $B \cap \mathbb{Z}_+$ *is infinite.*
(b) $P(\lim X_n = -\infty) = 1$ *and* $B \cap \mathbb{Z}_-$ *is infinite.*
(c) $P(\limsup X_n = +\infty$ *and* $\liminf X_n = -\infty) = 1$ *and* $B \neq \varnothing$.

In all other cases $P(X_n \in B \text{ i.o.}) = 0$.

Moreover for almost all $s(dM)$ bounded harmonic functions for $p_s(x, y)$ are constant.

Before stating our last result, we introduce some notation: Let S be the transformation induced on the set of probability measures on Z by the mapping $y \to -y$. Let σ be the transformation on Γ defined by $\sigma(\{s_x\}) = \{t_x\}$, where $t_x = S(s_{-x})$; the same symbol σ will denote the dual operator acting on the set of probability measures on Γ. We can now give a sufficient condition for recurrence.

COROLLARY 1.9. *Suppose M is invariant under σ and satisfies* (i)–(iii). *Then* $P(X_n = 0 \text{ i.o.}) = 1$.

Note that M is invariant under σ if the random measures s_x are independent under M and s_0 has the same distribution as $S(s_0)$. A class of examples to which this last corollary can be applied, but the s_x are not independent, can be constructed like this: Let A be a finite set and $p'(x, y)$ an irreducible probability matrix on A. Fix $0 < \alpha < 1$ and to each element (x, a) of $Z \times A$ assign the random transition probabilities

$$p_s((x,a),(y,b)) = \begin{cases} \alpha p'(a,b), & \text{if } x = y \text{ and } a \neq b, \\ (1-\alpha) s_{(x,a)}(y-x), & \text{if } a = b \text{ and } x \neq y, \\ \alpha p'(a,a) + (1-\alpha) s_{(x,a)}(0), & \text{if } a = b \text{ and } x = y, \\ 0, & \text{otherwise,} \end{cases}$$

where $s_{(x,a)}$ are iid random distributions on Z which are invariant under S and satisfy (1.2)–(1.4) a.s. Fix an element $a \in A$ and identify Z with $\mathbb{Z} \times \{a\}$. Let τ_i be the successive hitting times of $\mathbb{Z} \times \{a\}$. Then $(0, a)$, $x_{\tau_1}, x_{\tau_2}, \dots,$ is a one dimensional random walk in a random environment which satisfies all the conditions of Corollary 1.9.

This paper is organized as follows: In Section 2 we prove two preliminary lemmas concerning Markov chains on $\mathbb{Z}$. In Section 3 these lemmas are used to prove Theorem 1.5, which is then used in Section 4 to prove Theorem 1.8. Throughout these proofs τ_C will denote the hitting time of a subset C of $\mathbb{Z}$. If C reduces to a singleton x, then we will write τ_x rather than $\tau_{\{x\}}$.

2. Preliminary results. Lemma 2.1 gives an upper bound for the probability of jumping over a long interval. Its proof is contained in the proof of Corollary 7 of Cocozza-Thivent and Roussignol (1983). It is given here for the sake of completeness.

LEMMA 2.1. *Let s be an environment and $n \in \mathbb{N}$. If* (1.3) *holds, then for all x_0, $y \in \mathbb{Z}$ such that $x_0 < y$,*

$$P_{x_0}^s\big(\tau_{[y+rn,\infty)} < \tau_{[y,y+rn)}\big) \le (1-c)^n P_{x_0}^s\big(\tau_{[y,\infty)} < \infty\big).$$

If (1.4) *holds, then for all x_0, $y \in \mathbb{Z}$ such that $x_0 > y$,*

$$P_{x_0}^s\big(\tau_{(-\infty,y-rn]} < \tau_{(y-rn,y]}\big) \le (1-c)^n P_{x_0}^s\big(\tau_{(-\infty,y]} < \infty\big).$$

PROOF. We only write the proof of the first statement because it can be easily adapted to the second one. The left-hand side of the inequality to be proved is equal to

$$\sum_{z<y} \sum_{u=0}^{\infty} P_{x_0}^s\big(\tau_{[y,\infty)} > u,\ X_u = z,\ X_{u+1} \ge y + rn\big).$$

Hence by the Markov property,

$$\begin{aligned} &P_{x_0}^s\big(\tau_{[y+rn,\infty)} < \tau_{[y,y+rn)}\big) \\ &\quad = \sum_{z<y} \left(\sum_{u=0}^{\infty} P_{x_0}^s\big(\tau_{[y,\infty)} > u,\ X_u = z\big)\right) P_z^s(X_1 \ge y + rn). \end{aligned} \tag{2.2}$$

