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ASYMPTOTICS OF A CLASS OF MARKOV PROCESSES WHICH ARE NOT IN GENERAL IRREDUCIBLE

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Let α_n be a sequence of i.i.d. nondecreasing random maps on a subset S of $\mathbb{R}^k$ into itself and let X_0 be a random variable with values in S independent of the sequence α_n. Then $X_n \equiv \alpha_n \cdots \alpha_1 X_0$ is a Markov process. Conditions for the existence of unique invariant probabilities are obtained for such Markov processes which are not in general irreducible, extending earlier results of Dubins and Freedman to multidimensional and noncompact state spaces. In addition, a functional central limit theorem is obtained. These yield new results in time series and economic models.

1. Introduction. One way to study discrete parameter Markov processes is the following [Kifer (1986)]. Let $(S, \mathscr{S})$ be a measurable space, Γ a set of measurable maps on S into itself. Endow Γ with a σ-field $\mathscr{C}$ such that the map $(\gamma, x) \to \gamma(x)$ on $\Gamma \times S$ into S is $\mathscr{C} \otimes \mathscr{S} | \mathscr{S}$-measurable. Let P be a probability measure on $(\Gamma, \mathscr{C})$. On some probability space $(\Omega, \mathscr{F}, Q)$ define a sequence of i.i.d. random maps $\alpha_1, \alpha_2, \dots$ with common distribution P. For a given random variable X_0, independent of the sequence α_n, define $X_1 = \alpha_1 X_0, \dots, X_n = \alpha_n X_{n-1} = \alpha_n \cdots \alpha_1 X_0 \cdots$. Then X_n is a Markov process with transition probability $p(x, dy)$ given by

$$(1.1) \qquad p(x, B) = P(\{\gamma \in \Gamma : \gamma(x) \in B\}), \qquad x \in S,\ B \in \mathscr{S}.$$

We shall often write $X_n(x)$ for X_n in case $X_0 = x$. Denote by P^n the joint distribution of $\alpha_1, \dots, \alpha_n$, i.e., $P^n = P \times P \times \cdots \times P$ on $(\Gamma^n, \mathscr{C}^{\otimes n})$.

Let $\mathbb{B}(S)$ denote the linear space of all real-valued bounded measurable functions on S. The transition operator T on $\mathbb{B}(S)$ is defined by

$$(1.2) \qquad (Tf)(x) = \int f(y) p(x, dy), \qquad f \in \mathbb{B}(S).$$

Its adjoint is T^* defined on the space $\mathscr{M}(S)$ of all finite signed measures on $(S, \mathscr{S})$ by

$$(1.3) \qquad (T^*\mu)(B) = \int p(x, B) \mu(dx), \qquad \mu \in \mathscr{M}(S).$$

Let $\mathscr{P}(S) \subset \mathscr{M}(S)$ denote the set of all probability measures on $(S, \mathscr{S})$. Recall that a probability measure π on $(S, \mathscr{S})$ is said to be invariant for p if it is a fixed point of T^*: $T^*\pi = \pi$.

We shall write $p^{(n)}(x, dy)$ for the n-step transition probability, with $p^{(1)} = p$. Then $p^{(n)}(x, dy)$ is the distribution of $\alpha_n \cdots \alpha_1 x$.

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The transition probability p may not be φ-irreducible for any nonzero σ-finite measure φ. Recall that p is φ-irreducible if $\varphi(B) > 0$ implies that for each x there exists n such that $p^{(n)}(x, B) > 0$. There is an extensive literature on the asymptotic properties of Markov processes with φ-irreducible transition probabilities. See, e.g., Jain and Jamison (1967), Orey (1971), Tweedie (1974), (1975) and Revuz (1984). There is, however, no general theory for the nonirreducible case. In the present context, the latter arises, for example, when P is discrete. For some examples of nonirreducible models in biology and economics, see Reed (1974), Bhattacharya and Majumdar (1984), (1988) and Rosenblatt (1980). Our main interest in this article is to look at one such class of Markov processes, to find general conditions under which there exist unique invariant probabilities π, to study the stability of such measures and to identify broad classes of functions f in $L^2(S, \pi)$ for which the functional central limit theorem (FCLT) holds, i.e., the sequence of stochastic processes

$$Y_n(t) \equiv n^{-1/2}\left[\sum_{j=0}^{[nt]}\left(f(X_j) - \int f\,d\pi\right) + \left(t - \frac{[nt]}{n}\right)\left(f(X_{[nt]+1}) - \int f\,d\pi\right)\right] \tag{1.4}$$

converges in distribution to a Brownian motion under every initial distribution.

In the class of problems considered in this article, S is a topologically complete subspace of $\mathbb{R}^k$, i.e., the relativized topology on S may be metrized so as to make S complete. The Borel σ-field of S is $\mathscr{B}(S)$. For Γ one takes a set of measurable monotone nondecreasing functions $\gamma = (\gamma^{(1)}, \dots, \gamma^{(k)})$ on S into itself. In other words, $\gamma^{(i)}(x^{(1)}, \dots, x^{(k)})$ is monotone nondecreasing in each coordinate $x^{(1)}, \dots, x^{(k)}$. Make the assumption on P:

There exists x_0 and a positive integer m such that

$$Q(X_m(x) \le x_0 \;\forall\, x) > 0, \qquad Q(X_m(x) \ge x_0 \;\forall\, x) > 0. \tag{1.5}$$

It is then shown that there exists a unique invariant probability to which $p^{(n)}(x, dy)$ converges exponentially fast in a metric stronger than the Kolmogorov distance; this convergence is uniform for all $x \in S$ (Theorem 2.1). This generalizes an earlier result of Dubins and Freedman (1966) and Yahav (1975) who considered the case $k = 1$, S a compact interval. A necessary condition for compact S and arbitrary k is given by Lemma 2.6. Theorem 3.1 provides an FCLT of the type mentioned earlier. Section 4 contains two applications, one to mathematical economics and the other to nonlinear autoregressive models; both are new results.

2. Existence of a unique invariant probability. Let $S \subset \mathbb{R}^k$ be topologically complete in its relativized Euclidean topology and let Γ be a set of measurable monotone nondecreasing maps $\gamma = (\gamma^{(1)}, \dots, \gamma^{(k)})$ on S into S. We shall often write γx for $\gamma(x)$.

