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DOOB'S CONDITIONED DIFFUSIONS AND THEIR LIFETIMES[1]

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We study the lifetime of a conditioned diffusion (or h-path) on a bounded C^∞ domain G in $\mathbb{R}^d$. Making use of results of Donsker and Varadhan, we show that the tail of the distribution of the lifetime decays exponentially; in fact, the decay constant is the same as that for the exponential decay of the tail of the distribution of the first time the unconditioned diffusion exits G. In the case of Brownian motion and bounded domains (not necessarily C^∞) we describe some sufficient conditions to ensure the previously described asymptotic results hold here too.

Introduction. Let $a^{ij}(x)$ and $b^i(x) \in C^\infty(\mathbb{R}^d)$, $1 \le i, j \le d$, and for $f \in C^2(\mathbb{R}^d)$ define

$$Lf(x) := \frac{1}{2}\sum_{i,j} a^{ij}(x)\frac{\partial^2 f}{\partial x_i\,\partial x_j}(x) + \sum_i b^i(x)\frac{\partial f}{\partial x_i}(x). \tag{0.1}$$

We will always assume that L is strictly elliptic, i.e., for some $\lambda > 0$, $\sum_{i,j} a^{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2$ for $x, \xi \in \mathbb{R}^d$. Here $|\cdot|$ is the usual Euclidean norm. We will only be concerned with bounded domains in $\mathbb{R}^d$ so that it will be no loss to assume that $a^{ij}(x) = \delta^{ij}$ and $b^i(x) = 0$ for $|x|$ sufficiently large. These hypotheses guarantee that L uniquely determines a diffusion process $\{X_t: t \ge 0\}$ on $\mathbb{R}^d$ with a transition density $p(t, x, y)$ (with respect to Lebesgue measure on $\mathbb{R}^d$).

For any bounded domain $G \subseteq \mathbb{R}^d$ with C^∞ boundary, define $\tau_G := \inf\{t > 0: X_t \notin G\}$. The process X_t^G obtained by killing X_t at ∂G is a diffusion with state space G. Under our hypotheses on L, X_t^G will have a (substochastic) transition density $p_G(t, x, y)$ with respect to Lebesgue measure on $\mathbb{R}^d$. We call $h \in C^2(G)$ *harmonic for* L *on* G if $Lh = 0$ on G. It is well known that any strictly positive function h harmonic for L on G is excessive for X_t^G.

Furthermore, such an h determines a new diffusion Z_t^h on G having transition density (with respect to Lebesgue measure on G)

$$p_G^h(t, x, y) = h(x)^{-1}p_G(t, x, y)h(y), \qquad (x, y) \in G \times G. \tag{0.2}$$

Doob calls Z_t^h a conditioned diffusion or h-path. See his book [8, pages 566–567] for more information. It is not hard to see from (0.2) that the generator L_h of Z_t^h is an extension of

$$L_h f := \frac{1}{h}L(hf), \qquad f \in C^2(G). \tag{0.3}$$

We will denote by τ_G^h the lifetime of Z_t^h.

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In the case of Brownian motion (viz. $L = \frac{1}{2}\Delta$), several authors have studied the lifetime τ_G^h of the conditioned process. Cranston and McConnell [4] proved that in dimension $d = 2$, there is a universal constant $c > 0$ such that

$$E_x\tau_G^h \le c|G|, \qquad x \in G,$$

for any bounded open set $G \subseteq \mathbb{R}^2$. Here $|G|$ is the Lebesgue measure of G and E_x denotes expectation for the process starting at x. Their proof was simplified by Chung [1]. Cranston [3] extended the result to higher dimensions; he showed that if $G \subseteq \mathbb{R}^d$ ($d \ge 3$) is Lipschitz, then for some $c(G) > 0$,

$$E_x\tau_G^h \le c(G), \qquad x \in G.$$

For a certain class of Lipschitz domains $G \subseteq \mathbb{R}^d$ ($d \ge 2$), a series expansion for $P_x(\tau_G^h \ge t)$ was obtained in [5]. As a corollary of this result, it was shown there that

$$\text{(0.4)} \qquad \lim_{t\to\infty} \frac{1}{t}\log P_x\left(\tau_G^h > t\right) = -\lambda_G = \lim_{t\to\infty} \frac{1}{t}\log P_x(\tau_G \ge t),$$

where $\lambda_G > 0$ is the first positive eigenvalue of $\frac{1}{2}\Delta$ on G: for some $m_G \in C^2(G) \cap C^0(\overline{G})$, $\frac{1}{2}\Delta m_G = -\lambda_G m_G$ on G and $m_G > 0$ on G.

