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KILLING A MARKOV PROCESS UNDER A STATIONARY MEASURE INVOLVES CREATION[1]

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We construct the Kuznetsov measure associated with an excessive measure, m, and the semigroup of a process killed according to a multiplicative functional, M, in terms of the Kuznetsov measure corresponding to m and the original semigroup and certain functionals coming from M. We also obtain an explicit decomposition of m into its invariant and purely excessive parts relative to the killed semigroup. This decomposition involves objects related to the fine structure of M that have been introduced in recent years.

1. Introduction. One of the oldest constructions in the theory of Markov processes is the killing of a process by means of a multiplicative functional. The standard methods for this construction go back to Dynkin and Meyer and are described in Section III-3 of [2]. In recent years there has been considerable interest in studying a Markov process under a stationary (generally not finite) measure. See, for example, [4], [5], [13] and [18]. The purpose of this paper is to study the killing construction in this framework. If M is the multiplicative functional in question, then it is not surprising that the killing rate is (in some sense) given by $-dM_t$, nor is it surprising that if stationarity is to be preserved, then the killing must be balanced by birthing (or creation). What is surprising is that the creation rate is (in some sense) given by $d(1/M_t)$.

To be more precise let X be a Borel right process and m be a σ-finite excessive measure for X. Then there exists a stationary measure Q_m on two-sided paths Y with the same transition mechanism as X. This is described in Section 2. Let M be a multiplicative functional of X and let $K = (K_t)$ be the semigroup of the process killed according to M; that is, $K_t f(x) = P^x[f \circ X_t M_t]$. Then there exists a stationary measure Q^* corresponding to m and K, and the purpose of this paper is to express Q^* in terms of Q_m. It turns out that Q^* is obtained from Q_m by killing and birthing. The construction of Q^* in terms of Q_m is carried out in Section 4 and the interpretation as killing at rate $-dM$ and creation at rate $d(1/M)$ is given in Section 5. Of course, this must be appropriately interpreted since M may vanish and M is only defined over X (one-sided paths). Roughly speaking, the killing and creation rates are given by $-dM$ and $d(1/M)$ appropriately shifted in each interval on which the shifted M does not vanish. See Section 5 for the precise statements.

This investigation was suggested by the following observation of P. J. Fitzsimmons (oral communication). Let $M_t = e^{-qt}$, $0 < q < \infty$. Let U and V be "random variables" independent of Y under Q_m with distribution

$$Q_m(U \in du, V \in dv) = q^2 e^{-q(v-u)} 1_{\{u<v\}}\, du\, dv.$$

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Then if (P_t) is the semigroup of X, the stationary measure corresponding to m and $K_t = e^{-qt}P_t$ is obtained from Q_m by "birthing Y at U and killing it at V." The results in Sections 4 and 5 are the extension of this observation to general M. See especially the particular case described at the beginning of Section 5.

Section 2 contains the precise assumptions we make on the process X. It also contains the description of the associated stationary process (Y, Q_m) and its relationship to X. In Section 3 we give the definition of a multiplicative functional and then describe its extension to a functional over (Y, Q_m). This follows Mitro [16], [17] and is more or less known. The main results are in Sections 4 and 5. Actually, the stationary measure Q^* described above and in Sections 4 and 5 corresponds to the semigroup $K = (K_t)$ and m^* the restriction of m to the set E_M of permanent points of M rather than K and m. The measure m^* is (K_t) excessive and in Section 6 we describe the decomposition of m^* into its (K_t) invariant part, m_i^*, and its (K_t) purely excessive part, m_p^*. The expression for m_p^* in Theorem 6.16 is quite interesting as it involves several quantities related to the structure of M. Finally, there is an Appendix in which some of the results in [10] about homogeneous random sets and measures associated with a multiplicative functional are extended to the stationary process (Y, Q_m). These results are used in Sections 5 and 6 and should be reviewed before reading these sections. Logically, the Appendix might come directly after Section 3 but the results it contains are not needed in Section 4.

Most likely there should be analogous results for the "birthing" construction associated with a comultiplicative functional as described in [7]. This is currently being investigated by my student E. Toby.

Our notation is for the most part standard. For example, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}^+$ denote the rationals, the reals, the positive (i.e., nonnegative) reals, respectively. If $(E, \mathscr{E})$ is a measurable space $b\mathscr{E}$ $(p\mathscr{E})$ denotes the bounded (positive) $\mathscr{E}$-measurable functions while $\mathscr{E}^*$ denotes the σ-algebra of universally measurable sets over $(E, \mathscr{E})$. If μ is a measure on $(E, \mathscr{E})$ and $f \in p\mathscr{E}$, then $f\mu$ or $f \cdot \mu$ denotes the measure $f(x)\mu(dx)$. If $(F, \mathscr{F})$ is another measurable space and ϕ: $E \to F$, we write $\phi \in \mathscr{E}/\mathscr{F}$ if ϕ is a measurable mapping $(\phi^{-1}(\mathscr{F}) \subset \mathscr{E})$, and in this case $\phi(\mu)$ is the image of μ under ϕ; that is, the measure on $(F, \mathscr{F})$ defined by $\phi(\mu)(A) = \mu(\phi^{-1}(A))$ for $A \in \mathscr{F}$. If E is a topological space, $\mathscr{B}(E)$ denotes the σ-algebra of Borel subsets in E; that is, the σ-algebra generated by the open subsets of E. The end of a proof is marked by the symbol "□."

2. Preliminaries. Let E be a Borel subset of a compact metric space and $\mathscr{E}$ the σ-algebra of Borel subsets of E. Let a and b be distinct points not in E and define $E^a = E \cup \{a\}$, $E_b = E \cup \{b\}$ and $E_b^a = E \cup \{a, b\}$. Topologize E_b^a so that E has its original topology and a and b are isolated points in E_b^a. We regard E^a and E_b as subspaces of E_b^a and we let $\mathscr{E}^a$, $\mathscr{E}_b$ and $\mathscr{E}_b^a$ denote the Borel σ-algebras in these spaces. A function f on E is automatically extended to E_b^a by $f(a) = f(b) = 0$ unless explicitly stated otherwise.

Let Ω be the set of all right-continuous trajectories ω: $\mathbb{R}^+ \to E_b$ with b as cemetery. As usual $X_t(\omega) = \omega(t)$, $\theta_t\omega(s) = \omega(t + s)$, $\mathscr{F}^0 = \sigma(X_t, t \geq 0)$ and $\mathscr{F}_t^0 = \sigma(X_s,\ 0 \leq s \leq t)$. We assume given a Borel right process $X = (\Omega, \mathscr{F}^0, \mathscr{F}_t^0, X_t, \theta_t, P^x)$ in the sense of [6]. Let $(P_t)_{t \geq 0}$ and $(U^q)_{q \geq 0}$ denote the transition semigroup and resolvent of X, respectively. Here $P_0 = I$ and we write $U = U^0$. Let $\zeta = \inf\{t: X_t = b\}$ be the lifetime of X and P^b denote unit mass at $[b]$—the trajectory that is identically equal to b.

Denote by W the set of all maps $w: \mathbb{R} \to E_b^a$ such that there exists a nonvoid open interval $]\alpha(w), \beta(w)[$ on which w is E-valued and right continuous, with $w(t) = a$ if $t \le \alpha(w)$ and $w(t) = b$ if $t \ge \beta(w)$. In addition, we suppose that W contains the constant maps $[a]$ and $[b]$. (Note that $[b]$ is used in two senses: $[b] \in \Omega$ is the constant map defined for $t \ge 0$ while $[b] \in W$ is the constant map defined for $-\infty < t < \infty$. This should cause no confusion as the meaning will be clear from the context.) Let $Y_t(w) = w(t)$ be the coordinate maps on W and $\theta_t w(s) = w(t+s)$. Here Y_t and θ_t are defined for $t \in \mathbb{R}$. Note again that θ_t is used for the shift in W and in Ω. Let $\mathscr{G}^0 = \sigma(Y_t;\ t \in \mathbb{R})$ and $\mathscr{G}_t^0 = \sigma(Y_s,\ -\infty < s \le t)$. Observe that $\alpha(w) = \sup\{t: w(t) = a\}$ and $\beta(w) = \inf\{t: w(t) = b\}$, where the infimum and the supremum of the empty set are $+\infty$ and $-\infty$, respectively. Taking this as the definition of α and β when $w = [a]$ or $[b]$ gives $\alpha([a]) = +\infty = \beta([a])$, $\alpha([b]) = -\infty = \beta([b])$. In particular, α and β are $(\mathscr{G}_{t+}^0)$ stopping times in the sense that $\{\alpha < t\}$ and $\{\beta < t\}$ are in $\mathscr{G}_t^0$ for each $t \in \mathbb{R}$. Clearly, $\alpha \circ \theta_t = \alpha - t$ and $\beta \circ \theta_t = \beta - t$ for $t \in \mathbb{R}$. For typographical convenience we shall sometimes write $Y(t)$ for Y_t and $X(t)$ for X_t.

The spaces Ω and W are related by the mappings $\gamma_t: W \to \Omega$ defined for $t \in \mathbb{R}$ as

$$(2.1) \qquad \begin{aligned} \gamma_t w(s) &= w(t+s), &&\text{for } s \ge 0 \text{ if } \alpha(w) < t,\\ &= b, &&\text{for } s \ge 0 \text{ if } \alpha(w) \ge t. \end{aligned}$$

Clearly, if $t \in \mathbb{R}$, $\gamma_t = \gamma_0 \circ \theta_t$. If $\alpha < t$, $X_s \circ \gamma_t = Y_{s+t}$, and if $\alpha < t < \beta$, $\zeta \circ \gamma_t = \beta \circ \theta_t$. Note that γ_t is $\mathscr{G}_{t+s}^0/\mathscr{F}_s^0$ measurable for each $s \ge 0$ and $t \in \mathbb{R}$. One easily checks the following useful identities:

$$(2.2) \qquad \begin{aligned} &\text{(i)} \quad \gamma_t \circ \theta_s = \gamma_{t+s}, &&\text{on } W \text{ for all } s, t \in \mathbb{R},\\ &\text{(ii)} \quad \theta_s \circ \gamma_t = \gamma_{t+s}, &&\text{on } \{\alpha < t\} \text{ for } s \ge 0,\ t \in \mathbb{R}. \end{aligned}$$

If m is an excessive measure for X (that is, m is σ-finite and $mP_t \le m$ for each $t > 0$), then it follows from a theorem of Kuznetsov [12], see also [13] or [9], that there exists a unique measure Q_m on $(W, \mathscr{G}^0)$ not charging $[a]$ or $[b]$ such that if $t \in \mathbb{R}$ and $F \in p\mathscr{F}^0$, then

$$(2.3) \qquad Q_m\big(F \circ \gamma_t | \mathscr{G}_{t+}^0\big) = P^{Y(t)}(F), \quad \text{on } \{\alpha < t\}.$$

