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CENTRAL LIMIT THEOREM FOR AN INFINITE LATTICE SYSTEM OF INTERACTING DIFFUSION PROCESSES

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A central limit theorem for interacting diffusion processes is shown. The proof is based on an infinite-dimensional stochastic integral representation of smooth functionals of diffusion processes. Exponential decay of correlations and the equation of the fluctuation field are also obtained.

1. Introduction. Consider an infinite-dimensional diffusion process $X = (X_t^i, 0 \le t \le 1)_{i \in I}$ of the form

$$X_t^i = \xi^i + \int_0^t b^i(X_s)\,ds + W_t^i, \qquad i \in I,$$

where $I = Z^d$ is the d-dimensional lattice and $(W^i)_{i \in I}$ is a collection of independent Wiener processes. Such systems have been considered recently by several authors [for conditions to guarantee existence and uniqueness, see, e.g., Doss and Royer (1979), Shiga and Shimizu (1980), Fritz (1982) or Leha and Ritter (1984)]. We shall suppose that $(b^i)_{i \in I}$ is a stationary family of smooth drifts with bounded derivatives and that the process starts off with a translation invariant ergodic measure on R^I. The purpose of this paper is to derive a central limit theorem for a class of smooth functionals on the trajectory space $C[0,1]^I$. More precisely, let F be a Fréchet-differentiable functional on $C[0,1]^I$, Θ^i the usual shift transformation and V_n the cube $[-n, n]^d$ in Z^d. Then we show that the law of the standardized sum

$$S_n^*(F) := |V_n|^{-1/2} \sum_{i \in V_n} \left(F\Theta^i - E[F]\right)$$

is asymptotically normal with variance $\sigma^2(F) = \sum_i \operatorname{cov}(F, F\Theta^i)$. In particular we derive the distribution-valued stochastic differential equation associated to the Gaussian field of fluctuation [see, e.g., Holley and Stroock (1978) or Itô (1983)]. Similar results have been obtained for a mean-field interaction [cf. Shiga and Tanaka (1985) and Sznitman (1983)] and for some short range interacting spin-flip processes [cf. Holley and Stroock (1979, 1981)]. However, in this paper we restrict ourselves to finite time and it would be interesting to investigate the limit procedure when the time is speeded up simultaneously with the averaging over space as in Section 6 of Holley and Stroock (1979) or along hydrodynamical limes [cf. Rost (1983)].

The proof of the central limit theorem is based on an infinite-dimensional version of the Haussmann formula which gives an explicit representation for

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smooth functionals of diffusion processes as stochastic integrals [cf. Haussmann (1978, 1979), Davis (1980) and Ocone (1984)]. This representation also provides information on the rate of decay of covariance similar to that obtained by the Dobrushin contraction technique [cf. Deuschel (1987)].

In Section 2 we introduce the gradient process associated to the diffusion (X_t, $0 \le t \le 1$) and apply the method of variation of parameters to a linear perturbation of X. In Section 3 we show the Haussmann formula for finite dimensions with a perturbation argument using the Girsanov transformation [cf. Bismut (1981) and Blum (1986)]. Section 4 contains a proof of the central limit theorem by standard martingale techniques. In Section 5 we determine the equation of the fluctuation field.

2. Diffusion process with smooth drift. In this section we review the basic properties of the solution of an infinite system of stochastic differential equations with differentiable drifts.

Let us first introduce some notation. $I = Z^d$ is the d-dimensional lattice, $(\Theta^i)_{i\in I}$ the shift operation on R^I. For $x \in R^I$ put $|x|_1 = \Sigma_i |x^i|$, respectively, $|x|_\infty = \sup_i |x^i|$, and define as usual

$$l^p(I) = \left\{(x^i)_{i\in I}\colon |x|_p < \infty\right\}, \qquad p = 1, \infty,$$

and

$$l^2(\alpha) = \left\{(x^i)_{i\in I}\colon |x|_{\alpha,2} = \left(\sum_i \alpha^i |x^i|^2\right)^{1/2} < \infty\right\},$$

where $(\alpha^i)_{i\in I}$ is a positive sequence in $l^1(I)$. Let $(W_t^i,\ 0 \le t \le 1)_{i\in I}$ be a collection of independent real-valued Wiener processes on a probability space $(\Omega, \mathscr{O}, P)$. Denote by $(\mathscr{F}_t, 0 \le t \le 1)$, $\mathscr{F}_t := \sigma(W_s^i, 0 \le s \le t, i \in I)$, the filtration generated by $(W_t^i,\ 0 \le t \le 1)_{i\in I}$ and by $\|\ \|$ the supremum norm on $C[0,1]$. Consider a diffusion process $X = (X_t^i, 0 \le t \le 1)_{i\in I}$ of the form

$$X_t^i = \xi^i + \int_0^t b^i(X_s)\,ds + W_t^i, \qquad i \in I, \tag{2.1}$$

where $(b^i)_{i\in I}$ is a stationary family of differentiable drifts. More precisely, the b^i, $i \in I$, are real-valued functions defined on some Hilbert space $l^2(\gamma)$, $\gamma \in l^1(I)$, to be specified later, which satisfy the following Conditions:

1. The family is *stationary* with respect to the shift Θ^k on R^I,

$$b^{i+k}(x) = b^i(\Theta^k x) \quad \text{for all } i, k \in I.$$

2. For each $i \in I$, the function b^i is *differentiable* with respect to the norm $|\ |_\infty$ on $l^\infty(I)$: For all $x \in l^2(\gamma)$ and $h \in l^\infty(I)$,

$$b^i(x+h) - b^i(x) = \sum_j \partial_j b^i(x) h^j + r^i(x,h),$$

where $\partial b^i(x) = (\partial_j b^i(x))_{j\in I}$ is a continuous linear mapping on $l^\infty(I)$ and $|r^i(x,h)| = O(|h|_\infty^{1+\delta})$ for some $0 < \delta \le 1$.

3. For each $i \in I$, the norm of $\partial b^i(x)$ is uniformly *bounded* in x,

$$\sup_x |\partial_j b^i(x)| \le a^{i-j} \quad \text{with } |a|_1 < \infty.$$

REMARK 2.2. (i) Conditions 2 and 3 are naturally satisfied by a family of smooth drifts of finite range with bounded derivatives. Take $b^i \in C_b^2(R \otimes R^{N(i)})$, where $N(i)$ is a finite neighborhood of i. Then

$$b^i(x+h) - b^i(x) = \sum_{j \in N(i)} \partial_j b^i(x) h^j + \tfrac{1}{2} \sum_{j,\, k \in N(i)} \partial_j \partial_k b^i(y) h^j h^k$$

for some y between $x + h$ and x.