In this paper we extend (0.4) to more general domains G and we also consider the analogous problem for other diffusions (see Theorems 5.1 and 6.4). In essence, we evaluate the Donsker–Varadhan I-function explicitly enough to derive our conclusion. The interested reader should look at the Pinsky articles [13, 14]. In [13] he evaluates the I-function explicitly for diffusions with boundaries and in [14] he describes nice conditions that determine whether or not

$$E_x\exp\left(\int_0^{\tau_D} q(x(s))\,ds\right)$$

is finite [here D is a bounded C^2 domain and $q \in C(\overline{D})$].

We feel compelled to point out an intuitive connection between (0.4) and Falkner's conditional gauge theorem [10, Theorem 2.1, page 22]. This tie was pointed out by the referee, to whom I am grateful. Since $\lambda_G = \sup\{\lambda\colon E_x e^{\lambda\tau_G} < \infty\}$ and since $E_x e^{\lambda\tau_G^h} < \infty$ iff $E_x e^{\lambda\tau_G} < \infty$ by the conditional gauge theorem, we have $\lambda_G = \sup\{\lambda\colon E_x e^{\lambda\tau_G^h} < \infty\}$. Thus $P_x(\tau_G^h \ge t)$ should act like $e^{-\lambda_G t}$ when t is large.

The paper is organized as follows. In Section 1 we discuss some preliminaries. Various properties of Donsker–Varadhan's $\bar{I}$-function comprise the content of Section 2. Sections 3 and 4 give upper and lower bounds (respectively) for $\log P_x(\tau_G^h \ge t)$ in the case of general (non-self-adjoint) operators L. In Section 5 we prove the analogue to (0.4) for general L, and in Section 6 we extend (0.4) for Brownian motion and Lipschitz domains to more general domains.

1. Preliminaries, self-adjoint operators and smooth domains. We recall some properties of the transition densities $p(t, x, y)$ and $p_G(t, x, y)$ from the

Introduction. First, for some positive constants M and α,

$$0 < p_G(t, x, y) \le p(t, x, y) \le Mt^{-d/2}\exp(-\alpha|y - x|^2/t) \tag{1.1}$$

for $t > 0$, $(x, y) \in G \times G$. The last inequality is valid for $t > 0$ and $x, y \in \mathbb{R}^d$. Both $p_G(t, x, y)$ and $p(t, x, y)$ are jointly continuous on $(0, \infty) \times G \times G$ and $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, respectively. Also, for fixed $t > 0$ and $y \in G$,

$$\lim_{x \to a} p_G(t, x, y) = 0 \quad \text{for any } a \in \partial G. \tag{1.2}$$

(For these, see Dynkin [9, volume 2, Theorem 0.6, pages 230–231].)

For any topological space D define

$$\begin{aligned} C(D) &:= \text{all continuous real valued functions on } D,\\ B(D) &:= \text{all bounded real-valued Borel functions on } D,\\ C_b(D) &:= \{ f \in C(D) \colon f \text{ is bounded}\},\\ C_0(D) &:= \{ f \in C(D) \colon \operatorname{supp} f \text{ is a compact subset of } D\}. \end{aligned} \tag{1.3}$$