Moreover, Q_m is σ-finite. Of crucial importance is the fact that Q_m is *stationary*; that is, $\theta_t(Q_m) = Q_m$ for each $t \in \mathbb{R}$. In addition, Y is strong Markov under Q_m. To be precise let $\mathscr{G}^m$ be the completion of $\mathscr{G}^0$ under Q_m, and for $t \in \mathbb{R}$ let $\mathscr{G}_t^m$ be the σ-algebra generated by $\mathscr{G}_t^0$ and the ideal of all Q_m null sets in $\mathscr{G}^m$. The filtration $(\mathscr{G}_t^m)$ is right continuous. A map τ from W to $\mathbb{R} \cup \{-\infty, +\infty\}$ is a Q_m stopping time provided $\{\tau \le t\} \in \mathscr{G}_t^m$ for each $t \in \mathbb{R}$, and for such τ let $\mathscr{G}_\tau^m$ be the associated σ-algebra (i.e., all sets $G \in \mathscr{G}^m$ such that $G \cap \{\tau \le t\} \in \mathscr{G}_t^m$ for each $t \in \mathbb{R}$). Then if $F \in p\mathscr{F}^0$ and $G \in p\mathscr{G}_\tau^m$, one has

$$(2.4) \qquad Q_m[F \circ \gamma_\tau G;\ \alpha < \tau] = Q_m\big[P^{Y(\tau)}(F)G;\ \alpha < \tau\big],$$

and Q_m is σ-finite on the trace of $\mathscr{G}_\tau^m$ on $\{\alpha < \tau < \beta\}$. This may be proved as in [15]. See also [13].

We shall call Q_m the Kuznetsov measure corresponding to (P_t) and m.

3. Multiplicative functionals. Let $M = (M_t)_{t \geq 0}$ be a right-continuous exact multiplicative functional of X with $0 \leq M_t(\omega) \leq 1$ for each $t \geq 0$ and $\omega \in \Omega$. In light of Meyer's master perfection theorem [14], we may suppose that M has the following properties:

(3.1) $t \to M_t(\omega)$ is decreasing, right continuous and has values in $[0,1]$ for each $\omega \in \Omega$;

(3.2) $$M_t(\omega) = 0, \text{ if } t \leq \zeta(\omega) \quad \text{and} \quad M_t([b]) = 0, \quad \text{for all } t;$$

(3.3) $$M_{t+s}(\omega) = M_t(\omega)M_s(\theta_t\omega), \quad \text{for each } t, s \geq 0 \text{ and } \omega \in \Omega;$$

(3.4) $$M_t \in \mathscr{F}_{t+}^* = \bigcap_{s>t} \mathscr{F}_s^*, \quad \text{for each } t \text{ where } \mathscr{F}_s^* = \left(\mathscr{F}_s^0\right)^*;$$

(3.5) $$\lim_{s \downarrow 0} M_{t-s}(\theta_s\omega) = M_t(\omega), \quad \text{for each } t > 0 \text{ and } \omega \in \Omega.$$

In the sequel a functional satisfying (3.1)–(3.4) will be called a multiplicative functional (MF) and an exact MF is an MF which satisfies (3.5) in addition. It is immediate from (3.3) and the zero–one law that for each x either $P^x(M_0 = 1) = 1$ or $P^x(M_0 = 0) = 1$. The set E_M of *permanent points* of M is defined by $E_M = \{x \in E\colon\ P^x(M_0 = 1) = 1\}$. Then x is not in E_M if and only if almost surely P^x, $M_t = 0$ for all $t \geq 0$. Also (3.3) and (3.5) imply that $s \to M_{t-s}(\theta_s\omega)$ is increasing and right continuous on $[0, t[$ for each $t > 0$ and $\omega \in \Omega$.

We are going to define certain functionals on W in terms of M. We suppose that M is an MF of X, but we do not assume that M is exact for the moment. Define for $\alpha(w) < s \leq t$,

(3.6) $$\begin{aligned} N(s,t) = N(s,t;w) &= M_{t-s}(\gamma_s w), \quad && \text{if } \alpha(w) < s < t, \\ &= 1, && \text{if } \alpha(w) < s = t. \end{aligned}$$

The following lemma collects some elementary properties of $N(s, t)$.

(3.7) LEMMA. (i) *$s \to N(s,t)$ is increasing on $]\alpha, t[$, and $N(s,t) = 0$ if $\alpha < s < \beta \leq t$.*

(ii) *If $\alpha < s \leq t \leq u$, then $N(s,u) = N(s,t)N(t,u)$.*

(iii) *Let $\mathscr{G}_t^* = (\mathscr{G}_t^0)^*$. Then $N(s,t) \in \mathscr{G}_{t+}^*$.*

(iv) *If, in addition, M is exact, then $s \to N(s,t)$ is right continuous on $]\alpha, t[$.*

PROOF. We leave the straightforward verification of the first three properties to the reader. For (iv) suppose $\alpha < r < s < t$. Using (2.2)(ii)

$$M_{t-s} \circ \gamma_s = M_{t-r-(s-r)}(\theta_{s-r} \circ \gamma_r) \to M_{t-r} \circ \gamma_r,$$

as s decreases to r since M is exact. Hence $s \to N(s,t)$ is right continuous on $]\alpha, t[$, proving (iv). □

We next define a functional $N_t(w)$ for $t \in \mathbb{R}$ that we regard as the extension of M from Ω to W. For $t \in \mathbb{R}$ define

$$(3.8)\qquad \begin{aligned} N_t &= \inf\{N(s,t);\ \alpha < s < t\} = \lim_{s \downarrow \alpha} N(s,t), \qquad \alpha < t, \\ &= 1, \qquad t \le \alpha. \end{aligned}$$

Thus $N_t = N(\alpha + , t)$ if $\alpha < t$ and we extend the definition of $N(s,t)$ by setting

$$(3.9)\qquad \begin{aligned} N(\alpha, t) &= N_t = N(\alpha + , t), \quad \text{if } \alpha < t, \\ N(s,t) &= 0, \qquad\qquad\qquad \text{if } s < \alpha < t. \end{aligned}$$

Note that $N_t = 0$ if $t \ge \beta$ because of (3.2). It follows from (3.7)(ii) that

$$(3.10)\qquad N_t = N_s N(s,t), \quad \text{if } \alpha < s < t.$$

In particular, $t \to N_t$ is decreasing, $N_t = 1$ if $t \le \alpha$, and $N_t = 0$ if $t \ge \beta$. Since $t \to N(s,t) = M_{t-s} \circ \gamma_s$ is right continuous on $[s, \infty[$, it follows from (3.10) that $t \to N_t$ is right continuous on $]\alpha, \infty[$.

(3.11) PROPOSITION. *The function $t \to N_t$ is right continuous and decreasing on $]\alpha, \infty[$, it equals one on $]-\infty, \alpha]$ and zero on $[\beta, \infty[$. For each t, N_t is $\mathscr{G}^*_{t+}$-measurable. Let $N_{\alpha+} = \lim_{s\downarrow\alpha} N_s = \sup_{s>\alpha} N_s$. Then $N_{\alpha+}$ is either zero or one and if $N_{\alpha+} = 0$, one has $N_t = 0$ for all $t > \alpha$. If $\alpha < s + t$,*

$$(3.12)\qquad N_{t+s} = N_t \circ \theta_s.$$

PROOF. The assertions in the first sentence have already been checked and the measurability assertion is immediate from (3.7)(iii). Let s decrease to α in (3.10) to obtain $N_t = N_{\alpha+} N_t$ and letting t decrease to α, $N_{\alpha+} = (N_{\alpha+})^2$. Clearly, $N_t = 0$ for all $t > \alpha$ if $N_{\alpha+} = 0$. If $\alpha < t + s$, then $\alpha \circ \theta_s < t$ and so

$$\begin{aligned} N_t \circ \theta_s &= \lim_{r \downarrow \alpha \circ \theta_s} M_{t-r} \circ \gamma_r \circ \theta_s = \lim_{r \downarrow \alpha - s} M_{t+s-(s+r)} \circ \gamma_{s+r} \\ &= \lim_{u \downarrow \alpha} M_{t+s-u} \circ \gamma_u = N_{t+s}. \end{aligned}$$ □

We next give two examples. Let $q > 0$ and $M_t = e^{-qt} 1_{[0,\zeta[}(t)$. If $\alpha < t$, then

$$N_t = \lim_{s \downarrow \alpha} e^{-q(t-s)} 1_{[0, \zeta \circ \gamma_s[}(t - s).$$

Since $\alpha(w) < t$, $w \ne [a]$. If $w = [b]$, then $\zeta \circ \gamma_s[b] = 0$ so $N_t([b]) = 0$. If $w \ne [b]$, then $\alpha < s < \beta$ for s close to α and $\zeta \circ \gamma_s = \beta \circ \theta_s$, so

$$(3.13)\qquad N_t = e^{q(\alpha - t)} 1_{\{\alpha < t < \beta\}}, \quad \text{if } \alpha < t,$$

in all cases because $\alpha([b]) = -\infty$. Note that $N_t = 0$ on $\{\alpha = -\infty\}$.

For our second example let R be a perfect terminal time for X. To be precise we suppose, as we may without loss of generality, that $\{R \le t\} \in \mathscr{F}^*_{t+}$ and $t + R \circ \theta_t = R$ on $\{t < R\}$ for each $t \in \mathbb{R}^+$. Note that $R([b])$ is either zero or infinity. Clearly, $M_t = 1_{[0, R \wedge \zeta[}(t)$ is an MF of X. Following [5] define

$$(3.14)\qquad R^*(w) = \inf\{t + R(\gamma_t w) \colon \alpha(w) < t\}.$$

In [5], Fitzsimmons and Maisonneuve use $\tilde{R}$ in place of R^*. One checks easily that $R^* \geq \alpha$, $t + R^* \circ \theta_t = R^*$ for all $t \in \mathbb{R}$, and that R^* is a $(\mathscr{G}_{t+}^*)$ stopping time. The next proposition gives some additional properties of R^* and relates it to N.