Now note that

$$\begin{aligned}(2.3)\qquad &\sum_{z<y}\sum_{u=0}^{\infty} P^s_{x_0}\big(\tau_{[y,\infty)} > u,\, X_u = z\big)P^s_z(X_1 \geq y)\\ &\quad = \sum_{u=0}^{\infty} P^s_{x_0}\big(\tau_{[y,\infty)} = u+1\big) = P^s_{x_0}\big(\tau_{[y,\infty)} < \infty\big).\end{aligned}$$

Multiply and divide the right-hand side of (2.2) by $P^s_z(X_1 \geq y)$ using the convention $0/0 = 0$. Then apply (2.3) to conclude that

$$\begin{aligned}&P^s_{x_0}\big(\tau_{[y+rn,\infty)} < \tau_{[y,y+rn)}\big)\\ &\quad\leq P^s_{x_0}\big(\tau_{[y,\infty)} < \infty\big)\sup_{z<y}\frac{P^s_z(X_1 \geq y+rn)}{P^s_z(X_1 \geq y)}\\ &\quad= P^s_{x_0}\big(\tau_{[y,\infty)} < \infty\big)\sup_{z<y}\left[\frac{P^s_z(X_1 \geq y+rn)}{P^s_z(X_1 \geq y+r(n-1))}\cdots\frac{P^s_z(X_1 \geq y+r)}{P^s_z(X_1 \geq y)}\right].\end{aligned}$$

Since by (1.3) the last expression is bounded above by $P^s_{x_0}(\tau_{[y,\infty)} < \infty)(1-c)^n$, Lemma 2.1 follows. □

Lemma 2.4. *Let A be a shift invariant event, s an environment satisfying* (1.2)–(1.4) *and α_s the harmonic function associated to A and s through* (1.1). *Then either $\alpha_s(x) \equiv 0$ or there exists an integer l which depends only on c and r with the following property: For all $\varepsilon > 0$ there exists a strictly monotone sequence of integers $\{y_i\}$ satisfying*

$$|y_{i+1} - y_i| \leq l \quad \forall\, i \in \mathbb{N}$$

and

$$\alpha_s(y_i) \geq 1 - \varepsilon \quad \forall\, i \in \mathbb{N}.$$

Proof. Suppose $\alpha_s(z_0) > 0$ for some $z_0 \in \mathbb{Z}$ and let $\varepsilon > 0$. Due to the Markov property and the shift invariance of A, we have $P^s_{z_0}(\lim_n \alpha_s(X_n) = 1) = \alpha_s(z_0)$. Hence there exists $x_0 \in \mathbb{Z}$ such that $\alpha_s(x_0) > 1 - \varepsilon^2$. Consider the Markov chain with transition probabilities given by s. Let this chain start at x_0 and define the following stopping time (by convention $\inf \varnothing = +\infty$),

$$\tau = \inf\{k\colon \alpha_s(X_k) \leq 1 - \varepsilon\}.$$

Now

$$1 - \varepsilon^2 < \alpha_s(x_0) = E^s_{x_0}\alpha_s(X_{\tau\wedge n}) \leq 1 - \varepsilon P^s_{x_0}(\tau \leq n),$$

where the equality follows from the fact that $\alpha_s(X_k)$ is a $P^s_{x_0}$ martingale. Taking limits as n goes to infinity, we obtain

$$1 - \varepsilon^2 < \alpha_s(x_0) \leq 1 - \varepsilon P^s_{x_0}(\tau < \infty).$$

Hence $P^s_{x_0}(\tau = \infty) > 1 - \varepsilon$ and

$$P^s_{x_0}\Big(\inf_k \alpha_s(X_k) \geq 1 - \varepsilon\Big) > 1 - \varepsilon.$$

Since (1.2) implies that $P^s_{x_0}(\limsup|X_n| = +\infty) = 1$, we may assume that either

$$P^s_{x_0}\Big(\inf_k \alpha_s(X_k) \geq 1 - \varepsilon,\ \limsup_k X_k = +\infty\Big) > \tfrac{1}{2} - \varepsilon \tag{2.5}$$

or

$$P^s_{x_0}\Big(\inf_k \alpha_s(X_k) \geq 1 - \varepsilon,\ \liminf_k X_k = -\infty\Big) > \tfrac{1}{2} - \varepsilon. \tag{2.6}$$