Let $\mathscr{C}$ be a σ-field on Γ such that the map $(\gamma, x) \to \gamma x$ is measurable on $(\Gamma \times S, \mathscr{C} \otimes \mathscr{B}(S))$ into $(S, \mathscr{B}(S))$.

Let $\mathscr{A}$ be the class of all sets $A \subset S$ of the form

$$A = \{y \in S: \gamma(y) \le x\}, \tag{2.1}$$

where γ varies over the class of all continuous monotone nondecreasing functions on S into itself and x varies over $\mathbb{R}^k$.

On the space $\mathscr{P}(S)$ of all probability measures on $(S, \mathscr{B}(S))$, define the distance d by

$$d(\mu, \nu) = \sup\{|\mu(A) - \nu(A)|: A \in \mathscr{A}\}, \qquad \mu, \nu \in \mathscr{P}(S). \tag{2.2}$$

This defines a topology on $\mathscr{P}(S)$ that is stronger than the weak-star topology.

Our first main result is

THEOREM 2.1. *Suppose there exists a positive integer m and some $x_0 \in S$ such that* (1.5) *holds. Then there exists a unique invariant probability π and*

$$\sup\{d(p^{(n)}(x, dy), \pi(dy)): x \in S\} \to 0 \tag{2.3}$$

exponentially fast as $n \to \infty$.

First let us show

LEMMA 2.2. *The space $\mathscr{P}(S)$ is complete under the distance d defined by* (2.2).

PROOF. It is known that $\mathscr{P}(S)$ is topologically complete under the weak-star topology [see Parthasarathy (1967), page 46], which is weaker than its topology under d. Hence if μ_n is a sequence in $\mathscr{P}(S)$ such that $d(\mu_n, \mu_m) \to 0$ as $n, m \to \infty$, then there exists $\mu \in \mathscr{P}(S)$ such that μ_n converges weak-star to μ. Fix a continuous monotone nondecreasing γ on S into S and write F_n and F for the cumulative distribution functions of $\mu_n \circ \gamma^{-1}$ and $\mu \circ \gamma^{-1}$, respectively. Then $F_n(x)$ converges to $F(x)$ at all points x of continuity of F. On the other hand, $\sup\{|F_n(x) - F_m(x)|: x \in \mathbb{R}^k\} \le d(\mu_n, \mu_m)$. Hence F_n converges uniformly to a function that is necessarily right continuous. This implies that this limit function is F and that $F_n(x)$ converges to $F(x)$ uniformly for all x. This being true for every continuous nondecreasing γ, $\mu_n(A)$ converges to $\mu(A)$ for every $A \in \mathscr{A}$. But μ_n converges uniformly on $\mathscr{A}$. Hence $d(\mu_n, \mu) \to 0$. □

We now introduce a distance d_1 stronger than d. For $a \ge 0$, let $\mathscr{G}_a$ denote the class of all real-valued Borel measurable nondecreasing functions f on S satisfying $0 \le f(x) \le a$ for all $x \in S$. Define

$$d_a(\mu, \nu) = \sup\left\{\left|\int f\,d\mu - \int f\,d\nu\right|: f \in \mathscr{G}_a\right\}, \qquad \mu, \nu \in \mathscr{P}(S). \tag{2.4}$$

Clearly, $d_a(\mu, \nu) = a d_1(\mu, \nu)$ for all $a \ge 0$.

Let the linear map $T^{*n} = (T^n)^*$ be defined on $\mathscr{M}(S)$ by

$$(T^{*n}\mu)(B) = \int p^{(n)}(x, B)\mu(dx), \qquad n \ge 1, \mu \in \mathscr{M}(S), B \in \mathscr{B}(S). \tag{2.5}$$

In order to state the next lemma, fix $x_0 \in S$ and a positive integer m. Write

$$\begin{aligned} \Gamma_1 &= \{(\gamma_1,\dots,\gamma_m) \in \Gamma^m\colon \gamma_m \cdots \gamma_1 x \le x_0 \ \forall\, x\}, \\ \Gamma_2 &= \{(\gamma_1,\dots,\gamma_m) \in \Gamma^m;\ \gamma_m \cdots \gamma_1 x \ge x_0 \ \forall\, x\}. \end{aligned} \tag{2.6}$$

Lemma 2.3. *If* Γ_1, Γ_2 *are defined by* (2.6), *then*

$$d_1(T^{*n}\mu, T^{*n}\nu) \le \delta^{[n/m]} d_1(\mu, \nu), \tag{2.7}$$

where

$$\delta = \max\{1 - P^m(\Gamma_1), 1 - P^m(\Gamma_2)\}. \tag{2.8}$$

If (1.5) *holds, then* $\delta < 1$.

Proof. Let $f \in \mathscr{G}_1$. Then

$$\begin{aligned} 0 &\le h_1(x) \equiv \int_{\Gamma_1 \setminus (\Gamma_1 \cap \Gamma_2)} f(\gamma_m \cdots \gamma_1 x) P^m(d\gamma_1 \cdots d\gamma_m) \\ &\le f(x_0)(P^m(\Gamma_1) - P^m(\Gamma_1 \cap \Gamma_2)), \\ 0 &\le h_2(x) \equiv \int_{\Gamma_2 \setminus (\Gamma_1 \cap \Gamma_2)} (1 - f(\gamma_m \cdots \gamma_1 x)) P^m(d\gamma_1 \cdots d\gamma_m) \\ &\le (1 - f(x_0))(P^m(\Gamma_2) - P^m(\Gamma_1 \cap \Gamma_2)), \\ 0 &\le h_3(x) \equiv \int_{\Gamma \setminus (\Gamma_1 \cup \Gamma_2)} f(\gamma_m \cdots \gamma_1 x) P^m(d\gamma_1 \cdots d\gamma_m) \\ &\le 1 - P^m(\Gamma_1 \cup \Gamma_2), \\ &\int_{\Gamma_1 \cap \Gamma_2} f(\gamma_m \cdots \gamma_1 x) P^m(d\gamma_1 \cdots d\gamma_m) = f(x_0) P^m(\Gamma_1 \cap \Gamma_2). \end{aligned} \tag{2.9}$$

Now,

$$\begin{aligned} &\int f\, dT^{*m}\mu - \int f\, dT^{*m}\nu \\ &\quad = \int h_1(x)\mu(dx) - \int h_1(x)\nu(dx) + \int h_2(x)\nu(dx) \\ &\quad\quad - \int h_2(x)\mu(dx) + \int h_3(x)\mu(dx) - \int h_3(x)\nu(dx). \end{aligned} \tag{2.10}$$

