

ON THE UNIQUENESS OF THE INFINITE OCCUPIED CLUSTER IN DEPENDENT TWO-DIMENSIONAL SITE PERCOLATION

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We consider dependent site percolation on the two-dimensional square lattice, the underlying probability measure being invariant and ergodic under each of the translations and invariant under axis reflections. If this measure satisfies the FKG condition and if percolation occurs, then we show that the infinite occupied cluster is unique with probability 1, and that all vacant star-clusters are finite.

1. Introduction. Consider a probabilistic situation in which each of the sites of the two-dimensional square lattice (i.e., each point in Z^2) is either occupied or vacant, the stochastic nature being specified by a probability measure μ on the set of all such configurations. Regarding the nearest-neighbour bonds (i.e., the line segments of length 1 joining two points of Z^2) as connections, the set of occupied sites of a given configuration falls apart into maximal connected subsets called occupied clusters. The theory of site percolation on the square lattice deals with the description of these clusters.

In this article we shall be concerned with the number N of occupied clusters that contain an infinite number of sites. Clearly, N is a random variable invariant under the group of transformations of configuration space induced by the group of translations of Z^2 . If we restrict our attention to those probability measures which are ergodic under this group, this is sufficient to ensure that N is constant with probability 1. Then if N is not 0, we say that percolation occurs.

The first investigations in percolation theory were of Bernoulli percolation, which arises when each site is occupied with probability p and vacant with probability $1 - p$, independent of the other sites. Then there is a critical value p_c for the parameter p , strictly between 0 and 1, below which N is 0 and above which N is nonzero. More recently, it has been shown that N is 0 for $p = p_c$ [14]. The exact value of p_c is unknown; the best lower bound is 0.503478 ([15]) and heuristic calculations ([4]) indicate that p_c is approximately 0.59. We remark that, for the same model in higher dimensions, the value of N at p_c is not known.

Early in the development of percolation theory, Harris [9] showed for a model similar to the preceding one, the independent bond percolation model on Z^2 , that $N = 1$ above p_c . Fisher [6] then noted that Harris' techniques work equally well for independent site percolation on Z^2 . Later, an article by Coniglio, Nappi, Peruggi and Russo [3], based on ideas in Miyamoto [12], led to $N = 1$ when μ

Received December 1986; revised June 1987.

¹Supported by CNR under Grant 203.01.38.

AMS 1980 *subject classification*. Primary 60K35.

Key words and phrases. Dependent percolation, ergodicity, FKG condition, uniqueness of the infinite cluster, multiple ergodic theorem.

describes a percolating Ising model with no external field. General references containing many recent results on independent percolation are Kesten [10] and [11].

Our goal in this article is to extend these uniqueness results to a wide class of probability measures μ . More precisely, suppose that

- (1) μ is invariant under horizontal translation, vertical translation, horizontal axis reflection and vertical axis reflection;
- (2) μ is ergodic with respect to horizontal translation and vertical translation (separately);
- (3) increasing events are positively correlated under μ ; and
- (4) N is nonzero.

(The second condition implies that N is constant with probability 1; the third is the so-called ferromagnetic or FKG condition [7]).

Then we shall show that $N = 1$ with probability 1. In fact, we obtain, as Harris did, a bit more, namely, that any finite set of sites is surrounded by an occupied circuit with probability 1. Now, if we consider two points star-connected when their distance is less than or equal to $\sqrt{2}$, we can define the connected components of the set of vacant sites as vacant star-clusters. Our result implies that all vacant star-clusters are finite with probability 1.

Examples of measures satisfying our conditions are given by extremal Gibbs states in Z^2 , in particular, Ising states with nonzero external fields, and restrictions of higher-dimensional Ising states to a suitable plane. However, our results are quite general and the question arises as to whether the set of given conditions is minimal, i.e., it is not possible to omit any single condition and obtain the same result. In this direction, it is not hard to see that the separate ergodicity is a necessary part of the set of conditions, as the ergodicity under the whole group is not sufficient. We do not know whether the FKG condition can be omitted.

We mention three related results. In [13], Newman and Schulman have shown, under conditions more general than ours and for arbitrary dimensions, that the only possible values for N are 0, 1 and ∞ . For Bernoulli percolation in all dimensions, a relation between uniqueness and qualitative properties of thermodynamic functions was first shown in [16]; more recently the combined work of Aizenman, Kesten and Newman [1] has strengthened this relation and has yielded a proof of uniqueness. A corresponding result under our conditions may well also hold, but other methods will certainly be necessary for this.

