*The Annals of Probability*
1988, Vol. 16, No. 3, 1184–1193

# MOMENT BOUNDS FOR ASSOCIATED SEQUENCES

BY THOMAS BIRKEL

*Universität Köln*

Let $\{X_j: j \in \mathbb{N}\}$ be a sequence of associated random variables with zero mean and let $r > 2$. We give two conditions—on the moments and on the covariance structure of the process—which guarantee that

$$\sup_{m \in \mathbb{N} \cup \{0\}} E\left|\sum_{j=m+1}^{m+n} X_j\right|^r = O(n^{r/2})$$

holds. Examples show that neither condition can be weakened.

**1. Introduction.** Let $\{X_j: j \in \mathbb{N}\}$ be a sequence of random variables (r.v.'s) with $EX_j = 0$ and let $r > 2$. For $n \in \mathbb{N}$ put $S_n = \sum_{j=1}^n X_j$.

It is well known that for i.i.d. r.v.'s with $E|X_1|^r < \infty$, $E|S_n|^r = O(n^{r/2})$ holds, and hence

$$\sup_{m \in \mathbb{N} \cup \{0\}} E|S_{n+m} - S_m|^r = O(n^{r/2}). \tag{1.1}$$

Bounds of this kind are potentially useful to obtain limit theorems, especially strong laws of large numbers, central limit theorems and laws of the iterated logarithm [see, for example, Serfling (1970) and Stout (1974), Chapter 3.7]. Conditions for relation (1.1) to hold have been investigated under various dependence structures of the process $\{X_j: j \in \mathbb{N}\}$, especially for mixing sequences [see Ibragimov (1962) and Herrndorf (1983) for $\varphi$-mixing sequences, Ibragimov (1975) and Herrndorf (1984) for $\rho$-mixing sequences and Yokoyama (1980) for strong mixing sequences and other dependence structures].

In this paper we consider sequences of associated r.v.'s. A finite family $\{X_1, \dots, X_m\}$ of r.v.'s is associated if for any two coordinatewise nondecreasing functions $f, g$ on $\mathbb{R}^m$

$$\operatorname{Cov}(f(X_1, \dots, X_m), g(X_1, \dots, X_m)) \geq 0$$

holds whenever the covariance is defined. An infinite family is associated if every finite subfamily is associated. In the last years there has been growing interest in this kind of dependence. There exist central limit theorems [see Newman (1980) and Cox and Grimmett (1984)] and laws of the iterated logarithm [see Dabrowski (1985)], as well as invariance principles [see Newman and Wright (1981)], for associated sequences. However, up to now, no general conditions are known which imply moment inequalities of the form (1.1) for associated processes. It is the purpose of our paper to fill this gap.

The known results about association show that the independence structure of an associated sequence is highly determined by its covariance structure, i.e., by

Received December 1986.
*AMS* 1980 *subject classifications*. Primary 60E15; secondary 62H20.
*Key words and phrases*. Moment bounds, partial sums of associated random variables.

the behaviour of the coefficient

$$u(n) = \sup_{k \in \mathbb{N}} \sum_{j: |j-k| \geq n} \operatorname{Cov}(X_j, X_k), \qquad n \in \mathbb{N} \cup \{0\}$$

[note that $u(n) = 2\sum_{j=n+1}^{\infty} \operatorname{Cov}(X_1, X_j)$ for stationary $X_j$]. It will turn out that moment bounds for partial sums of associated sequences also depend on the rate of decrease of $u(n)$. In Section 2 of our paper we give two natural conditions on the process $\{X_j: j \in \mathbb{N}\}$ which guarantee that (1.1) holds. In particular we assume $\sup_{j \in \mathbb{N}} E|X_j|^{r+\delta} < \infty$ for some $\delta > 0$, and $u(n) = O(n^{-\gamma})$ for some $\gamma = \gamma(r, \delta) > 0$. No stationarity is required. Examples (cf. Section 4) show that neither condition can be weakened.

The exact results are stated in Section 2. The proofs of our theorems as well as some lemmas will be given in Section 3.