Let a_1, a_2, a_3 denote the constants appearing on the right sides in (2.9) bounding h_1, h_2, h_3. Then, h_1, $a_2 - h_2$, h_3 belong to $\mathscr{G}_{a_i}$, $i = 1, 2, 3$. Therefore,

$$\begin{aligned} d_1(T^{*m}\mu, T^{*m}\nu) &\le \sup_{f \in \mathscr{G}_1} [\{ f(x_0)(P^m(\Gamma_1) - P^m(\Gamma_1 \cap \Gamma_2)) \\ &\qquad + (1 - f(x_0))(P^m(\Gamma_2) - P^m(\Gamma_1 \cap \Gamma_2)) \\ &\qquad + (1 - P^m(\Gamma_1 \cup \Gamma_2))\} d_1(\mu, \nu)] \\ &\le [\max\{P^m(\Gamma_1) - P^m(\Gamma_1 \cap \Gamma_2), P^m(\Gamma_2) - P^m(\Gamma_1 \cap \Gamma_2)\} \\ &\qquad + 1 - P^m(\Gamma_1) - P^m(\Gamma_2) + P^m(\Gamma_1 \cap \Gamma_2)] d_1(\mu, \nu) \\ &= \max\{1 - P^m(\Gamma_2), 1 - P^m(\Gamma_1)\} d_1(\mu, \nu). \end{aligned} \tag{2.11}$$

For the last equality, if $P^m(\Gamma_1) \geq P^m(\Gamma_2)$, then

$$\max\{P^m(\Gamma_1) - P^m(\Gamma_1 \cap \Gamma_2), P^m(\Gamma_2) - P^m(\Gamma_1 \cap \Gamma_2)\} = P^m(\Gamma_1) - P^m(\Gamma_1 \cap \Gamma_2).$$

Thus (the sum on) the left side of the equality in (2.11) is $1 - P^m(\Gamma_2)$. But the right side also equals $1 - P^m(\Gamma_2)$ in this case. The case $P^m(\Gamma_2) > P^m(\Gamma_1)$ is exactly similar. Hence

$$d_1(T^{*m}\mu, T^{*m}\nu) \leq \delta d_1(\mu, \nu). \tag{2.12}$$

Also,

$$\begin{aligned} d_1(T^*\mu, T^*\nu) &= \sup\left\{\left|\int\left[\int f(y)p(x, dy)\right]\mu(dx)\right.\right. \\ &\qquad \left.\left. - \int\left[\int f(y)p(x, dy)\right]\nu(dx)\right| : f \in \mathscr{G}_1\right\} \\ &\leq d_1(\mu, \nu). \end{aligned} \tag{2.13}$$

Combining (2.12) and (2.13) one arrives at (2.7). If (1.5) holds, it is trivial to check that $\delta < 1$. □

Since $d(\mu, \nu) \leq d_1(\mu, \nu) \leq 1$, the following is immediate from Lemma 2.3:

$$d(T^{*n}\mu, T^{*n}\nu) \leq \delta^{[n/m]}, \qquad n = 1, 2, \ldots. \tag{2.14}$$

Corollary 2.4 is a consequence of Lemma 2.2 and (2.14).

COROLLARY 2.4. *If* (1.5) *holds for some* $x_0 \in S$ *and some positive integer* m, *then there exists a unique probability measure* π *on* $(S, \mathscr{B}(S))$ *such that*

$$\sup_{x \in S} d\left(p^{(n)}(x, dy), \pi(dy)\right) \leq \delta^{[n/m]} \to 0 \quad \text{as } n \to \infty. \tag{2.15}$$

PROOF. For $n' > n$, one has

$$d\left(p^{(n)}(x, dy), p^{(n')}(x, dy)\right) = d(T^{*n}\mu, T^{*n}\nu) \leq \delta^{[n/m]}, \tag{2.16}$$

with $\mu = \delta_x$ (point mass at x) and $\nu = T^{*(n'-n)}\delta_x$. Hence $p^{(n)}(x, dy)$ is a Cauchy sequence in the metric d. Let π be its limit, which exists by Lemma 2.2. Letting $n' \to \infty$ in (2.16) one arrives at (2.15). □

If the probability measure π in Corollary 2.4 can be shown to be invariant, then the proof of Theorem 2.1 would be complete. The next result shows this.

LEMMA 2.5. (i) *Suppose there exist* $a = (a^{(1)}, \ldots, a^{(k)})$ *and* $b = (b^{(1)}, \ldots, b^{(k)})$ *in* S *such that* $a \leq x \leq b$ *for all* $x \in S$. *If, in this case,* $p^{(n)}(x, dy)$ *converges weakly to the same probability measure* $\pi(dy)$ *on* S *for every* $x \in S$, *then* π *is the unique invariant probability for* p.

(ii) *The probability measure* π *in Corollary* 2.4 *is the unique invariant probability for* p, *whether or not there exist* a *and* b *as in part* (i).

PROOF. (i) Let

$$X_n(x) = \alpha_n \cdots \alpha_1 x, \qquad Y_n(x) = \alpha_1 \cdots \alpha_n x, \qquad x \in S, \tag{2.17}$$

where $\alpha, \alpha_1, \alpha_2, \dots$ is an i.i.d. sequence of random maps (on S into S) with common distribution P and defined on some probability space $(\Omega, \mathscr{F}, Q)$. The distribution of $Y_n(x)$ is the same as that of $X_n(x)$, namely, $p^{(n)}(x, dy)$. Now $Y_n(a)$ increases and $Y_n(b)$ decreases, respectively, to $\underline{Y}$ and $\overline{Y}$, say. Since $Y_n(a) \le Y_n(b)$ for all n, $\underline{Y} \le \overline{Y}$. Under the hypothesis, however, the distributions of $\underline{Y}$ and $\overline{Y}$ are the same, namely, π. Hence $\underline{Y} = \overline{Y}$, almost surely. Therefore,

$$\begin{aligned} p^{(n+1)}(a, S \cap [a, x]) &= Q(\alpha Y_n(a) \le x) \ge Q(\alpha \underline{Y} \le x) \\ &= \int p(z, S \cap [a, x]) \pi(dz) \\ &= Q(\alpha \overline{Y} \le x) \ge Q(\alpha Y_n(b) \le x) \\ &= p^{(n+1)}(b, S \cap [a, x]). \end{aligned} \tag{2.18}$$