2. Preliminaries. We begin by fixing our notation for points and subsets of the discrete plane. Let $P = Z^2$ and set

$$\begin{aligned} 0 &= (0, 0), \\ x_m &= (m, 0), \quad m \in Z, \\ y_n &= (0, n), \quad n \in Z. \end{aligned}$$

We shall need the following subsets of Z^2 , described in terms of points $z = (z_1, z_2)$

belonging to them:

- (1) The upper and lower half-planes

$$H^+ = [z_2 \geq 0]$$

and

$$H^- = [z_2 \leq 0].$$

- (2) The horizontal strips

$$Q_m = [-m \leq z_2 \leq m], \quad m \geq 0.$$

- (3) The horizontal line and half-lines

$$L_m = [z_2 = m],$$

$$L_m^+ = [z_2 = m, z_1 \geq 0],$$

$$L_m^- = [z_2 = m, z_1 \leq 0], \quad m \in Z.$$

- (4) *Boxes* B of the form $[-m \leq z_1, z_2 \leq m]$, $m \geq 0$. The set of all such boxes is denoted by \mathbb{B} .

- (5) *Chains*, defined to be finite or infinite sequences of elements of P whose successive terms are at distance 1 from each other. A chain is *self-avoiding* if any two of its elements are distinct, and a finite chain is a *circuit* if its first and last element are at distance 1 from each other. A chain is said to *join* two (points or) subsets if it contains elements belonging to each, and it joins a subset to ∞ if it contains an infinite number of distinct terms, one of which belongs to the subset.

- (6) *Star-chains*, obtained by replacing "distance 1" in (5) by "distance 1 or $\sqrt{2}$."

In the following we shall make no distinction between z and $\{z\}$ for notational convenience, and occasionally we shall confuse a chain with the curve in R^2 obtained by connecting successive elements of the chain to each other with straight line segments. We shall also make use of the elementary properties of the *index* or *winding number* of a finite chain C around a point $z \in P$ not belonging to C . The index, denoted by

$$i(C, z),$$

is intuitively defined as $1/2\pi$ times the total change in the angle of the vector $z' - z$ as z' proceeds from the initial point of the chain to the final point along the curve corresponding to the chain. For the precise definition and the elementary properties, we refer to Beardon [2].

Next we introduce the probability space

$$\Omega = \{0, 1\}^P,$$

provided with the σ -algebra \mathbf{A} generated by all finite cylinders and a fixed probability measure μ . An element $\omega = (\omega_z)_{z \in P} \in \Omega$ is a *configuration*; $z \in P$ is said to be *vacant* or *occupied* in ω according to whether $\omega_z = 0$ or $\omega_z = 1$. An event $A \in \mathbf{A}$ is *increasing* if $\omega \in A$ and $\omega \leq \omega'$ imply $\omega' \in A$, where " \leq " denotes the coordinatewise partial ordering on Ω , and *decreasing* if the complement of A is increasing.

Our notation for connection by occupied site chains will be as follows. Let U , V and W be subsets of P , with possibly also $V = \infty$. Then

$$[U, V; W]$$

denotes the set of all configurations ω for which U and V are joined by a chain each of whose elements is occupied in ω and belongs to W . If $W = P$, then we simply write

$$[U, V].$$

It is easily seen that the occupied chain joining U and V may be assumed to start in U , to end in V , and to be self-avoiding.

If $U \subseteq P$ is finite, then U denotes the set of all configurations ω for which there exists a self-avoiding circuit C (disjoint from U) each of whose elements is occupied in ω and such that for each $z \in U$, $i(C, z) = +1$. It is well known that this is equivalent to requiring that U is not star-joined to ∞ by a chain of vacant elements (see, e.g., [10]).

If $\omega \in \Omega$, an *occupied cluster* in ω is a subset of P such that any two points of the subset can be joined by a chain all of whose elements are occupied, and such that it is maximal with respect to this property. The set of occupied clusters forms a partition of the set of occupied sites of ω , and we denote by $N(\omega)$ the number of occupied clusters in ω that contain an infinite number of sites.

