STATIONARY REGENERATIVE SETS AND SUBORDINATORS[1]

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In this paper we give a simple construction of the general stationary regenerative set, based on the stationary version of the associated subordinator (increasing Lévy process). We show that, in a certain sense, the closed range of such a Lévy process is a stationary regenerative subset of $\mathbb{R}$. The distribution of this regenerative set is σ-finite in general; it is finite iff the increments of the Lévy process have finite expectation.

1. Introduction. A regenerative set is a random subset of $\mathbb{R}$ with the property that any stopping time in the set splits the set into two independent pieces, the right-hand piece (as viewed from the stopping time) having a distribution independent of the particular stopping time. Such sets arise naturally as the set of times when a strong Markov process visits a particular point in its state space. For background on regenerative sets the reader can consult [2, 5, 6, 8–11, 13 and 14].

It is known [8, 10 and 11] that any regenerative subset of $[0, +\infty[$, defined on a probability space, can be realized as the closed range of an appropriate subordinator (an increasing process with stationary independent increments). We propose to extend this correspondence to cover the case of *stationary* regenerative subsets of $\mathbb{R}$. Such regenerative sets have been considered by Taksar [13] and by Maisonneuve [10]. These authors confine themselves to the case where the increments of the associated subordinator have finite expectation. (Note however that the methods used in [10] carry over to the general case.)

In recent years a theory of stationary Markov processes on σ-finite (typically infinite) underlying measure spaces has been developed. The fundamental paper in this regard is Kuznetsov [7]. See also [3]. We use this theory to deal with stationary regenerative sets. We consider a stationary subordinator Y and its closed range M. Unfortunately the distribution of M is never σ-finite. But if T is an appropriate random time (say a passage time of Y), then the pair (T, M) has a σ-finite law which factors as the product of Lebesgue measure and a second measure P. The measure P is the (σ-finite) distribution of a regenerative set which is naturally associated with Y.

In the next section we set our notation and recall some basic facts about subordinators and regenerative sets. In Section 3 we consider the range of a stationary subordinator, as discussed previously. In the final section we use the result of Section 3 to give a simple proof that if M is a stationary regenerative set, then $-M$ has the same distribution as M.

2. Regenerative sets and subordinators. Our notation and the basic definition (2.1) are inspired by [10] and [2]. The definition of τ_t is slightly different from that of [10] and [2]; this change was suggested to us by Bernard Maisonneuve (private communication). See the note at the end of [2].

Let Ω^0 denote the class of closed subsets of $\mathbb{R}$. For $t \in \mathbb{R}$, $w^0 \in \Omega^0$, define

$$d_t(w^0) = \inf\{s > t: s \in w^0\}, \qquad r_t(w^0) = d_t(w^0) - t,$$

$$\tau_t(w^0) = \text{cl}\{s - t: s \in w^0 \cap \,]t, +\infty[\,\} = \text{cl}((w^0 - t) \cap \,]0, +\infty[).$$

Here cl denotes closure and $\inf \phi = +\infty$. Set $\mathscr{G}^0 = \sigma\{r_s: s \in \mathbb{R}\}$, $\mathscr{G}_t^0 = \sigma\{r_s: s \leq t\}$. Clearly $(d_t: t \in \mathbb{R})$ is an increasing cadlag process adapted to $(\mathscr{G}_t^0: t \in \mathbb{R})$, and $d_t \geq t$ for all $t \in \mathbb{R}$.

A *random set* is a measurable mapping M from a measurable space $(\Omega, \mathscr{F})$ into $(\Omega^0, \mathscr{G}^0)$. Associated to a random set M are several processes: $D_t = d_t \circ M$, $R_t = r_t \circ M$, $M_t = \tau_t \circ M$.

Let $(\Omega, \mathscr{F}, P)$ be a σ-finite measure space and let $(\mathscr{F}_t: t \in \mathbb{R})$ be a right continuous filtration in $(\Omega, \mathscr{F})$. Let M be a random set defined on $(\Omega, \mathscr{F})$.