For $f\colon D \to \mathbb{R}$ we will write $f(\infty) = a$ if for each $\varepsilon > 0$ there is a compact set $K \subseteq D$ such that $|f(x) - a| < \varepsilon$ whenever $x \notin K$. Then define

$$\hat{C}(D) := \{ f \in C(D) \colon f(\infty) = 0\}. \tag{1.4}$$

Now $p(t, x, y)$ and $p_G(t, x, y)$ define the semigroups

$$T_t f(x) := \int p(t, x, y) f(y)\, dy, \qquad f \in B(\mathbb{R}^d), \tag{1.5}$$

$$T_t^G f(x) := \int_G p_G(t, x, y) f(y)\, dy, \qquad f \in B(G), \tag{1.6}$$

and these semigroups satisfy

$$T_t\colon B(\mathbb{R}^d) \to C_b(\mathbb{R}^d), \tag{1.7}$$

$$T_t\colon \hat{C}(\mathbb{R}^d) \to \hat{C}(\mathbb{R}^d), \tag{1.8}$$

$$T_t^G\colon B(G) \to \hat{C}(G), \tag{1.9}$$

$$T_t^G\colon B(G) \to C^\infty(G), \tag{1.10}$$

$$\text{for } f \in C_b(G) \text{ and } x \in G,\ T_t^G f(x) \to f(x) \text{ as } t \to 0, \tag{1.11}$$

$$\text{for } f \in C_0^2(G),\ \lim_{t \to 0} \sup_{x \in G} |f(x) - T_t^G f(x)| = 0. \tag{1.12}$$

For (1.7) and (1.8) see Dynkin [9, volume 1, Theorem 5.11 and its proof on pages 162–163]. (1.9) may be found in Dynkin [9, volume 2, Theorem 13.18 and its proof, pages 53–54]. (1.10) is true because for any $f \in B(G)$ $u(t, x) := T_t^G f(x)$ satisfies $\partial u/\partial t = Lu$ for $(t, x) \in (0, \infty) \times G$ (cf. Il'in, Kalashnikov and Oleinik [11, Section 4.3, pages 84–88]). Theorem 3 of Il'in, Kalashnikov and Oleinik [11, page 85] gives (1.1). Finally, (1.12) may be found in the proof of Theorem 13.18 of Dynkin [9, volume 2] [see especially (13.76) on page 54].

REMARK 1.1. In the case of Brownian motion ($L = \frac{1}{2}\Delta$), (1.1), (1.2) and (1.7)–(1.12) hold for any bounded open set $G \subseteq \mathbb{R}^d$ with regular boundary.

LEMMA 1.2. *Let $h > 0$ be harmonic for the operator L [in (0.1)] on G, where $G \subseteq \mathbb{R}^d$ is a bounded domain with C^∞ boundary. Then $\int_G h(x)\,dx < \infty$.*

PROOF. Let $E \subseteq \mathbb{R}^d$ be closed with $E \subseteq G$. It suffices to show that for some $C > 0$, $\int_G h(x)\,dx \le C\int_E h(x)\,dx$.

The formal adjoint L^* of L is strictly elliptic on $\mathbb{R}^d$ and it has C^∞ coefficients. Moreover, the Green functions g_G and g_G^* for L and L^*, respectively, are given by

$$g_G(x, y) = g_G^*(y, x) = \int_0^\infty p_G(t, x, y)\,dt.$$

Our hypotheses on L^* and G ensure that for some $C := C(E) > 0$,

$$\int_G g_G^*(x, y)\,dy \le C\int_E g_G^*(x, y)\,dy. \tag{1.13}$$

This result may be found in Krasnosel'skii [12, Lemma 7.2, page 258].

For any $f \in B(G)$ write

$$\mathscr{G}f := \int_G g_G(\cdot, y) f(y)\,dy \quad \text{and} \quad \mathscr{G}^*f := \int_G g_G^*(\cdot, y) f(y)\,dy.$$

Since G is bounded with C^∞ boundary, for some $\varepsilon > 0$, $\sup_{x \in G} E_x e^{\varepsilon\tau_G} < \infty$ and hence $\sup_{x \in G} E_x\tau_G < \infty$. In particular, by Fubini's theorem,

$$\infty > \int_G \Big[\sup_{x \in G} E_x\tau_G\Big]\,dy > \int_G \int_0^\infty \int_G p_G(t, x, y)\,dy\,dt\,dx = \int_G\int_G g_G(x, y)\,dx\,dy.$$

Hence if $(\cdot, \cdot)$ is the usual inner product on $L^2(G, dx)$, then $(\mathscr{G}g, f) = (g, \mathscr{G}^*f)$ for any $g, f \in B(G)$.