**2. Results.**

THEOREM 1. *Let $\{X_j: j \in \mathbb{N}\}$ be a sequence of associated r.v.'s satisfying $EX_j = 0$ and*

(2.1) $$\sup_{j \in \mathbb{N}} E|X_j|^{r+\delta} < \infty \quad \textit{for some } r > 2 \textit{ and } \delta > 0.$$

*Assume*

(2.2) $$u(n) = O(n^{-(r-2)(r+\delta)/2\delta}).$$

*Then there is a constant B not depending on n such that for all $n \in \mathbb{N}$*

(2.3) $$\sup_{m \in \mathbb{N} \cup \{0\}} E|S_{n+m} - S_m|^r \leq Bn^{r/2}.$$

If the $X_j$ are uniformly bounded we obtain

THEOREM 2. *Let $\{X_j: j \in \mathbb{N}\}$ be a sequence of associated r.v.'s satisfying $EX_j = 0$ and $|X_j| \leq C < \infty$ for $j \in \mathbb{N}$. Assume*

(2.4) $$u(n) = O(n^{-(r-2)/2}).$$

*Then* (2.3) *holds.*

Example 1 demonstrates that condition (2.2) cannot be improved: Condition (2.1) together with a slightly weakened form of (2.2), namely

$$u(n) = O\left(a_n n^{-(r-2)(r+\delta)/2\delta}\right) \quad \text{for some } a_n \uparrow \infty,$$

does not imply (2.3).

If (2.1) is valid only for $\delta = 0$, bounds for $E|S_{n+m} - S_m|^r$ do not depend on the rate of decrease of $u(n)$: Example 2 shows that even if $u(n)$ is decreasing to 0 as quickly as you want, the moment condition $\sup_{j \in \mathbb{N}} E|X_j|^r < \infty$ does not imply (2.3). In particular Example 2 shows that under the assumption $\sup_{j \in \mathbb{N}} E|X_j|^r < \infty$ only the trivial bound $E|S_n|^r = O(n^r)$ can be obtained [if $u(n) > 0$ for all $n \in \mathbb{N}$].

From Theorem 1 we get

COROLLARY. *Let* $\{X_j: j \in \mathbb{N}\}$ *be a sequence of associated r.v.'s with* $EX_j = 0$. *Assume*

(2.5) $$\sup_{j \in \mathbb{N}} E|X_j|^s < \infty \quad \text{for some } s > 2$$

*and*

(2.6) $$u(n) = O(n^{-\rho}) \quad \text{for some } \rho > 0.$$

*Then there exists* $r > 2$ *such that* (2.3) *holds.*

PROOF. Put $r = 2s(1+\rho)/(s+2\rho)$, $\delta = s - r$. Then we have $r > 2$, $\delta > 0$ and $\rho = (r-2)(r+\delta)/2\delta$. Hence (2.5), (2.6) and Theorem 1 imply the assertion. □

Example 3 shows that in our corollary even for uniformly bounded $X_j$, $j \in \mathbb{N}$ [whence (2.5) holds for all $s$], condition (2.6) cannot be weakened.

**3. Proofs.** We will begin the proof of our results with Lemma 1, concerning truncated variables.

LEMMA 1. *Let* $X_1, X_2$ *be associated r.v.'s with finite variance and let* $-\infty \le a < b \le \infty$. *Then the following inequalities hold for* $X_1^* = \max\{a, \min\{X_1, b\}\}$, $X_1^+ = \max\{X_1, 0\}$, $X_1^- = \max\{-X_1, 0\}$:

(i) $$0 \le \mathrm{Cov}(X_1^*, X_2) \le \mathrm{Cov}(X_1, X_2),$$

(ii) $$0 \le \mathrm{Cov}(X_1^+, X_2) \le \mathrm{Cov}(X_1, X_2),$$
$$0 \ge \mathrm{Cov}(X_1^-, X_2) \ge -\mathrm{Cov}(X_1, X_2).$$

PROOF. For $t \in \mathbb{R}$ put $f(t) = \max\{a, \min\{t, b\}\}$, $g(t) = t$. Then $f$ and $g - f$ are nondecreasing. Since $X_1$ and $X_2$ are associated, we obtain $0 \le \mathrm{Cov}(f(X_1), X_2) \le \mathrm{Cov}(g(X_1), X_2)$. This proves (i).

Applying (i) with $a = 0$, $b = \infty$ and $a = -\infty$, $b = 0$, we obtain (ii). □

Lemma 2 is the main tool for our results. We assume all occurring moments to be finite.

LEMMA 2. *Let* $X_1 \ge 0$, $X_2$ *be associated r.v.'s and let* $\rho > 0$.

