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ASYMPTOTIC FORMS FOR THE DERIVATIVES OF ONE-SIDED STABLE LAWS

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For the derivatives $f_\alpha^{(k)}(x)$ of the one-sided stable density of index $\alpha \in (0,1)$ asymptotic formulas are computed as $k \to \infty$ thereby exhibiting the detailed analytic structure for large orders of derivatives. The results extend those for the well-known case $\alpha = \frac{1}{2}$ which may be expressed in terms of Laguerre polynomials (formulas of Plancherel–Rotach type).

1. Introduction and summary. Usually stable laws are considered via their characteristic functions. The latter are well known to be expressible in terms of "elementary" functions (see, e.g., [2, 4, 6, 18]), a property which is not known for the corresponding stable densities except for the three cases of the normal, Cauchy, and the one-sided stable density of index $\frac{1}{2}$. In general, only series and integral representations are available. This lack of explicitness often causes difficulties even in proving simple properties of the general stable density such as, e.g., unimodality which after several vain attempts of various authors finally has been established by Yamazato [15]. For a sharpening see [3]. Further, there is a variety of papers dealing with the analytic structure of stable densities (e.g., [1, 2, 5, 6, 10, 16–18] and the references therein).

In continuation of these investigations the present paper primarily is concerned with derivatives of stable laws. For orientation, we consider the one-sided stable density of index $\frac{1}{2}$ given by

$$f_{1/2}(x) := \begin{cases} \dfrac{1}{2\sqrt{\pi}\,x^{3/2}} e^{-1/4x}, & x > 0, \\ 0, & x \le 0 \end{cases} \tag{1.1}$$

([4], page 171, and [6], page 143), the derivatives of which can be written as

$$f_{1/2}^{(k)}(x) = \frac{(-1)^k k!}{x^k} L_k^{(1/2)}(1/4x) f_{1/2}(x), \qquad k \in \mathbb{N}_0 \tag{1.2}$$

([11], page 388, problem 73, and [3], page 241), where $L_k^{(\beta)}$ is Laguerre's polynomial, the definition of which we take from [11], page 100. Now it is well known from the theory of orthogonal polynomials that finer analytic properties of $L_k^{(\beta)}$ become apparent in studying its behaviour as k gets large. The best asymptotic description of $L_k^{(\beta)}(x)$ is known as the formulas of Plancherel–Rotach type ([11], Theorem 8.22.8, page 200).

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It is the main purpose of this paper to extend these classical results for (1.2) to the general one-sided stable law of index $\alpha \in (0,1)$ which we assume to be defined by its Laplace transform e^{-t^α} ([2], page 448). Denoting by $f_\alpha(x)$ the associated stable density we prove asymptotic formulas for the derivatives $f_\alpha^{(k)}(x)$ as $k \to \infty$, where according to the case $\alpha = \frac{1}{2}$ we distinguish the following regions for x depending on k:

(1) oscillation interval (a_k, ∞) (Theorem 1);
(2) monotonicity interval $(0, a_k)$ (Theorem 2);
(3) neighbourhood of the turning point a_k (Theorem 3);
(4) neighbourhood of infinity $k^s = O(x)$, $k \to \infty$ (Theorem 4),

where $s > 0$ suitably and

$$a_k := \left(\frac{k}{1-\alpha}\alpha^{(\alpha+1)/(\alpha-1)}\right)^{(\alpha-1)/\alpha}, \qquad k \in \mathbb{N}. \tag{1.3}$$

The proofs depend on integral representations obtained by Laplace inversion and on the asymptotic evaluation of the resulting contour integrals by the saddle-point method and its various modifications.

2. Auxiliary results. In this section we collect technical details and prepare some preliminary results which are basic for our main theorems in the next section.

Starting from the Laplace transform e^{-t^α}, by the inversion formula ([14], Chapter 2, Section 7) we obtain

$$f_\alpha(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{tx-t^\alpha}\,dt, \qquad c \geq 0,\ x \geq 0, \tag{2.1}$$

for the one-sided stable density of index $\alpha \in (0,1)$. Here and throughout the power t^α denotes the principal value, that is $t^\alpha = \exp(\alpha \log t)$, where $\log t = \log|t| + i \arg t$, $-\pi < \arg t < \pi$, for t in the set

$$\mathbb{C}_- := \{t = \sigma + i\tau \mid \text{if } \sigma \leq 0, \text{then } \tau \neq 0\}. \tag{2.2}$$

In view of the integrability properties of e^{-t^α} an application of the Schwarz reflection principle and Cauchy's theorem implies

LEMMA 1. *If $k \in \mathbb{N}_0$, $c, x \geq 0$, then*

(i) $$f_\alpha^{(k)}(x) = \frac{1}{\pi}\operatorname{Im}\int_c^{c+i\infty} t^k e^{tx-t^\alpha}\,dt,$$

(ii) $$f_\alpha^{(k)}(x) = \frac{(-1)^{k+1}}{\pi}\operatorname{Im}\int_0^\infty t^k e^{-tx-e^{i\pi\alpha}t^\alpha}\,dt.$$

In evaluating $f_\alpha^{(k)}(x)$ asymptotically we transform the integrals such that the method of steepest descent is applicable. To this end in Lemma 1(i) we put $c = 0$

and substitute $\tau = tx^{1/(\alpha-1)}$, which gives (cf. [6], Section 5.9, page 151)

$$f_\alpha^{(k)}(x) = \frac{x^{(k+1)/(\alpha-1)}}{\pi}\operatorname{Im}\int_0^{i\infty}\tau^k e^{x^{\alpha/(\alpha-1)}(\tau-\tau^\alpha)}\,d\tau.$$

Putting

$$N := 4k + 3, \tag{2.3}$$

$$x := \left(\frac{N}{4}\zeta - b_k y\right)^{(\alpha-1)/\alpha}, \tag{2.4}$$

$$p(t) := (t^\alpha - t)\zeta - \log t, \tag{2.5}$$

where $\zeta, b_k > 0$ and $y \in \mathbb{R}$ have to be chosen suitably below, we get the following representation in

LEMMA 2. *With the notation above for $x > 0$ we have*

$$f_\alpha^{(k)}(x) = \frac{x^{(k+1)/(\alpha-1)}}{\pi}\operatorname{Im}\int_0^{i\infty} e^{-(N/4)p(t)}e^{-b_k y(t-t^\alpha)}t^{-3/4}\,dt. \tag{2.6}$$