The proof will be completed assuming the first of these alternatives holds; the other case can be treated analogously. We also assume without loss of generality that $\varepsilon < \frac{1}{4}$. It follows now from (2.5) that for all $y > x_0$,

$$P^s_{x_0}\Big(\inf_k \alpha_s(X_k) \geq 1 - \varepsilon,\ \tau_{[y,\infty)} < \infty\Big) > \tfrac{1}{4} \tag{2.7}$$

and it follows from Lemma 2.1 that there exists an l, which depends only on c and r, such that for all $y > x_0$,

$$P^s_{x_0}\Big(\tau_{[y,y+l)} < \infty \,|\, \tau_{[y,\infty)} < \infty\Big) > \tfrac{3}{4}.$$

This and (2.7) imply that

$$P^s_{x_0}\Big(\inf_k \alpha_s(X_k) \geq 1 - \varepsilon,\ \tau_{[y,y+l)} < \infty\Big) > 0.$$

Hence for all $y > x_0$, there exists a y_0 such that $0 \leq y_0 - y < l$ and $\alpha_s(y_0) \geq 1 - \varepsilon$. This proves Lemma 2.4. □

3. Proof of Theorem 1.5. Let $\alpha_s(x)$ be defined by (1.1). It follows from (1.2) that for almost all $s(dM)$ either $\alpha_s(x) \equiv 0$ or $\alpha_s(x) \equiv 1$ or $0 < \alpha_s(x) < 1$ for all $x \in \mathbb{Z}$. We will first show that the third alternative is satisfied by a set of M measure 0. Let $\varepsilon > 0$ and $\delta = M\{s\colon 0 < \alpha_s(0) < 1\}$. Suppose s satisfies (1.2)–(1.4) and $0 < \alpha_s(0) < 1$. By Lemma 2.4 we have

$$\liminf_n \frac{\#\{x\colon -n \leq x \leq n \text{ and } 1 - \varepsilon < \alpha_s(x) < 1\}}{2n+1} \geq \frac{1}{2l}.$$

Therefore there exists a positive integer $n_0 = n_0(\varepsilon)$ such that

$$M\left\{s\colon \#\{x\colon -n_0 \leq x \leq n_0 \text{ and } 1 - \varepsilon < \alpha_s(x) < 1\} \geq \frac{2n_0 + 1}{3l}\right\} > \frac{\delta}{2}.$$

This implies that for some $x_0 = x_0(\varepsilon) \in \mathbb{Z}$,

$$M\{s\colon 1 - \varepsilon < \alpha_s(x_0) < 1\} \geq \frac{\delta}{2}\frac{1}{3l} = \frac{\delta}{6l}.$$

Since M and A are invariant under translations on Z, we obtain

$$M\{s\colon 1 - \varepsilon < \alpha_s(0) < 1\} \geq \frac{\delta}{6l}$$

and letting ε go to 0, we conclude that $\delta = 0$. Now we know that for almost all $s(dM)$, $\alpha_s(x)$ is independent of x. Since A is invariant under T, this implies that the function $s \to \alpha_s(0)$ is invariant under translations on Γ. Since M is ergodic, $\alpha_s(0)$ is constant a.s. (dM) and Theorem 1.5 is proved.

4. Proofs of Theorem 1.8 and Corollary 1.9. Note that if M satisfies the hypothesis of Corollary 1.9, then $P(\lim X_n = +\infty) = P(\lim X_n = -\infty)$. By Corollary 1.6 these probabilities are equal to 0 or 1; therefore, they must both be equal to 0 and $P(\limsup X_n = +\infty \text{ and } \liminf X_n = -\infty) = 1$. Hence Corollary 1.9 is a special case of Theorem 1.8.

PROOF OF THEOREM 1.8. We will consider three different cases, according to which of the alternatives of Corollary 1.6 holds.

FIRST CASE. $P(\lim X_n = +\infty) = 1$. If $B \cap \mathbb{Z}_+$ is finite, then we obviously have $P(X_n \in B \text{ i.o.}) = 0$. Hence we may assume that $B \cap \mathbb{Z}_+$ is infinite. To prove that $P(X_n \in B \text{ i.o.}) = 1$, it suffices to show that for almost all $s(dM)$, $P^s(X_n \in B \text{ i.o.}) = 1$.