(i) *If* $X_1 \le R < \infty$, *then*
$$\mathrm{Cov}(X_1^{1+\rho}, X_2) \le (1+\rho)R^\rho \mathrm{Cov}(X_1, X_2).$$

(ii) *If* $|X_2| \le R < \infty$ *and* $p > 1 + \rho$, *then*
$$\mathrm{Cov}(X_1^{1+\rho}, X_2) \le (1 + \rho + 2R)(E|X_1|^p)^{\rho/(p-1)}(\mathrm{Cov}(X_1, X_2))^{(p-1-\rho)/(p-1)}.$$

(iii) *If* $\gamma > 0$ *and* $p, q > 1$ *with* $1/p + 1/q = 1$, *then*
$$\mathrm{Cov}(X_1^{1+\rho}, X_2) \le (3+\rho)\left(E|X_1|^{p(1+\rho+\gamma)}\right)^{\rho/p(\rho+\gamma)} \times (E|X_2|^q)^{\rho/q(\rho+\gamma)}(\mathrm{Cov}(X_1, X_2))^{\gamma/(\rho+\gamma)}.$$

(iv) *If* $\delta > 0$, *then*

$$\operatorname{Cov}\big(X_1^{1+\rho}, X_2\big) \le (3+\rho)\big(E|X_1|^{2+\rho}\big)^{\rho(1+\rho+\delta)/(\delta+\rho(2+\rho+\delta))} \times \big(E|X_2|^{2+\rho+\delta}\big)^{\rho/(\delta+\rho(2+\rho+\delta))}(\operatorname{Cov}(X_1, X_2))^{\delta/(\delta+\rho(2+\rho+\delta))}.$$

PROOF. (i) For $t \in \mathbb{R}$ put

$$f(t) = t^{1+\rho} 1_{\{0 \le t \le R\}},$$
$$g(t) = (1+\rho)R^\rho t 1_{\{0 \le t \le R\}} + \rho R^{1+\rho} 1_{\{t > R\}}.$$

Then $g - f$ is nondecreasing. As $X_1$ and $X_2$ are associated, we obtain $\operatorname{Cov}(f(X_1), X_2) \le \operatorname{Cov}(g(X_1), X_2)$. This proves (i).

(ii) Let $N > 0$ be fixed. Then we have

$$\operatorname{Cov}\big(X_1^{1+\rho}, X_2\big) = \operatorname{Cov}\big(f(X_1)^{1+\rho}, X_2\big) + \operatorname{Cov}\big(X_1^{1+\rho} - f(X_1)^{1+\rho}, X_2\big), \tag{3.1}$$

where for $t \in \mathbb{R}$,

$$f(t) = t 1_{\{0 \le t \le N\}} + N 1_{\{t > N\}}.$$

Since nondecreasing functions of associated r.v.'s are associated [see property $(P_4)$ of Esary, Proschan and Walkup (1967)], it follows that $f(X_1)$ and $X_2$ are associated. Using (i) and Lemma 1(i), we therefore obtain

$$\operatorname{Cov}\big(f(X_1)^{1+\rho}, X_2\big) \le (1+\rho)N^\rho \operatorname{Cov}(X_1, X_2). \tag{3.2}$$

The second term on the right-hand side of (3.1) is bounded by

$$\begin{aligned} &\operatorname{Cov}\big(X_1^{1+\rho} - f(X_1)^{1+\rho}, X_2\big) \\ &\quad \le 2RE|X_1^{1+\rho} 1_{\{X_1 > N\}}| \le 2RN^{-p+1+\rho}E|X_1|^p. \end{aligned} \tag{3.3}$$

Since uncorrelated associated r.v.'s are independent [see Corollary 3 of Newman (1984)], we may assume w.l.o.g. that $\operatorname{Cov}(X_1, X_2) > 0$. Choosing $N = (E|X_1|^p/\operatorname{Cov}(X_1, X_2))^{1/(p-1)}$, (3.1), (3.2) and (3.3) imply (ii).

(iii) Let $N > 0$ be fixed. We proceed as in (ii). By Hölder's inequality the second term on the right-hand side of (3.1) is bounded by

$$\begin{aligned} \operatorname{Cov}\big(X_1^{1+\rho} - f(X_1)^{1+\rho}, X_2\big) &\le 2\Big(E|X_1^{p(1+\rho)} 1_{\{X_1 > N\}}|\Big)^{1/p}\big(E|X_2|^q\big)^{1/q} \\ &\le 2N^{-\gamma}\big(E|X_1|^{p(1+\rho+\gamma)}\big)^{1/p}\big(E|X_2|^q\big)^{1/q}. \end{aligned} \tag{3.4}$$

Again we assume $\operatorname{Cov}(X_1, X_2) > 0$. Then, choosing

$$N = \Big[\big(E|X_1|^{p(1+\rho+\gamma)}\big)^{1/p}\big(E|X_2|^q\big)^{1/q}/\operatorname{Cov}(X_1, X_2)\Big]^{1/(\rho+\gamma)},$$

(3.1), (3.2) and (3.4) imply (iii).