(3.15) PROPOSITION. *Let R and R^* be as above and let $M_t = 1_{[0, R \wedge \zeta[}(t)$. Then*:

(i) *$t \to t + R \circ \gamma_t$ is increasing on $]\alpha, \infty[$ and so $R^* = \lim_{t \downarrow \alpha}(t + R \circ \gamma_t)$.*
(ii) *$R^* = t + R \circ \gamma_t$ if $\alpha < t < R^*$.*
(iii) *$N_t = 1_{]\alpha, R^* \wedge \beta[}(t)$ if $\alpha < t$.*

PROOF. Note first that if $t \geq 0$, then $t + R \circ \theta_t \geq R$. Therefore if $\alpha < s < t$, one has

$$t + R \circ \gamma_t = s + (t - s) + R \circ \theta_{t-s} \circ \gamma_s \geq s + R \circ \gamma_s,$$

which proves (i). Now if $\alpha < t$, it follows from (3.14) that $R^* \leq t + R \circ \gamma_t$. If $\alpha < s < t < R^*$, then $0 \leq t - s < R^* - s \leq R \circ \gamma_s$ and so

$$t + R \circ \gamma_t = s + (t - s) + R \circ \theta_{t-s} \circ \gamma_s = s + R \circ \gamma_s,$$

because R is a terminal time. This and (i) establish (ii). For (iii) suppose first that $\alpha < t < R^*$. Then

$$\begin{aligned} N_t &= \lim_{s \downarrow \alpha} M_{t-s} \circ \gamma_s = \lim_{s \downarrow \alpha} 1_{[s, s + (R \wedge \zeta) \circ \gamma_s[}(t) \\ &= \lim_{s \downarrow \alpha} 1_{[s, R^* \wedge \beta[}(t) = 1_{]\alpha, R^* \wedge \beta[}(t), \end{aligned}$$

where the third equality follows from (ii). If $\alpha < t$ and $R^* < t$, then $t > s + R \circ \gamma_s$ for all s "close to α." Therefore

$$N_t = \lim_{s \downarrow \alpha} M_{t-s} \circ \gamma_s = \lim_{s \downarrow \alpha} 1_{[0, (R \wedge \zeta) \circ \gamma_s[}(t - s) = 0.$$

Hence $t \to N_t$ and $t \to 1_{]\alpha, R^* \wedge \beta[}(t)$ are right continuous and agree on $]\alpha, \infty[$ except possibly at R^* by the above. This establishes (iii). □

We now fix an *exact* MF, M. We recall some basic facts from [2]. Let $(K_t)_{t \geq 0}$ be the semigroup of the process (X, M), that is, X killed according to M. Thus

$$K_t f(x) = P^x[f \circ X_t M_t]. \tag{3.16}$$

Let (V^q) be the resolvent of (X, M) or (K_t) so that

$$V^q f(x) = \int_0^\infty e^{-qt} K_t f(x)\, dt = P^x \int_0^\infty e^{-qt} f \circ X_t M_t\, dt. \tag{3.17}$$

Since M is exact, $V^q f$ is nearly Borel and finely continuous on all of E. Hence $E_M = \{V^1 1 > 0\}$ is nearly Borel and finely open. Also $1_{E_M}(x) = P^x(M_0)$. It is clear that $K_t(x, \cdot) = 0$ if $x \notin E_M$, and it follows that $K_t f = K_t(f \cdot 1_{E_M})$. Consequently, the measure $K_t(x, \cdot)$ is carried by E_M for each $t \geq 0$ and $x \in E$. Let m be an excessive measure (for X), and let $m^* = 1_{E_M} \cdot m$. Then $m^* K_t = m K_t \uparrow m^*$ as $t \downarrow 0$ and so m^* is excessive for (K_t).

In the next section we shall express the Kuznetsov measure Q^* corresponding to (K_t) and m^* in terms of Q_m and M. Strictly speaking, one cannot apply Kuznetsov's theorem to construct Q^* on the space of two-sided paths in E_M since E_M need not be Borel. [The remark (1.1) on page 1397 in [13] is incorrect.] However, one may apply the methods of [9] to construct Q^* on the space W defined in Section 2 (two-sided paths in E). Then Q^* is the unique measure on $(W, \mathscr{G}^0)$ not charging $[a]$ or $[b]$ such that for $t_1 < \cdots < t_n$,

$$\begin{aligned} &Q^*(\alpha < t_1, Y_{t_1} \in dx_1, \dots, Y_{t_n} \in dx_n, t_n < \beta) \\ &\quad = m^*(dx_1) K_{t_2 - t_1}(x_1, dx_2) \cdots K_{t_n - t_{n-1}}(x_{n-1}, dx_n). \end{aligned} \tag{3.18}$$

Alternatively, one may regard Theorem 4.9 as another method for constructing Q^*.

4. The construction. In this section we fix an exact MF, M, of X and an excessive measure m. We use the notation in the last two paragraphs of Section 3. As explained there we are going to represent Q^* in terms of Q_m and M. In this section we give the construction and in Section 5 we investigate the mechanism underlying the construction.

Recall from (3.7) that $s \to N(s, t)$ is increasing and right continuous on $]\alpha, t[$ and has the value 1 for $s = t$. Define for each $t \in \mathbb{R}$,

$$\rho_t(ds) = \rho_t(w, ds) = 1_{]\alpha, t]}(s)\, d_s N(s, t). \tag{4.1}$$

Thus ρ_t is a measure carried by $]\alpha, t]$ of total mass $\rho_t(1) = 1_{\{\alpha < t\}}(1 - N_t)$ since $\lim_{s \downarrow \alpha} N(s, t) = N_t$. Also ρ_t is a kernel from $(W, \mathscr{G}^*_{t+})$ to $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ in light of (3.7)(iii). Next if $\alpha < t$, $u \to N(t, u) = M_{u-t} \circ \gamma_t$ is right continuous and decreasing on $]t, \infty[$ with $N(t, \beta) = 0$ and $N(t, t+) = M_0 \circ \gamma_t$. For each $t \in \mathbb{R}$ define a measure $\rho^t(du) = \rho^t(w, du)$ on $]t, \beta]$ by

$$\rho^t(du) = 1_{\{\alpha < t\}} 1_{]t, \beta]}(u)[-d_u N(t, u)]. \tag{4.2}$$

Thus ρ^t is carried by $]t, \beta]$ and has total mass $M_0 \circ \gamma_t 1_{\{\alpha < t < \beta\}}$. Since $M_0 \circ \gamma_t$ is either zero or one, ρ^t is either a probability or zero. Note that ρ^t may charge $+\infty$ when $\beta = +\infty$. If $\alpha < t < \beta$, $\rho^t(\{\beta\}) = N(t, \beta -) = \lim_{u \uparrow \beta} M_{u-t} \circ \gamma_t = M_{\zeta -} \circ \gamma_t$, because $\beta \circ \theta_t = \zeta \circ \gamma_t$ in this case. Moreover, ρ^t is a kernel from $(W, \mathscr{G}^*)$ to $(]t, \infty], \mathscr{B}(]t, \infty]))$. Here $\mathscr{G}^* = (\mathscr{G}^0)^*$.

If $s < t$ let $R(s, t) = R(s, t, w) =]\alpha, s] \times]t, \beta]$. Define a measure $\rho^{s,t}(du, dv) = \rho^{s,t}(w, du, dv)$ on $R(s, t)$ by

$$\rho^{s,t}(du, dv) = 1_{\{\alpha < s < t < \beta\}} N(s, t) \rho_s(du) \rho^t(dv). \tag{4.3}$$

It sometimes will be convenient to regard $\rho^{s,t}$ as a measure on $]-\infty, s] \times]t, +\infty]$ that is carried by $R(s, t)$. Note that if F is a positive Borel function on $\mathbb{R} \times]-\infty, \infty]$, then $\rho^{s,t}(F)$ is $\mathscr{G}^*$ measurable as a function of w.

(4.4) LEMMA. *Let $s < t$ and $\sigma < \tau$. Then $\rho^{s,t} = \rho^{\sigma,\tau}$ on $R(s, t) \cap R(\sigma, \tau)$.*

PROOF. Since $R(s,t) \cap R(\sigma,\tau) = R(s \wedge \sigma, t \vee \tau)$ it suffices to show if $\alpha < s < t < \beta$ and $\alpha < \sigma < \tau < \beta$,

(i) $$\rho^{s,t} = \rho^{\sigma,t} \quad \text{on } R(\sigma,t) \text{ if } \sigma < s < t,$$

(ii) $$\rho^{s,t} = \rho^{s,\tau} \quad \text{on } R(s,\tau) \text{ if } s < t < \tau.$$

For (i) suppose $\alpha < u_1 < u_2 \leq \sigma$ and $t < v_1 < v_2$, and let $\Gamma =]u_1, u_2] \times]v_1, v_2]$. Then

$$(4.5) \quad \rho^{\sigma,t}(\Gamma) = N(\sigma,t)[N(u_2,\sigma) - N(u_1,\sigma)] \times [N(t,v_1) - N(t,v_2)].$$

But $N(\sigma,t) = N(\sigma,s)N(s,t)$ by (3.7) and so using (3.7) again $\rho^{\sigma,t}(\Gamma) = \rho^{s,t}(\Gamma)$. Consequently, by the monotone class theorem $\rho^{\sigma,t} = \rho^{s,t}$ on $R(\sigma,t)$. The argument for (ii) is similar. □

Since ($\mathbb{Q}$ denotes the rationals)

$$H = \{(u,v): \alpha < u < v \leq \beta\} = \bigcup_{s<t;\ s,t\in\mathbb{Q}} R(s,t),$$

it follows that there exists a unique σ-finite measure $\rho(du,dv)$ on H such that for any $s < t$, $\rho(du,dv) = \rho^{s,t}(du,dv)$ on $R(s,t)$. We shall usually regard ρ as a measure on $]-\infty,\infty[\times]-\infty,\infty]$ that is carried by H.