DEFINITION 2.1. $M = (\Omega, \mathscr{F}, \mathscr{F}_t, M, P)$ is a regenerative set provided

  (i) (D_t) is adapted to $(\mathscr{F}_t)$;
  (ii) $P(D_t = +\infty) = 0, \forall\, t \in \mathbb{R}$;
  (iii) $R_t(P) \equiv P(R_t \in \cdot)$ is a σ-finite measure on $[0, +\infty[$ for each $t \in \mathbb{R}$;
  (iv) there is a probability measure P^0 on $(\Omega^0, \mathscr{G}^0)$ such that for all $t \in \mathbb{R}$,

$$(2.2) \qquad P\big(f(M_{D_t})\,|\mathscr{F}_t\big) = P^0(f), \qquad \forall f \in (\mathscr{G}^0)^+.$$

In this case P^0 is called the regeneration law of M.

REMARK 2.3.   (a) Definition 2.1(ii) means that M is unbounded on the right, a.e. P. This hypothesis is made to simplify the exposition and could be dispensed with.

  (b) Since $\sigma(R_t) \subset \mathscr{F}_t$, Definition 2.1(iii) implies that P restricted to $\mathscr{F}_t$ is a σ-finite measure. Thus the conditional expectation required in Definition 2.1(iv) is well defined.

Let M^0 denote the identity map on Ω^0. It is easy to check that $(\Omega^0, \mathscr{G}^0, \mathscr{G}_{t+}^0, M^0, P^0)$ is a regenerative set (with regeneration law P^0). In addition P^0 satisfies

$$(2.4) \qquad P^0\big(M^0 \subset [0, +\infty[\big) = 1.$$

It is well known that any such regenerative law P^0 arises as the distribution of the closed range of some *subordinator* (see [8, 10 and 11]). That is, on some probability space there is defined a subordinator $X = (X_t: t \geq 0)$ (an increasing, right continuous $\mathbb{R}$-valued process with stationary independent increments) such that the distribution of the random set $\text{cl}\{X_t - X_0: t \geq 0, X_t > X_0\}$ is precisely P^0. Note that because of Definition 2.1(ii), $P^0(M^0$ is unbounded$) = 1$; it follows that $X_t \uparrow +\infty$ as $t \uparrow +\infty$, almost surely. Let $\mathbb{P}^x$ denote the law of X under the

condition $X_0 = x$ $(x \in \mathbb{R})$. Clearly $\mathbb{P}^x$ is the same as the law of $(X_t + x: t \geq 0)$ under $\mathbb{P}^0$. Because X has stationary independent increments, the laws $\mathbb{P}^x$ (and so also the regeneration law P^0) are completely determined by

$$(2.5) \qquad \mathbb{P}^x(e^{-\alpha X_t}) = e^{-\alpha x}\mathbb{P}^0(e^{-\alpha X_t}) = \exp(-\alpha x - tg(\alpha)), \qquad \alpha > 0,$$

where the *Lévy exponent g* is given by

$$(2.6) \qquad g(\alpha) = \lambda\alpha + \int_0^\infty (1 - e^{-\alpha x})\Pi(dx).$$

Here $\lambda \geq 0$ is the *drift* of X and Π, the *Lévy measure* of X, is a measure on $]0, +\infty[$ such that the right side of (2.6) is finite for all $\alpha > 0$. Let $U(x, A)$ denote the potential kernel for X, namely,

$$(2.7) \qquad U(x, A) = \mathbb{P}^x\left(\int_0^\infty 1_A(X_t)\, dt\right).$$

Write $U(A)$ for $U(0, A)$; by spatial invariance $U(x, A) = U(A - x)$. Using (2.5) we have

$$(2.8) \qquad \int_0^\infty e^{-\alpha y}U(dy) = 1/g(\alpha), \qquad \alpha > 0.$$

Now define a measure π on $[0, +\infty[$ by

$$(2.9) \qquad \pi(A) = \lambda\varepsilon_0(A) + \int_A \Pi(]x, +\infty[)\, dx, \qquad A \in \mathscr{B}_{[0, +\infty[}.$$