(iv) follows from (iii), putting $\gamma = \delta/(2+\rho+\delta)$ and $p = (2+\rho)/(1+\rho+\gamma)$. □

Proof of Theorem 1. Let $r = l + \rho$, where $l \in \mathbb{N}$, $l \geq 2$ and $\rho \in (0,1]$. We proceed by induction on $l$. For simplicity we introduce the following notation:

$$S_{m,n} = S_{n+m} - S_m,$$
$$a_n = \sup_{m \in \mathbb{N} \cup \{0\}} E|S_{m,n}|^r.$$

We shall show that there exist $C_1 < \infty$ and $\varepsilon \in (0,1)$, such that for all $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$,

$$E|S_{m,2n}|^r \leq 2a_n + C_1 a_n^{1-\rho} n^{\rho r/2} + C_1 a_n^{1-\varepsilon} n^{\varepsilon r/2}. \tag{3.5}$$

From (3.5) we obtain

$$a_{2n} \leq 2a_n + C_1 a_n^{1-\rho} n^{\rho r/2} + C_1 a_n^{1-\varepsilon} n^{\varepsilon r/2} \quad \text{for } n \in \mathbb{N},$$

and by induction there exists $C < \infty$ such that $a_n \leq C n^{r/2}$ for all $n \in \{2^\nu : \nu \in \mathbb{N} \cup \{0\}\}$. Then (2.3) follows from the proof of Lemma 7.4 of Doob (1953).

To prove (3.5) we show the following inequalities for $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$:

$$E|S_{m,2n}|^r \leq 2a_n + 2^{l+1}\left(E|S_{m,n}|^\rho |S_{m+n,n}|^l + E|S_{m,n}|^l |S_{m+n,n}|^\rho\right); \tag{3.6}$$

$$\begin{aligned} E|S_{m,n}|^\rho |S_{m+n,n}|^l &\leq a_n^{1-\rho}\left(E|S_{m,n}||S_{m+n,n}|^{l-1+\rho}\right)^\rho, \\ E|S_{m,n}|^l |S_{m+n,n}|^\rho &\leq a_n^{1-\rho}\left(E|S_{m,n}|^{l-1+\rho}|S_{m+n,n}|\right)^\rho; \end{aligned} \tag{3.7}$$

$$\begin{aligned} E|S_{m,n}||S_{m+n,n}|^{l-1+\rho} &\leq \left|\operatorname{Cov}\left(|S_{m,n}|, |S_{m+n,n}|^{l-1+\rho}\right)\right| + C_2 n^{r/2}, \\ E|S_{m,n}|^{l-1+\rho}|S_{m+n,n}| &\leq \left|\operatorname{Cov}\left(|S_{m,n}|^{l-1+\rho}, |S_{m+n,n}|\right)\right| + C_2 n^{r/2}; \end{aligned} \tag{3.8}$$

$$\begin{aligned} \left|\operatorname{Cov}\left(|S_{m,n}|, |S_{m+n,n}|^{l-1+\rho}\right)\right| &\leq C_3 a_n^{1-\gamma} n^{\gamma r/2}, \\ \left|\operatorname{Cov}\left(|S_{m,n}|^{l-1+\rho}, |S_{m+n,n}|\right)\right| &\leq C_3 a_n^{1-\gamma} n^{\gamma r/2}, \end{aligned} \tag{3.9}$$

where $\gamma = (r - 2 + \delta)/s$, $s = \delta + (r-2)(r+\delta)$.

Then (3.5) follows from (3.6)–(3.9), putting $\varepsilon = \rho\gamma$. Therefore it remains to prove (3.6)–(3.9).

*To prove* (3.6): Elementary estimates yield

$$\begin{aligned} E|S_{m,2n}|^r &= E|S_{m,n} + S_{m+n,n}|^{l+\rho} \\ &\leq 2a_n + \sum_{j=1}^{l} \binom{l}{j}\left(E|S_{m,n}|^{l-j+\rho}|S_{m+n,n}|^j + E|S_{m,n}|^j|S_{m+n,n}|^{l-j+\rho}\right) \\ &\leq 2a_n + 2\sum_{j=1}^{l} \binom{l}{j}\left(E|S_{m,n}|^\rho|S_{m+n,n}|^l + E|S_{m,n}|^l|S_{m+n,n}|^\rho\right) \\ &\leq 2a_n + 2^{l+1}\left(E|S_{m,n}|^\rho|S_{m+n,n}|^l + E|S_{m,n}|^l|S_{m+n,n}|^\rho\right). \end{aligned}$$

This proves (3.6).