For each $w \in W$ define a measure $\lambda(du,dv) = \lambda(w,du,dv)$ on $\{(u,v): -\infty \leq u < v \leq \infty\}$ by

$$(4.6) \qquad \lambda(du,dv) = 1_{\{-\infty \leq u < v \leq \infty\}}[\varepsilon_\alpha(du)(-dN_v) + \rho(du,dv)].$$

Recall that $v \to N_v$ is right continuous and decreasing on $]\alpha,\infty[$ and $N_v = 0$ if $v \geq \beta$. As a result λ is carried by $\{(u,v): \alpha \leq u < v \leq \beta\}$. It will turn out that Q^* is obtained from Q_m by "birthing and killing Y" according to λ. More exactly, let $\tilde{W} = W \times \{(u,v): -\infty \leq u < v \leq \infty\}$, and write $\tilde{w} = (w,u,v)$ for the generic element in $\tilde{W}$. Define $U(\tilde{w}) = U(w,u,v) = u$ and $V(\tilde{w}) = V(w,u,v) = v$. For each $t \in \mathbb{R}$ define

$$(4.7) \qquad \begin{aligned} \tilde{Y}_t(\tilde{w}) = \tilde{Y}_t(w,u,v) &= Y_t(w), && \text{if } \alpha \vee u < t < \beta \wedge v,\\ &= a, && \text{if } t \leq \alpha \vee u < \beta \wedge v,\\ &= b, && \text{if } \alpha \vee u < \beta \wedge v \leq t,\\ &= a, && \text{if } \alpha \geq v,\\ &= b, && \text{if } \beta \leq u. \end{aligned}$$

Note that if $u < v$, then $\beta \wedge v \leq \alpha \vee u$ if and only if either $\alpha \geq v$ or $\beta \leq u$. Also that for each $\tilde{w}$ the function $t \to \tilde{Y}_t(\tilde{w})$ is in W and if $\tilde{\alpha} = \sup\{t: \tilde{Y}_t = a\}$ and $\tilde{\beta} = \inf\{t: \tilde{Y}_t = b\}$, then $\tilde{\alpha} = \alpha \vee U$ and $\tilde{\beta} = \beta \wedge V$ provided $\alpha \vee U < \beta \wedge V$.

Let $\mathscr{B}$ denote the Borel σ-algebra in $\{(u,v): -\infty \leq u < v \leq \infty\}$ and let $\tilde{\mathscr{G}} = \mathscr{G}^* \times \mathscr{B}$. We define a measure $\tilde{Q} = \tilde{Q}_m$ on $(\tilde{W}, \tilde{\mathscr{G}})$ by

$$(4.8) \qquad \tilde{Q}(F) = \int\left(\int F(w,u,v)\lambda(w,du,dv)\right)Q_m(dw),$$

for $F \in p(\tilde{\mathscr{G}})$. Since $\tilde{Q}$ is carried by $(\alpha \leq U < V \leq \beta\}$ one has $\tilde{\alpha} = U$ and $\tilde{\beta} = V$

a.s. $\tilde{Q}$. The process $(\tilde{Y}_t)_{t\in\mathbb{R}}$ under the measure $\tilde{Q}$ is the precise meaning of Y under Q_m birthed and killed according to λ. Here is the main result of this section.

(4.9) THEOREM. *The processes* $(\tilde{Y}_t, \tilde{Q})$ *and* (Y_t, Q^*) *are equivalent; that is, if* $t_1 < \cdots < t_n$, *then* $\tilde{Q}(\tilde{\alpha} < t_1, \tilde{Y}_{t_1} \in dx_1, \ldots, \tilde{Y}_{t_n} \in dx_n, t_n < \tilde{\beta})$ *is given by the right-hand side of* (3.18).

(4.10) REMARK. Perhaps a better way of stating (4.9) is that if $\Phi\colon \tilde{W} \to W$ by $\Phi\tilde{w}(t) = \tilde{Y}_t(\tilde{w})$, then $\Phi(\tilde{Q}) = Q^*$. This, of course, follows from (4.9) and the uniqueness of Q^* subject to (3.18) once one observes that $\Phi(\tilde{Q})$ does not charge $[a]$ or $[b]$. One may also regard $\Phi(\tilde{Q})$ as an alternative construction of Q^*.

We prepare several lemmas for the proof of Theorem 4.9. The first of them is the key technical fact that we need.

(4.11) LEMMA. *For each* $t \in \mathbb{R}$, $\tilde{Q}(U = t) = 0$.

PROOF. First $\tilde{Q}(U = t \le \alpha) \le Q_m(\alpha = t) = 0$. Thus it suffices to show $\tilde{Q}(\alpha < t = U) = 0$. But

$$\begin{aligned}\tilde{Q}(\alpha < t = U) &= \tilde{Q}(\alpha < U = t < V \le \beta)\\ &\equiv Q_m\left[\int_{\{\alpha<u}\int_{<v\le\infty\}} 1_{\{t\}}(u)\rho(du, dv);\ \alpha < t < \beta\right]\\ &= Q_m\left[\rho(\{t\},]t, \infty]);\ \alpha < t < \beta\right]\\ &= Q_m\left[\lim_{u\downarrow t}\rho(\{t\},]u, \infty]);\ \alpha < t < \beta\right].\end{aligned}$$

Now if $\alpha < t < u < \beta$ one has

$$\begin{aligned}\rho(\{t\},]u, \infty]) &= N(t, u)\rho_t(\{t\})\rho^u(]u, \infty])\\ &= N(t, u)[1 - N(t-, t)]M_0\circ\gamma_u\\ &= M_{u-t}\circ\gamma_t M_0\circ\gamma_u\left[1 - \lim_{s\uparrow t} M_{t-s}\circ\gamma_s\right],\end{aligned}$$

and $M_{u-t}\circ\gamma_t M_0\circ\gamma_u = (M_{u-t}M_0\circ\theta_{u-t})\circ\gamma_t = M_{u-t}\circ\gamma_t$. Therefore letting u decrease to t we obtain

$$\tilde{Q}(\alpha < t = U) = Q_m\left[M_0\circ\gamma_t\left(1 - \lim_{s\uparrow t} M_{t-s}\circ\gamma_s\right);\ \alpha < t < \beta\right], \tag{4.12}$$

since $M_{u-t}\circ\gamma_t$ increases to $M_0\circ\gamma_t$ as u decreases to t. Next choose $f \in \mathscr{E}$ with $0 < f \le 1$ and $m(f) < \infty$. Let $g = U^1 f$. Then $0 < g \le 1$, g is finely continuous, and $m(g) = mU^1(f) \le m(f) < \infty$. Observe that

$$Q_m[g\circ Y_t M_0\circ\gamma_t] = Q_m\left[P^{Y(t)}(M_0 g\circ X_0)\right] = m(1_{E_M} g) = m^*(g).$$

Also $s \to M_{t-s} \circ \gamma_s$ is increasing on $]\alpha, t[$ and so

$$Q_m\left[g \circ Y_t M_0 \circ \gamma_t \lim_{s \uparrow t} M_{t-s} \circ \gamma_s\right]$$
$$= \lim_{s \uparrow t} Q_m[g \circ Y_t M_0 \circ \gamma_t M_{t-s} \circ \gamma_s;\ \alpha < s]$$
$$= \lim_{s \uparrow t} Q_m[g \circ X_{t-s} \circ \gamma_s (M_0 \circ \theta_{t-s} M_{t-s}) \circ \gamma_s;\ \alpha < s]$$
$$= \lim_{s \uparrow t} Q_m\left[P^{Y(s)}(g \circ X_{t-s} M_{t-s})\right] = \lim_{s \uparrow t} m K_{t-s} g.$$

But $K_{t-s} g \to g \cdot 1_{E_M}$ as $s \uparrow t$ since g is finely continuous and bounded. Also since $g = U^1 f$, $K_r g \le P_r g \le e^r g \le eg$ if $r \le 1$. But $m(g) < \infty$ and so $mK_{t-s} g \to m(g \cdot 1_{E_M}) = m^*(g)$ as s increases to t. Combining these facts with (4.12) and the fact that $g > 0$ we obtain $\tilde{Q}(\alpha < t = U) = 0$, proving (4.11). □

(4.13) REMARK. One may also show that $\tilde{Q}(V = t) = 0$. But this will be an immediate consequence of Theorem 4.9 since $\tilde{\beta} = V$ a.s. $\tilde{Q}$.

(4.14) LEMMA. (i) *Almost surely* Q_m *on* $\{\alpha < s < t < \beta\}$, $\lambda(U < s, t < V) = N(s, t)$. (ii) *Almost surely* Q_m *on* $\{\alpha < s < \beta\}$, $\lambda(U < s < V) = M_0 \circ \gamma_s$.

PROOF. Since $Q_m[\lambda(U = s)] = \tilde{Q}(U = s) = 0$ by (4.11), it suffices to compute for $\alpha < s < t < \beta$,

$$\lambda(U \le s, t < V) = N_t + \int_{]\alpha, s]} \int_{]t, \infty]} \rho(du, dv)$$
$$= N_t + N(s, t)\rho_s(1)\rho^t(1) = N_t + N(s, t)(1 - N_s) M_0 \circ \gamma_t.$$

But $N(s, t)(1 - N_s) = N(s, t) - N_t$. Letting t decrease to s in (3.10) and then replacing s by t one finds $N_t = N_t M_0 \circ \gamma_t$. Consequently,

$$\lambda(U \le s, t < V) = N(s, t) M_0 \circ \gamma_t = M_{t-s} \circ \gamma_s M_0 \circ \theta_{t-s} \gamma_s$$
$$= M_{t-s} \circ \gamma_s = N(s, t),$$

proving (i). For (ii)

$$\lambda(U < s < V) = \lim_{t \downarrow s} \lambda(U < s, t < V) = \lim_{t \downarrow s} M_{t-s} \circ \gamma_s = M_0 \circ \gamma_s. \qquad \square$$

We are now ready to prove (4.9). If $f \in p\mathscr{E}$,

$$\tilde{Q}(f \circ \tilde{Y}_t) = Q_m[f \circ Y_t \lambda(U < t < V)]$$
$$= Q_m[f \circ Y_t M_0 \circ \gamma_t] = Q_m\left[f \circ Y_t P^{Y(t)}(M_0)\right] = m^*(f).$$

Next suppose $s < t$ and $f, g \in p(\mathscr{E})$. Then

$$\tilde{Q}\left[f \circ \tilde{Y}_s g \circ \tilde{Y}_t\right] = Q_m[f \circ Y_s g \circ Y_t;\ U < s, t < V]$$
$$= Q_m[f \circ Y_s g \circ X_{t-s} \circ \gamma_s M_{t-s} \circ \gamma_s]$$
$$= Q_m[f \circ Y_s K_{t-s} g \circ Y_s] = m(f K_{t-s} g).$$

But $K_{t-s}(x, \cdot) = 0$ if $x \notin E_M$ and so this equals $m^*(f K_{t-s} g)$.