Points and subsets of P , configurations and events in Ω , will be moved around using the horizontal and vertical translations

$$S(z) = z + (1, 0),$$

$$T(z) = z + (0, 1).$$

Note that S translates points and subsets of P to the right, but configurations and events to the left:

$$(S\omega)_z = \omega_{S(z)}, \quad z \in P.$$

T acts in a similar fashion.

We can now state our assumptions A concerning the probability measure μ on (Ω, \mathbf{A}) .

(A.1) μ is invariant under horizontal and vertical translations and axis reflection.

(A.2) μ is ergodic (separately) under horizontal and vertical translation.

(A.3) For any increasing events E and F ,

$$\mu(E \cap F) \geq \mu(E)\mu(F).$$

(A.4) $0 < \mu([0, \infty]) < 1$.

A few remarks are useful. Assumption (A.2) is more than enough to imply that N is constant with probability 1, and assumption (A.4) rules out $N = 0$, as well as the trivial measure for which all sites are occupied with probability 1. Assumption (A.3), the ferromagnetic or FKG condition, implies the same in-

equality if E and F are both decreasing, and the reverse inequality if one is decreasing and the other increasing, as follows from the definition of decreasing events.

THEOREM. *If assumptions (A.1)–(A.4) hold, then*

$$\mu(N = 1) = 1.$$

Moreover, any finite set of sites is surrounded by an occupied circuit with probability 1 and, equivalently, all vacant star-clusters are finite with probability 1.

The main part of the proof will be given in the next section. The remainder of this section will be devoted to preparations for the proof. Our first lemma allows us to prove a bit less. In the sequel we always assume that (A.1)–(A.4) hold.

BOX LEMMA. *Suppose that there exists a positive number δ such that for each box $B \in \mathbb{B}$,*

$$\mu(\mathbf{B}) \geq \delta.$$

Then any finite set of sites is surrounded by an occupied circuit with probability 1 and

$$\mu(N = 1) = 1.$$

PROOF. Any finite set is contained in a box $B \in \mathbb{B}$ and the event \mathbf{B} decreases as the size of B increases. Therefore,

$$\mu\left(\bigcap_{B \in \mathbb{B}} \mathbf{B}\right) \geq \delta,$$

and since this event is translation invariant, it has by ergodicity measure 1, and only $\mu(N = 1) = 1$ remains to be shown. Let $z, z' \in P$. Then the set $\{z, z'\}$ is surrounded by an occupied circuit with probability 1, so the probability that z and z' belong to different infinite occupied clusters is 0. Since there are only countably many pairs $z, z' \in P$, we conclude that $\mu(N = 1) = 1$. \square

Next we show that percolation cannot occur in a strip.

STRIP LEMMA. *For each $z \in Q_m$ we have*

$$\mu([z, \infty; Q_m]) = 0.$$

PROOF. By (A.4), vacant sites occur with positive probability, and then by (A.3) line segments of vacant sites also occur with positive probability, since occurrence of a vacant site is a decreasing event. Hence by ergodicity of S the strip Q_m will be closed off infinitely often in each direction by a line segment of vacant sites with probability 1, and the lemma follows. \square

The following lemma is ergodic-theoretic in nature and is a mild adaptation of an interesting theorem of Furstenberg [8].

MULTIPLE ERGODIC LEMMA. *If A_0, A_1 and A_2 are monotonic (i.e., increasing or decreasing) events, then*

$$D\text{-}\lim_{N \rightarrow \infty} \mu(A_0 \cap S^{-N}A_1 \cap S^{-2N}A_2) = \mu(A_0)\mu(A_1)\mu(A_2),$$

where $D\text{-}\lim_{N \rightarrow \infty} \alpha_N = \alpha$ if α_N tends to α along a sequence of density 1.

PROOF. First suppose that $A_2 = \Omega$. Then by ergodicity of S ,

$$\lim_{N \rightarrow \infty} N^{-1} \sum_{n=0}^{N-1} \mu(A_0 \cap S^{-n}A_1) = \mu(A_0)\mu(A_1),$$

and by (A.3) the sign of

$$\mu(A_0 \cap S^{-n}A_1) - \mu(A_0)\mu(A_1)$$

is constant. Thus the lemma follows for $A_2 = \Omega$ by standard arguments. Now imitate the proof on page 85 of [8], noting that the events obtained by translation of monotonic events and the intersection of monotonic events of the same type are still monotonic. \square

Clearly, the same lemma holds for T . The result leads to a lower bound independent of the box size for percolation probabilities outside large boxes which are far away. We shall use it in the following form.