(ii) Condition 1 implies the stationarity of $(b^i)_{i \in I}$,

$$\partial_{j+k} b^{i+k}(x) = \partial_j b^i(\Theta^k x) \quad \text{for all } i, j, k \in I. \tag{2.3}$$

(iii) We could as well introduce diffusion coefficients to our system (2.1),

$$X_t^i = \xi^i + \int_0^t b^i(X_s)\, ds + \int_0^t \sigma(X_s^i)\, dW_s^i, \qquad i \in I,$$

and obtain the same kind of results as long as σ^2 is a strictly positive differentiable function with a bounded derivative.

By the mean value theorem we have

$$|b^i(x) - b^i(y)| \le \sum_j a^{i-j}|x^j - y^j| = (a * |x-y|)^i \tag{2.4}$$

for all $x - y \in l^\infty(I)$. Following Leha and Ritter (1984) we define the strictly positive sequence $(\gamma^k)_{k \in I} \in l^1(I)$,

$$\gamma := \sum_{n=1}^{\infty} \tilde{a}^{*n}/K^n,$$

where $\tilde{a}$ is a strictly positive sequence with $\tilde{a}^k \ge a^{-k}$ and $|a|_1 < K$. Since γ is superharmonic with respect to the kernel $\tilde{a}/K$, $\tilde{a} * \gamma \le K\gamma$, inequality (2.4) implies

$$\begin{aligned} |b(x) - b(y)|_{\gamma,2}^2 &= \sum_i \gamma^i |b^i(x) - b^i(y)|^2 \\ &\le |a|_1 \sum_{i,j} \gamma^i a^{i-j} |x^j - y^j|^2 \le |a|_1 K |x-y|_{\gamma,2}^2, \end{aligned}$$

i.e., the mapping $b\colon x \to (b^i(x))_{i \in I}$ defines a Lipschitz continuous function from $l^2(\gamma)$ to $l^2(\gamma)$. Rewriting (2.1) as an $l^2(\gamma)$-valued stochastic differential equation,

$$X_t = \xi + \int_0^t b(X_s)\, ds + W_t, \tag{2.1'}$$

we have by Theorem 3.3 and Remark 4.6 of Leha and Ritter (1984)

PROPOSITION 2.5. *Assume Conditions* 1–3. *Then for each* $\xi \in l^2(\gamma)$, (2.1′) *has a unique strong solution* $X = (X_t^i(\xi, W),\ 0 \le t \le 1)_{i \in I} \in C([0,1], l^2(\gamma))$, *which satisfies*

$$E^P\left[\sum_k \gamma^k |X_t^k|^2\right] < \infty. \tag{2.6}$$

REMARK 2.7. (i) The stationarity of $(b^i)_{i\in I}$ and the uniqueness of the solution imply the stationarity of $X(\xi, W)$,

$$\Theta^k X(\xi, W) = X\Theta^k(\xi, W), \quad P \text{ a.s. for all } k \in I.$$

(ii) Using Doob's inequality we can generalize inequality (2.6),

$$E^P\left[\sum_k \gamma^k \|X^k\|^2\right] < \infty. \tag{2.6$'$}$$

We shall now see how the smoothness of the drifts implies the smoothness of the solution of (2.1). Note that, although the mapping b: $(x^i)_{i\in I} \to (b^i(x))_{i\in I}$ is Lipschitz continuous with respect to $|\ |_{\gamma,2}$, it is not necessarily differentiable with respect to $|\ |_{\gamma,2}$. In order to cover such examples as in Remark 2.2(i), we have to introduce the norm $|\ |_\infty$. This makes the extension of finite-dimensional results more delicate. Denote by $B(l^\infty(I))$ the space of bounded linear operators from $l^\infty(I)$ to $l^\infty(I)$ and by $\|\ \|_B$ the operator norm. The matrix $(\partial_j b^i(x))_{(j,i)\in I\otimes I}$ can be considered as an element of $B(l^\infty(I))$ with

$$\|\partial b(x)\|_B \le |a|_1 \tag{2.8}$$

by Condition 3. Let $\Phi = (\Phi_{t,s}^{i,k}(X),\ 0 \le s \le t \le 1)_{(i,k)\in I\otimes I}$ be the fundamental solution in $B(l^\infty(I))$ of the linear equation

$$\Phi_{t,s}^{i,k} = \delta^{i,k} + \int_s^t \sum_j \partial_j b^i(X_u)\Phi_{u,s}^{j,k}\,du, \qquad (i,k) \in I \otimes I, \tag{2.9}$$

respectively, in operator form,

$$\Phi_{t,s} = E + \int_s^t \partial b(X_u)\Phi_{u,s}\,du, \tag{2.9$'$}$$

where E denotes the identity. The matrix $\Phi_{t,s}$ is given by $\Phi_{t,s} = Z_t Z_s^{-1}$, where $(Z_t,\ 0 \le t \le 1)$ is the solution of

$$Z_t = E + \int_0^t \partial b(X_u) Z_u\,du$$

and Z_s^{-1} is the inverse of Z_s [cf. Daleckiĭ and Kreĭn (1974)]. Note that (2.8) implies

$$\|\Phi_{t,s}(X)\|_B \le e^{|a|_1(t-s)}; \tag{2.10}$$

moreover, by (2.3), $\Phi_{t,s}$ is stationary,

$$\Phi_{t,s}^{i+j,k+j}(X) = \Phi_{t,s}^{i,k}(\Theta^j X) \quad \text{for all } i, k, j \in I.$$

Thus defining the positive coefficients

$$c^{i-k}(s) := \sup_X \sup_{s\le t\le 1} \left|\Phi_{t,s}^{i,k}(X)\right|,$$

by (2.10) we have

$$\sum_k c^k(s) \le e^{|a|_1(1-s)}. \tag{2.11}$$

The process $\Phi_{t,s}^{i,k}$ plays the role of the gradient of X_t^i with respect to X_s^k. If we interpret $c^{i-k}(s)$ as a weight of the influence of the kth coordinate on the trajectory of $(X_t^i,\ s \le t \le 1)$, inequality (2.11) roughly says that the total influence remains bounded. If we have nonconstant diffusion coefficients $(\sigma(x^i))_{i\in I}$, the gradient process is of the form

$$\Phi_{t,s}^{i,k} = \delta^{i,k} + \int_s^t \sum_j \partial_j b^i(X_u)\Phi_{u,s}^{j,k}\,du + \int_s^t \sigma'(X_u^i)\Phi_{u,s}^{i,k}\,dW_u^i, \qquad (i,k) \in I \otimes I,$$

and $c^{k-i}(s)$ should be replaced by

$$c^{k-i}(s) = \sup_X E^P\left[\sup_{s\le t\le 1} |\Phi_{t,s}^{i,k}|\,\middle|\,\mathscr{F}_s\right].$$

We conclude this section with an application of the variation of parameters method to the following linear perturbation of X.