Now suppose as an induction hypothesis that we have established that (3.18) holds for a fixed $n-1$ with Q^* replaced by $\tilde{Q}$ and Y by $\tilde{Y}$. Given $t_1 < \cdots < t_{n-1} < t_n$ and $f_1, \ldots, f_{n-1}, f_n \in p\mathscr{E}$,

$$\tilde{Q}\left[\prod_{j=1}^{n} f_j \circ \tilde{Y}_{t_j}\right] = Q_m\left[\prod_{j=1}^{n} f_j \circ Y_{t_j}; U < t_1, t_n < V\right]$$
$$= Q_m\left[\prod_{j=1}^{n} f_j \circ Y_{t_j} N(t_1, t_n)\right]$$
$$= Q_m\left[\prod_{j=1}^{n-1} f_j \circ Y_{t_j} N(t_1, t_{n-1}) f_n \circ Y_{t_n} N(t_{n-1}, t_n)\right].$$

But $f_n \circ Y_{t_n} N(t_{n-1}, t_n) = (f_n \circ X_{t_n - t_{n-1}} M_{t_n - t_{n-1}}) \circ \gamma_{t_{n-1}}$ and the remaining terms are $\mathscr{G}^*_{t_{n-1}+}$-measurable, and so this last expression equals

$$Q_m\left[\prod_{j=1}^{n-1} f_j \circ Y_{t_j} K_{t_n - t_{n-1}} f_n \circ Y_{t_{n-1}} N(t_1, t_{n-1})\right]$$
$$= \int \cdots \int m^*(dx_1) K_{t_2 - t_1}(x_1, dx_2) \cdots K_{t_n - t_{n-1}}(x_{n-1}, dx_n) f_1(x_1) \cdots f_n(x_n),$$

where the last equality follows from the induction hypothesis. □

(4.15) Remark. One may easily check that for each $t \in \mathbb{R}$, $\rho(du, dv) \circ \theta_t = \rho(du + t, dv + t)$ and that λ satisfies the same identity. Thus ρ and λ are examples of what Dynkin in [3] calls homogeneous additive functionals of order 2. (Dynkin requires that his functionals are carried by $]\alpha, \beta[\times]\alpha, \beta[$.)

5. The interpretation of λ. In order to motivate the main result of this section we shall consider first a very special case. We fix an exact MF, M, of X as in Section 4. We suppose first that the associated N does not vanish on $]\alpha, \beta[$. The example (3.13) shows that this is stronger than assuming that M does not vanish on $[0, \zeta[$. Under this assumption $N(s, t) = N_t/N_s$ on $\alpha < s < t < \beta$ according to (3.10). Therefore if $\alpha < s < t < \beta$ one has $\rho_s(du) = 1_{]\alpha, s]}(u) N_s\, d(1/N_u)$ and $\rho^t(dv) = 1_{]t, \beta]}(v)(N_t)^{-1}(-dN_v)$. Consequently, $\rho(du, dv) = 1_{\{\alpha < u < v \le \beta\}}\, d(1/N_u)(-dN_v)$ and hence

$$\lambda(du, dv) = 1_{\{\alpha \le u < v \le \beta\}}\left(\varepsilon_\alpha(du) + d(N_u)^{-1}\right)(-dN_v). \tag{5.1}$$

This says that "particles representing $\tilde{Y}$" are born at α and then killed at rate $-dN_v$ and, in addition, are born in the interval $]\alpha, \beta[$ at rate $d(N_u)^{-1}$ and then killed at rate $-dN_v$.

We turn now to the general case. Let $S = \inf\{t: M_t = 0\}$. Then $S \le \zeta$ and S is a terminal time which need not be exact. Let H and J be the homogeneous random sets on Ω and W associated with S as defined in the Appendix. See (A.3) and (A.1). Let G be the set of left endpoints in $]\alpha, \beta[$ of the contiguous intervals of J in $]\alpha, \beta[$; that is, the maximal open intervals in $]\alpha, \beta[\setminus J$. If $d = \inf\{u: u \in J\} \wedge \beta$ as defined in (A.5) and if $\alpha < d$, then $]\alpha, d[$ is also a contiguous interval, but $\alpha \notin G$. In any case we may write

$$]\alpha, \beta[= J \cup]\alpha, d[\cup \bigcup_{r \in G}]r, d_r[, \tag{5.2}$$

where $d_t = \inf\{u > t: u \in J\} \wedge \beta$ is defined in (A.5). Of course, $]s, t[$ is empty if $t \le s$. It will be convenient to define $G_0 = G$ if $\alpha = d$ and $G_0 = G \cup \{\alpha\}$ if $\alpha < d$. Thus G_0 is the set of *all* left endpoints of the intervals contiguous to J in $]\alpha, \beta[$. Note that $d_\alpha = d$ and that $-\infty$ is in G_0 when $\alpha = -\infty < d$. Let $\Lambda(dt)$ be the optional HRM (of X) carried by $]0, \zeta[\backslash H$ such that

$$\Lambda(dt) = -\frac{dM_t}{M_{t-}} = M_t\, d\left(\frac{1}{M_t}\right), \quad \text{on }]0, S[, \tag{5.3}$$

introduced in the Appendix. See (A.8) and (A.9). Let κ be its extension to W as defined in (A.10). Then κ is carried by $]\alpha, \beta[\backslash J$. Moreover, if $\alpha < r < d_r$ and $r \notin J$, then on $[r, d_r[$ one has

$$\begin{aligned}\kappa(du) &= \Lambda(\gamma_r, du - r) = -\left(M_{(u-r)-} \circ \gamma_r\right)^{-1} d_u M_{u-r} \circ \gamma_r \\ &= M_{u-r} \circ \gamma_r\, d_u\left(M_{u-r} \circ \gamma_r\right)^{-1},\end{aligned} \tag{5.4}$$

because $u - r < d_r - r = D \circ \gamma_r = S \circ \gamma_r$ by (A.6)(ii). Finally, (5.4) also holds if $r \in J$ provided S is exact so that $S = D$.

Let us return for a moment to the situation in which N_t does not vanish on $]\alpha,\beta[$. Then J is empty and $G_0 = \{\alpha\}$. Using (5.4) we see that if $\alpha < r < \beta$, then $d_r = \beta$ and on $[r, \beta[$ one has

$$\kappa(du) = N_u\, d\left(\frac{1}{N_u}\right), \qquad \kappa(dv) = -(N_{v-})^{-1} dN_v.$$

Recall from (3.9) that $N(\alpha, u) = N(\alpha +, u) = N_u$ and note that the mass of $-dN_v$ at β is $N_{\beta-}$ since $N_\beta = 0$. Therefore we may write (5.1) in the form

$$\lambda(du, dv) = \left[\varepsilon_\alpha(du) + \kappa(du)\right] N(u, v-)\left[\kappa(dv) + \varepsilon_\beta(dv)\right], \tag{5.5}$$

on $u < v$. It is this formula that we shall extend to the general case.

The basic calculations are contained in the proposition. We need the definition

$$g_t = \sup\{u \le t: u \in J\} \vee \alpha, \tag{5.6}$$

where the supremum of the empty set is minus infinity. Since J is closed in $]\alpha, \beta[$, $g_t \in J$ if $\alpha < g_t < \beta$, and $\alpha \le g_t \le t$.

(5.7) PROPOSITION. *Let $\alpha < s < \beta$. Then ρ_s is carried by $[g_s, s]$ and*

$$\rho_s(du) = N(u, s)\left[1_{]g_s, s]}(u)\kappa(du) + \varepsilon_{g_s}(du) 1_{\{g_s > \alpha\}}\right]. \tag{5.8}$$

Let $\alpha < t < \beta$. Then ρ^t is carried by $]t, d_t]$. If $t \notin J$, then

$$\rho^t(dv) = N(t, v-)\left[1_{]t, d_t[}(v)\kappa(dv) + 1_{\{t < d_t\}}\varepsilon_{d_t}(dv)\right]. \tag{5.9}$$

If S is exact (5.9) *also holds for $t \in J$.*

PROOF. Suppose $u < g_s$. If $g_s = \alpha$, then $N(u, s) = 0$. If $\alpha < u < g_s$, then $d_u \le g_s \le s$ and so $s - u \ge d_u - u = D \circ \gamma_u \ge S \circ \gamma_u$. Therefore $N(u, s) = 0$ whenever $u < g_s$. In particular, ρ_s is carried by $[g_s, s]$. We next claim that

$$1_{]g_s, s]}(u)\rho_s(du) = N(u, s)\kappa(du) 1_{]g_s, s]}(u). \tag{5.10}$$

If $s \in J$, $g_s = s$ and this is clear. Suppose $s \notin J$. If $g_s < t < s < \beta$, then $s < d_t$ since J is closed and $t \notin J$, and so for $t < u \leq s$, $u - t < d_t - t = D \circ \gamma_t = S \circ \gamma_t$. Therefore $N(t, u) \neq 0$ and $N(u, s) = N(t, s)[N(t, u)]^{-1}$. Hence on $]t, s]$ using (5.4) we obtain

$$\rho_s(du) = N(t, s)\, d_u[N(t, u)]^{-1} = N(u, s)\kappa(du),$$

establishing (5.10) since $t \in]g_s, s[$ is arbitrary. But $\rho_s(\{g_s\}) = N(g_s, s)$ since $N(u, s) = 0$ if $u < g_s$. Combining this with (5.10) we obtain (5.8) because ρ_s is also carried by $]\alpha, s]$.

If $v \geq d_t$, then $v - t \geq D \circ \gamma_t \geq S \circ \gamma_t$ and hence $N(t, v) = 0$. Therefore ρ^t is carried by $]t, d_t]$—recall ρ^t is carried by $]t, \beta]$. Suppose $t \notin J$. Then by (5.4)

$$\rho^t(dv) = N(t, v-)\kappa(dv), \quad \text{on }]t, d_t[,$$

and this is true for $t \in J$ if S is exact. But $\rho^t(\{d_t\}) = 1_{\{t < d_t\}} N(t, d_t -)$ since $N(t, d_t) = 0$ as seen in the first sentence of this paragraph. This proves (5.9). □

We come now to the main result of this section recall that we have set $N(\alpha, u) = N_u$. It will be convenient to let I_t denote the interval $]t, d_t[$ and also to denote the indicator of this interval, so $I_t(u) = 1_{]t, d_t[}(u)$. Using this notation we have

(5.11) **Theorem.** *The measure λ has the following expression*:

$$\begin{aligned}\lambda(du, dv) &= 1_{\{u<v\}} \sum_{r \in G_0} [I_r(u)\kappa(du) + \varepsilon_r(du)][-d_v N(u, v)] \\ &= 1_{\{u<v\}} N(u, v-) \sum_{r \in G_0} [I_r(u)\kappa(du) + \varepsilon_r(du)] \\ &\quad \times \big[I_r(v)\kappa(dv) + \varepsilon_{d_r}(dv)\big].\end{aligned}$$

Proof. We shall first show that λ is given by the last expression in (5.11). Suppose first that $\alpha < s < t < \beta$ and $t \notin J$. Then from (3.7)(ii), (4.3) and (5.7) one has (recall κ does not charge J)

$$\begin{aligned}\rho^{s,t}(du, dv) &= N(u, v-)\Big[1_{]g_s, s]}(u)\kappa(du) + 1_{\{\alpha < g_s\}}\varepsilon_{g_s}(du)\Big] \\ &\quad \times \Big[I_t(v)\kappa(dv) + 1_{\{t < d_t\}}\varepsilon_{d_t}(dv)\Big] 1_{\{u<v\}}.\end{aligned} \tag{5.12}$$