*To prove* (3.7): We only prove the first inequality; the second follows similarly. W.l.o.g. we assume $\rho \in (0,1)$. Then by Hölder's inequality

$$E|S_{m,n}|^{\rho}|S_{m+n,n}|^{l} \le \left(E|S_{m,n}||S_{m+n,n}|^{l-1+\rho}\right)^{\rho}\left(E|S_{m+n,n}|^{(l-\rho(l-1+\rho))/(1-\rho)}\right)^{1-\rho}$$
$$\le a_n^{1-\rho}\left(E|S_{m,n}||S_{m+n,n}|^{l-1+\rho}\right)^{\rho}.$$

This proves (3.7).

*To prove* (3.8): Again we only prove the first inequality. We have

$$E|S_{m,n}||S_{m+n,n}|^{l-1+\rho} \le |\mathrm{Cov}(|S_{m,n}|, |S_{m+n,n}|^{l-1+\rho})| + \left(ES_{m,n}^2\right)^{1/2}E|S_{m+n,n}|^{l-1+\rho}. \tag{3.10}$$

Our assumptions imply that $u(0) < \infty$ and by definition of $u(0)$,

$$ES_{m,n}^2 \le u(0)n, \qquad ES_{m+n,n}^2 \le u(0)n. \tag{3.11}$$

If $l = 2$, (3.8) follows from (3.10), (3.11) and Hölder's inequality. If $l > 2$, we inductively assume

$$E|S_{m+n,n}|^{l-1+\rho} \le C_4 n^{(l-1+\rho)/2}.$$

Then (3.10) and (3.11) yield (3.8).

*To prove* (3.9): We have

$$\begin{aligned}\left|\mathrm{Cov}\left(|S_{m,n}|, |S_{m+n,n}|^{l-1+\rho}\right)\right| &= \left|\mathrm{Cov}\left(|S_{m,n}|, |S_{m+n,n}|^{r-1}\right)\right| \\ &\le \left|\mathrm{Cov}\left(S_{m,n}^+, (S_{m+n,n}^+)^{r-1}\right)\right| + \left|\mathrm{Cov}\left(S_{m,n}^+, (S_{m+n,n}^-)^{r-1}\right)\right| \\ &\quad + \left|\mathrm{Cov}\left(S_{m,n}^-, (S_{m+n,n}^+)^{r-1}\right)\right| + \left|\mathrm{Cov}\left(S_{m,n}^-, (S_{m+n,n}^-)^{r-1}\right)\right|.\end{aligned} \tag{3.12}$$

Since $S_{m,n}^+$ and $(S_{m+n,n}^+)^{r-1}$ are nondecreasing and $S_{m,n}^-$ and $(S_{m+n,n}^-)^{r-1}$ are nonincreasing functions of $X_{m+1}, \dots, X_{m+2n}$, and since $X_{m+1}, \dots, X_{m+2n}$ are associated, we obtain

$$\mathrm{Cov}\left(S_{m,n}^+, (S_{m+n,n}^+)^{r-1}\right) \ge 0,$$
$$\mathrm{Cov}\left(S_{m,n}^+, (S_{m+n,n}^-)^{r-1}\right) \le 0,$$
$$\mathrm{Cov}\left(S_{m,n}^-, (S_{m+n,n}^+)^{r-1}\right) \le 0,$$
$$\mathrm{Cov}\left(S_{m,n}^-, (S_{m+n,n}^-)^{r-1}\right) \ge 0.$$

Hence by (3.12),

$$\left|\mathrm{Cov}\left(|S_{m,n}|, |S_{m+n,n}|^{l-1+\rho}\right)\right| \le \sum_{j=m+1}^{m+n} \mathrm{Cov}\left(X_j, (S_{m+n,n}^+)^{r-1}\right) + \sum_{j=m+1}^{m+n} \mathrm{Cov}\left(-X_j, (S_{m+n,n}^-)^{r-1}\right). \tag{3.13}$$