The choices (2.3) and (2.4) have been made in light of the case $\alpha = \frac{1}{2}$ (cf. [11], Theorem 8.22.8, page 200). The asymptotic evaluation of (2.6) below requires a solution of the saddle-point equation $p'(t) = 0$, that is,

$$(\alpha t^\alpha - t)\zeta = 1. \tag{2.7}$$

This is a transcendental equation for which we have to determine solutions $t \in \mathbb{C}_-$ [see (2.2)]. In order to get explicit representations for possible solutions we try to find a parametrization of the positive parameter ζ by starting with the setup $t = r(\phi)e^{2i\phi}$. Equating real and imaginary parts in (2.7) this gives

$$r(\phi) = \left(\frac{\alpha\sin 2\alpha\phi}{\sin 2\phi}\right)^{1/(1-\alpha)}, \qquad 0 < \phi < \pi/2, \tag{2.8}$$

and

$$\zeta = \frac{1}{\alpha r(\phi)^\alpha}\frac{\sin 2\phi}{\sin 2(1-\alpha)\phi}.$$

This observation suggests considering

$$\rho(\phi) := \frac{1}{\alpha r(\phi)^\alpha}\frac{\sin 2\phi}{\sin 2(1-\alpha)\phi}, \qquad 0 < \phi < \pi/2, \tag{2.9}$$

with $r(\phi)$ defined in (2.8). Further, we put

$$r^*(\phi) := r(i\phi) = \left(\frac{\alpha\sinh 2\alpha\phi}{\sinh 2\phi}\right)^{1/(1-\alpha)}, \qquad \phi > 0, \tag{2.10}$$

$$\rho^*(\phi) := \rho(i\phi) = \frac{1}{\alpha r^*(\phi)^\alpha}\frac{\sinh 2\phi}{\sinh 2(1-\alpha)\phi}, \qquad \phi > 0, \tag{2.11}$$

and generally for ζ in (2.7) we choose the parametrizations

$$\zeta := \begin{cases} \rho(\phi), & 0 < \phi < \pi/2, \\ \rho(0), & \\ \rho^*(\phi), & \phi > 0, \end{cases} \tag{2.12}$$

where

$$r(0) = r^*(0) = \alpha^{2/(1-\alpha)} \tag{2.13}$$

and

$$\rho(0) = \rho^*(0) = \frac{1}{1-\alpha}\alpha^{((\alpha+1)/(\alpha-1))} \tag{2.14}$$

are obtained from (2.8) and (2.9) by continuity. Before discussing the saddle-point equation (2.7) in detail we need some lemmas for trigonometric and hyperbolic functions.

LEMMA 3. *If* $\gamma \in (0,1)$, *then*

(i) $\gamma \sin x < \sin \gamma x$, *for* $0 < x < \pi$,
(ii) $\operatorname{cotan} x < \gamma \operatorname{cotan} \gamma x$, *for* $0 < x < \pi$,
(iii) $\sin \gamma x / \sin x$ *is strictly increasing on* $(0, \pi)$,
(iv) $(\sin \alpha x / \sin(1-\alpha)x) \cos x$ *is strictly decreasing on* $(0, \pi/2)$,
(v) $((1-\alpha)\sin \alpha x / \alpha \sin(1-\alpha)x)\sin x - x$ *is strictly decreasing on* $(0, \pi)$,
(vi) $1 - \alpha - (\alpha \operatorname{cotan} \alpha x - \operatorname{cotan} x)\operatorname{cotan} x > 0$, *for* $0 < x < \pi$.

PROOF. Parts (i)–(v) are proved successively by considering derivatives. Part (vi) is immediate if $\pi/2 \le x < \pi$. For $0 < x < \pi/2$ we use the expansion

$$\operatorname{cotan} x - \frac{1}{x} = -\frac{2}{x}\sum_{1}^{\infty} \zeta(2\mu)\left(\frac{x}{\pi}\right)^{2\mu}, \qquad 0 < x < \pi \tag{2.15}$$

(e.g., [9], page 143), where ζ here denotes Riemann's ζ-function. Now (2.15) implies that

$$\begin{aligned} \alpha \operatorname{cotan} \alpha x - \operatorname{cotan} x &= \frac{2}{x}\sum_{1}^{\infty} \zeta(2\mu)\left(\frac{x}{\pi}\right)^{2\mu}(1-\alpha^{2\mu}) \\ &\le (1-\alpha)\frac{4}{x}\sum_{1}^{\infty}\zeta(2\mu)\mu\left(\frac{x}{\pi}\right)^{2\mu}. \end{aligned} \tag{2.16}$$

If $0 < x \le \pi/4$, then we use (2.15) and (2.16) to obtain

$$\begin{aligned} &1 - \alpha - (\alpha \operatorname{cotan} \alpha x - \operatorname{cotan} x)\operatorname{cotan} x \\ &\quad > (1-\alpha)\left(1 - \frac{4}{x^2}\sum_{1}^{\infty}\zeta(2\mu)\mu\left(\frac{x}{\pi}\right)^{2\mu}\right) \\ &\quad \ge (1-\alpha)\left(1 - \frac{2}{3}\sum_{1}^{\infty}\frac{\mu}{16^{\mu-1}}\right) = (1-\alpha)\left(1 - \frac{2}{3}\frac{16^2}{15^2}\right) > 0. \end{aligned}$$

If $\pi/4 < x < \pi/2$, then we have $\operatorname{cotan} x < 2 - (4/\pi)x$, by convexity, and further [use (2.15) and (2.16) again] we get

$$\begin{aligned}
&1 - \alpha - (\alpha \operatorname{cotan} \alpha x - \operatorname{cotan} x)\operatorname{cotan} x \\
&> (1-\alpha)\left(1 - \frac{4}{3}\sum_{1}^{\infty}\frac{\mu}{\pi^{2(\mu-1)}}\left(1 - \frac{2}{\pi}x\right)x^{2\mu-1}\right) \\
&\geq (1-\alpha)\left(1 - \frac{2}{3}\sum_{1}^{\infty}\frac{1}{\pi^{2(\mu-1)}}\left(\frac{\pi}{2}\frac{2\mu-1}{2\mu}\right)^{2\mu-1}\right) \\
&\geq (1-\alpha)\left(1 - \frac{2\pi}{3}\sum_{1}^{\infty}\frac{1}{4^{\mu}}\right) = (1-\alpha)\left(1 - \frac{2\pi}{9}\right) > 0,
\end{aligned}$$

where we have used the fact that the function $(1 - (2/\pi)x)x^{2\mu-1}$ on $[\pi/4, \pi/2]$ attains its maximum at $x = (\pi/2)((2\mu - 1)/2\mu)$. □

As a simple consequence of Lemma 3(iii) we obtain [see (2.8), (2.9), (2.13) and (2.14)]

LEMMA 4. (i) *r is strictly increasing on $[0, \pi/2)$ with* $r([0, \pi/2)) = [\alpha^{2/(1-\alpha)}, \infty)$.