The expression (4.3) for $\rho^{s,t}$ contains the factor $N(s, t)$. Suppose $]s, t] \cap J$ is not empty. Then $s \leq d_s \leq t$ and so $t - s \geq d_s - s \geq S \circ \gamma_s$. Therefore $N(s, t) = 0$. In other words, $\rho^{s,t} \neq 0$ only if there exists a unique $r \in G_0$ with $r \leq s < t < d_r$. But then $g_s = r$ and $d_t = d_r$. Combining this with (5.12) we obtain

$$\begin{aligned}\rho(du, dv) &= 1_{\{u<v\}} N(u, v-) \\ &\quad \times \sum_{r \in G_0} \big[I_r(u)\kappa(du) + 1_{\{\alpha<r\}}\varepsilon_r(du)\big]\big[I_r(v)\kappa(dv) + \varepsilon_{d_r}(dv)\big].\end{aligned} \tag{5.13}$$

We claim next that

$$-dN_v = N_{v-}\big[1_{]\alpha, d[}(v)\kappa(dv) + \varepsilon_d(dv)\big]. \tag{5.14}$$

Let $T = \inf\{t\colon N_t = 0\}$. From (A.7), $T \le d$ and clearly $-dN_v$ is carried by $]\alpha, T]$. But from (A.7), $T = d$ if $T > \alpha$. Thus it suffices to prove (5.14) when N does not vanish on $]\alpha, d[$. Obviously, both sides of (5.14) put the same mass, namely N_{d-}, at d. If $\alpha < t < v < d$, then $N_t > 0$ and $N_{v-} = N_t N(t, v-)$. Consequently, (5.4) implies that on $]t, d[$ one has

$$N_{v-}\kappa(dv) = -N_t\, d_v N(t, v) = -dN_v,$$

which verifies (5.14) on $]t, d[$ and, hence, on $]\alpha, d[$. Finally, from (5.13), (5.14) and the definition (4.6) we see that λ is equal to the last expression in (5.11).

For the other, note that if $r \in G_0$ and $r = \alpha$, then (5.14) states [recall $N(\alpha, v) = N_v$]

$$-d_v N(r, v) = dN_v = N(r, v-)\big[\kappa(dv) + \varepsilon_{d_r}(dv)\big],$$

and a similar argument shows that this holds for each $r \in G$. If $r < u < d_r$, then observing that $-d_v N(u, v)$ is carried by $\{v\colon N(u, v-) > 0\}$ the same argument shows that

$$-d_v N(u, v) = N(u, v-)\big[\kappa(dv) + \varepsilon_{d_r}(dv)\big],$$

on $]u, d_r]$. Substituting these observations into the last expression in (5.11) yields the first equality. □

The first expression for λ in (5.11) states that in order to obtain $\tilde{Y}$ from Y, "particles" are created at the left endpoint of each of the intervals $I_r =]r, d_r[$, $r \in G_0$, and in the interior of I_r at rate $\kappa(du)$ and then particles created at time u are killed at rate $-d_v N(u, v)$.

We turn now to another interpretation of λ. Let $r \in G_0$ and $r < t < d_r$. Then if $t < u < d_r$, $u - t < d_r - t = d_t - t = S \circ \gamma_t$ and this also holds at $t = r$ if S is exact. Hence $N(t, u) > 0$. Therefore from (5.4)

$$\kappa(du) = \Lambda(\gamma_t, du - t) = N(t, u)\, d_u[N(t, u)]^{-1},$$

on $]t, d_r[$ and even on $]r, d_r[$ if S is exact. Thus from (5.11) for $r \in G_0$ and $r < t < d_r$,

$$I_{]t, d_r[}(u)\lambda(du, dv) = 1_{\{u<v\}}\, d_u[N(t, u)]^{-1}[-d_v N(t, v)], \tag{5.15}$$

and if S is exact one may put $t = r$ in (5.15). This states that particles are created at rate $d_u(N(t, u))^{-1}$ and killed at rate $-d_v N(t, v)$ in each such interval $]t, d_r[$. Finally, if S is exact substituting (5.15) with $t = r$ into (5.11) gives [recall $N(\alpha, u) = N_u$]

$$\lambda(du, dv) = 1_{\{u<v\}} \sum_{r \in G_0} \big[I_r(u)\, d_u[N(r, u)]^{-1} + \varepsilon_r(du)\big][-d_v N(r, v)]. \tag{5.16}$$

This formula is the clearest expression of the fact that particles are being born (or created) according to the reciprocal of M and killed according to M. Finally, (5.11) and (5.16) should be compared with the special cases (5.5) and (5.1).

6. The decomposition of m^*. As in Sections 4 and 5 we fix an exact MF, M, of X and an excessive measure m. Then m has a unique decomposition

$m = m_i + m_p$, where m_i is *invariant* ($m_i P_t = m_i$ for each $t > 0$) and m_p is *purely excessive* (m_p is excessive and $m_p P_t$ decreases to zero as t increases to infinity). In fact, it is shown in [5], that for each $t \in \mathbb{R}$,

$$m_i(f) = Q_m[f \circ Y_t;\ \alpha = -\infty], \tag{6.1}$$

$$m_p(f) = Q_m[f \circ Y_t;\ \alpha > -\infty]. \tag{6.2}$$

In addition, there exists a unique entrance law, $\eta = (\eta_t)_{t>0}$, for P_t such that $m_p = \int_0^\infty \eta_t\, dt$, and from [5] for each $t > 0$,

$$\eta_t(f) = Q_m[f \circ Y_{\alpha+t};\ 0 < \alpha < 1]. \tag{6.3}$$

Recall that $m^* = 1_{E_M} \cdot m$ is the restriction of m to E_M and that m^* is excessive for the semigroup (K_t) generated by M. See (3.16). We are going to investigate the decomposition of m^* into its (K_t) invariant, m_i^*, and purely excessive, m_p^*, parts. First note that it follows from (3.10) that $N_0 = N_0 M_0 \circ \gamma_0$ on $\alpha < 0 < \beta$ and so

$$\begin{aligned} Q_m[f \circ Y_0 N_0] &= Q_m\left[f \circ Y_0 N_0 P^{Y(0)}(M_0)\right] \\ &= Q_m\left[f \circ Y_0 N_0 1_{E_M} \circ Y_0\right]. \end{aligned}$$

Thus the measure $f \to Q_m[f \circ Y_0 N_0]$ is carried by E_M. We now define three measures,

$$\begin{aligned} m_1(f) &= Q_m[f \circ Y_0 N_0;\ \alpha = -\infty], \\ m_2(f) &= Q_m[f \circ Y_0 N_0;\ \alpha > -\infty], \\ m_3(f) &= Q_m[f \circ Y_0(1 - N_0);\ Y_0 \in E_M]. \end{aligned} \tag{6.4}$$

Clearly, $m^* = m_1 + m_2 + m_3$ and we shall interpret each of these three pieces.

It is not surprising that m_1 is the (K_t) invariant part, m_i^*, of m^*. To see this use (6.1), (4.9), (4.14)(i), and $N_t = \lim_{s \downarrow \alpha} N(s, t)$ to compute

$$\begin{aligned} m_i^*(f) &= \tilde{Q}\left[f \circ \tilde{Y}_t;\ \tilde{\alpha} = -\infty\right] \\ &= \tilde{Q}[f \circ Y_t;\ -\infty = U < t < V,\ \alpha = -\infty] \\ &= Q_m[f \circ Y_t N_t;\ \alpha = -\infty] = m_1(f). \end{aligned}$$

It is also easy to verify by direction calculation that $m_1 K_t = m_1$, if one prefers.

To the entrance law η one may associate a unique σ-finite measure Q_η on $(W, \mathcal{G}^0)$ such that $Q_\eta(\alpha \neq 0) = 0$ and if $0 < t_1 < \cdots < t_n$,

$$\begin{aligned} &Q_\eta\left(Y_{t_1} \in dx_1, \dots, Y_{t_n} \in dx_n,\ t_n < \beta\right) \\ &\quad = \eta_{t_1}(dx_1) P_{t_2 - t_1}(x_1, dx_2) \cdots P_{t_n - t_{n-1}}(x_{n-1}, dx_n). \end{aligned} \tag{6.5}$$

Moreover, checking finite-dimensional distributions it is easy to see that

$$Q_{m_p} = \int_{-\infty}^{\infty} \theta_t(Q_\eta)\, dt. \tag{6.6}$$

See, for example, [9]. Now define

$$\eta_t^M(f) = Q_\eta[f \circ Y_t N_t], \qquad t > 0. \tag{6.7}$$

(6.8) PROPOSITION. *$\eta^M = (\eta_t^M)_{t>0}$ is an entrance law for (K_t) and $m_2 = \int_0^\infty \eta_t^M\,dt$.*

PROOF. If $t > 0$, $s \geq 0$ and $f \in p\mathscr{E}$, then

$$\begin{aligned}\eta_t^M(K_s f) &= Q_\eta[K_s f \circ Y_t N_t]\\ &= Q_\eta[P^{Y(t)}[f \circ X_s M_s] N_t]\\ &= Q_\eta[f \circ Y_{t+s} M_s \circ \gamma_t N_t]\\ &= Q_\eta[f \circ Y_{t+s} N_{t+s}] = \eta_{t+s}^M(f),\end{aligned}$$

where the third equality uses the fact that $(Y_t)_{t>0}$ is Markov under Q_η with semigroup (P_t) and the fourth equality follows from (3.10). Next for $f \in p\mathscr{E}$, since $Q_\eta(\alpha \neq 0) = 0$ we have

$$\begin{aligned}\int_0^\infty \eta_t^M(f)\,dt &= \int_0^\infty Q_\eta[f \circ Y_t N_t]\,dt\\ &= \int_{-\infty}^\infty Q_\eta[f \circ Y_t N_t]\,dt = \int_{-\infty}^\infty \theta_t(Q_\eta)[f \circ Y_0 N_0]\,dt\\ &= Q_{m_p}(f \circ Y_0 N_0) = m_2(f),\end{aligned}$$

where we have used (6.6) and (6.2) for the last two equalities. Since $\eta_t^M \leq \eta_t$ it is σ-finite. □

(6.9) REMARK. Since $\eta_t^M = \eta_t^M K_0$ it follows that η_t^M is carried by E_M. One may also show that

$$\eta_t^M(f) = Q_m[f \circ Y_{\alpha+t} N_{\alpha+t}; 0 < \alpha < 1], \tag{6.10}$$

but we leave the proof of (6.10) to the interested reader.