Let $U = (U_t^k,\ 0 \le t \le 1)_{k\in I} \in C([0,1], l^\infty(I))$ be an $(\mathscr{F}_t)$-adapted process of the form

$$U_t^k = U_0^k + \int_0^t u_s^k\,ds, \qquad k \in I,$$

such that

$$E^P\big[\|U\|_\infty^{1+\delta}\big] = k(U,\delta) < \infty, \tag{2.12}$$

with $\|U\|_\infty := \sup_k \|U^k\|$. Denote by $X(\varepsilon) \in C([0,1], l^2(\gamma))$ the solution of

$$X_t(\varepsilon) = \xi + \int_0^t b(X_s(\varepsilon))\,ds + \varepsilon U_t + W_t. \tag{2.13}$$

LEMMA 2.14. *The following inequalities hold*:

$$E^P\big[\|X(\varepsilon) - X\|_\infty^{1+\delta}\big] \le \varepsilon^{1+\delta} k(U,\delta) e^{(1+\delta)|a|_1}, \tag{2.15}$$

$$E^P\big[\|X(\varepsilon) - X - \varepsilon\Psi\|_\infty\big] = o(\varepsilon), \tag{2.16}$$

where $\Psi = (\Psi_t^k,\ 0 \le t \le 1)_{k\in I}$ *is the process given by*

$$\Psi_t = \Phi_{t,0}U_0 + \int_0^t \Phi_{t,s}u_s\,ds.$$

PROOF. Let $\eta = (\eta_t^k,\ 0 \le t \le 1)_{k\in I} \in C([0,1], l^\infty(I))$ be the positive solution of the linear equation

$$\eta_t = \int_0^t a * \eta_s\,ds + \varepsilon|U_t|.$$

By (2.4), $X(\varepsilon) - X$ satisfies the inequality

$$|X_t(\varepsilon) - X_t| \le \int_0^t a * |X_s(\varepsilon) - X_s|\,ds + \varepsilon|U_t|;$$

hence, $\eta - |X(\varepsilon) - X|$ is in the cone of the nonnegative functions $C([0,1], l_+^2(\gamma))$ [cf. Daleckiĭ and Kreĭn (1974)], i.e., $X(\varepsilon) - X \in C([0,1], l^\infty(I))$ with

$$\|X(\varepsilon) - X\|_\infty \le \|\eta\|_\infty \le \varepsilon e^{|a|_1}\|U\|_\infty.$$

This implies the first inequality by (2.12). For the second inequality, note that $\Psi \in C([0,1], l^\infty(I))$ is the solution of the inhomogeneous equation

$$\Psi_t = U_0 + \int_0^t \partial b(X_s)\Psi_s\, ds + \int_0^t u_s\, ds$$

[cf. Daleckii and Krein (1974)]. Put $\zeta = X(\varepsilon) - X - \varepsilon\Psi$; then $E^P[\|\zeta\|_\infty] < \infty$ by (2.12) and (2.15) and

$$\zeta_t = \int_0^t \partial b(X_s)\zeta_s\, ds + R_t(\varepsilon)$$

with

$$R_t(\varepsilon) = \int_0^t \{b(X_s(\varepsilon)) - b(X_s) - \partial b(X_s)(X_s(\varepsilon) - X_s)\}\, ds.$$

Condition 2 and (2.15) yield

$$E^P[\|R(\varepsilon)\|_\infty] \le KE^P[\|X(\varepsilon) - X\|_\infty^{1+\delta}] = O(\varepsilon^{1+\delta}).$$

Hence, by (2.8), we obtain

$$E^P\Big[\sup_{0\le u\le t} |\zeta_u|_\infty\Big] \le |a|_1 \int_0^t E^P\Big[\sup_{0\le u\le s} |\zeta_u|_\infty\Big]\, ds + O(\varepsilon^{1+\delta}),$$

which implies (2.16) by Gronwall's lemma. □

3. The infinite-dimensional Haussmann formula. Let F be a measurable functional on $C[0,1]^I$ with $E^P[F^2(X)] < \infty$. Then, since $X = X(\xi, W)$ is a strong solution of (2.1), $F(X)$ can be written as

$$F(X) = E^P[F(X)|\mathscr{F}_0] + \sum_{i\in I}\int_0^1 h_t^i\, dW_t^i, \tag{3.1}$$

where $(h_t^i,\ 0 \le t \le 1)_{i\in I}$ is a sequence of $(\mathscr{F}_t)$-adapted processes with

$$E^P\Big[\big(F(X) - E^P[F(X)|\mathscr{F}_0]\big)^2\Big] = \sum_i \int_0^1 E^P\big[(h_s^i)^2\big]\, ds$$

[cf. Hitsuda and Watanabe (1976)]. The purpose of this section is to generalize the Haussmann formula for infinite dimensions which gives an explicit representation for the integrands $(h^i)_{i\in I}$ of a smooth functional F in terms of the gradient process Φ and the derivative of F [cf. Haussmann (1978, 1979) and Blum (1986)].

Consider $C([0,1], l^\infty(I))$ as a Banach space endowed with the supremum norm $\|\ \|_\infty$. We introduce the class $\mathscr{C}^1$ of real-valued functionals F on $C([0,1], l^2(\gamma))$ which are Fréchet-differentiable with respect to the norm $\|\ \|_\infty$ on $C([0,1], l^\infty(I))$:

4. For all $X \in C([0,1], l^2(\gamma))$ and $H \in C([0,1], l^\infty(I))$,

$$F(X+H) - F(X) = \sum_j D_jF(X)H^j + R(X,H),$$

where $DF(X) = (D_jF(X))_{j\in I}$ is a continuous linear functional on $C([0,1], l^\infty(I))$ with norm uniformly bounded in X and $|R(X,H)| = O(\|H\|_\infty^{1+\delta})$ for some $0 < \delta \le 1$.