(ii) *ρ is strictly decreasing on $[0, \pi/2)$ with*

$$\rho([0, \pi/2)) = \left(0, \frac{1}{1-\alpha}\alpha^{((\alpha+1)/(\alpha-1))}\right].$$

The next lemma again is verified successively by considering derivatives. Therefore we omit its simple proof.

LEMMA 5. *If $\gamma \in (0,1)$, then*

(i) $\sinh \gamma x < \gamma \sinh x$, *for* $x > 0$,
(ii) $\gamma \operatorname{cotanh} \gamma x < \operatorname{cotanh} x$, *for* $x > 0$,
(iii) $\sinh \gamma x / \sinh x$ *is strictly decreasing on* $(0, \infty)$,
(iv) $(\sinh \gamma x / \sinh x)\, e^{(1-\gamma)x}$ *is strictly increasing on* $(0, \infty)$.

For the functions r^* and ρ^* [see (2.10) and (2.11)] this immediately gives

LEMMA 6. (i) *r^* is strictly decreasing on $[0, \infty)$ with* $r^*([0, \infty)) = (0, \alpha^{2/(1-\alpha)}]$,

(ii) *ρ^* is strictly increasing on $[0, \infty)$ with*

$$\rho^*([0, \infty)) = \left[\frac{1}{1-\alpha}\alpha^{((\alpha+1)/(\alpha-1))}, \infty\right).$$

Now we are in a position to discuss the saddle-point equation (2.7) through

LEMMA 7. (i) *Suppose that in* (2.12) $\zeta = \rho(\phi)$, *where* $\phi \in (0, \pi/2)$ *is fixed. Then* (2.7) *in* $\mathbb{C}_-$ *has only the solutions* t_0 *and* $\bar{t}_0$ *with*

$$t_0 = t_0(\phi) = r(\phi)e^{2i\phi}. \tag{2.17}$$

(ii) *Suppose that in* (2.12) $\zeta = \rho^*(\phi)$, *where* $\phi \in [0, \infty)$ *is fixed. Then* (2.7) *in* $\mathbb{C}_-$ *has only the solutions* t_0^* *and* t_0^{**} *with*

$$t_0^* = t_0^*(\phi) = r^*(\phi)e^{2\phi}, \qquad t_0^{**} = t_0^*(-\phi) \tag{2.18}$$

and $0 < t_0^{**} \le t_0^*$, *where* $t_0^{**} = t_0^*$ *iff* $\phi = 0$.

PROOF. Since (2.7) has real coefficients it suffices to consider $t \in \mathbb{C}_-$ with $\operatorname{Im} t \ge 0$. Further, it is readily verified that t_0, t_0^* and t_0^{**} satisfy (2.7) in the cases stated. Thus it remains to investigate the question of uniqueness.

(i) Suppose that $t = t(\psi) = s(\psi)e^{2i\psi}$, $0 \le \psi < \pi/2$, $s(\psi) > 0$, satisfies (2.7). Then equating imaginary and real parts in (2.7) gives $s(\psi) = r(\psi)$ and $\rho(\psi) = \rho(\phi)$ which implies $\psi = \phi$, by Lemma 4(ii). Hence we get $t = t_0$.

(ii) Assume that $s(\psi)e^{2i\psi}$, $0 < \psi < \pi/2$, is a solution of (2.7). Then we conclude $\rho(\psi) = \rho^*(\phi)$ as above which contradicts Lemma 4(ii) and Lemma 6(ii). Thus the only solutions in $\mathbb{C}_-$ are real and positive. Since (2.7) is equivalent to

$$p'(t) = -\frac{1}{t} + \rho^*(\phi)(\alpha t^{\alpha-1} - 1) = 0,$$

we conclude from [8], page 48, problem 75, that p' has at most 2 zeros on $(0, \infty)$ including multiplicities. Obviously, we have $t_0^{**} \le t_0^*$ [see (2.10) and (2.18)] and further, by Lemma 8 below, we get $p''(t_0^*) \le 0$ with equality iff $\phi = 0$, which completes the proof. □

Straightforward calculations lead to the following formulas in

LEMMA 8. (i) *If* $\zeta = \rho(\phi)$, $0 < \phi < \pi/2$, *in* (2.5), *then*

$$p(t_0) = -\log r(\phi) - 2i\phi + \frac{1}{\alpha} + \left(\frac{1}{\alpha} - 1\right)\frac{\sin 2\alpha\phi}{\sin 2(1-\alpha)\phi}e^{2i\phi},$$

$$p'(t_0) = 0,$$

$$p''(t_0) = \frac{\alpha}{t_0^2}\left(1 - \frac{1-\alpha}{\alpha}\frac{\sin 2\alpha\phi}{\sin 2(1-\alpha)\phi}e^{2i\phi}\right).$$

(ii) *If* $\zeta = \rho^*(\phi)$, $0 \le \phi < \infty$, *in* (2.5), *then*

$$p(t_0^*) = -\log r^*(\phi) - 2\phi + \frac{1}{\alpha} + \left(\frac{1}{\alpha} - 1\right)\frac{\sinh 2\alpha\phi}{\sinh 2(1-\alpha)\phi}e^{2\phi},$$

$$p'(t_0^*) = 0,$$

$$p''(t_0^*) = -\frac{1}{t_0^{*2}}\left((1-\alpha)\frac{\sinh 2\phi}{\sinh 2(1-\alpha)\phi}e^{2\alpha\phi} - 1\right) \le 0,$$

with equality iff $\phi = 0$, *and*

$$p'''\left(t_0^*(0)\right) = -\alpha^{(5+\alpha)/(\alpha-1)}.$$

PROOF. We only mention that for the assertion concerning $p''(t_0^*)$ Lemma 5(iii) can be used. □

3. Main results. These consist of generalizations of the formulas of Plancherel–Rotach type for Laguerre polynomials (cf. [11], Theorem 8.22.8, page 200). Thereby the detailed analytic structure of derivatives of one-sided stable laws is exhibited in terms of elementary functions for large orders of derivatives. According to the partition of the positive real axis introduced in Section 1 first we consider the oscillation interval for $f_\alpha^{(k)}$ in