In order to discuss m_3 we need some more machinery. Recall from Section 5 and the Appendix the homogeneous random sets H over Ω and $\mathscr{J}$ over W associated with M and the homogeneous random measures Λ over Ω and κ over W. Also $\nu = \nu_\Lambda$ denotes the characteristic measure of Λ. See (A.11), (A.12) and (A.16). We shall also need the exit system $({}^*P^x, B)$ associates with the homogeneous random set H. Here ${}^*P^x(\cdot)$ is a kernel (of σ-finite measures) from $(E, \mathscr{E}^*)$ to $(\Omega, \mathscr{F}^*)$ and B is an (adapted) additive functional of X with a bounded one potential. Let ν^B be the characteristic measure of B, that is, ν^B is defined by (A.11) with $\Lambda(ds)$ replaced by dB_s. The property of the exit system $({}^*P^x, B)$ that we need is contained in Theorem 6.8 of [5]. It states that if $F = F(t, x, \omega) \geq 0$ is universally measurable over $\mathscr{B}(\mathbb{R}) \times \mathscr{E} \times \mathscr{F}^0$, then

$$Q_m \sum_{r \in G} F(r, Y_r, \gamma_r) = \iint_{\mathbb{R} \times E} dt\, \nu^B(dx)\, {}^*P^x[F(t, x, \cdot)], \tag{6.11}$$

where G (defined in Section 5) is the set of left endpoints in $]\alpha, \beta[$ of the contiguous intervals of $\mathscr{J}$ in $]\alpha, \beta[$.

Define

$$q_t(x, f) = {}^*P^x[f \circ X_t M_t], \qquad t > 0. \tag{6.12}$$

Using the fact that $(X_t)_{t>0}$ is Markov under $*P^x$ with transition semigroup (P_t), it is easy to see that for each x, $t > 0$, and $s \geq 0$, $q_{t+s}(x, \cdot) = \int q_t(x, dy)K_s(y, \cdot)$. Therefore defining $\nu^B q_t(\cdot) = \int \nu^B(dx)q_t(x, \cdot)$ one has $\nu^B q_{t+s} = \nu^B q_t K_s$. [Actually, $q_t(x, \cdot)$ and $\nu^B q_t$ are entrance laws for (K_t), the required σ-finiteness following from the properties of exit systems. However, we do not need this general result for Theorem 6.16 below.] We may now describe m_3.

(6.13) PROPOSITION. *Using the above notation*

$$m_3 = \int_0^\infty \left(\nu K_t + \nu^B q_t\right) dt.$$

PROOF. Recall the definitions of d and d_t in (A.5) and g_t in (5.6). Let $f \in p\mathscr{E}$ vanish off E_M. Since $1 - N_0 = \rho_0(1)$ when $\alpha < 0$ we may write

$$m_3(f) = Q_m\left[f(Y_0)\int_{]\alpha,0]}\rho_0(du)\right].$$

By (5.7), ρ_0 is carried by $[g_0, 0]$. If $d_0 = 0$ and $t > 0$, then $g_0 = 0 < g_t$ and so $N(g_0, t) = 0$ because $N(u, s) = 0$ whenever $u < g_s$. See the first two sentences of the proof of (5.7). Hence, by right continuity, $N(g_0, 0) = 0$ if $d_0 = 0$. Therefore from (5.8) we have

$$\begin{aligned} m_3(f) = {} & Q_m\left[f \circ Y_0 \int_{]\alpha,0]} N(u,0)\kappa(du)\right] \\ & + Q_m\left[f \circ Y_0 N(g_0, 0);\ \alpha < g_0, 0 < d_0\right]. \end{aligned} \tag{6.14}$$

The first term on the right-hand side of (6.14) equals

$$\begin{aligned} & Q_m\left[f \circ Y_0 \int_{]\alpha,0]} M_{-u} \circ \gamma_u \kappa(du)\right. \\ & = Q_m \int_{-\infty}^0 (f \circ X_{-u} M_{-u}) \circ \gamma_u 1_{\{\alpha < u\}} \kappa(du) \\ & = Q_m \int_{-\infty}^0 P^{Y(u)}[f \circ X_{-u} M_{-u}]\kappa(du) \\ & = Q_m \int_{-\infty}^0 K_{-u} f \circ Y_u \kappa(du) \\ & = \int_{-\infty}^0 \nu K_{-u}(f)\, du = \int_0^\infty \nu K_u(f)\, du, \end{aligned} \tag{6.15}$$

where the second equality follows from (A.14) and the fourth from (A.16). For the second term on the right-hand side of (6.14) observe that $\alpha < g_0$ and $0 < d_0$ if and only if $r = g_0$ satisfies $r \in G$ and $r \leq 0 < d_r$. Therefore using (6.11) the second term of the right-hand side of (6.14) equals

$$\begin{aligned} & Q_m \sum_{r \in G} 1_{\{r \leq 0 < d_r\}} (f \circ X_{-r} M_{-r}) \circ \gamma_r \\ & = \int_{-\infty}^0 dt \int \nu^B(dx) {*P^x}[f \circ X_{-t} M_{-t}] = \int_0^\infty \nu^B q_t(f)\, dt. \end{aligned}$$

Combining these results gives (6.13). □

(6.16) **Theorem.** *Let* $\eta_t^* = \eta_t^M + \nu K_t + \nu^B q_t$. *Then* η^* *is an entrance law for* (K_t) *and*

$$m^* = m_1 + \int_0^\infty \eta_t^*\, dt. \tag{6.17}$$

Proof. The only thing remaining to be proved is that each η_t^* is σ-finite since we know that $\eta_t^* K_s = \eta_{t+s}^*$ and that (6.17) holds. Since m is σ-finite let $f \in \mathscr{E}$ be strictly positive with $m(f) < \infty$. But then

$$\begin{aligned}\infty > m(f) \geq \int_t^\infty \eta_s^*(f)\, ds &= \int_0^\infty \eta_{t+s}^*(f)\, ds\\ &= \int_0^\infty \eta_t^* K_s(f)\, ds = \eta_t^*(Vf).\end{aligned}$$

But $Vf > 0$ on E_M and since $\eta_t^* = \eta_t^* K_0$, each η_t^* is carried by E_M. Hence η_t^* is σ-finite. □

(6.18) **Remarks.** The entrance laws η^M and $\nu^B q_t$ may be decomposed further. Let $m_p = \mu U + \psi$ be the Riesz decomposition of m_p into a potential μU and a harmonic excessive measure ψ. See (3.7) in [5]. Let $\xi = (\xi_t)_{t>0}$ be the entrance law [for P_t)] such that $\psi = \int_0^\infty \xi_t\, dt$. Then $\eta_t = \mu P_t + \xi_t$ and it is readily verified that $\eta_t^M = \mu K_t + \xi_t^M$, where ξ^M is defined by (6.7) with η replaced by ξ. Let F be the set of regular points of H; that is, $F = \{x\colon P^x(D = 0) = 1\}$. It is known that ${}^*P^x$ is a multiple, say $h(x)$, of P^x when $x \notin F$. See, for example, (6.6) in [5]. Let $\mu_1 = 1_F \nu^B$ and $\mu_2 = h \cdot 1_{E-F}\nu^B$. Then it follows that $\nu^B q_t = \mu_1 q_t + \mu_2 K_t$. Combining these observations we may write

$$\eta_t^* = (\nu + \mu + \mu_2)K_t + \xi_t^M + \mu_1 q_t. \tag{6.19}$$

APPENDIX

In this Appendix we collect some facts about homogeneous random sets in Ω and their extension to W. Suppose that H is a homogeneous random set (HRS) contained in $]0, \zeta[$; that is, for each ω, $H(\omega)$ is a subset of $]0, \zeta(\omega)[$ such that if $s > 0$ and $t \geq 0$, then $s + t \in H(\omega)$ if and only if $s \in H(\theta_t \omega)$. We shall also use $H = (H_t(\omega))$ to denote the indicator of $\{(t, \omega)\colon t \in H(\omega)\} \subset]0, \infty[\times \Omega$. Then the homogeneity condition becomes $H_{t+s} = H_t \circ \theta_s$ for $t > 0$, $s \geq 0$. Following Mitro [17] we extend H to W by

$$J(w) = \bigcup_{\alpha < t < \beta} \{t + H \circ \gamma_t\}. \tag{A.1}$$

Formally (A.1) differs from Mitro's definition but it will become clear that, in fact, it is the same. The following proposition gives some properties of J. We leave their straightforward verification to the reader.

(A.2) **Proposition.** *Let H and J be as above.*

(i) *If* $\alpha < s < t < \beta$, *then* $t + H \circ \gamma_t = (s + H \circ \gamma_s) \cap]t, \infty[$.
(ii) $J \subset]\alpha, \beta[$.

(iii) *Let* $J_t(w)$ *denote the indication of* $\{(t, w): t \in J(w)\} \subset \mathbb{R} \times W$. *Then* $J_u = H_{u-t} \circ \gamma_t$ *if* $\alpha < t < u < \beta$.

(iv) $J_{u+v} = J_u \circ \theta_v$ *for* $u, v \in \mathbb{R}$.

(v) *If* H *is closed in* $]0, \zeta[$, *then* J *is closed in* $]\alpha, \beta[$. *If* H *is optional, then* J *is optional relative to the filtration* $(\mathcal{G}_t^m)$.

Let M be an exact MF as defined in Section 3. Define $S = \inf\{t: M_t = 0\}$. Note that $S = \inf\{t > 0: M_t = 0\}$ since $M_t = 0$ for all t if $M_0 = 0$. It is easily checked that S is a perfect terminal time as defined above (3.14) and $S \leq \zeta$ since $M_t = 0$ for $t \geq \zeta$. However, S need not be exact. Let $S_t = t + S \circ \theta_t$. Then $t \to S_t$ is increasing and $S_t \leq \zeta$ if $t < \zeta$. We are going to recall some results from [10]. First if $t \leq u < S_t$, then $S_u = S_t$. For each ω define

$$H(\omega) = \{S_t(\omega);\ t > 0\}^{-} \cap]0, \zeta(\omega)[, \tag{A.3}$$

where "$^{-}$" denotes closure. It is shown in [10], that H is a closed optional HRS. For $t \geq 0$ define

$$D_t = \inf\{u > t: u \in H\} \wedge \zeta. \tag{A.4}$$

Then $t \to D_t$ is increasing and right continuous and $S_{t+} = D_t$. Moreover, if $t < S_t$, then $S_t = D_t$. Let $D = D_0$. Then D is the exact regularization of S and $D_t = t + D \circ \theta_t$. In particular, $D = S$ when $S > 0$. Let J be defined in terms of H by (A.1). Then J is closed in $]\alpha, \beta[$. The following notation differs from that in [10] (which deals only with functionals on Ω) but will be convenient for us. Define (the infimum of the empty set in $+\infty$)

$$\begin{aligned} d_t &= \inf\{u > t: u \in J\} \wedge \beta, \\ d = d_\alpha &= \inf\{u \in J\} \wedge \beta. \end{aligned} \tag{A.5}$$

Then $t \leq d_t \leq \beta$. Of course, d and d_t bear the same relation to J as D and D_t to H.