COROLLARY. *Let $z \in P, U$ and V be finite subsets of P , and W an infinite subset of P . Then there exists a positive integer N such that*

$$\mu([z, \infty; W \setminus (S^{-N}U \cup S^N V)]) \geq \frac{1}{2} \mu([z, \infty; W]).$$

PROOF. Define the events

$$A_0 = [\text{all sites in } U \text{ are vacant}],$$

$$A_1 = [z, \infty; W],$$

$$A_2 = [\text{all sites in } V \text{ are vacant}],$$

$$\tilde{A}_N = [z, \infty; W \setminus (S^{-N}U \cup S^N V)].$$

Then A_0 and A_2 are decreasing, A_1 and \tilde{A}_N are increasing, and clearly

$$S^N A_0 \cap A_1 \cap S^{-N} A_2 = S^N A_0 \cap \tilde{A}_N \cap S^{-N} A_2.$$

Then (A.3) yields

$$\begin{aligned} \mu(\tilde{A}_N) \mu(S^N A_0 \cap S^{-N} A_2) &\geq \mu(S^N A_0 \cap \tilde{A}_N \cap S^{-N} A_2) \\ &= \mu(S^N A_0 \cap A_1 \cap S^{-N} A_2). \end{aligned}$$

By the multiple ergodic lemma, we have

$$D\text{-}\lim_{N \rightarrow \infty} \mu(S^N A_0 \cap A_1 \cap S^{-N} A_2) = \mu(A_0)\mu(A_1)\mu(A_2)$$

and

$$D\text{-}\lim_{N \rightarrow \infty} \mu(S^N A_0 \cap S^{-N} A_2) = \mu(A_0)\mu(A_2).$$

Hence

$$D\text{-}\limsup_{N \rightarrow \infty} \mu(\tilde{A}_N) \geq \mu(A_1)$$

and the corollary follows. \square

Our last lemma will provide a tool for constructing circuits around boxes.

TOPOLOGICAL LEMMA. *Let $\bar{z} \in B \in \mathbb{B}$, and let z_1 and z_2 be two different points outside B . Suppose that C_1 and C_2 are finite chains both starting at z_1 , ending at z_2 and disjoint from B . If $i(C_1, \bar{z})$ is different from $i(C_2, \bar{z})$, then there exists a self-avoiding circuit C (disjoint from B) such that*

$$i(C, \bar{z}) = 1$$

and such that any site of C is a site of either C_1 or C_2 .

PROOF. Construct the circuit \bar{C} by going from z_1 to z_2 via C_1 and then from z_2 back to z_1 via C_2 . By additivity of the index,

$$i(\bar{C}, \bar{z}) = i(C_1, \bar{z}) - i(C_2, \bar{z})$$

and this last term is different from 0.

It then follows that the component of $R^2 \setminus \bar{C}$ containing \bar{z} is bounded and contains B . Moreover, it is simply connected and its boundary, traversed in the correct direction, yields the desired self-avoiding circuit C . \square

3. Proof of the theorem. We divide the proof into two parts, according to whether percolation occurs in the upper half-plane or not. A proof of the first part can be obtained by adapting Harris [9], as was done in Ferrari [5], but the proof we give here is more closely related to assumptions (A.1)–(A.4) and indicates the direction to be followed in the second part of the proof.

First part of the proof. Assume that percolation occurs in the upper half-plane, that is,

$$\mu([0, \infty; H^+]) = p > 0.$$

We intend to apply the box lemma of the preceding section. Choose any box $B \in \mathbb{B}$. Using the corollary with $z = 0$, $U = \emptyset$, $V = B$, $W = H^+$ and T in place of S , we obtain a positive integer N such that