REMARK 3.2. The $D_jF(X)$, $j \in I$, denote the partial derivatives of F with respect to X^j. By the Riesz representation theorem there exists a (signed-) measure $D_jF(X, dt)$ on [0,1], such that

$$D_jF(X)H^j = \int_0^1 D_jF(X, dt)H_t^j.$$

The total variation of $D_jF(X, \cdot)$ equals the operator norm of $D_jF(X)$ and is uniformly bounded in X,

$$|D_jF| := \sup_X \|D_jF(X, \cdot)\|_{\mathrm{var}} < \infty,$$

with $|DF|_1 = \sum_j |D_jF| < \infty$ by condition 4.

The mean value theorem implies the Lipschitz continuity of F,

$$|F(X) - F(Y)| \le \sum_j |D_jF| \|X^j - Y^j\|; \tag{3.3}$$

hence, $F(X) \in L^2(P)$ as

$$E^P\left[F^2(X)\right] \le 2|DF|_1 \sum_j |D_jF| E^P\left[\|X^j\|^2\right] + 2F^2(0) < \infty,$$

by (2.6′) and (2.7). Lemma 3.4 is an immediate consequence of the chain rule.

LEMMA 3.4. *Let $X(\varepsilon)$ be the solution of the perturbed equation* (2.13) *and $F \in \mathscr{C}^1$; then the function $F(X(\varepsilon))$ is differentiable in $L^1(P)$ with respect to $\varepsilon \in [0,1]$, with*

$$\frac{d}{d\varepsilon} E^P[F(X(\varepsilon))]_{\varepsilon=0} = E^P[DF(X)\Psi] = E^P\left[\sum_j \int_0^1 D_jF(X, dt)\Psi_t^j\right]. \tag{3.5}$$

PROOF. By Condition 4 we have

$$\begin{aligned} F(X(\varepsilon)) - F(X) = \varepsilon DF(X)\Psi &+ DF(X)(X(\varepsilon) - X - \varepsilon\Psi) \\ &+ R(X, X(\varepsilon) - X). \end{aligned}$$

Thus

$$\begin{aligned} E^P[|F(X(\varepsilon)) - F(X) - \varepsilon DF(X)\Psi|] \le\ & |DF|_1 E^P\left[\|X(\varepsilon) - X - \varepsilon\Psi\|_\infty\right] \\ &+ KE^P\left[\|X(\varepsilon) - X\|_\infty^{1+\delta}\right], \end{aligned}$$

which implies (3.5) by (2.15) and (2.16). □

We can state the main result of this section.

THEOREM 3.6 (Haussmann formula). *Let F be in the class $\mathscr{C}^1$. Then*

$$F(X) = E^P[F(X)|\mathscr{F}_0] + \sum_i \int_0^1 E^P\left[DF(X)\Phi_{\cdot,s}^{\cdot,i}|\mathscr{F}_s\right] dW_s^i, \tag{3.7}$$

with

$$DF(X)\Phi_{\cdot,s}^{\cdot,i} = \sum_j \int_s^1 D_j F(X, dt)\Phi_{t,s}^{j,i}.$$

PROOF. We follow the argument of Bismut (1981) using the Girsanov transformation. Assume $E^P[F(X)|\mathscr{F}_0] = 0$. Then (3.7) is equivalent to

$$E^P\left[F(X)\sum_i \int_0^1 u_s^i\, dW_s^i\right] = E^P\left[\int_0^1 \sum_i DF(X)\Phi_{\cdot,s}^{\cdot,i} u_s^i\, ds\right] \tag{3.8}$$

for all adapted $(u_s^i, 0 \le s \le 1)_{i\in I}$ with bounded $\sum_i \int_0^1 (u_s^i)^2\, ds$. By (3.1) and Itô's formula we have

$$E^P\left[F(X)\sum_i \int_0^1 u_s^i\, dW_s^i\right] = E^P\left[\int_0^1 \sum_i h_s^i u_s^i\, ds\right],$$

which implies $h_s^i = E^P[DF(X)\Phi_{\cdot,s}^{\cdot,i}|\mathscr{F}_s]$ by a standard projection argument. The $(W_t^i(\varepsilon), 0 \le t \le 1)_{i\in I}$ satisfy

$$dW_t^i(\varepsilon) = dW_t^i - \varepsilon u_t^i\, dt, \qquad i \in I.$$

Then $(W^i(\varepsilon))_{i\in I}$ is a collection of independent Wiener processes under $(\Omega, \mathscr{O}, P^{(\varepsilon)})$, where $P^{(\varepsilon)}$ is given by the Girsanov transformation

$$dP^{(\varepsilon)} = G(\varepsilon)\, dP,$$

with $G(\varepsilon) = \exp(\varepsilon\sum_i\int_0^1 u_s^i\, dW_s^i - \frac{1}{2}\varepsilon^2\sum_i\int_0^1(u_s^i)^2\, ds)$ [cf. Theorem 4.1 of Hitsuda and Watanabe, (1976)], i.e.,

$$E^P[F(X)G(\varepsilon)] = E^P[F(X(\varepsilon))]. \tag{3.9}$$

Since $\sum_i\int_0^1(u_s^i)^2\, ds$ is bounded, we can differentiate the left side of (3.9) in the expectation and obtain

$$\frac{d}{d\varepsilon}E^P[F(X)G(\varepsilon)]\bigg|_{\varepsilon=0} = E^P\left[F(X)\sum_i \int_0^1 u_s^i\, dW_s^i\right].$$

By Lemma 3.4 the right side of (3.9) is differentiable with

$$\begin{aligned}\frac{d}{d\varepsilon}E^P[F(X(\varepsilon))]\bigg|_{\varepsilon=0} &= E^P[DF(X)\Psi] = E^P\left[\sum_{j,i}\int_0^1 D_jF(X, dt)\int_0^t \Phi_{t,s}^{j,i}u_s^i\, ds\right]\\ &= E^P\left[\sum_i\int_0^1\left(\sum_j\int_s^1 D_jF(X, dt)\Phi_{t,s}^{j,i}\right)u_s^i\, ds\right]\end{aligned}$$

by partial integration. □

The following example illustrates a potential theoretic approach to the Haussmann formula [cf. Davis (1980)].