THEOREM 1. *Suppose that* $\phi \in (0, \pi/2)$ *is fixed*, $N = 4k + 3$, *and*

$$x = \left\{ \frac{N}{4\alpha} \left(\frac{\sin 2\phi}{\sin 2\alpha\phi} \right)^{\alpha/(1-\alpha)} \frac{\sin 2\phi}{\sin 2(1-\alpha)\phi} \right\}^{(\alpha-1)/\alpha} \tag{3.1}$$

Then

$$f_\alpha^{(k)}(x) = \frac{2}{\sqrt{\pi N/2}} \times \frac{\left(\dfrac{N}{4} \dfrac{\sin 2\phi}{\alpha \sin 2(1-\alpha)\phi} \right)^{(k+1)/\alpha}}{\left(\alpha^2 - 2\alpha(1-\alpha)\dfrac{\sin 2\alpha\phi}{\sin 2(1-\alpha)\phi}\cos 2\phi + (1-\alpha)^2 \dfrac{\sin^2 2\alpha\phi}{\sin^2 2(1-\alpha)\phi} \right)^{1/4}}$$

$$\times \exp\left\{ -\frac{N}{4}\left(\frac{1}{\alpha} + \left(\frac{1}{\alpha} - 1 \right) \frac{\sin 2\alpha\phi}{\sin 2(1-\alpha)\phi} \cos 2\phi \right) \right\} \tag{3.2}$$

$$\times \left\{ \sin\left(\frac{N}{4}\left(2\phi - \frac{1-\alpha}{\alpha} \frac{\sin 2\alpha\phi}{\sin 2(1-\alpha)\phi} \sin 2\phi \right) + b(\phi) \right) + O\left(\frac{1}{k} \right) \right\},$$

as $k \to \infty$, *where*

$$b(\phi) = \frac{\phi}{2} + \frac{1}{2}\arctan\left\{ \frac{\dfrac{1-\alpha}{\alpha} \dfrac{\sin 2\alpha\phi}{\sin 2(1-\alpha)\phi} \sin 2\phi}{1 - \dfrac{1-\alpha}{\alpha} \dfrac{\sin 2\alpha\phi}{\sin 2(1-\alpha)\phi} \cos 2\phi} \right\}. \tag{3.3}$$

REMARK. Arctan denotes the principal branch that is $-\pi/2 < \arctan \xi < \pi/2$, for real ξ. Further, observe that

$$1 - \frac{1-\alpha}{\alpha} \frac{\sin 2\alpha\phi}{\sin 2(1-\alpha)\phi} \cos 2\phi > 0, \qquad 0 < \phi < \pi/2,$$

by Lemma 3(iv).

PROOF OF THEOREM 1. We use representation (2.6) in Lemma 2 with $y = 0$ and $\zeta = \rho(\phi)$ [see (2.12)]. This gives (3.1) instead of (2.4) and

$$f_\alpha^{(k)}(x) = \frac{x^{(k+1)/(\alpha-1)}}{\pi}\operatorname{Im}\int_0^{i\infty} e^{-(N/4)p(t)}t^{-3/4}\,dt, \tag{2.6$'$}$$

which in the sequel we treat by the method of steepest descent (see, e.g., [7], Chapter 4, Section 7). Lemmas 7(i) and 8(i) show that t_0 defined in (2.17) is the only saddle point in the upper half-plane, being more precisely a simple one. The crucial point is to find a path on which $\operatorname{Re}(p(t) - p(t_0)) \geq 0$ with equality iff $t = t_0$. A partial study of the conformal mapping $p(t)$ shows that we could deal with the path $\{t | \arg t = 2\phi\}$ in case $\pi/4 \leq \phi < \pi/2$, a choice which unfortunately does not work for $0 < \phi < \pi/4$. Thus for every $\phi \in (0, \pi/2)$ in (2.6′) we consider the contour

$$C_\phi := \{t(\psi) := r(\psi)e^{2i\psi} | 0 \leq \psi < \pi/2\}, \tag{3.4}$$

where $r(\psi)$ is given by (2.8). Obviously, C_ϕ passes through the saddle point t_0. Using the reality of the integrand and Cauchy's theorem (2.6′) is rewritten as

$$f_\alpha^{(k)}(x) = \frac{x^{(k+1)/(\alpha-1)}}{\pi}\operatorname{Im}\int_{C_\phi} e^{-(N/4)p(t)}t^{-3/4}\,dt. \tag{3.5}$$

Use Lemma 4(i) for an existence proof of the integral.

Among the various assumptions of the saddle-point method we only verify in detail the most important reality condition mentioned above. The other ones are immediate (see [7], page 127). From (2.5), (2.8), (2.9) and (3.4) we get

$$\begin{aligned}
&\frac{d}{d\psi}\operatorname{Re}(p(t(\psi)) - p(t_0))\\
&\quad = \operatorname{Re} p'(t(\psi))t'(\psi)\\
&\quad = \operatorname{Re}\big(-1 + \rho(\phi)\big(\alpha t(\psi)^\alpha - t(\psi)\big)\big)\left(\frac{r'(\psi)}{r(\psi)} + 2i\right)\\
&\quad = \operatorname{Re}\left(-1 + \rho(\phi)\alpha r(\psi)^\alpha \frac{\sin 2(1-\alpha)\psi}{\sin 2\psi}\right)\left(\frac{r'(\psi)}{r(\psi)} + 2i\right)\\
&\quad = \left(\frac{\rho(\phi)}{\rho(\psi)} - 1\right)\frac{r'(\psi)}{r(\psi)},
\end{aligned}$$

which in turn, by Lemma 4, gives

$$\frac{d}{d\psi}\operatorname{Re}(p(t(\psi)) - p(t_0))\left\{\begin{matrix} < \\ = \\ > \end{matrix}\right\}0, \quad \text{if}\ \begin{cases} \psi \in (0, \phi), \\ \psi = \phi, \\ \psi \in (\phi, \pi/2). \end{cases}$$

Since $t(\phi) = t_0$, it follows that

$$\operatorname{Re}(p(t) - p(t_0)) > 0, \quad \text{for } t \in C_\phi - \{t_0\}.$$

Now an application of Theorem 7.1 in [7], page 127, to (3.5) yields

$$f_\alpha^{(k)}(x) = \frac{2x^{(k+1)/(\alpha-1)}}{\sqrt{\pi N/4}} \operatorname{Im} \frac{e^{-(N/4)p(t_0)}}{\sqrt{2p''(t_0)}} t_0^{-3/4}\left(1 + O\left(\frac{1}{k}\right)\right). \tag{3.6}$$