Since nondecreasing (nonincreasing) functions of associated r.v.'s are associated, $X_j$ and $S_{m+n,n}$, $X_j$ and $S^+_{m+n,n}$ and $-X_j$ and $S^-_{m+n,n}$ are associated r.v.'s for every $j \in \{m+1, \dots, m+n\}$. Applying Lemma 2(iv) (with $r-2$ instead of $\rho$ and $X_1 = S^+_{m+n,n}$ resp. $X_1 = S^-_{m+n,n}$, $X_2 = X_j$ resp. $X_2 = -X_j$), Lemma 1(ii) and our assumptions (2.1) and (2.2), we obtain from (3.13)

$$\begin{aligned}
&|\mathrm{Cov}(|S_{m,n}|, |S_{m+n,n}|^{l-1+\rho})| \\
&\quad \le 2(r+1) \sup_{i \in \mathbb{N}} \left(E|X_i|^{r+\delta}\right)^{(r-2)/s} a_n^{1-\gamma} \sum_{j=m+1}^{m+n} \left(\mathrm{Cov}(X_j, S_{m+n,n})\right)^{\delta/s} \\
&\quad \le C_5 a_n^{1-\gamma} \sum_{j=1}^{n} u(j)^{\delta/s} j^{(r-2)(r+\delta)/2s} j^{-(r-2)(r+\delta)/2s} && [\text{by } (2.1)] \\
&\quad \le C_6 a_n^{1-\gamma} \sum_{j=1}^{n} j^{-(r-2)(r+\delta)/2s} && [\text{by } (2.2)] \\
&\quad \le C_7 a_n^{1-\gamma} n^{\gamma r/2}.
\end{aligned}$$

This proves the first inequality in (3.9).

The second inequality follows similarly and thus the proof of Theorem 1 is complete. □

PROOF OF THEOREM 2. Under the assumptions of Theorem 2, (3.9) holds with $\gamma = 1/(r-1)$. This can be proved, using Lemma 2(ii) [instead of Lemma 2(iv)] and (2.4). The proof of Theorem 2 then follows along the lines of the proof of Theorem 1. □

**4. Examples.** All examples in this section have a common structure: Let $P$ be a probability measure concentrated on $\mathbb{Z}$ which fulfills $P(\{k\}) = P(\{-k\}) = p_k$, $k \in \mathbb{N}$. For $j \in \mathbb{N}$ let the r.v. $X_j$ be defined by

$$X_j(k) = \begin{cases} 0, & |k| < j \\ \alpha_j \beta_k, & k \ge j \\ -\alpha_j \beta_{|k|}, & k \le -j \end{cases}, \qquad k \in \mathbb{Z}, \tag{4.1}$$

where $\{\alpha_j: j \in \mathbb{N}\}$ and $\{\beta_k: k \in \mathbb{N}\}$ are sequences of positive numbers such that $\{\beta_k: k \in \mathbb{N}\}$ is nondecreasing and $\sum_{k=1}^{\infty} \beta_k^2 p_k < \infty$. Since nondecreasing functions of a single r.v. are associated [see properties ($P_3$) and ($P_4$) of Esary, Proschan and Walkup (1967)], $\{X_j: j \in \mathbb{N}\}$ is a sequence of associated r.v.'s. By construction we have $EX_j = 0$ for $j \in \mathbb{N}$ and

$$E|X_j|^\gamma = 2\alpha_j^\gamma \sum_{k=j}^{\infty} \beta_k^\gamma p_k \quad \text{for } j \in \mathbb{N}, \gamma > 0, \tag{4.2}$$

$$\mathrm{Cov}(X_i, X_{i+j}) = 2\alpha_i \alpha_{i+j} \sum_{k=i+j}^{\infty} \beta_k^2 p_k \quad \text{for } i, j \in \mathbb{N}, \tag{4.3}$$

$$S_n(k) = \beta_k \sum_{j=1}^{n} \alpha_j \quad \text{for } k \ge n \in \mathbb{N}. \tag{4.4}$$

EXAMPLE 1. For every $r > 2$ and $\delta > 0$ and for every sequence of real numbers $0 < a_n \uparrow \infty$ there exists a sequence $\{X_j: j \in \mathbb{N}\}$ of associated r.v.'s with $EX_j = 0$ such that (2.1) and

$$u(n) = O\left(a_n n^{-(r-2)(r+\delta)/2\delta}\right) \tag{4.5}$$

are fulfilled, but

$$\limsup_{n \in \mathbb{N}} E|S_n|^r / n^{r/2} = \infty \quad \text{holds.} \tag{4.6}$$

PROOF. Let $f: (0, \infty) \to (0, \infty)$ have the following properties:

(i) $f$ is continuously differentiable;
(ii) $f$ is nondecreasing and $\lim_{x \to \infty} f(x) = \infty$;
(iii) $x \to f(x)/x^{\gamma}$ is nonincreasing and $\lim_{x \to \infty} f(x)/x^{\gamma} = 0$ for every $\gamma > 0$;
(iv) $f(n) \le a_n^{1/(r-1+\delta)}$ for $n \in \mathbb{N}$;
(v) $f'(x) \le 1/x$ for $x > 0$.