One checks that $\int_0^\infty e^{-\alpha x}\pi(dx) = g(\alpha)/\alpha$. It follows from (2.8) that

$$(2.10) \qquad \int_0^\infty e^{-\alpha x}\pi(dx)\int_0^\infty e^{-\alpha y}U(dy) = 1/\alpha, \qquad \alpha > 0.$$

Inverting Laplace transforms in (2.10) we arrive at

$$(2.11) \qquad \pi U(A) \equiv \int_{[0, +\infty[} \pi(dx)U(x, A) = m(A \cap]0, +\infty[),$$

where m denotes Lebesgue measure on $\mathbb{R}$. Clearly π is a σ-finite measure. It is known (see, e.g., [10]) that π is an invariant measure for the "residual life" process $(r_t: t \geq 0)$ which under P^0 is a strong Markov process. This fact is also an easy consequence of the construction of the next section; see Theorem 3.5(i). Note that by (2.9) we have

$$\pi[0, +\infty[= \lambda + \int_0^\infty x\Pi(dx).$$

On the other hand, by (2.5) and (2.6),

$$\mathbb{P}^0(X_t)/t = \lambda + \int_0^\infty x\Pi(dx).$$

Thus $\pi[0, +\infty[< \infty$ iff $\mathbb{P}^0(X_t) < \infty$ for all $t > 0$.

3. Stationary subordinators and regenerative sets. We shall say that a regenerative set M is *stationary* provided $M - t \equiv \{s - t: s \in M\}$ has the same

law as M, for each $t \in \mathbb{R}$. More formally, M is stationary if its distribution [a measure on $(\Omega^0, \mathscr{G}^0)$] is invariant under the family of transformations θ_t: $w^0 \to w^0 - t$, $t \in \mathbb{R}$, $w^0 \in \Omega^0$.

In view of the relationship noted in the last section between subordinators and regenerative subsets of $[0, +\infty[$, one might think that each stationary regenerative set could be obtained as the closed range of the appropriate stationary subordinator. Unfortunately the distribution of the range of a stationary subordinator is *never* a σ-finite measure: If it gives an event positive measure, then that event has infinite measure. This problem can be sidestepped by means of a trick; the complete story is contained in Theorem 3.5.

To begin, let P^0 be the regeneration law of some regenerative set as in Definition 2.1. Let $X = (X_t: t \geq 0; \mathbb{P}^x: x \in \mathbb{R})$ be the associated subordinator with Lévy exponent g as in (2.6). Thus P^0 is the law [on $(\Omega^0, \mathscr{G}^0)$] of $\text{cl}\{X_t - X_0: t > 0, X_t > X_0\}$ under any of the laws $\mathbb{P}^x$.

Note that the Lebesgue measure m on $\mathbb{R}$ is an invariant measure for X. Proposition 3.1 is therefore a special case of a theorem of Kuznetsov [7]. See also [3] for related matters. Let W denote the space of paths w: $\mathbb{R} \to \mathbb{R}$ which are increasing and right continuous. Let $Y_t(w) = w(t)$, $\mathscr{G} = \sigma\{Y_s: s \in \mathbb{R}\}$, $\mathscr{G}_t = \sigma\{Y_s: s \leq t\}$.

PROPOSITION 3.1. *There is a unique σ-finite measure Q on $(W, \mathscr{G})$ under which $Y = (Y_t: t \in \mathbb{R})$ is a strong Markov process with one-dimensional distributions all Lebesgue measure m and with the same transition function as X. More precisely,*

(i) $Q(Y_t \in A) = m(A)$, $t \in \mathbb{R}$, $A \in \mathscr{B}$;

(ii) $Q(f(Y_{T+s})|\mathscr{G}_{T+}) = \mathbb{P}^{Y_T}(f(X_s))$, $s \geq 0$, $f \in \mathscr{B}^+$, *whenever* T: $W \to \mathbb{R} \cup \{+\infty, -\infty\}$ *is a stopping time of the filtration $(\mathscr{G}_{t+}: t \in \mathbb{R})$.*

REMARK 3.2. (a) It is implicit in Proposition 3.1(ii) that Q restricted to $\mathscr{G}_{T+} \cap \{T \in \mathbb{R}\}$ is σ-finite.