In forming $(p''(t_0))^{1/2}$ the branch of $\omega_0 = \arg p''(t_0)$ must satisfy

$$|\omega_0 + 2\omega| \le \pi/2, \quad \text{where } \omega = \lim_{\psi \to \phi + 0} \arg(t(\psi) - t_0).$$

From Lemma 8(i) we get

$$\omega_0 = -4\phi + \arg\left(1 - \frac{1-\alpha}{\alpha} \frac{\sin 2\alpha\phi}{\sin 2(1-\alpha)\phi} e^{2i\phi}\right). \tag{3.7}$$

Deforming C_ϕ locally at $t = t_0$ such that $\arg t = 2\phi$ near t_0 we have $\omega = 2\phi$ and the argument in (3.7) is determined by the principal branch of arctan as defined in the remark above. Now using (2.4), (2.12) and Lemma 8(i) in (3.6) the proof is completed. □

Looking at (3.2) again, by Lemma 3(v), we observe that the function

$$2\phi - \frac{1-\alpha}{\alpha} \frac{\sin 2\alpha\phi}{\sin 2(1-\alpha)\phi} \sin 2\phi$$

is strictly increasing on $(0, \pi/2)$ with increment π. This together with the following theorem confirms asymptotically the property of $f_\alpha^{(k)}$ to have exactly k (simple) zeros on $(0, \infty)$ (cf. Theorem 2(iv), page 236, in [3]). The monotonicity interval is treated by

THEOREM 2. *Suppose that $\phi > 0$ is fixed, $N = 4k + 3$, and*

$$x = \left\{\frac{N}{4\alpha}\left(\frac{\sinh 2\phi}{\alpha \sinh 2\alpha\phi}\right)^{\alpha/(1-\alpha)} \frac{\sinh 2\phi}{\sinh 2(1-\alpha)\phi}\right\}^{(\alpha-1)/\alpha}. \tag{3.8}$$

Then

$$f_\alpha^{(k)}(x) = \frac{\left(\dfrac{N}{4} \dfrac{\sinh 2\phi}{\alpha \sinh 2(1-\alpha)\phi}\right)^{(k+1)/\alpha} e^{\phi/2}}{\sqrt{\pi N/2}\left((1-\alpha)\dfrac{\sinh 2\phi}{\sinh 2(1-\alpha)\phi} e^{2\alpha\phi} - 1\right)^{1/2}} \times \exp\left\{-\frac{N}{4}\left(\frac{1}{\alpha} + \left(\frac{1}{\alpha} - 1\right) \times \frac{\sinh 2\alpha\phi}{\sinh 2(1-\alpha)\phi} e^{2\phi} - 2\phi\right)\right\}\left(1 + O\left(\frac{1}{k}\right)\right), \tag{3.9}$$

as $k \to \infty$.

PROOF. Again we use representation (2.6) in Lemma 2 with $y = 0$, however now with $\zeta = \rho^*(\phi)$ [see (2.12)]. Then clearly (2.4) reads as (3.8) and, by the

Schwarz reflection principle, (2.6) becomes

$$f_\alpha^{(k)}(x) = \frac{x^{(k+1)/(\alpha-1)}}{2\pi i} \int_{-i\infty}^{i\infty} e^{-(N/4)p(t)} t^{-3/4}\, dt.$$

As above a discussion of the function $p(t)$ in this case shows that according to Lemma 7(ii) t_0^* is an appropriate choice of a saddle point rather than t_0^{**}. Thus we put

$$C_\phi^* := \{t(\psi) = r(\psi) t_0^* \alpha^{(2/(\alpha-1))} e^{2i\psi} | -\pi/2 < \psi < \pi/2\} \tag{3.10}$$

$[t(0) = t_0^*$ by (2.13)] and by Cauchy's theorem we obtain

$$f_\alpha^{(k)}(x) = \frac{x^{(k+1)/(\alpha-1)}}{2\pi i} \int_{C_\phi^*} e^{-(N/4)p(t)} t^{-3/4}\, dt, \tag{3.11}$$

which again we are going to treat by the saddle-point method. By symmetry it suffices to check the key condition $\operatorname{Re}(p(t(\psi)) - p(t_0^*)) > 0$ for $0 < \psi < \pi/2$.

First we note that

$$1 < t_0^* \alpha^{(2/(\alpha-1))} =: c, \tag{3.12}$$

by (2.10), (2.13), (2.18) and Lemma 5(iv). From (3.10), (2.9) and (2.5) we get

$$\begin{aligned}
&\frac{d}{d\psi}\operatorname{Re}(p(t(\psi)) - p(t_0^*)) \\
&\quad = \operatorname{Re} p'(t(\psi))t'(\psi) \\
&\quad = \operatorname{Re}\bigl(-1 + \rho^*(\phi)(\alpha t(\psi)^\alpha - t(\psi))\bigr)\left(\frac{r'(\psi)}{r(\psi)} + 2i\right) \\
&\quad = \frac{r'(\psi)}{r(\psi)}\bigl(-1 + \rho^*(\phi)(\alpha c^\alpha r(\psi)^\alpha \cos 2\alpha\psi - c r(\psi)\cos 2\psi)\bigr) \\
&\qquad -2\rho^*(\phi)(\alpha c^\alpha r(\psi)^\alpha \sin 2\alpha\psi - c r(\psi)\sin 2\psi) \\
&\quad = \frac{r'(\psi)}{r(\psi)}\left(-1 + \frac{\alpha\rho^*(\phi)c^\alpha r(\psi)^\alpha}{\sin 2\psi}(\sin 2(1-\alpha)\psi - (c^{1-\alpha} - 1)\sin 2\alpha\psi \cos 2\psi)\right) \\
&\qquad +2\alpha\rho^*(\phi)c^\alpha r(\psi)^\alpha \sin 2\alpha\psi(c^{1-\alpha} - 1) \\
&\quad = \frac{r'(\psi)}{r(\psi)}\left(-1 + c^\alpha\frac{\rho^*(\phi)}{\rho(\psi)}\right) \\
&\qquad +\rho^*(\phi)\alpha c^\alpha r(\psi)^\alpha \sin 2\alpha\psi(c^{1-\alpha} - 1)\left\{2 - \frac{r'(\psi)}{r(\psi)}\operatorname{cotan} 2\psi\right\}.
\end{aligned}$$