For $j, k \in \mathbb{N}$ put

$$\alpha_j = j^{\rho}/f(j), \qquad \beta_k = 1,$$
$$p_k = Cf(k+1)^{r+\delta}/(k+1)^{1+\rho(r+\delta)},$$

where $\rho = r/2\delta$. Let $\{X_j: j \in \mathbb{N}\}$ be given by (4.1). Then by (4.2) we have for $j \in \mathbb{N}$,

$$E|X_j|^{r+\delta} = 2C\left(j^{\rho(r+\delta)}/f(j)^{r+\delta}\right) \sum_{k=j+1}^{\infty} f(k)^{r+\delta}/k^{1+\rho(r+\delta)}. \tag{4.7}$$

Using integration by parts, (i), (ii), (iii) and (v) imply

$$\begin{aligned} \sum_{k=j+1}^{\infty} f(k)^{r+\delta}/k^{1+\rho(r+\delta)} &\le \int_j^{\infty} f(x)^{r+\delta}/x^{1+\rho(r+\delta)}\,dx \\ &\le C_1 f(j)^{r+\delta}/j^{\rho(r+\delta)}. \end{aligned} \tag{4.8}$$

Then (2.1) follows from (4.7) and (4.8). Applying (4.3), (4.8), (ii) and (iii), we obtain for $i, j \in \mathbb{N}$,

$$\begin{aligned} \operatorname{Cov}(X_i, X_{i+j}) &\le (2CC_1/f(1)) f(i+j)^{r-1+\delta}/(i+j)^{\rho(r-2+\delta)} \\ &\le C_2 f(j+1)^{r-1+\delta}/(j+1)^{1+(r-2)(r+\delta)/2\delta}. \end{aligned}$$

Hence,

$$\begin{aligned} u(n) &\le C_3 \sum_{j=n+1}^{\infty} f(j)^{r-1+\delta}/j^{1+(r-2)(r+\delta)/2\delta} \\ &\le C_4 f(n)^{r-1+\delta}/n^{(r-2)(r+\delta)/2\delta} && \text{[similar to (4.8)]} \\ &\le C_4 a_n n^{-(r-2)(r+\delta)/2\delta} && \text{[according to (iv)]}. \end{aligned}$$

This proves (4.5).

Since by (4.4) and (ii), $S_n(k) \geq C_5 n^{1+\rho}/f(n)$ for $k \geq n$, we get

$$E|S_n|^r \geq \sum_{k=n}^{\infty} |S_n(k)|^r p_k \geq C_6 f(n)^{\delta} n^{r/2}.$$

Then (4.6) follows from (ii). □

EXAMPLE 2. For every $r > 2$ and for every sequence of real numbers $0 < a_n \downarrow 0$ there exists a sequence $\{X_j\colon j \in \mathbb{N}\}$ of associated r.v.'s with $EX_j = 0$ such that

$$\sup_{j \in \mathbb{N}} E|X_j|^r < \infty \tag{4.9}$$

and

$$u(n) = O(a_n) \tag{4.10}$$

are satisfied, but (4.6) holds.

PROOF. For $j \in \mathbb{N}$ put $b_j = (j/a_j)^{1/(r-2)}$. Then we have

$$0 < b_j \uparrow \infty. \tag{4.11}$$

For $j, k \in \mathbb{N}$ put

$$\alpha_j = j,\ \beta_k = b_k, \qquad p_k = C/\left(b_k^r k^{r+1}\right)$$

and let $\{X_j\colon j \in \mathbb{N}\}$ be given by (4.1). Then (4.9) follows from (4.2). Applying (4.3) and (4.11), we obtain for $i, j \in \mathbb{N}$,

$$\operatorname{Cov}(X_i, X_{i+j}) \leq C_1/\left(b_j j\right)^{r-2} = C_1 a_j/j^{r-1},$$

and hence

$$u(n) \leq C_2 \sum_{j=n}^{\infty} a_j/j^{r-1} \leq C_3 a_n.$$

This proves (4.10).