EXAMPLE 3.10. Take a smooth function f of finite range with bounded derivatives: $f \in C_b^2(R \otimes R^{N(i)})$, $|N(i)| < \infty$, and consider the $\mathscr{C}^1$-functional

$F(X) := f(X_1)$. Let $(P_t,\ 0 \le t \le 1)$ denote the semigroup associated to $(X_t,\ 0 \le t \le 1)$ and define a space–time harmonic function v: $(0,1) \otimes R^I \to \mathrm{R}$,

$$v(t, X_t) := P_{1-t} f(X_t) = E^P[f(X_1)|\mathscr{F}_t], \tag{3.11}$$

by the Markov property of $(X_t,\ 0 \le t \le 1)$ [cf. Leha and Ritter (1984)]. By Lemma 3.4, $v(t, x)$ is differentiable in x with

$$\partial_i v(t, X_t) = E^P\left[\sum_j \partial_j f(X_1) \Phi_{1,t}^{j,i} | \mathscr{F}_t\right]. \tag{3.12}$$

Computing the quadratic variation of the martingale $(v(t, X_t),\ 0 \le t \le 1)$ and $(W_t^i,\ 0 \le t \le 1)$, we have

$$d\langle v, W^i \rangle_t = \partial_i v(t, X_t)\, dt.$$

Hence,

$$f(X_1) = E^P[f(X_1)|\mathscr{F}_0] + \sum_i \int_0^1 \partial_i v(t, X_t)\, dW_t^i,$$

which is equivalent to the Haussmann formula (3.7).

4. Central limit theorem and decay of covariance. Let the process $X = (X_t^i,\ 0 \le t \le 1)_{i \in I}$ start off with a measure μ carried by $l^2(\gamma)$ and let Q denote the law of X on $C([0,1], l^2(\gamma))$. Assuming that μ is shift invariant with respect to $(\Theta^i)_{i \in I}$, ergodic and weakly mixing, then Proposition 2.5 and Remark 2.7 imply the shift invariance and ergodicity of Q [cf. Kornfeld, Sinai and Fomin (1981)]. Moreover, for all $F \in L^1(Q)$, we have

$$\lim_{n \to \infty} |V_n|^{-1} \sum_{k \in V_n} F\Theta^k = E^Q[F(X)], \quad Q \text{ a.s.}, \tag{4.1}$$

where $V_n = [-n, n]^d$ [cf. Wiener (1939)]. We shall derive a central limit theorem for $\mathscr{C}^1$-functionals using the Haussmann formula and standard martingale techniques.

For a measurable function F on $C([0,1], l^2(\gamma))$ with $E^Q[F^2(X)] < \infty$, put

$$S_n^*(F) := |V_n|^{-1/2} \sum_{k \in V_n} \{F\Theta^k - E^Q[F(X)]\}.$$

Let $L(R^I)$ be the class of Lipschitz continuous functions on R^I,

$$|f(x) - f(y)| \le \sum_i \delta_i(f)|x^i - y^i| \quad \text{and} \quad \sum_i \delta_i(f) < \infty,$$

where $\delta_i(f) = \sup\{|f(x) - f(y)|/|x^i - y^i|: x^k = y^k,\ k \ne i\}$. We assume that the central limit theorem holds for the initial measure μ:

5. For all $f \in L(R^I)$, $S_n^*(f)(X_0)$ converges in law to a centered Gaussian random variable $Y_0(f)$ with variance

$$\sigma_0^2(f) = \sum_k \mathrm{cov}_\mu(f, f\Theta^k) \ge 0.$$

This condition is satisfied if, e.g., under μ the X_0^i, $i \in I$, are positively associated [cf. Newman (1980) and Fessler (1986)] or weakly dependent [cf. Bolthausen (1982) and Künsch (1982, 1984)]. Theorem 4.2 gives sufficient conditions for the central limit theorem.

THEOREM 4.2. *Let μ be a shift invariant ergodic measure satisfying Condition 5 and let F be an $L^2(Q)$-functional on $C([0,1], l^2(\gamma))$ of the form* (3.1)

$$F(X) = E^Q[F(X)|\mathscr{F}_0] + \sum_i \int_0^1 h_s^i \, dW_s^i,$$

such that

$$PF(X_0) := E^Q[F(X)|\mathscr{F}_0] \in L(R^I), \tag{4.3}$$

$$\int_0^1 E^Q\left[\left(\sum_i |h_s^i \Theta^{-i}|\right)^2\right] ds < \infty. \tag{4.4}$$

Then $S_n^*(F)$ converges in law to a centered Gaussian random variable $Y(F)$ with variance

$$\sigma^2(F) = \sum_k \operatorname{cov}_Q(F, F\Theta^k) \geq 0.$$

PROOF. We show that $S_n^*(F)$ converges in law to

$$Y(F) = Y_0(PF) + \int_0^1 E^Q\left[\left(\sum_i h_s^i \Theta^{-i}\right)^2\right]^{1/2} d\beta_s,$$

where $(\beta_s,\ 0 \leq s \leq 1)$ is a Wiener process independent of $Y_0(PF)$. This is equivalent to the preceding statement, since by (3.1) and (4.4),

$$\begin{aligned}\sum_k \operatorname{cov}_Q(F, F\Theta^k) &= \sum_k \operatorname{cov}_\mu(PF, PF\Theta^k) \\ &\quad + \sum_k E^Q\left[\left(\sum_i \int_0^1 h_s^i \, dW_s^i\right)\left(\sum_j \int_0^1 h_s^j \Theta^k \, dW_s^{j+k}\right)\right] \\ &= \sigma_0^2(PF) + \int_0^1 E^Q\left[\sum_{i,j} h_s^i h_s^j \Theta^{i-j}\right] ds = \operatorname{var}(Y(F)).\end{aligned}$$

Let $\varepsilon > 0$. By (4.4) we can choose $m \in N$ such that

$$\int_0^1 E^Q\left[\left(\sum_{i \notin V_m} |h_s^i \Theta^{-i}|\right)^2\right] ds < \varepsilon. \tag{4.5}$$

Define a continuous martingale $(M_t^{(m,n)},\ 0 \leq t \leq 1)$ by

$$M_t^{(m,n)} = S_n^*(PF)(X_0) + |V_n|^{-1/2} \sum_{k \in V_n} \int_0^t \left(\sum_{i \in V_m} h_s^i \Theta^{-i}\right) \Theta^k \, dW_s^k.$$