(b) Q is stationary in the sense that $\sigma_t(Q) = Q$, $\forall$ $t \in \mathbb{R}$, where $(\sigma_t w)(s) = w(t + s)$. Also, it is clear from Proposition 3.1(i) and (2.5) that Q is invariant under the spatial translations ϕ_x defined by $(\phi_x w)(t) = w(t) - x$.

Let $M = M(w) = \text{cl}\{Y_t(w): t \in \mathbb{R}\}$ denote the closed range of Y. Clearly M is a random set on $(W, \mathscr{G})$. We leave it to the reader to check that, a.e. Q,

$$(3.3) \qquad \lim_{t \to -\infty} Y_t = -\lim_{t \to +\infty} Y_t = -\infty.$$

Thus M is almost surely unbounded on both sides. Note that $M(\phi_x w) = \theta_x(M(w))$, where $\theta_x w^0 = w^0 - x$ as before.

Let us say that a $\mathscr{G}$-measurable random time T: $W \to \mathbb{R} \cup \{+\infty, -\infty\}$ is an *intrinsic time* if $Q(T \notin \mathbb{R}) = 0$ and if

$$t + T(\sigma_t w) = T(w), \qquad \forall t \in \mathbb{R}, \forall w \in W.$$

For example, the passage times L_s defined by

$$(3.4) \qquad\qquad L_s = \inf\{t \in \mathbb{R}: Y_t > s\}$$

are intrinsic times. Note that $L_s(\phi_x w) = L_{s+x}(w)$.

THEOREM 3.5. *Let T be any intrinsic time.* (i) *Define a measure P on* $(\Omega^0, \mathscr{G}^0)$ *by*

$$(3.6) \qquad\qquad P(A) = Q(0 < T \le 1; M \in A).$$

Then $(\Omega^0, \mathscr{G}^0, \mathscr{G}^0_{t+}, M^0, P)$ is a stationary regenerative set with regeneration law P^0. In addition, $P(r_t \in \cdot) = \pi$ for all $t \in \mathbb{R}$, where π is given by (2.9). *In particular, P is independent of T.*
  (ii) $Q((T, M) \in \cdot) = m \otimes P$.

From (ii) we see that $Q(M \in A) = (+\infty)P(A)$ for any $A \in \mathscr{G}^0$, justifying the remark in the second paragraph of this section.

The proof of Theorem 3.5 requires a lemma. This lemma is a special case of a "switching identity" of Neveu [12]. We give a short proof for completeness.

LEMMA 3.7. *Let S and T be intrinsic times, and let $A \in \mathscr{G}^+$ be (σ_t)-invariant. Let ψ and φ be positive Borel functions on $\mathbb{R}$ with $0 < m(\psi) = m(\varphi) < \infty$. Then*

$$Q(\psi(S)A) = Q(\varphi(T)A).$$

PROOF. We use Fubini to compute

$$m(\varphi)Q(\psi(S)A) = Q\left(\psi(S)\int_{\mathbb{R}} \varphi(T + u)\, du\, A\right)$$

$$= \int_{\mathbb{R}} Q(\psi(S \circ \sigma_u)\varphi(T \circ \sigma_u + u)A \circ \sigma_u)\, du$$

$$= \int_{\mathbb{R}} Q(\psi(S - u)\varphi(T)A)\, du = m(\psi)Q(\varphi(T)A). \qquad \square$$

PROOF OF THEOREM 3.5. (i) By Lemma 3.7, the measure P is independent of T. Taking $T = L_t$ we see that for $t \in \mathbb{R}$, $A \in \mathscr{G}^0$,

$$P(\theta_t \circ M^0 \in A) = Q(0 < L_t \le 1; \theta_t \circ M \in A)$$

$$(3.8) \qquad\qquad = Q(0 < L_0 \circ \phi_t \le 1; M \circ \phi_t \in A)$$

$$= Q(0 < L_0 \le 1; M \in A) = P(A).$$

This proves the stationarity assertion; also, since $r_0 \circ M = Y_{L_0}$, $\nu \equiv P(r_t \in \cdot) = Q(0 < L_0 \le 1; Y_{L_0} \in \cdot)$. But by formula 2.8 of [3], since Y is increasing and since $Q(L_0 = 0) = 0 = Q(Y_0 = 0)$,