From (4.4) it follows that $S_n(k) \geq C_4 b_k n^2$ for $k \geq n$. Hence we get

$$E|S_n|^r \geq C_5 n^{2r} \sum_{k=n}^{\infty} 1/k^{r+1} \geq C_6 n^r,$$

which implies (4.6). □

EXAMPLE 3. For every sequence of real numbers $0 < a_n \downarrow 0$, fulfilling $\limsup_{n \in \mathbb{N}} n^{\rho} a_n = \infty$ for every $\rho > 0$, there exists a sequence $\{X_j\colon j \in \mathbb{N}\}$ of uniformly bounded associated r.v.'s with $EX_j = 0$ such that (4.10) is valid, but (4.6) holds for every $r > 2$.

PROOF. For $j \in \mathbb{N}$ put $b_j = a_j/\log(j+1)^2$. Then we have

$$0 < b_j \downarrow 0 \tag{4.12}$$

and

$$\limsup_{n \in \mathbb{N}} n^{\rho} b_n = \infty \quad \text{for every } \rho > 0. \tag{4.13}$$

For $j, k \in \mathbb{N}$ put $\alpha_j = b_j$, $\beta_k = 1$, $p_k = C/k^2$, and let $\{X_j\colon j \in \mathbb{N}\}$ be given by

(4.1). According to (4.1) and (4.12), the r.v.'s are uniformly bounded. Applying (4.3) and (4.12), we obtain for $i, j \in \mathbb{N}$,

$$\operatorname{Cov}(X_i, X_{i+j}) \le C_1 b_j/j,$$

and hence

$$u(n) \le C_2 \sum_{j=n}^{\infty} b_j/j \le C_3 a_n.$$

This proves (4.10).

From (4.4) and (4.12) it follows that $S_n(k) \ge nb_n$ for $k \ge n$. Hence we get

$$E|S_n|^r \ge C_4 n^{r/2} n^{(r-2)/2} b_n^r,$$

which together with (4.13) implies (4.6) for every $r > 2$. □

**Acknowledgment.** This work is taken from my dissertation written at the University of Cologne under the direction of Professor D. Landers. I am deeply indebted to him for his constant interest and for several useful comments which led to a considerable improvement of the presentation.

## REFERENCES

COX, J. T. and GRIMMETT, G. (1984). Central limit theorems for associated random variables and the percolation model. *Ann. Probab.* **12** 514–528.

DABROWSKI, A. R. (1985). A functional law of the iterated logarithm for associated sequences. *Statist. Probab. Lett.* **3** 209–212.

DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

ESARY, J., PROSCHAN, F. and WALKUP, D. (1967). Association of random variables with applications. *Ann. Math. Statist.* **38** 1466–1474.

HERRNDORF, N. (1983). The invariance principle for $\varphi$-mixing sequences. *Z. Wahrsch. verw. Gebiete* **63** 97–108.

HERRNDORF, N. (1984). A functional central limit theorem for $\rho$-mixing sequences. *J. Multivariate Anal.* **15** 141–146.

IBRAGIMOV, I. A. (1962). Some limit theorems for stationary processes. *Theory Probab. Appl.* **7** 349–382.

IBRAGIMOV, I. A. (1975). A note on the central limit theorem for dependent random variables. *Theory Probab. Appl.* **20** 135–141.

NEWMAN, C. M. (1980). Normal fluctuations and the FKG inequalities. *Comm. Math. Phys.* **74** 119–128.

NEWMAN, C. M. (1984). Asymptotic independence and limit theorems for positively and negatively dependent random variables. In *Inequalities in Statistics and Probability* (Y. L. Tong, ed.) 127–140. IMS, Hayward, Calif.

NEWMAN, C. M. and WRIGHT, A. L. (1981). An invariance principle for certain dependent sequences. *Ann. Probab.* **9** 671–675.

SERFLING, R. J. (1970). Convergence properties of $S_n$ under moment restrictions. *Ann. Math. Statist.* **41** 1235–1248.

STOUT, W. F. (1974). *Almost Sure Convergence.* Academic, New York.

YOKOYAMA, R. (1980). Moment bounds for stationary mixing sequences. *Z. Wahrsch. verw. Gebiete* **52** 45–57.

MATHEMATISCHES INSTITUT DER
UNIVERSITÄT KÖLN
WEYERTAL 86–90
D-5000 KÖLN 41
FEDERAL REPUBLIC OF GERMANY