Let $a_j(s) = \inf_{1 \le i \le r} P^s_{j-i}(\tau_j < \infty)$, $j = 0, 1, 2, \dots$. This is a sequence of identically distributed random variables and it follows from (1.2) that given $\varepsilon > 0$, there exists a $\delta > 0$ such that $M\{s\colon a_i(s) > \delta\} \ge 1 - \varepsilon$, $i = 0, 1, 2, \dots$. Hence

$$M\left(\bigcap_{n=1}^{\infty}\left[\bigcup_{\substack{i \in B \\ i \ge n}} \{s\colon a_i(s) > \delta\}\right]\right) \ge 1 - \varepsilon.$$

Since ε is arbitrary, this shows that for almost all $s(dM)$, there exists a $\delta(s) > 0$ such that

$$C(s) = \{i \in B \cap \mathbb{N}\colon a_i(s) > \delta(s)\} \quad \text{is infinite.}$$

In the sequel we will fix an s for which $C(s)$ is infinite, the Markov chain associated to it is irreducible, (1.3) holds and $P^s(\lim X_n = +\infty) = 1$. It now suffices to show that for such an environment s, $P^s(X_n \in C(s) \text{ i.o.}) = 1$. To simplify the notation, we will write δ and C for $\delta(s)$ and $C(s)$, respectively. Now, due to the irreducibility of s, $P^s(X_n \in C \text{ i.o.}) = 1$ is equivalent to $P^s_x(\tau_C < \infty) = 1$ for all $x \in \mathbb{Z}$.

Let $\beta(x) = P^s_x(\tau_C < \infty)$. It follows from Çinlar [(1975), Chapter 6, Proposition 4.5] that $1 - \beta(x) \equiv 0$ or $\sup_x 1 - \beta(x) = 1$. Hence it suffices to prove that

$$\inf_x \beta(x) > 0. \tag{4.1}$$

Fix an arbitrary $x \in \mathbb{Z}$ and let $y \in C \cap [x + r, \infty)$. Then

$$\begin{aligned} \beta(x) > P^s_x(\tau_y < \infty) &\ge P^s_x(\tau_y < \infty, \tau_{[y-r, y)} < \infty) \\ &\ge P^s_x(\tau_{[y-r, y)} < \infty) \inf_{1 \le i \le r} P^s_{y-i}(\tau_y < \infty). \end{aligned} \tag{4.2}$$

It follows from $P^s(\lim X_n = +\infty) = 1$ and the irreducibility of s that $P^s_x(\tau_{[y,\infty)} < \infty) = 1$. This and Lemma 2.1 imply that $P^s_x(\tau_{[y-r, y)} < \infty) \ge c$. It follows now from (4.2) that $\beta(x) \ge c\delta$. Since neither c nor δ depends on x, (4.1) is proved.

To show that bounded harmonic functions for this same environment s are constant, let $\alpha(x)$ be such a function and consider a strictly increasing sequence $\{x_i\}$, contained in $C(s)$ and such that $\lim_i \alpha(x_i) = a$ for some real number a.

Given $\varepsilon > 0$, let $C(\varepsilon) = \{x \in \mathbb{Z}: x = x_i$ for some i and $|\alpha(x) - a| < \varepsilon\}$. Now note that the irreducibility of s and $P^s(\lim X_n = +\infty) = 1$ imply that $P_z^s(\lim X_n = +\infty) = 1$ for all $z \in \mathbb{Z}$. This and the fact that $C(\varepsilon) \cap \mathbb{Z}_+$ is an infinite subset of $C(s)$ allow us, just as before, to prove that $P_z^s(\tau_{C(\varepsilon)} < \infty) = 1$ for all $z \in \mathbb{Z}$. This and the equality $\alpha(z) = E_z^s \alpha(X_{\tau_{C(\varepsilon)}})$, which follows from the fact that $\alpha(X_n)$ is a P_z^s martingale, imply that $|\alpha(z) - a| \le \varepsilon$ for all $z \in \mathbb{Z}$. Since ε is arbitrary, α must be constant.

SECOND CASE. $P(\lim X_n = -\infty) = 1$. This is treated as the first case.

THIRD CASE. $P(\limsup X_n = +\infty$ and $\liminf X_n = -\infty) = 1$. Let $a_0(s)$ be as in the first case. Then $a_0(s) > 0$ a.s. (dM). Consider an environment s such that $a_0(s) = \delta > 0$, the Markov chain associated to s is irreducible, (1.3) holds and $P^s(\limsup X_n = +\infty, \liminf X_n = -\infty) = 1$. An argument similar to the one used in the first case shows that for all $x \in \mathbb{Z}$, $P_x^s(\tau_0 < \infty) \ge c a_0(s)$; therefore, $P_x^s(\tau_0 < \infty) = 1$ for all $x \in \mathbb{Z}$. Hence the Markov chain associated to s is recurrent and bounded harmonic functions are constant. Since this holds for almost all $s(dM)$, Theorem 1.8 is proved. □

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