$$\mu([0, \infty; H^+ \setminus T^N B]) \geq p/2,$$

and T -invariance of μ then yields

$$\mu([y_{-N}, \infty; T^{-N} H^+ \setminus B]) \geq p/2.$$

The strip lemma implies that

$$\mu([y_{-N}, \infty; Q_N]) = 0,$$

so that we have

$$\mu([y_{-N}, L_N; Q_N \setminus B]) \geq p/2.$$

This event is the union of the events

$$[y_{-N}, L_N^+; (L_{-N} \cup Q_{N-1} \cup L_N^+) \setminus B]$$

and

$$[y_{-N}, L_N^-; (L_{-N} \cup Q_{N-1} \cup L_N^-) \setminus B],$$

which have the same measure by (A.1) since they are reflections of each other with respect to the vertical axis. Hence

$$\mu([y_{-N}, L_N^+; (L_{-N} \cup Q_{N-1} \cup L_N^+) \setminus B]) \geq p/4,$$

and by horizontal reflection then also

$$\mu([y_N, L_{-N}^+; (L_N \cup Q_{N-1} \cup L_{-N}^+) \setminus B]) \geq p/4.$$

The events in the last two inequalities are increasing, so that, by (A.3) their intersection J satisfies

$$\mu(J) \geq p^2/16.$$

We now claim that

$$J \subseteq [y_{-N}, y_N; P \setminus B],$$

since if $\omega \in J$, then there are occupied self-avoiding chains from y_{-N} to L_N^+ and from y_N to L_{-N}^+ , both lying in Q_N and avoiding B . Let $z_+ \in L_N^+$ and $z_- \in L_{-N}^+$ be their end-points. If y_N or z_- belongs to the chain from y_{-N} to L_N^+ , then there is the connection we required. Otherwise, this last chain can be enlarged to a self-avoiding circuit C with $i(C, y_N) = 0$ and $i(C, z_-) = \pm 1$, which the chain from y_N to z_- must then intersect in a point of the other chain, and we have verified the claim.

It follows that

$$\mu([y_{-N}, y_N; P \setminus B]) \geq p^2/16.$$

Now choose any $x \in B$ and note that the index of a chain from y_{-N} to y_N avoiding B around x is an odd multiple of $\frac{1}{2}$, and changes sign under reflection with respect to the vertical axis. Hence if we define J^+ (J^-) to be the event that there is a chain of occupied sites from y_{-N} to y_N outside B with positive (negative) index around x , then

$$\mu(J^+) = \mu(J^-) \geq p^2/32,$$

and since both events are increasing,

$$\mu(J^+ \cap J^-) \geq p^4/1024.$$

But now the topological lemma yields

$$J^+ \cap J^- \subseteq \mathbf{B},$$

so

$$\mu(\mathbf{B}) \geq p^4/1024$$

independently of the size of B , and the box lemma with $\delta = p^4/1024$ is then applied to complete the proof.

Second part of the proof. Assume now that

$$\mu([0, \infty]) = p > 0$$

and

$$\mu([0, \infty; H^+]) = 0.$$

Together with translation invariance this implies in particular that with probability 1, any infinite self-avoiding chain of occupied sites intersects each horizontal line in an infinite number of distinct sites. Again we intend to apply the box lemma. Choose any box $B \in \mathbb{B}$. Using the corollary with $z = 0$, $U = V = B$ and $W = P$, we obtain an integer $R > 0$ such that

$$\mu([0, \infty; P \setminus (S^R B \cup S^{-R} B)]) \geq p/2.$$

REMARK. The full strength of the multiple ergodicity seems to be necessary here. The point is that the intersection of the events

$$[0, \infty; P \setminus S^R B]$$

and

$$[0, \infty; P \setminus S^{-R} B]$$

is not, contrary to what one might suppose at first sight, contained in the event

$$[0, \infty; P \setminus (S^R B \cup S^{-R} B)].$$

Our penultimate aim now is to bound $\mu([0, x_{2R}; P \setminus S^R B])$ from below. By horizontal invariance,

$$\mu([x_{2R}, \infty; P \setminus (S^R B \cup S^{3R} B)]) \geq p/2.$$

Next, let

$$I = \{x_m; 0 \leq m \leq 4R\}$$

and apply the corollary again with $z = x_{2R}$, $U = V = I$ and $W = P \setminus (S^R B \cup S^{3R} B)$. This yields a positive integer M such that

$$\mu([x_{2R}, \infty; K_{2R}]) \geq p/4,$$

where for notational convenience

$$K_{2R} = P \setminus (S^R B \cup S^{3R} B \cup S^M I \cup S^{-M} I).$$

Put

$$K = P \setminus (S^R B \cup S^{-R} B).$$

Now if $\omega \in [0, \infty; K]$, then there is an infinite self-avoiding chain of occupied