Next, using

$$\frac{r'(\psi)}{r(\psi)} = \frac{2}{1-\alpha}(\alpha \operatorname{cotan} 2\alpha\psi - \operatorname{cotan} 2\psi), \tag{3.13}$$

by (3.12), Lemmas 4, 6 and 3(vi) it follows that

$$\begin{aligned}
&\frac{d}{d\psi}\operatorname{Re}(p(t(\psi)) - p(t_0^*)) \\
&\quad \geq \frac{r'(\psi)}{r(\psi)}\left(-1 + \frac{\rho^*(\phi)}{\rho(\psi)}\right) \\
&\qquad + \left\{\frac{2\alpha}{1-\alpha}\rho^*(\phi)c^\alpha r(\psi)^\alpha \sin 2\alpha\psi(c^{1-\alpha} - 1)\right. \\
&\qquad\qquad \left.\times(1 - \alpha - \operatorname{cotan} 2\psi(\alpha \operatorname{cotan} 2\alpha\psi - \operatorname{cotan} 2\psi))\right\} \\
&\quad > 0, \quad \text{if } 0 < \psi < \pi/2.
\end{aligned}$$

Thus we have $\operatorname{Re}(p(t) - p(t_0^*)) > 0$ for $t \in C_\phi^* - \{t_0^*\}$. Now another application of Theorem 7.1 in [7], page 127, to (3.11) implies that

$$f_\alpha^{(k)}(x) = \frac{x^{(k+1)/(\alpha-1)}e^{-(N/4)p(t_0^*)}}{i\sqrt{\pi N/4}\sqrt{2p''(t_0^*)}}t_0^{*\,-3/4}\left(1 + O\left(\frac{1}{k}\right)\right).$$

Further, from (3.13) and (2.16) we have

$$\lim_{\psi \to 0+} \arg(t(\psi) - t_0^*) = \pi/2.$$

Hence the branch of $\omega_0 = \arg p''(t_0^*)$ must satisfy $|\omega_0 + \pi| \leq \pi/2$, giving $\omega_0 = -\pi$ in view of Lemma 8(ii). Thus it follows that ($e^{i\omega_0/2} = e^{-i\pi/2} = -i$)

$$f_\alpha^{(k)}(x) = \frac{x^{(k+1)/(\alpha-1)}e^{-(N/4)p(t_0^*)}}{\sqrt{\pi N/2}\,|p''(t_0^*)|^{1/2}}t_0^{*\,-3/4}\left(1 + O\left(\frac{1}{k}\right)\right),$$

and finally the assertion is verified via (2.4), (2.12), (2.10), (2.18) and Lemma 8(ii). □

The gap between the two regions previously investigated is described by

THEOREM 3. *Suppose that $y \in \mathbb{R}$ is fixed, $N = 4k + 3$, and*

$$x = \left\{\frac{N/4}{\alpha^{(1+\alpha)/(1-\alpha)}(1-\alpha)} - y\frac{\alpha}{1-\alpha}\left(\frac{N/4}{6\alpha^{(5+\alpha)/(1-\alpha)}}\right)^{1/3}\right\}^{(\alpha-1)/\alpha}. \tag{3.14}$$

Then

$$\begin{aligned}
f_\alpha^{(k)}(x) &= \frac{x^{(k+1)/(\alpha-1)}}{\pi}\alpha^{(2k/(1-\alpha))}\left(\frac{6\alpha^{(5+\alpha)/(1-\alpha)}}{N/4}\right)^{1/3} \\
&\quad \times e^{x^{\alpha/(\alpha-1)}\alpha^{2\alpha/(1-\alpha)}(\alpha^2-1)}\left(A(y) + O(k^{-1/3})\right),
\end{aligned} \tag{3.15}$$

as $k \to \infty$, where

$$A(y) := \operatorname{Im} e^{2\pi i/3}\int_0^\infty e^{-\rho^3 - \rho e^{2\pi i/3}y}\,d\rho \tag{3.16}$$

is Airy's function (see [11], Section 1.81 and problem 2, page 377).

PROOF. This runs along the lines for that of Theorem 8.22.8c in [11], pages 232–234. Therefore we restrict examination of the details to some essential steps. Obviously now in (2.4) and (2.12) we have the limit case $\phi = 0$. Choosing $y = 0$ Lemma 8(ii) shows that $t_0^*(0)$ is a second-order saddle point for the integral (2.6). The parametrization (3.14) produces a small neighbourhood of the turning point $x = a_k$ in (1.3).

Once more we use (2.6) with $\zeta = \rho(0)$ in (2.4), (2.12) and

$$b_k := \frac{\alpha}{1-\alpha}\left(\frac{N/4}{6\alpha^{(5+\alpha)/(1-\alpha)}}\right)^{1/3} \tag{3.17}$$

leading from (2.4) to (3.14). However, now we may not apply the saddle-point method directly but a modified version. Suppose that $\frac{1}{9} < \delta < \frac{1}{6}$ and

$$c_k := \frac{\alpha}{1-\alpha} b_k^{-1}, \qquad \tau_0 := t_0^*(0) = \alpha^{2/(1-\alpha)} \tag{3.18}$$

[see (2.13) and (2.18)]. Then Lemma 2 in connection with Cauchy's theorem implies that

$$f_\alpha^{(k)}(x) = \frac{x^{(k+1)/(\alpha-1)}}{\pi}(H_k + R_k), \tag{3.19}$$

where

$$H_k := \operatorname{Im} \int_{\tau_0}^{\tau_0 + c_k k^\delta e^{i\pi/3}} e^{-(N/4)p(t) - b_k y(t-t^\alpha)} t^{-3/4}\, dt \tag{3.20}$$

and

$$R_k := \operatorname{Im} \int_{\gamma_k} e^{-(N/4)p(t) - b_k y(t-t^\alpha)} t^{-3/4}\, dt, \tag{3.21}$$

with

$$\gamma_k := \{t(\psi) = r(\psi)\tau_0 \alpha^{2/(\alpha-1)} e^{2i\psi} + c_k k^\delta e^{\pi i/3} | 0 \le \psi < \pi/2\} \tag{3.22}$$