Then by (3.1) we have for $n > m$,

$$\begin{aligned}S_n^*(F) &= S_n^*(PF)(X_0) + S_n^*\left(\sum_{i \in V_m} \int_0^1 h_s^i \, dW_s^i\right) + S_n^*\left(\sum_{i \notin V_m} \int_0^1 h_s^i \, dW_s^i\right) \\ &= M_1^{(m,n)} + R_1^{(m,n)} + S_n^*\left(\sum_{i \notin V_m} \int_0^1 h_s^i \, dW_s^i\right),\end{aligned}$$

where $R_1^{(m,n)}$ is given by

$$\begin{aligned} R_1^{(m,n)} &= |V_n|^{-1/2} \sum_{k \in V_n} \left(\sum_{i \in V_m} \int_0^1 h_s^i \Theta^k \, dW_s^{i+k} \right) \\ &\quad - |V_n|^{-1/2} \sum_{k \in V_n} \left(\sum_{i \in V_m} \int_0^1 h_s^i \Theta^{k-i} \, dW_s^k \right) \\ &= |V_n|^{-1/2} \sum_{k \in \partial_m(V_n)} \int_0^1 \left(\sum_{i \in V_m} h_s^i \Theta^{k-i} \{ \chi_{V_n}(k-i) - \chi_{V_n}(k) \} \right) dW_s^k, \end{aligned}$$

with $\partial_m(V_n) := V_{n+m} - V_{n-m}$. Both $R_1^{(m,n)}$ and $S_n^*(\sum_{i \notin V_m} \int_0^1 h_s^i \, dW_s^i)$ can be neglected,

$$E^Q\left[\left(R_1^{(m,n)} \right)^2 \right] \le |\partial_m(V_n)| / |V_n| \int_0^1 E^Q\left[\left(\sum_i |h_s^i \Theta^{-i}| \right)^2 \right] ds \underset{n \to \infty}{\to} 0,$$

by (4.4). On the other hand we have

$$S_n^*\left(\sum_{i \notin V_m} \int_0^1 h_s^i \, dW_s^i \right) = |V_n|^{-1/2} \sum_{k \in V_n} \sum_{i \notin V_m} \int_0^1 h_s^i \Theta^k \, dW_s^{i+k}.$$

Thus (4.5) implies

$$\begin{aligned} &E^Q\left[\left(S_n^*\left(\sum_{i \notin V_m} \int_0^1 h_s^i \, dW_s^i \right) \right)^2 \right] \\ &\quad \le |V_n|^{-1} \sum_{k \in V_n} \int_0^1 E^Q\left[\sum_{i, j \notin V_m} |h_s^i \Theta^k| \, |h_s^j \Theta^{i+k-j}| \right] ds < \varepsilon. \end{aligned}$$

It remains to prove the convergence of $M_1^{(m,n)}$ as $n \to \infty$. Computing the quadratic variation of $M^{(m,n)}$, by (4.1) we have

$$\begin{aligned} \langle M^{(m,n)} \rangle_t &= |V_n|^{-1} \sum_{k \in V_n} \int_0^t \left(\sum_{i \in V_m} h_s^i \Theta^{-i} \right)^2 \Theta^k \, ds \\ &\underset{n \to \infty}{\to} \int_0^t E^Q\left[\left(\sum_{i \in V_m} h_s^i \Theta^{-i} \right)^2 \right] ds, \quad Q\text{. a.s.} \end{aligned}$$

This together with the convergence of $M_0^{(m,n)}$ by (4.3) and Condition 5 implies the convergence in law of $M_1^{(m,n)}$ as $n \to \infty$ to

$$Y^{(m)}(F) = Y_0(PF) + \int_0^1 E^Q\left[\left(\sum_{i \in V_m} h_s^i \Theta^{-i} \right)^2 \right]^{1/2} d\beta_s$$

[cf. Shiryayev (1981)] and we obtain (4.5) with $m \to \infty$. □

As a direct consequence we obtain our main result.

COROLLARY 4.6. *The central limit theorem holds for functionals of the class* $\mathscr{C}^1$.

PROOF. We first show the Lipschitz continuity of PF with the mean value theorem. Take a path $\xi(\varepsilon) = x + \varepsilon(y - x)$, $0 \le \varepsilon \le 1$, from x to y in $l^2(\gamma)$. Applying Lemma 3.4 with $U_t \equiv y - x$ and $\Psi_t = \Phi_{t,0}(y - x)$, we have $(d/d\varepsilon)PF(\xi(\varepsilon)) = (d/d\varepsilon)E^P[F(X(\xi(\varepsilon), W))] = \sum_i E^P[DF(X(\varepsilon))\Phi_{\cdot,0}^{\cdot,i}](y^i - x^i)$, where $E^P[DF(X(\varepsilon))\Phi_{\cdot,0}^{\cdot,i}] = E^P[\sum_j \int_0^1 D_j F(X(\varepsilon), dt)\Phi_{t,0}^{j,i}]$ is bounded in $l^1(I)$,

$$\sup_{\varepsilon} \left| E^P\left[DF(X(\varepsilon))\Phi_{\cdot,0}^{\cdot,i}\right] \right| \le \sum_j |D_j F| c^{j-i}(0).$$

Hence by the mean value theorem, $PF \in L(R^I)$ with

$$\delta_i(PF) \le \sum_j |D_j F| c^{j-i}(0). \tag{4.7}$$

Condition (4.4) follows immediately from the Haussmann formula,

$$|h_s^i \Theta^{-i}| \le \sum_j |D_j F| E^P\left[\|\Phi_{\cdot,s}^{j,i}\| \,|\mathscr{F}_s\right]\Theta^{-i} \le \sum_j |D_j F| c^{j-i}(s). \tag{4.8}$$

Hence

$$\int_0^1 E^Q\left[\left(\sum_i |h_s^i \Theta^{-i}|\right)^2\right] ds \le |DF|_1^2 (e^{2|a|_1} - 1)(2|a|_1)^{-1}$$

by (2.11). □

The Haussmann formula also provides information on the decay of covariances. Suppose that the following estimate holds:

6. $|\mathrm{cov}_\mu(f, g)| \le \sum_{j,k} \delta_j(f) d^{k-j}(0) \delta_k(g)$
for $f, g \in L(R^I)$, where $(d^k(0))_{k \in I}$ is a positive sequence in $l^1(I)$ [see, e.g., Künsch (1982) and Föllmer (1982)].

Put $\tilde{c}^k(s) := c^{-k}(s)$ and define a positive sequence $(d^k)_{k \in I} \in l^1(I)$ by

$$d := \tilde{c}(0) * d(0) * c(0) + \int_0^1 \tilde{c}(s) * c(s)\, ds,$$

with $|d|_1 \le e^{2|a|_1} |d(0)|_1 + (e^{2|a|_1} - 1)(2|a|_1)^{-1}$ [cf. (2.11)].