$$\nu U = Q(L_0 < 0; Y_0 \in \cdot)$$

$$= Q(Y_0 > 0; Y_0 \in \cdot) = m(\cdot \cap \,]0, \infty[) = \pi U,$$

by (2.11). Thus $v = \pi$; in particular, $P(r_t \in \cdot)$ is σ-finite. It only remains to check (2.2) of Definition 2.1; since $\theta_t(P) = P$ we need only consider the case $t = 0$. But $M_{D_0} = \mathrm{cl}\{Y_t - Y_{L_0}: Y_t > Y_{L_0}\}$ and L_0 is a stopping time of $(\mathscr{G}_{t+})$. It follows from Proposition 3.1(ii) that the conditional distribution of M_{D_0} given $\mathscr{G}_{L_0+}$ is the same as the $\mathbb{P}^0$ distribution of $\mathrm{cl}\{X_t: t > 0, X_t > 0\}$, namely, P^0. Now (2.2) follows since M is $\mathscr{G}_{L_0+}/\mathscr{G}_{0+}^0$-measurable.

(ii) The measure $Q((T, M) \in \cdot)$ is easily seen to be translation invariant in its first coordinate, and it is σ-finite since P is σ-finite. The assertion now follows by the argument used for (2.4) of [3]. See also [4]. $\square$

4. An application.

As an application of the construction of the last section we shall give a short proof of the distributional equality $M =_d - M$, whenever M is a stationary regenerative set.

Thus, let M be a stationary regenerative set with regeneration law P^0. Let $\tilde{P}$ be the distribution of M on $(\Omega^0, \mathscr{G}^0)$. We claim that, up to a constant multiple, $\tilde{P}$ must be identical with the particular stationary regenerative set constructed in the last section, starting from P^0. Indeed, the residual life process (r_t), which under P^0 is a strong Markov process, is easily seen to be a finely recurrent process with a single recurrence class. By [1], the process (r_t) has a σ-finite invariant measure which is *unique* up to a constant multiple. But each of the laws $\pi = r_0(P)$ and $\tilde{\pi} = r_0(\tilde{P})$ is a σ-finite invariant measure for (r_t). Thus π and $\tilde{\pi}$ are multiples of one another. It follows that P and $\tilde{P}$ are multiples, as claimed.

THEOREM 4.1. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, M, P)$ be a stationary regenerative set. Then for any $A \in \mathscr{G}^0$, $P(M \in A) = P(-M \in A)$. That is, $M =_d - M$.*

PROOF. By the discussion preceding the theorem, we may work with the canonical realization $(\Omega^0, \mathscr{G}^0, \mathscr{G}_{t+}^0, M^0, P)$. Moreover we can assume that P arises via Theorem 3.5. Thus let (Y_t) be the stationary subordinator and M the closed range of Y, as in Theorem 3.5. Consider the process $\tilde{Y}_t = -Y_{(-t)-}$, $t \in \mathbb{R}$. Clearly $\tilde{Y}$ is increasing and right continuous. We leave it to the reader to check that $(\tilde{Y}_t)$ under Q has the same distribution as (Y_t) under Q. Note that the closed range $\tilde{M}$ of $\tilde{Y}$ is just $-M$. Moreover $\tilde{L}_0 = \inf\{t: \tilde{Y}_t > 0\}$ is an intrinsic time (of Y). Thus

$$P(A) = Q(0 < L_0 \leq 1, M \in A)$$

$$= Q(0 < \tilde{L}_0 \leq 1, \tilde{M} \in A)$$

$$= Q(0 < \tilde{L}_0 \leq 1, -M)$$

$$= P(\tilde{A}),$$

where $\tilde{A} = \{w^0: -w^0 \in A\}$. The identity $P(A) = P(\tilde{A})$, $\forall A \in \mathscr{G}^0$ is a formal version of the statement of the theorem, so we are done. $\square$

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