If x is a point of continuity of the cumulative distribution function (c.d.f.) of π, then the two extreme sides of (2.18) have the same limit $\pi(S \cap [a, x])$. Hence one must have the equality

$$\pi(S \cap [a, x]) = \int p(z, S \cap [a, x]) \pi(dz). \tag{2.19}$$

Since the class of sets $S \cap [a, x]$ for which (2.19) holds is closed under finite intersections and generates $\mathscr{B}(S)$, it follows that [see, e.g., Billingsley (1979), Theorem 3.3, page 34]

$$\pi(B) = \int p(z, B) \pi(dz) \qquad \forall\, B \in \mathscr{B}(S), \tag{2.20}$$

i.e., π is invariant for p. If π' is also invariant, then

$$\pi'(S \cap [a, x]) = \int p^{(n)}(z, S \cap [a, x]) \pi'(dz) \qquad \forall\, x. \tag{2.21}$$

In particular, for points x of continuity of the c.d.f. of π, one may take limits to get $\pi'(S \cap [a, x]) = \pi(S \cap [a, x])$. This implies $\pi' = \pi$, proving uniqueness.

(ii) First consider the case $m = 1$. In case there do not exist a and/or b as in part (i), reduce the problem to that of a bounded S, by an increasing homeomorphism. Let now $a^{(i)} = \inf\{x^{(i)}: x \in S\}$ and $b^{(i)} = \sup\{x^{(i)}: x \in S\}$, $1 \le i \le k$. Write $a = (a^{(1)}, \dots, a^{(k)})$, $b = (b^{(1)}, \dots, b^{(k)})$. Let $\bar{S} = S \cup \{a, b\}$. For $\gamma \in \Gamma_1$, set $\gamma(a) = a$ and $\gamma(b) = x_0$; for $\gamma \in \Gamma_2$, set $\gamma(a) = x_0$ and $\gamma(b) = b$; for $\gamma \notin \Gamma_1 \cup \Gamma_2$, set $\gamma(a) = a$ and $\gamma(b) = b$. Then the hypothesis (1.5), with $m = 1$, still applies on the new state space $\bar{S}$. Therefore, by Corollary 2.4 and the preceding part (i), there exists a unique invariant probability $\bar{\pi}(dy)$ to which $p^{(n)}(x, dy)$ converges in the d-metric, for all $x \in S$. Since $p^{(n)}(x, dy)$ converges to $\pi(dy)$ for all $x \in S$, $\bar{\pi} = \pi$.

To deal with the case $m > 1$, take for Γ the set $\Gamma^{(m)}$ of all compositions $\gamma_m \cdots \gamma_1$ with $\gamma_i \in \Gamma$, $1 \le i \le m$. For the σ-field $\mathscr{C}^{(m)}$ on $\Gamma^{(m)}$, take the class of all sets B whose inverse images under the map $(\gamma_1 \cdots \gamma_m) \to \gamma_m \cdots \gamma_1$ are in $\mathscr{C}^m$. Let $P^{(m)}$ be the induced probability measure on $(\Gamma^{(m)}, \mathscr{C}^{(m)})$. The (one-step) transition probability arising from the map $(\gamma, x) \to \gamma x$ on $\Gamma^{(m)}$ into S is then $p^{(m)}(x, dy)$, with the associated adjoint operator T^{*m}. By the preceding paragraph, π is the unique fixed point of T^{*m}: $T^{*m}\pi = \pi$. Also, one has

$$T^*\pi = T^*(T^{*mn}\pi) = T^{*(mn+1)}\pi \to \pi$$

in the d-metric. Hence $T^*\pi = \pi$. □

This completes the proof of Theorem 2.1.

REMARK 2.5.1. If one can show that $\mathscr{P}(S)$ is complete in the metric d_1 defined by (2.4), then the contraction mapping theorem immediately yields $T^*\pi = \pi$. This is true for $k = 1$ and we are uncertain for $k > 1$.

REMARK 2.5.2. Theorem 2.1 and its proof go over to a topologically complete $S \subset \mathbb{R}^\infty$.

In case $k = 1$ and S is compact, the hypothesis of Theorem 2.1 is also necessary, leaving aside the case $P(\{\gamma(M) = M\}) = 1$ for some unique M [see Dubins and Freedman (1966) for the continuous case].

More generally, one has the following result. As before, S is always taken to be topologically complete.

LEMMA 2.6. *Let $S \subset \mathbb{R}^k$, Γ a set of measurable nondecreasing functions on S and let P be a probability measure on $(\Gamma, \mathscr{C})$ such that $p^{(n)}(x, dy)$ converges weakly for each x to the same probability $\pi(dy)$. Assume that there are two points $a, b \in S$ such that $a \le x \le b$ for all $x \in S$. Then* (1.5) *holds for some x_0 and some m, provided there are two points $c = (c^{(1)}, \dots, c^{(k)})$ and $d = (d^{(1)}, \dots, d^{(k)})$ in the support of $\pi(dy)$ such that $c^{(i)} < d^{(i)}$ for $1 \le i \le k$.*

PROOF. Let $Y_n(a) \uparrow \underline{Y}$, $Y_n(b) \downarrow \overline{Y}$ [see (2.17)]. Since $p^{(n)}(a, dy)$ and $p^{(n)}(b, dy)$ converge weak-star to the same limit, $\underline{Y} = \overline{Y}$ a.s. Choose $\theta > 0$ such that $c^{(i)} + \theta < d^{(i)} - \theta$ for $1 \le i \le k$. Writing $e = (1, 1, \dots, 1)$, there exists a positive integer m such that $\text{prob}(X_m(b) \le c + \theta e) = \text{prob}(Y_m(b) \le c + \theta e) > 0$ and $\text{prob}(X_m(a) \ge d - \theta e) = \text{prob}(Y_m(a) \ge d - \theta e) > 0$. Then (1.5) holds for this m and any $x_0 \in [c + \theta e, d - \theta e]$. □

3. A functional central limit theorem. One of the principal objectives in this article is to obtain functional central limit theorems for

$$Y_n(t) \equiv n^{-1/2} \sum_{j=0}^{[nt]} \left(f(X_j) - \int f \, d\pi \right), \qquad 0 \le t < \infty, \tag{3.1}$$

or its polygonal version defined by (1.4), for broad classes of functions f in $L^2(S, \pi)$ under the general assumptions made in Section 2. In many situations, especially when P is discrete, the Markov processes X_n considered here are not φ-irreducible with respect to any nontrivial σ-finite measure φ. As a consequence, the processes, even though ergodic, are not even strongly mixing. Indeed, the tail σ-field may be nontrivial [see Rosenblatt (1980) for an example].