sites from the origin in ω , which by assumption must (with full probability) intersect the union of half-lines $S^M L_0^+ \cup S^{-M} L_0^-$ infinitely often, and hence at least once. The first intersection can lie either in $S^M L_0^+$ or $S^{-M} L_0^-$ and vertical axis reflection then yields

$$\mu([0, S^M L_0^+; K \setminus S^{-M} L_0^-]) \geq p/4.$$

Similarly,

$$\mu([x_{2R}, S^{-M} L_0^-; K_{2R} \setminus S^{M+4R} L_0^+]) \geq p/8,$$

by reflection around the vertical axis at $2R$. Note that here the symmetry argument depends essentially on the presence of the “spurious” boxes $S^{-R} B$ and $S^{3R} B$, and that the length of I has been chosen to preserve the necessary symmetries.

Next, let A^+ denote the event that there exists an occupied chain C from 0 to $S^M L_0^+$ contained in $K \setminus S^{-M} L_0^-$, such that it either contains x_{2R} or is such that

$$i(C, x_{2R}) > 0,$$

and denote by A^- the corresponding event with negative index. By horizontal axis reflection,

$$\mu(A^+) = \mu(A^-) \geq p/8,$$

since

$$A^+ \cup A^- = [0, S^M L_0^+; K \setminus S^{-M} L_0^-]$$

has measure at least $p/4$. Now (A.3) yields

$$\mu(A^+ \cap A^- \cap [x_{2R}, S^{-M} L_0^-; K_{2R} \setminus S^{M+4R} L_0^+]) \geq p^3/512,$$

as these three events are increasing. We now claim that

$$A^+ \cap A^- \cap [x_{2R}, S^{-M} L_0^-; K_{2R} \setminus S^{M+4R} L_0^+] \subseteq [0, x_{2R}; P \setminus S^R B].$$

Indeed, if ω is a configuration belonging to the intersection on the left, then there exist occupied chains C^+ , C^- and \tilde{C} such that

- (1) C^+ and C^- begin at 0 and end in $S^M L_0^+$;
- (2) either x_{2R} belongs to $C^+ \cup C^-$ or $i(C^+, x_{2R}) > 0$ and $i(C^-, x_{2R}) < 0$;
- (3) \tilde{C} begins at x_{2R} and ends in $S^{-M} L_0^-$;
- (4) C^+ and C^- do not intersect $S^R B \cup S^{-M} L_0^-$;
- (5) \tilde{C} does not intersect $S^M I \cup S^{M+4R} L_0^+ \cup S^R B = S^R B \cup S^M L_0^+$.

Now form a circuit \bar{C} by first traversing C^+ , then the straight line segment J in $S^M L_0^+$ between the terminal elements of C^+ and C^- , and then returning to 0 via C^- in the reverse direction. Since the index is additive and since $i(J, x_{2R})$ is clearly 0, we have that either x_{2R} belongs to $C^+ \cup C^-$ or

$$i(\bar{C}, x_{2R}) = i(C^+, x_{2R}) - i(C^-, x_{2R}) \geq 1.$$

In the latter case x_{2R} belongs to a bounded component of $R^2 \setminus \bar{C}$ and $S^{-M} L_0^-$ lies in the unbounded component of $R^2 \setminus \bar{C}$, as C^+ and C^- do not intersect $S^{-M} L_0^-$. Hence \tilde{C} , which connects x_{2R} and $S^{-M} L_0^-$, must intersect \bar{C} , and since it

does not intersect $S^M L_0^+$, it must intersect C^+ or C^- . Thus in both cases 0 and x_{2R} are connected by a chain lying in $\tilde{C} \cup C^+ \cup C^-$ and noting that these chains do not intersect $S^R B$ then yields $\omega \in [0, x_{2R}; P \setminus S^R B]$. It follows that

$$\mu([0, x_{2R}; P \setminus S^R B]) \geq p^3/512,$$

and an application of the topological lemma as in the first part of the proof gives

$$\mu(\mathbf{B}) = \mu(\mathbf{S}^R \mathbf{B}) \geq p^6/2^{20}.$$

Taking $\delta = p^6/2^{20}$ in the box lemma finishes the proof. \square

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