[cf. (3.10)]. In (3.20) the integration is performed on a straight line. Putting

$$t - \tau_0 = c_k \rho e^{i\pi/3}, \qquad 0 \le \rho \le k^\delta,$$

in (3.20) and using Taylor expansions at τ_0 [see Lemma 8(ii), (3.17) and (3.18)] we obtain

$$\begin{aligned} H_k &= e^{-(N/4)p(\tau_0) - b_k y(\tau_0 - \tau_0^\alpha)} \tau_0^{-3/4} c_k \\ &\quad \times \operatorname{Im} e^{i\pi/3} \int_0^{k^\delta} \exp\left\{-\frac{N}{4}\left(\frac{-c_k^3}{6} p'''(\tau_0)\rho^3 + \cdots\right)\right\} \\ &\quad \times \exp\{-b_k y((1 - \alpha\tau_0^{\alpha-1}) c_k \rho e^{i\pi/3} + \cdots)\} \\ &\quad \times \left\{1 - \frac{3}{4}\frac{c_k \rho}{\tau_0} e^{i\pi/3} + \cdots\right\} d\rho \\ &= e^{-(N/4)p(\tau_0) - b_k y(\tau_0 - \tau_0^\alpha)} \tau_0^{-3/4} c_k \\ &\quad \times \operatorname{Im} e^{i\pi/3} \int_0^{k^\delta} e^{-\rho^3 + y\rho e^{i\pi/3}} \left(1 + \sum_{\nu=1}^{\infty} u_\nu(\rho) k^{-\nu/3}\right) d\rho. \end{aligned}$$

This is an asymptotic expansion in the usual sense and the u_ν are polynomials independent of k. Since obviously

$$A(y) = \operatorname{Im} e^{i\pi/3} \int_0^\infty e^{-\rho^3 + y\rho e^{i\pi/3}}\, d\rho$$

[see (3.16)] and

$$\int_{k^\delta}^\infty e^{-\rho^3 + y\rho e^{i\pi/3}} u_\nu(\rho)\, d\rho = O(e^{-ck^{3\delta}}), \qquad k \to \infty,$$

for some $c > 0$, we get

$$\text{(3.23)} \quad H_k = c_k \tau_0^{-3/4} \exp\left(-\frac{N}{4} p(\tau_0) - b_k y(\tau_0 - \tau_0^\alpha)\right)\left(A(y) + O(k^{-1/3})\right),$$

as $k \to \infty$. For handling R_k in (3.21) we write

$$\text{(3.24)} \quad \begin{aligned} R_k &= \exp\left\{-\left(\frac{N}{4} - \frac{b_k y}{\zeta}\right) p(\tau_0)\right\} \\ &\quad \times \operatorname{Im} \int_{\gamma_k} \exp\left\{-\left(\frac{N}{4} - \frac{b_k y}{\zeta}\right)(p(t) - p(\tau_0))\right\} t^{b_k y/\zeta - 3/4}\, dt, \end{aligned}$$

and further for the integral we get the bound

$$\text{(3.25)} \qquad O\left(e^{ck^{1/3}} \int_{\gamma_k} e^{-(k/2)\operatorname{Re}(p(t) - p(\tau_0))}\, |dt|\right),$$

as $k \to \infty$ ($c > 0$) since the integrand decays exponentially in γ_k. Finally, for the very same reason there exists a $\psi_0 \in (0, \pi/2)$ such that

$$\operatorname{Re}(p(t(\psi)) - p(\tau_0)) \ge c|t|, \qquad \psi \in [\psi_0, \pi/2),$$

and, by continuity, from the proof of Theorem 2 we take

$$\frac{d}{d\psi} \operatorname{Re}(p(t(\psi)) - p(\tau_0)) > 0, \qquad \psi \in (0, \psi_0]$$

[see (3.10) and (3.22)]. This together with the relation [see Lemma 8(ii)]

$$\operatorname{Re}(p(t(\psi)) - p(\tau_0)) = -\tfrac{1}{6} p'''(\tau_0) c_k^3 k^{3\delta} + O(c_k^4 k^{4\delta}), \qquad k \to \infty,$$

gives

$$O\left(e^{ck^{1/3} - c'k^{3\delta}}\right) = O\left(e^{-c''k^{3\delta}}\right)$$

($c', c'' > 0$, $\delta > \frac{1}{9}$) as a bound for (3.25). Now combining (3.24), (3.23), (3.19), (3.18), (3.17), (3.14), (2.4) and (2.5), formula (3.15) is derived completely. □

Tedious but straightforward calculations show that Theorems 1, 2 and 3 reduce to the formulas of Plancherel–Rotach type for the Laguerre polynomials $L_k^{(1/2)}$ (e.g., [11], Theorem 8.22.8, page 200, [13], Chapter VI, Section 2, and [12], Chapter III) via (1.1), (1.2) and Stirling's formula. A refined analysis in the proof of Theorem 3 gives the better error bound $O(k^{-2/3})$ precisely in the case $\alpha = \frac{1}{2}$ which is consistent with the known results of the just mentioned literature.

Finally, we investigate $f_\alpha^{(k)}(x)$ for large x in

THEOREM 4. *Suppose that $r < 1/2\alpha$, $\sigma = \min(1, 1 - \alpha r)$ and $k/x = O(k^r)$. Then*

$$f_\alpha^{(k)}(x) = \sqrt{\frac{2}{k\pi}}(-1)^k\left(\frac{k}{x}\right)^{k+1} e^{-k}$$

$$\times \exp\left\{-\left(\frac{k}{x}\right)^\alpha \cos\alpha\pi + \left(\frac{k}{x}\right)^{2\alpha}\frac{\alpha^2}{2k}\cos 2\alpha\pi\right\} \tag{3.26}$$

$$\times\left\{\sin\left(\left(\frac{k}{x}\right)^\alpha \sin\alpha\pi - \left(\frac{k}{x}\right)^{2\alpha}\frac{\alpha^2}{2k}\sin 2\alpha\pi\right) + O(k^{-\sigma})\right\},$$

as $k \to \infty$.

PROOF. Again we use ideas in [11], 8.72, pages 225–227, and start with Lemma 1(ii) which can be rewritten in the form

$$f_\alpha^{(k)}(x) = \frac{(-1)^{k+1}}{\pi}\left(\frac{k}{x}\right)^{k+1} e^{-k}\operatorname{Im}\int_0^\infty (te^{1-t})^k e^{-(tk/x)^\alpha e^{i\pi\alpha}}\,dt. \tag{3.27}$$

Next, let $0 < \delta < \frac{1}{4}$,

$$\xi := -\left(\frac{k}{x}\right)^\alpha e^{i\pi\alpha}, \tag{3.28}$$

and

$$J_\delta := \left[1 - \frac{k^\delta}{\sqrt{k}}, 1 + \frac{k^\delta}{\sqrt{k}}\right]. \tag{3.29}$$

The aim is to apply a modified version of the method of steepest descent to (3.27). An approximation of the saddle point is located in J_δ. Therefore we split the integral in (3.27) as

$$\int_0^\infty (te^{1-t})^k e^{\xi t^\alpha}\,dt = \int_{J_\delta} + \int_{J_\delta^c} =: H_k + R_k, \tag{3.30}$$

say. First we verify R_k is a remainder term.