The process Y_n defined by (3.1) or (1.4) takes values in the space $D[0, \infty)$ of real-valued right continuous functions on $[0, \infty)$ having left-hand limits with the Skorohod topology. The distribution of Y_n is then a probability measure on the Borel σ-field of $D[0, \infty)$, and its convergence in distribution to a Brownian motion means the weak-star convergence of this sequence of distributions to a Wiener measure [see, e.g., Parthasarathy (1967), Chapter 7].

THEOREM 3.1. *Let the hypothesis of Theorem* 2.1 *hold.*

(a) *Then for every f that may be expressed as the difference between two monotone nondecreasing functions in $L^2(S, \pi)$, $f - \int f\,d\pi$ belongs to the range of $T - I$.*

(b) *Whatever the initial distribution, the functional central limit theorem holds if f is as in part* (a), *and the variance parameter of the limiting Brownian motion is given by $\int g^2\,d\pi - \int (Tg)^2\,d\pi$, where g is an element of $L^2(S, \pi)$ satisfying $(T - I)g = f - \int f\,d\pi$.*

For the proof let us begin with two simple but crucial lemmas. Let $\|\cdot\|_2$ denote the norm in $L^2(S, \pi)$.

LEMMA 3.2. *Let μ be a probability measure on $(\mathbb{R}^1, \mathscr{B}(\mathbb{R}^1))$ such that $\int x^2\mu(dx) < \infty$. Then*

$$\int x^2\mu(dx) - \left(\int x\mu(dx)\right)^2 = \tfrac{1}{2}\int\int (x - y)^2\mu(dx)\mu(dy).$$

PROOF. Expand the right-hand side and integrate. □

LEMMA 3.3. *Let $f \in L^2(S, \pi)$ and write*

$$\bar{f} = \int f\,d\pi. \tag{3.2}$$

If $\sum_{n=0}^{\infty}\|(T^n(f - \bar{f}))\|_2 < \infty$, then $f - \bar{f}$ belongs to the range of $T - I$; indeed, $(T - I)g = f - \bar{f}$, where

$$g = -\sum_{n=0}^{\infty} T^n(f - \bar{f}). \tag{3.3}$$

PROOF. Apply $T - I$ to the right side of (3.3). □

PROOF OF THEOREM 3.1. Let $f \in L^2(S, \pi)$ be monotone nondecreasing. By Lemma 3.2,

$$
\begin{aligned}
&\|T^m(f - \bar{f})\|_2^2 \\
&\quad = \int\left(\int (f(y) - \bar{f})p^{(m)}(x, dy)\right)^2 \pi(dx) \\
&\quad = \int\left[\int (f(y) - \bar{f})^2 p^{(m)}(x, dy)\right. \qquad (3.4)\\
&\qquad \left. - \tfrac{1}{2}\int\int (f(y) - f(z))^2 p^{(m)}(x, dy)p^{(m)}(x, dz)\right]\pi(dx) \\
&\quad = \|f - \bar{f}\|_2^2 - \tfrac{1}{2}\int\left[\int\int (f(y) - f(z))^2 p^{(m)}(x, dy)p^{(m)}(x, dz)\right]\pi(dx).
\end{aligned}
$$

Now

$$
\begin{aligned}
&\int\int (f(y) - f(z))^2 p^{(m)}(x, dy)p^{(m)}(x, dz) \\
&\quad \geq \int_{\{z \geq x_0\}}\int_{\{y \leq x_0\}} (f(y) - f(x_0))^2 p^{(m)}(x, dy)p^{(m)}(x, dz) \\
&\qquad + \int_{\{z \leq x_0\}}\int_{\{y > x_0\}} (f(y) - f(x_0))^2 p^{(m)}(x, dy)p^{(m)}(x, dz) \qquad (3.5)\\
&\quad \geq P^m(\Gamma_2)\int_{\{y \leq x_0\}} (f(y) - f(x_0))^2 p^{(m)}(x, dy) \\
&\qquad + P^m(\Gamma_1)\int_{\{y > x_0\}} (f(y) - f(x_0))^2 p^{(m)}(x, dy) \\
&\quad \geq \min\{P^m(\Gamma_1), P^m(\Gamma_2)\}\int (f(y) - f(x_0))^2 p^{(m)}(x, dy),
\end{aligned}
$$

where Γ_1 and Γ_2 are defined by (2.6). Hence

$$
\begin{aligned}
&\int\left[\int\int (f(y) - f(z))^2 p^{(m)}(x, dy)p^{(m)}(x, dz)\right]\pi(dx) \\
&\quad \geq \min\{P^m(\Gamma_1), P^m(\Gamma_2)\}\int\left[\int (f(y) - f(x_0))^2 p^{(m)}(x, dy)\right]\pi(dx) \qquad (3.6)\\
&\quad = \min\{P^m(\Gamma_1), P^m(\Gamma_2)\}\int (f(y) - f(x_0))^2 \pi(dy) \\
&\quad \geq \min\{P^m(\Gamma_1), P^m(\Gamma_2)\}\|f - \bar{f}\|_2^2 \geq (1 - \delta)\|f - \bar{f}\|_2^2,
\end{aligned}
$$

where δ, defined by (2.8), is less than 1. Using (3.6) in (3.4) one gets

$$\|T^m(f - \bar{f})\|_2 \leq c\|f - \bar{f}\|_2, \qquad (3.7)$$

where

$$c = \left(1 - \tfrac{1}{2}(1 - \delta)\right)^{1/2} < 1. \tag{3.8}$$

Next note that if f is monotone nondecreasing, so is Tf and therefore $T^m f$. Hence iteration of (3.7) yields

$$\|T^{jm}(f - \bar{f})\|_2 \le c^j \|f - \bar{f}\|_2, \qquad j = 1, 2, \dots. \tag{3.9}$$

Since T is a contraction on $L^2(S, \pi)$, one has, finally,

$$\|T^n(f - \bar{f})\|_2 \le c^{[n/m]} \|f - \bar{f}\|_2 \qquad \forall\, n. \tag{3.10}$$

It now follows from Lemma 3.3 that $f - \bar{f}$ belongs to the range of $T - I$. This proves part (a).