(A.6) **Proposition.** *Let* M *be an exact MF. Then using the above notation*:

(i) $J = (\bigcup_{\alpha < t < \beta}\{t + S \circ \gamma_t\})^{-} \cap]\alpha, \beta[$.

(ii) *If* $\alpha < t < \beta$, *then* $d_t = t + D \circ \gamma_t$, *and if* $t \notin J$, *then* $D \circ \gamma_t = S \circ \gamma_t$.

(iii) *If* $\alpha < t < d$, *then* $d = t + D \circ \gamma_t = t + S \circ \gamma_t$.

Proof. Let $\alpha < t < \beta$ and choose s with $\alpha < s < t < \beta$. Then $r = t - s > 0$ and $r < \beta - s = \beta \circ \theta_s = \zeta \circ \gamma_s$. Now $r + S \circ \theta_r \in H$ provided $r + S \circ \theta_r < \zeta$ or, equivalently, $S \circ \theta_r < \zeta \circ \theta_r$. Thus $r + S \circ \theta_r \circ \gamma_s \in H \circ \gamma_s$ provided $S \circ \theta_r \circ \gamma_s < \zeta \circ \theta_r \circ \gamma_s$. Since $\theta_r \circ \gamma_s = \gamma_{s+r}$ this gives $(t = s + r)$, $t + S \circ \gamma_t \in s + H \circ \gamma_s \subset J$ provided $S \circ \gamma_t < \zeta \circ \gamma_t = \beta \circ \theta_t$. But $S \circ \gamma_t < \beta \circ \theta_t$ when $t + S \circ \gamma_t < \beta$. Thus if $\alpha < t < \beta$ and $t + S \circ \gamma_t < \beta$, then $t + S \circ \gamma_t \in J$. Since J is closed in $]\alpha, \beta[$ this yields

$$\left(\bigcup_{\alpha < t < \beta}\{t + S \circ \gamma_t\}\right)^{-} \cap]\alpha, \beta[\subset J.$$

Conversely, let $u \in J$. Then $\alpha < u < \beta$ and if $\alpha < t < u < \beta$, $u - t \in H \circ \gamma_t$. From the definition of H, there exist $r_n > 0$ with $r_n + S \circ \theta_{r_n} \circ \gamma_t \to u - t$ or $t + r_n + S \circ \gamma_{t+r_n} \to u$. This establishes (i).

For (ii) suppose $\alpha < t < \beta$. Then

$$\begin{aligned}\inf\{u > t\colon u \in J\} &= t + \inf\{u > 0\colon u + t \in J\}\\ &= t + \inf\{u > 0\colon u \in H \circ \gamma_t\},\end{aligned}$$

where the last equality follows from (A.2)(iii) because $u + t < \beta$ if $u + t \in J$ and $u < \zeta \circ \gamma_t = \beta - t$ if $u \in H \circ \gamma_t$. Therefore $d_t = t + D \circ \gamma_t$. Note that if the sets in braces in the last display are empty, then $d_t = \beta$ and $t + D \circ \gamma_t = t + \zeta \circ \gamma_t = \beta$. If $t \notin J$, then from (i), $S \circ \gamma_t \neq 0$ and so $D \circ \gamma_t = S \circ \gamma_t$ proving (ii). If $\alpha < t < d$, then $d = d_t$ and $t \notin J$, so (iii) follows from (ii). □

Remark. The same argument as that used to prove (A.6)(i) shows that

$$J = \Bigg(\bigcup_{\alpha < t < \beta} \{t + D \circ \gamma_t\}\Bigg)^{-} \cap \,]\alpha, \beta[.$$

If $M_t = e^{-qt} 1_{[0,\zeta[}(t)$, then H and J are empty and hence $d = \beta$. If $\alpha = -\infty$, then from (3.13), $N_t = 0$ on $]\alpha, d[$. The next proposition shows that, in general, N does not vanish on $]\alpha, d[$ unless it vanishes identically on $]\alpha, \infty[$.

(A.7) Proposition. *Let N be the functional defined in* (3.8) *and let* $T = \inf\{t\colon N_t = 0\}$. *Then either $T = \alpha$ or $T = d$.*

Proof. Clearly, $T \geq \alpha$ since $N_t = 1$ if $t \leq \alpha$. If $\alpha < t < d \leq v$, then $v - t \geq d - t = S \circ \gamma_t$ by (A.6)(iii). Therefore $N(t, v) = 0$ and so $N_v = 0$. Thus $T \leq d$. Suppose $\alpha < T < d$. Let $\alpha < t < T < v < d$. Then $v - t < d - t = S \circ \gamma_t$ so $N(t, v) > 0$. But from (3.10), $N_v = N_t N(t, v)$, and this contradicts the fact that $N_t > 0$ and $N_v = 0$. □

It was shown in Sections 3 and 4 of [10] that there exists a unique optional homogeneous random measure (HRM), $\Lambda(dt) = \Lambda(\omega, dt)$ carried by $H^c \cap \,]0, \zeta[$ such that

$$\Lambda(dt) = -dM_t/M_{t-}, \quad \text{on }]0, S[. \tag{A.8}$$

In the present situation because (3.3) holds identically, the argument in [10] shows that

$$\Lambda(\theta_t \omega, B) = \Lambda(\omega, B + t)$$

identically in $t \geq 0$, $\omega \in \Omega$ and $B \in \mathscr{B}(R^+)$. It is well known and an immediate consequence of the integration by parts formula that $(M_{t-})^{-1}\, dM_t = -M_t\, d(1/M_t)$, and so (A.8) may be written

$$\Lambda(dt) = M_t\, d(1/M_t), \quad \text{on }]0, S[. \tag{A.9}$$

There is a standard method of extending Λ to a homogeneous random measure, $\kappa(dt) = \kappa(w, dt)$ on W that is carried by $]\alpha, \beta[$ and satisfies for

$B \subset]t, \infty[$ and $\alpha(w) < t$,

$$\kappa(w, B) = \Lambda(\gamma_t w, B - t). \tag{A.10}$$

See, for example, [11] or [16]. In the present situation $\kappa(\theta_t w, B) = \kappa(w, B + t)$ identically in $t \in \mathbb{R}$, $w \in W$ and $B \subset \mathscr{B}(\mathbb{R})$. Moreover, κ does not charge J. Let $\nu = \nu_\Lambda$ be the characteristic measure of Λ; that is, for $f \in p\mathscr{E}^*$,

$$\nu(f) = \lim_{t \downarrow 0} t^{-1} P^m \int_{]0, t]} f \circ X_s \Lambda(ds). \tag{A.11}$$

It is known (for example, Section 12 of [11]) that

$$\nu(f) = Q_m \int_0^1 f \circ Y_t \kappa(dt). \tag{A.12}$$

We shall also need a projection result. First of all there exists a $(\mathscr{G}_{t+}^*)$ optional process $Z^* = (Z_t^*(w))$ that is bounded, strictly positive on $]\alpha, \beta[$, and for which $Q_m \int Z_t^* \kappa(dt) < \infty$. To see this let $Z_t(\omega) = \sum_{n \geq 0} 2^{-n} Z_t^n(\omega)$, where Z^n is defined at the bottom of page 146 of [10]. Under the present hypotheses on M, each Z^n is $(\mathscr{F}_{t+}^*)$ optional. It was shown in [10] that $0 \leq Z_t^n \leq 1$, $\int Z_t^n \Lambda(dt) \leq 1$ and $\bigcup_n \{Z_t^n > 0\} = [0, \infty[$. Consequently, if $t > 0$, $0 < Z_t \leq 2$ and $\int Z_t \Lambda(dt) \leq 2$. Choose $f \in \mathscr{E}$ with $0 < f \leq 1$ and $m(f) \leq 1$. For each rational q let $0 < c(q)$ with $\sum c(q) < \infty$. Define for each q,

$$^*Z_t^q = 1_{]q, \infty[}(t) f \circ Y_q Z_{t-q} \circ \gamma_q 1_{\{\alpha < q\}}. \tag{A.13}$$

Then $^*Z^q$ is optional over $(\mathscr{G}_{t+}^*)$ and it follows from (A.10) that $Q_m \int {}^*Z_t^q \kappa(dt) \leq 2m(f) < \infty$. Finally, $Z^* = \sum_{q \in \mathbb{Q}} c(q)\, {}^*Z^q$ has the desired properties. It now follows that $t \to \int_\alpha^t Z_s^* \kappa(ds)$ is $(\mathscr{G}_t^m)$-optional. Hence by standard projection results (see, for example, [1]) if $F_u(\omega) \in p(\mathscr{B}(\mathbb{R}) \times \mathscr{F}^*)$, then

$$Q_m \int F_u(\gamma_u) \kappa(du) = Q_m \int P^{Y(u)}(F_u) \kappa(du). \tag{A.14}$$

Finally, we need an extension of (A.12). Let η be the measure on $\mathbb{R} \times E$ defined by

$$\eta(F) = Q_m \int F(t, Y_t) \kappa(dt), \qquad F \in p(\mathscr{B}(\mathbb{R}) \times \mathscr{E}). \tag{A.15}$$

If $F_s(t, x) = F(s + t, x)$, then

$$\begin{aligned} \eta(F_s) &= Q_m \int F(t + s, Y_t) \kappa(dt) = Q_m \int F(t, Y_{t-s}) \kappa(dt - s) \\ &= Q_m \left(\int F(t, Y_t) \kappa(dt) \right) \circ \theta_{-s} = \eta(F), \end{aligned}$$

since $\theta_{-s}(Q_m) = Q_m$. From (A.12) one has $\eta([0,1] \times A) = \nu(A)$ for $A \in \mathscr{E}$ and using the process Z^* constructed in the previous paragraph it is clear that η is a countable sum of finite measures. Therefore by [8] one has

$$\begin{aligned} Q_m \int F(t, Y_t) \kappa(dt) &= \int_{-\infty}^{\infty} dt \int_E F(t, x) \nu(dx) \\ &= \int_E \nu(dx) \int_{-\infty}^{\infty} F(t, x)\, dt. \end{aligned} \tag{A.16}$$

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