PROPOSITION 4.9. *The covariance of any two functionals F and G in $\mathscr{C}^1$ satisfies*

$$|\mathrm{cov}_Q(F, G)| \le \sum_{j,k} |D_j F| d^{k-j} |D_k G|. \tag{4.10}$$

PROOF. By (3.7) we have

$$\mathrm{cov}_Q(F, G) = \mathrm{cov}_\mu(PF, PG) + \int_0^1 \sum_i E^Q\left[h_s^i \tilde{h}_s^i\right] ds,$$

with $h_s^i = E^P[DF\Phi_{\cdot,s}^{\cdot,i}|\mathscr{F}_s]$ and $\tilde{h}_s^i = E^P[DG\Phi_{\cdot,s}^{\cdot,i}|\mathscr{F}_s]$. Since PF and $PG \in L(R^I)$, we can apply Condition 6 and with (4.7) obtain

$$\begin{aligned}\left|\operatorname{cov}_\mu(PF, PG)\right| &\le \sum_{i,l} \delta_i(PF) d^{l-i}(0)\delta_l(PG)\\ &\le \sum_{j,k} |D_jF| \Big(\sum_{i,l} c^{j-i}(0) d^{l-i}(0) c^{k-l}(0)\Big)|D_kG|\\ &= \sum_{j,k} |D_jF| (\tilde{c}(0)_* d(0)_* c(0))^{k-j} |D_kG|.\end{aligned}$$

On the other hand, inequality (4.8) yields

$$\begin{aligned}\left|\int_0^1 \sum_i E^Q[h_s^i \tilde{h}_s^i]\, ds\right| &\le \sum_{j,k} |D_jF| \int_0^1 \sum_i c^{j-i}(s) c^{k-i}(s)\, ds |D_kG|\\ &= \sum_{j,k} |D_jF| \left(\int_0^1 \tilde{c}(s)_* c(s)\, ds\right)^{k-j} |D_kG|.\end{aligned}$$

This, together with the preceding inequality, implies (4.10). □

REMARK 4.11. Estimate (4.10) implies an exponential decay of correlation [cf. Künsch (1982) and Föllmer (1982)]: Let $r(i, j) = r(|i - j|)$ be a shift invariant metric on I and put $|x|_{e^r,1} := \sum_i e^{r(i)}|x^i|$. If we suppose that both $|a|_{e^r,1} < \infty$ and $|d(0)|_{e^r,1} < \infty$, then $|d|_{e^r,1}$ is finite with

$$|d|_{e^r,1} \le \exp(2|a|_{e^r,1})|d(0)|_{e^r,1} + \left(\exp(2|a|_{e^r,1}) - 1\right)(2|a|_{e^r,1})^{-1}.$$

Applying (4.10) and the triangle inequality for r, we obtain

$$\sum_i |\operatorname{cov}_Q(F, G\Theta^i)| e^{r(i)} \le |d|_{e^r,1} |DF|_{e^r,1} |DG|_{e^r,1}. \tag{4.12}$$

Finally we derive an estimate for the Laplace asymptotics,

$$\Lambda_n^Q(F) := |V_n|^{-1} \log\left(E^Q\left[\exp\left(\sum_{k\in V_n} F\Theta^k(X)\right)\right]\right), \qquad F \in \mathscr{C}^1,$$

which can be used for the identification of the rate of the large deviation of the measure $|V_n|^{-1}\sum_{k\in V_n}\delta_{\Theta^k}$ [cf. Stroock (1984)].

PROPOSITION 4.13. *Assume that the initial distribution* μ *satisfies*

7. $\lim_{n\to\infty} \Lambda_n^\mu(f) = \Lambda^\mu(f)$ *for* $f \in L(R^I)$.

Then the following estimate holds for $F \in \mathscr{C}^1$:

$$\limsup_{n\to\infty} \Lambda_n^Q(F) \le \Lambda^\mu(PF) + \tfrac{1}{2}|DF|_1^2 (e^{2|a|_1} - 1)(2|a|_1)^{-1}. \tag{4.14}$$

PROOF. Let $F(X)$ be written in the form (3.1) and define the martingale $(M_t^{(n)}, 0 \le t \le 1)$

$$M_t^{(n)} := \sum_{k \in V_n} PF\Theta^k(X_0) + \sum_{k \in V_n} \left(\sum_i \int_0^t h_s^i \, dW_s^i \right) \Theta^k.$$

By (4.8) and (2.1), the quadratic variation of $M^{(n)}$ is bounded by

$$\frac{d\langle M^{(n)} \rangle_t}{dt} \le |V_n| \, |DF|_1^2 e^{2|\alpha|_1(1-t)}.$$

Hence Itô's formula implies (4.14) as

$$E^Q\left[\exp\left(M_1^{(n)}\right)\right] \le E^\mu\left[\exp\left(M_0^{(n)}\right)\right] \exp\left(\frac{|V_n|}{2} |DF|_1^2 \int_0^1 e^{2|\alpha|_1(1-t)} \, dt \right). \qquad \square$$

5. The fluctuation field. This section illustrates how the central limit theorem can be applied to the derivation of the fluctuation field.

Let $\mathscr{S}_n := \mathscr{S}(R^{V_n})$ be the Schwarz space of rapidly decreasing smooth functions on R^{V_n} and $\mathscr{S}_n' := \mathscr{S}(R^{V_n})$ the space of tempered distributions. Put $\mathscr{S}_\infty := \oplus_n \mathscr{S}_n$ for the direct sum viewed as a nuclear space [cf. Yamazaki (1985)] and $\mathscr{S}_\infty' := \otimes_n \mathscr{S}_n'$ for the dual. We introduce a continuous $\mathscr{S}_\infty'$-valued process $Y^{(n)} = (Y_t^{(n)}(\psi), 0 \le t \le 1)_{\psi \in \mathscr{S}_\infty}$,

$$Y_t^{(n)}(\psi) := S_n^*(\psi)(X_t) = |V_n|^{-1/2} \sum_{k \in V_n} \left\{ \psi\Theta^k(X_t) - E^Q[\psi(X_t)] \right\}.$$

The central limit theorem of the last section implies

PROPOSITION 5.1. *The process $Y^{(n)}$ converges in law to a continuous $\mathscr{S}'$-valued Gaussian process $Y = (Y_t(\psi), 0 \le t \le 1)_{\psi \in \mathscr{S}_\infty}$ with variance*

$$\sigma_t^2(\psi) = \sum_k \operatorname{cov}_Q\left(\psi(X_t), \psi\Theta^k(X_t)\right).$$