In order to prove part (b), let $(T - I)g = f - \bar{f}$. Then

$$\begin{aligned} \sum_{j=0}^{n} \left(f(X_j) - \bar{f} \right) &= \sum_{j=0}^{n} \left(Tg(X_j) - g(X_j) \right) \\ &= \sum_{j=1}^{n+1} \left(Tg(X_{j-1}) - g(X_j) \right) + \left(g(X_{n+1}) - g(X_0) \right). \end{aligned} \tag{3.11}$$

Since $Tg(X_{j-1}) - g(X_j)$, $j \ge 0$, is (under the initial distribution π) a stationary ergodic sequence of martingale differences, the functional central limit theorem follows [see Billingsley (1968), Theorem 23.1; Gordon and Lifsic (1978) and Bhattacharya (1982), Theorem 2.1]. In this case the variance parameter of the limiting Brownian motion is $E(Tg(X_{j-1}) - g(X_j))^2 = \|g\|_2^2 - \|Tg\|_2^2$.

It remains to prove the functional central limit theorem starting from an arbitrary initial state x. Let $f \in L^2(S, \pi)$ be monotone nondecreasing. Let $\{X_j\}$ denote the process with initial distribution π. Write

$$\begin{aligned} S_{m, m'}(x) &= n^{-1/2} \sum_{j=m}^{m'} \left(f(X_j(x)) - \bar{f} \right), \\ S_{m, m'} &= n^{-1/2} \sum_{j=m}^{m'} \left(f(X_j) - \bar{f} \right). \end{aligned} \tag{3.12}$$

Then $S_{0,n}(x) = S_{0, n_0 - 1}(x) + S_{n_0, n}(x)$. Now, for every n_0,

$$S_{0, n_0 - 1}(x) \to 0 \quad \text{a.s. as } n \to \infty. \tag{3.13}$$

Also, for all $r \in \mathbb{R}^1$,

$$Q\left(S_{n_0, n}(x) > r\right) = Eh_n\left(X_{n_0}(x)\right) = \int h_n(y) p^{(n_0)}(x, dy), \tag{3.14}$$

where Q is the probability measure on the basic probability space and

$$h_n(y) = Q\left(S_{0, n - n_0}(y) > r\right) \tag{3.15}$$

is an increasing function of y. Hence, by Lemma 2.3,

$$\sup_{n>n_0}\left|\int h_n(y)p^{(n_0)}(x,dy) - \int h_n(y)\pi(dy)\right| \to 0 \quad \text{as } n_0 \to \infty. \tag{3.16}$$

Therefore, given $\varepsilon > 0$, one may choose $n_0 = n_0(\varepsilon)$ such that the left side of (3.16) is less than $\varepsilon/3$. Then choose $n(\varepsilon)$ such that for all $n \ge n(\varepsilon)$,

$$\begin{aligned} &\left|Q\left(S_{n_0,n}(x) > r\right) - Q\left(S_{0,n}(x) > r\right)\right| < \varepsilon/3, \\ &\left|Q\left(S_{0,n} > r\right) - Q\left(S_{0,n-n_0} > r\right)\right| < \varepsilon/3. \end{aligned} \tag{3.17}$$

It follows that

$$\left|Q\left(S_{0,n}(x) > r\right) - Q\left(S_{0,n} > r\right)\right| < \varepsilon \qquad \forall\, n \ge n(\varepsilon). \tag{3.18}$$

Hence the distribution of $S_{0,n}(x)$ converges in the weak-star topology to the appropriate Gaussian law. In this manner one proves convergence of the finite dimensional distributions of $Y_n(t)$ to those of a Brownian motion when the initial state is x. It remains to prove that the distributions of Y_n, $n = 1, 2, \dots$, form a precompact set. To prove the latter, for an arbitrary set of positive integers $n_0 < n_1 < \cdots < n_{N+1} = n$ and a positive number r, write

$$\begin{aligned} A(y) &= \left\{\left[\max_{0\le i\le N} S_{n_i-n_0,\,n_{i+1}-n_0-1}(y)\right] > r\right\}, \\ B(y) &= \left\{\left[\max_{0\le i\le N} S_{n_i-n_0,\,n_{i+1}-n_0-1}(y)\right] \ge -r\right\}. \end{aligned} \tag{3.19}$$

Let A, B denote the corresponding events for the sequence $\{X_j\}$. Since $Q(A(y))$ and $Q(B(y))$ are increasing in y, Lemma 2.3 may be used again to show that, as $n_0 \to \infty$,

$$\begin{aligned} &Q\left[\left[\max_{0\le i\le N} S_{n_i,\,n_{i+1}-1}(x)\right] > r\right] - Q\left[\left[\max_{0\le i\le N} S_{n_i,\,n_{i+1}-1}\right] > r\right] \\ &\quad = \int Q(A(y))p^{(n_0)}(x,dy) - \int Q(A(y))\pi(dy) \to 0, \end{aligned} \tag{3.20}$$

uniformly for all N, n_i and r. A similar relation holds for the min and $B(y)$. Since the partial sum process under the initial distribution π converges to a Brownian motion, it now follows by Prohorov's theorem [see Billingsley (1968), Section 15] that Y_n converges in distribution to the same Brownian motion.

Finally, in case $f = f_1 - f_2$ with f_i monotone nondecreasing and in $L^2(S,\pi)$, $i = 1, 2$, the preceding argument easily extends to the joint distribution of the processes $Y_n^{(1)}$ and $Y_n^{(2)}$ associated with f_1 and f_2, respectively. Instead of the function (3.15), one now looks at $Q(S^{(1)}_{0,n-n_0}(y) > r_1, S^{(2)}_{0,n-n_0}(y) > r_2)$, where $S^{(1)}$ and $S^{(2)}$ are partial sums corresponding to f_1 and f_2, respectively. Hence $Y_n = Y_n^{(1)} - Y_n^{(2)}$ converges in distribution to the appropriate Brownian motion when $X_0 \equiv x$. It follows, on integration with respect to x, that this convergence holds under an arbitrary initial distribution. □