(i) Suppose that $0 < \alpha \le \frac{1}{2}$. Then $\operatorname{Re}\xi \le 0$, by (3.28), and

$$|R_k| \le \int_0^{1-k^\delta/\sqrt{k}} (te^{1-t})^k\,dt + \int_{1+k^\delta/\sqrt{k}}^\infty (te^{1-1})^k\,dt =: R_k' + R_k'',$$

say. A straightforward estimate yields $R_k' = O(e^{-ck^{2\delta}})$, $k \to \infty$, for some $c > 0$ and the very same bound for R_k'' is derived by writing

$$R_k'' = e^k k^{-k-1}\Gamma\left(k+1, \left(1 + \frac{k^\delta}{\sqrt{k}}\right)k\right)$$

and using the estimate (2.14) in [7], page 70, for the incomplete gamma function.

(ii) Suppose that $\frac{1}{2} < \alpha < 1$. Then $\operatorname{Re} \xi > 0$ and

$$|R_k| \leq \int_0^{1-k^\delta/\sqrt{k}} \left(te^{1-t}\right)^k e^{|\cos \alpha\pi|(k/x)^\alpha t^\alpha}\, dt$$

$$+ \int_{1+k^\delta/\sqrt{k}}^{\infty} \left(te^{1-t}\right)^k e^{|\cos \alpha\pi|(k/x)^\alpha t^\alpha}\, dt =: R_k' + R_k'',$$

say. Further, as above this gives

$$R_k' \leqq e^{-ck^{2\delta}+(k/x)^\alpha} \leq e^{-c'k^{2\delta}},$$

provided δ is sufficiently close to $\frac{1}{4}$ which is possible, since $r < 1/2\alpha$. For R_k'' we write

$$R_k'' \leq e^k \int_{1+k^\delta/\sqrt{k}}^{\infty} t^k e^{-k(1-\varepsilon)t+f_k(t)}\, dt,$$

with $f_k(t) := -\varepsilon tk + (k/x)^\alpha t^\alpha$. Choosing $\varepsilon := \alpha(k/x)^\alpha(1 + k^\delta/\sqrt{k})^{\alpha-1}/k$, f_k attains its maximum at the left endpoint. Thus it follows that

$$R_k'' \leqq \int_{1+k^\delta/\sqrt{k}}^{\infty} t^k e^{-k(1-\varepsilon)t}\, dt \exp\left\{k - \varepsilon k\left(1 + \frac{k^\delta}{\sqrt{k}}\right) + (k/x)^\alpha\left(1 + \frac{k^\delta}{\sqrt{k}}\right)^\alpha\right\}.$$

Substituting $\tau = (1 - \varepsilon)tk$ the latter integral becomes an incomplete gamma function which may be treated by the very same estimates as above finally leading to

$$R_k'' = O\left(e^{-ck^{2\delta}}\right), \qquad k \to \infty$$

($c > 0$). Hence for every $\alpha \in (0,1)$ in (3.30) we have

$$R_k = O\left(e^{-ck^{2\delta}}\right), \qquad k \to \infty. \tag{3.31}$$

Next, we evaluate H_k asymptotically, which is supposed to be the leading term in (3.30). To this end for $t \in J_\delta$ [see (3.29)] we put

$$t = 1 + \frac{\rho}{\sqrt{k}}, \qquad -k^\delta \leq \rho \leq k^\delta,$$

and obtain the expansion (note that $\delta < \frac{1}{2}$)

$$\left(te^{1-t}\right)^k = \exp\left\{k \sum_2^\infty \frac{(-1)^{\nu-1}}{\nu}(t-1)^\nu\right\}$$

$$= e^{-\rho^2/2} \exp\left\{k \sum_3^\infty \frac{(-1)^{\nu-1}}{\nu}\left(\frac{\rho}{\sqrt{k}}\right)^\nu\right\}$$

$$= e^{-\rho^2/2}\left\{1 + \frac{1}{3}\frac{\rho^3}{\sqrt{k}} + \sum_2^\infty \frac{u_\nu(\rho)}{\sqrt{k}^\nu}\right\},$$

as $k \to \infty$, where u_ν is a polynomial independent of k. Similarly, we get

$$e^{t^\alpha \xi} = \exp\{\xi + \xi((1 + t - 1)^\alpha - 1)\}$$
$$= e^{\xi + \xi\alpha\rho/\sqrt{k}} \exp\left\{\xi \sum_2^\infty \binom{\alpha}{\nu}\left(\frac{\rho}{\sqrt{k}}\right)^\nu\right\}.$$

The argument of the latter exponential is of order $O(k^{\alpha r + 2\delta - 1}) = o(1)$. Hence

$$e^{t^\alpha \xi} = e^{\xi + \xi\alpha\rho/\sqrt{k}}\left\{1 + \frac{\xi\rho^2}{k}O(1)\right\},$$

where again we have used the fact that $\xi\rho^2/k = O(k^{\alpha r + 2\delta - 1}) = o(1)$.

Substitution in (3.30) yields

$$H_k = \frac{e^\xi}{\sqrt{k}} \int_{-k^\delta}^{k^\delta} e^{-\rho^2/2 + \xi\alpha\rho/\sqrt{k}}\left(1 + \frac{1}{3}\frac{\rho^3}{\sqrt{k}} + \sum_2^\infty \frac{u_\nu(\rho)}{\sqrt{k}^\nu}\right)$$
$$\times\left(1 + \frac{\xi}{k}\rho^2 O(1)\right) d\rho$$
$$= \frac{1}{\sqrt{k}} e^{\xi + \xi^2\alpha^2/2k} \int_{-\infty}^{\infty} e^{-\rho^2/2}\, d\rho\left(1 + O\left(\frac{\xi}{k}\right) + O\left(\frac{1}{k}\right)\right).$$

In the last step observe that $\xi/\sqrt{k} = O(k^{\alpha r - 1/2}) = o(1)$. Now choosing δ close to $\frac{1}{4}$ (3.27), (3.28), (3.30) and (3.31) imply (3.26) and the proof is finished. □

Again Theorem 4 can be looked at as an extension of known results for Laguerre polynomials (cf. [11], Theorem 8.22.1 of Fejér, page 198, [13], page 220 and [12], Chapter 3.3), however with a less precise remainder term. Finally, we mention that our error terms also can be equipped with a uniformity condition and the asymptotic formulas easily can be extended to complete asymptotic expansions. But we do not pursue this question in this paper.

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