PROOF. Note that $\mathscr{S}_\infty \subset \mathscr{C}^1$, hence Corollary 4.6 implies the convergence of the finite-dimensional distributions of $Y^{(n)}$. By Theorem 5.3 of Mitoma (1983), it suffices to show the tightness of the laws of $(Y_t^{(n)}(\psi), 0 \le t \le 1)$ in $C[0,1]$ for fixed $\psi \in \mathscr{S}_\infty$. By Itô's formula we have

$$Y_t^{(n)}(\psi) = Y_0^{(n)}(\psi) + \int_0^t Y_s^{(n)}(\mathscr{L}\psi) \, ds + M_t^{(n)}(\psi), \tag{5.2}$$

where $\mathscr{L}\psi(x) = \sum_k \{ b^k(x) \partial_k \psi(x) + \frac{1}{2} \partial_k^2 \psi(x) \}$ is the generator of the diffusion $(X_t, 0 \le t \le 1)$ and $(M_t^{(n)}(\psi), 0 \le t \le 1)$ is the continuous martingale,

$$M_t^{(n)}(\psi) = |V_n|^{-1/2} \sum_{k \in V_n} \sum_i \int_0^t \partial_i \psi\Theta^k(X_s) \, dW_s^{k+i}.$$

We prove the inequalities

$$E^Q\left[\left(\int_s^t Y_u^{(n)}(\mathscr{L}\psi) \, du \right)^2 \right] \le k_1(\psi)|t-s|^2, \tag{5.3}$$

$$E^Q\left[\left(\int_s^t dM_u^{(n)}(\psi) \right)^4 \right] \le k_2(\psi)|t-s|^2 \tag{5.4}$$

for $0 \le s \le t \le 1$. This implies the tightness in $C[0,1]$ by Theorem 12.3 of Billingsley (1968). Note that

$$\int_s^t Y_u^{(n)}(\mathscr{L}\Psi)\,du = S_n^*\left(\int_s^t \mathscr{L}\psi(X_u)\,du\right),$$

where $\int_s^t \mathscr{L}\psi(X_u)\,du$ is a $\mathscr{C}^1$-functional with

$$\left|D\left(\int_s^t \mathscr{L}\psi(X_u)\,du\right)\right|_1 \le \sum_i |\partial_i(\mathscr{L}\psi)|\,|t-s|.$$

Applying inequality (4.12) with $r \equiv 0$, we obtain (5.3) as

$$E^Q\left[\left(S_n^*\left(\int_s^t \mathscr{L}\psi(X_u)\,du\right)\right)^2\right] \le \sum_k \left|\operatorname{cov}_Q\left(\int_s^t \mathscr{L}\psi(X_u)\,du, \int_s^t \mathscr{L}\psi\Theta^k(X_u)\,du\right)\right|$$

$$\le \left(\sum_i |\partial_i(\mathscr{L}\psi)|\right)^2 |d|_1 |t-s|^2.$$

For the second inequality it suffices to see that the quadratic variation of $M^{(n)}(\psi)$ remains bounded in n,

$$\sup_n d\langle M^{(n)}(\psi)\rangle_t/dt \le \left(\sum_i |\partial_i \psi|\right)^2$$

(cf. proof of Theorem 4.4). □

From now on we sharpen Condition 2 to:

2′. For all $i \in I$ there exists a finite neighborhood $N(i)$ of i such that $b^i \in \mathscr{S}(R \otimes R^{N(i)})$.

Then

$$\mathscr{L}\Psi(x) = \sum_k \left\{b^k(x)\partial_k\psi(x) + \tfrac{1}{2}\partial_k^2\psi(x)\right\},$$

$$\mathscr{D}\Psi(x) = \sum_k \partial_k\psi\Theta^{-k}(x)$$

define two linear operators from $\mathscr{S}_\infty$ to $\mathscr{S}_\infty$.

PROPOSITION 5.5. *The process* $(Y_t, 0 \le t \le 1)$ *satisfies the linear stochastic differential equation*

$$(5.6)\qquad Y_t(\psi) = Y_0(\psi) + \int_0^t Y_s(\mathscr{L}\psi)\,ds + \int_0^t dB_s(\mathscr{D}\psi), \qquad \psi \in \mathscr{S}_\infty,$$

where $B = (B_t(\psi), 0 \le t \le 1)_{\psi \in \mathscr{S}_\infty}$ *is a* $\mathscr{S}'_\infty$*-valued Wiener process with quadratic variation*

$$\langle B(\psi)\rangle_t = \int_0^t E^Q[\psi^2(X_s)]\,ds.$$

PROOF. Take $\psi \in \mathscr{S}_\infty$; by (5.2) we have

$$Y_t^{(n)}(\psi) - Y_0^{(n)}(\psi) - \int_0^t Y_s^{(n)}(\mathscr{L}\psi)\,ds = \int_0^t dM_s^{(n)}(\psi). \tag{5.2'}$$

Since $\psi(X_t) - \psi(X_0) - \int_0^t \mathscr{L}\psi(X_s)\,ds \in \mathscr{C}^1$, the left side of (5.2′) converges in law to

$$Y_t(\psi) - Y_0(\psi) - \int_0^t Y_s(\mathscr{L}\psi)\,ds$$

by Corollary 4.6 and Proposition 5.1. By the ergodic theorem we have

$$\langle M^{(n)}(\psi)\rangle_t \underset{n\to\infty}{\to} \int_0^t E^Q\left[(\mathscr{D}\psi(X_s))^2\right] ds, \quad Q \text{ a.s.}$$

(cf. proof of Theorem 4.2). This implies the convergence in law of the martingale $M^{(n)}(\psi)$ to the Wiener process $B(\mathscr{D}\psi)$ [cf. Shiryayev (1981)]. □

REMARK 5.7. Let $\psi \in \mathscr{S}_\infty$. Then the Haussmann formula yields

$$\psi(X_t) = P_t\psi(X_0) + \sum_i \int_0^t \partial_i P_{t-s}\psi(X_s)\,dW_s^i, \tag{5.8}$$

where $(P_t, 0 \le t \le 1)$ is the semigroup associated to $(X_t, 0 \le t \le 1)$ (cf. Example 3.10). Note that

$$\mathscr{D}(P_{t-s}\psi)(x) = \sum_k \partial_k P_{t-s}\psi\Theta^{-k}(x)$$

is well defined as

$$\left|\partial_k P_{t-s}\psi\Theta^{-k}\right| \le \sum_j |\partial_j\psi| c^{j-k}(1 - (t-s))$$

by (4.8). Applying Theorem 4.2 to (5.8), we obtain an explicit form for the solution of (5.6),

$$Y_t(\psi) = \overline{Y}_0(P_t\psi) + \int_0^t d\overline{B}_s(\mathscr{D}(P_{t-s}\psi)),$$

where $\overline{Y}_0$ and $(\overline{B}_t, 0 \le t \le 1)$ denote $L^2(Q)$-completions of Y_0 and $(B_t, 0 \le t \le 1)$.

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