4. Two examples.

EXAMPLE 4.1. We shall write vectors in bold face in this example in order to distinguish them from scalars. In mathematical economics it is quite common to take $S = (0, \infty)^k$, Γ a set of nondecreasing and continuously differentiable maps $\gamma = (\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(k)})$ such that each $\gamma^{(i)}$ is strictly concave, which may indicate, e.g., a law of diminishing returns. For simplicity, we take P to have finite support Γ. Assume, in addition, that for each $\gamma \in \Gamma$, (i) $\gamma(\mathbf{x}) \downarrow \mathbf{0}$ as $\mathbf{x} \downarrow \mathbf{0}$, (ii) $\lim_{\mathbf{x}\downarrow 0} D_i\gamma^{(i)}(\mathbf{x}) > 1$, $1 \le i \le k$, (iii) $\lim D_i\gamma^{(i)}(\mathbf{x}) < 1$ as $x^{(j)} \uparrow \infty$ for all j, $1 \le j \le k$, $1 \le i \le k$, (iv) $\lim D_i\gamma^{(i')}(\mathbf{x}) = 0$ for $i \ne i'$, as $x^{(j)} \uparrow \infty$ for all j, $1 \le j \le k$, $1 \le i \ne i' \le k$. Here $D_i = \partial/\partial x^{(i)}$.

Let us show that for each $\gamma \in \Gamma$ there exist two points $\mathbf{x}_1 < \mathbf{x}_2 \in S$ such that the range of γ on $[\mathbf{x}_1, \mathbf{x}_2]$ is contained in $[\mathbf{x}_1, \mathbf{x}_2]$. First note that $\gamma^{(i)}(\mathbf{x}) = \gamma^{(i)}(\mathbf{x}) - \gamma^{(i)}(\mathbf{0}) \ge \sum x^{(j)} D_j\gamma^{(i)}(\mathbf{x})$, which is greater than $x^{(i)}$ for all sufficiently small $\mathbf{x}$ in view of (ii). Hence $\gamma(\mathbf{x}) > \mathbf{x}$ for all sufficiently small $\mathbf{x}$. Choose $\mathbf{x}_1$ such that $\gamma(\mathbf{x}) > \mathbf{x}$ for all $\mathbf{x} \le \mathbf{x}_1$. Next, let the limit in (iii) be $\beta_i < 1$ and take $\beta = \max\{\beta_1, \dots, \beta_k\}$. Let $0 < \varepsilon < (1 - \beta)/2$. Choose $a > 0$ so that $D_i\gamma^{(i)}(\mathbf{x}) < \beta + \varepsilon/2k$ and $D_j\gamma^{(i)}(\mathbf{x}) < \varepsilon/2k$ for $i \ne j$, if $\mathbf{x} \ge (a, a, \dots, a)$. For all $b > a$, one has $\theta = \theta(a, b) \in [0, 1]$ so that

$$
\begin{aligned}
\gamma^{(i)}(b, \dots, b) &= \gamma^{(i)}(a, a, \dots, a) \\
&\quad + (b - a)\sum_j D_j\gamma^{(i)}(a + \theta(b - a), \dots, a + \theta(b - a)) \\
&\le \gamma^{(i)}(a, a, \dots, a) + (b - a)(\beta + \varepsilon/2) \\
&\le \gamma^{(i)}(a, a, \dots, a) + b(\beta + \varepsilon) \\
&\le \gamma^{(i)}(a, \dots, a) + b(1 + \beta)/2,
\end{aligned}
\tag{4.1}
$$

which is smaller than b for all sufficiently large b. Hence $\gamma(b, b, \dots, b) < (b, b, \dots, b)$ for all large b. Let $\mathbf{x}_2 = (b, \dots, b)$ for such a large b. Then $\mathbf{x}_1$ and $\mathbf{x}_2$ satisfy the requirement mentioned previously.

Using the Brouwer fixed point theorem on $[\mathbf{x}_1, \mathbf{x}_2]$, it follows that γ has a fixed point $\mathbf{x}_\gamma \in [\mathbf{x}_1, \mathbf{x}_2]$. If $\mathbf{x}_* \le \mathbf{x}_1$ and $\mathbf{x}^* \equiv (b, b, \dots, b) \ge \mathbf{x}_2$ for all $\gamma \in \Gamma$, then every γ maps $[\mathbf{x}_*, \mathbf{x}^*]$ into itself. In particular, $\mathbf{x}_\gamma \in [\mathbf{x}_*, \mathbf{x}^*]$ for all $\gamma \in \Gamma$. Since the range of γ^m on $[\mathbf{x}_*, \mathbf{x}^*]$ is contained in $[\gamma^m(\mathbf{x}_*), \gamma^m(\mathbf{x}^*)]$ and $\gamma^m(\mathbf{x}_*) \uparrow \mathbf{x}_\gamma$ and $\gamma^m(\mathbf{x}^*) \downarrow \mathbf{x}_\gamma$ as $m \uparrow \infty$, the distance between the range of γ^m and $\{\mathbf{x}_\gamma\}$ goes to zero as $m \to \infty$. Here we have used the fact that a strictly concave γ cannot have more than one fixed point in $[\mathbf{x}_1, \mathbf{x}_2]$ since $\gamma(\mathbf{0}) = \mathbf{0}$.

Assume finally that (v) there are $\gamma, \gamma' \in \Gamma$ such that $x_\gamma^{(i)} < x_{\gamma'}^{(i)}$ for $1 \le i \le k$. It follows from the preceding paragraph that if $\mathbf{x}_0$ is any given point in $(\mathbf{x}_\gamma, \mathbf{x}_{\gamma'})$, then the ranges of γ^m and γ'^m are contained in $[\mathbf{x}_*, \mathbf{x}_0]$ and $[\mathbf{x}_0, \mathbf{x}^*]$, respectively, for all sufficiently large m. Thus (1.5) holds. Hence, by Theorem 2.1, there exists a unique invariant probability π on the new state space $[\mathbf{x}_*, \mathbf{x}^*]$ such that $T^{*n}\mu$ converges in the d-metric to π uniformly for all probability measures μ on $[\mathbf{x}_*, \mathbf{x}^*]$. Since $\mathbf{x}_*$ can be taken arbitrarily small and $\mathbf{x}^*$ arbitrarily large, the invariant measure π is unique on $S = (0, \infty)^k$ and $T^{*n}\mu$ converges weakly to π for every μ on S, although an exponential rate of convergence may not hold in general, unless the support of μ is compact.

The assumption of finite Γ may be easily relaxed to the assumption of compactness of the support of P, where the topology on Γ is that of uniform convergence on compact subsets of $(0, \infty)^k$. In particular, it is enough to require, in addition to (i)–(v), that (vi) for some $x \in S$ the set $\{\gamma(x): \gamma \in \Gamma\}$ is bounded and (vii) the sets $\{D_j\gamma^{(i)}: \gamma \in \Gamma\}$, $1 \le i, j \le k$, are bounded on every compact subset of $(0, \infty)^k$.

The case $k = 1$ and Γ finite is known in mathematical economics and is described in Bhattacharya and Majumdar (1984) and Mirman (1980).

We now turn to nonlinear autoregressive models. An autoregressive process of order $q \ge 1$ is a sequence of random variables U_n with values in $\mathbb{R}^r$ satisfying a relationship of the form

$$U_{n+q} = \varphi(U_n, U_{n+1}, \dots, U_{n+q-1}) + \eta_{n+q}, \qquad n = 0, 1, \dots, \tag{4.2}$$

where φ is a measurable function on $(\mathbb{R}^r)^q$ into $\mathbb{R}^r$ and η_n, $n = q, q+1, \dots$, is an i.i.d. sequence with values in $\mathbb{R}^r$ independent of the initial variables $U_0, U_1, \dots, U_{q-1}$. Then the process $X_n = (U_n, U_{n+1}, \dots, U_{n+q-1})$, $n = 0, 1, \dots$, is a Markov process on the state space $S = (\mathbb{R}^r)^q$.

EXAMPLE 4.2 (Nonlinear autoregressive models with φ nondecreasing). Suppose $\varphi = (\varphi^{(1)}, \dots, \varphi^{(r)})$ is a bounded nondecreasing function of its arguments and that $a^{(i)} \le \varphi^{(i)} \le b^{(i)}$, $1 \le i \le r$. Assume

$$\begin{gathered} \operatorname{prob}\big(\eta_n \le (c^{(1)}, \dots, c^{(r)})\big) > 0, \\ \operatorname{prob}\big(\eta_n \ge (d^{(1)}, \dots, d^{(r)})\big) > 0, \end{gathered} \tag{4.3}$$

where the constants $c^{(i)}$ and $d^{(i)}$ satisfy

$$d^{(i)} - c^{(i)} \ge b^{(i)} - a^{(i)}, \qquad 1 \le i \le r. \tag{4.4}$$

Write $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ for $(a^{(1)}, \dots, a^{(r)}), (b^{(1)}, \dots, b^{(r)}), (c^{(1)}, \dots, c^{(r)}), (d^{(1)}, \dots, d^{(r)})$. Let us show that the Markov process $X_n = (U_n, \dots, U_{n+q-1})$ then admits a unique invariant probability and Theorems 2.1 and 3.1 apply. For $q = 1$ condition (1.5) applies with $m = 1$, since $\operatorname{prob}(X_1(x) \le \mathbf{b} + \mathbf{c}\ \forall\, x) > 0$, $\operatorname{prob}(X_1(x) \ge \mathbf{a} + \mathbf{d}\ \forall\, x) > 0$ and one may take any $x_0 \in [\mathbf{b} + \mathbf{c}, \mathbf{a} + \mathbf{d}]$. In general it may be shown that (1.5) holds with $m = q$. For example, in the case $q = 2$,

$$X_{n+1} = (U_{n+1}, U_{n+2}) = \psi(X_n) + \varepsilon_{n+1}, \tag{4.5}$$

where $\psi(x^{(1)}, x^{(2)}) = (x^{(2)}, \varphi(x))$ for $x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^r \times \mathbb{R}^r$ and $\varepsilon_{n+1} = (0, \eta_{n+1})$. Hence

$$\begin{aligned} X_0(x) &\equiv x = (x^{(1)}, x^{(2)}), \\ X_1(x) &= (x^{(2)}, \varphi(x) + \eta_2), \\ X_2(x) &= (\varphi(x) + \eta_2, \varphi(X_1(x)) + \eta_3) \\ &= \big(\varphi(x) + \eta_2, \varphi(x^{(2)}, \varphi(x) + \eta_2) + \eta_3\big), \end{aligned} \tag{4.6}$$

so that

$$\operatorname{prob}(X_2(x) \le (\mathbf{b} + \mathbf{c}, \mathbf{b} + \mathbf{c})\ \forall\, x) \ge \operatorname{prob}(\eta_2 \le \mathbf{c}, \eta_3 \le \mathbf{c}) = (\operatorname{prob}(\eta_2 \le \mathbf{c}))^2 > 0$$

and

$$\operatorname{prob}(X_2(x) \ge (\mathbf{a} + \mathbf{d}, \mathbf{a} + \mathbf{d})\ \forall\, x) \ge (\operatorname{prob}(\eta_2 \ge \mathbf{d}))^2 > 0.$$

Thus one may take x_0 to be any point of $(\mathbb{R}^r)^2$ in

$$[(\mathbf{b} + \mathbf{c}, \mathbf{b} + \mathbf{c}), (\mathbf{a} + \mathbf{d}, \mathbf{a} + \mathbf{d})] = [\mathbf{b} + \mathbf{c}, \mathbf{a} + \mathbf{d}]^2.$$

The general case is now clear.

Since U_n is a nondecreasing function of X_n, it follows that for every integer $s \ge 0$, as $n \to \infty$ the (joint) distribution of $(U_n, U_{n+1}, \dots, U_{n+s})$ under an arbitrary initial distribution of $(U_0, \dots, U_{q-1})$ converges in the d-metric on $(\mathbb{R}^r)^s$ to its steady state distribution (i.e., its distribution when the initial distribution is the invariant distribution π).

If, in addition to (4.3) and (4.4), one assumes that $E|\eta_n|^2 < \infty$, then by Theorem 3.1 applied to the function $f(x) = x^{(1)}$, $x = (x^{(1)}, x^{(2)}, \dots, x^{(q)}) \in (\mathbb{R}^r)^q$, the functional central limit theorem holds for the summands U_n.

It may be noted that (4.4) means that the error distribution is well spread out. Indeed, if η_n has a distribution whose support is unbounded in each coordinate (e.g., if it has full support $\mathbb{R}^r$), then this hypothesis is automatically satisfied and the support of the invariant probability in $(\mathbb{R}^r)^q$ is noncompact.

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Note added in proof. A recent unpublished manuscript by H. Hopenhayn and E. Prescott entitled "Invariant distributions for monotone Markov processes" has come to our attention. In this, the authors prove a result similar to our Theorem 2.1.

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