The rest of the proof is due to Falkner [10] (cf. his Lemma 2.11, page 26). For any nonnegative $\psi \in B(G)$,

$$\begin{aligned}
\int_G (\mathscr{G}\psi)(x)\,dx &= (\mathscr{G}\psi, 1) = (\psi, \mathscr{G}^*1) \\
&\le C(\psi, \mathscr{G}^*1_E) \qquad\qquad [\text{by } (1.13)] \\
&= C(\mathscr{G}\psi, 1_E) \\
&= C\int_E (\mathscr{G}\psi)(x)\,dx.
\end{aligned} \tag{1.14}$$

But for any nonnegative harmonic function h on G, we may find nonnegative $\psi_n \in B(G)$ so that $\mathscr{G}\psi_n \uparrow h$ on G as $n \to \infty$ (cf. Port and Stone [15, Theorem 2.1, page 164]). Hence by (1.14) and monotone convergence $\int_G h(x)\,dx \le C\int_E h(x)\,dx$ as desired. □

2. Properties of the Donsker–Varadhan $\bar{I}$-function. Let $\mathscr{M}(\mathbb{R}^d)$ be the set of (Borel) probability measures on $\mathbb{R}^d$ and let $C_+^\infty(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d)$: for some C_1, C_2, $0 < C_1 \le f(x) \le C_2$ for $x \in \mathbb{R}^d\}$.

Define the $\bar{I}$-function on $\mathscr{M}(\mathbb{R}^d)$ by

$$\bar{I}(\mu) := -\inf_{f \in C_+^\infty(\mathbb{R}^d)} \int \frac{Lf}{f}\, d\mu, \qquad \mu \in \mathscr{M}(\mathbb{R}^d) \tag{2.1}$$

[here L is given by (0.1) and satisfies the hypotheses in the Introduction]. Then $0 \le \bar{I}(\mu) \le \infty$ and $\bar{I}(\mu)$ is lower semicontinuous with respect to the topology of weak convergence on $\mathscr{M}(\mathbb{R}^d)$.

LEMMA 2.1. *Let $G \subseteq \mathbb{R}^d$ be a bounded domain with C^∞ boundary. If $\mu \in \mathscr{M}(\mathbb{R}^d)$ satisfies $\mu(\bar{G}) = 1$ and $\mu(\partial G) > 0$, then $\bar{I}(\mu) = \infty$.*

PROOF. Consider any $f \in C_+^\infty(\mathbb{R}^d)$ and let $G_n \subseteq \mathbb{R}^d$, $n \ge 1$, be bounded open sets with $G_n \downarrow \bar{G}$ and suppose ∂G_n is C^∞ for all n. Now

$$\frac{LT_t^{G_n} f}{T_t^{G_n} f} = \frac{(\partial/\partial t) T_t^{G_n} f}{T_t^{G_n} f} = \frac{\partial}{\partial t} \ln\left(\frac{T_t^{G_n} f}{f}\right)$$

so that

$$\int \frac{LT_t^{G_n} f}{T_t^{G_n} f}\, d\mu = \frac{d}{dt} \int \ln\left(\frac{T_t^{G_n} f}{f}\right) d\mu.$$

Since $(t, x) \in (0, \infty) \times G_n \to T_t^{G_n} f(x)$ is C^∞ and $\inf f > 0$, for each $t > 0$ we may extend $T_t^{G_n} f|_{\bar{G}}$ by some $g_t \in C_+^\infty(\mathbb{R}^d)$. Thus

$$\begin{aligned} -\bar{I}(\mu) &\le \int \frac{Lg_t}{g_t}\, d\mu = \int \frac{LT_t^{G_n} f}{T_t^{G_n} f}\, d\mu \qquad (\operatorname{supp} \mu \subseteq \bar{G}) \\ &= \frac{d}{dt} \int \ln\left(\frac{T_t^{G_n} f}{f}\right) d\mu. \end{aligned}$$

Integrating with respect to t from 0 to 1 gives

$$\begin{aligned} -\bar{I}(\mu) &\le \int \ln\left(\frac{T_1^{G_n} f}{f}\right) d\mu \\ &\le \int_G \ln \frac{\sup f}{\inf f}\, d\mu + \int_{\partial G} \ln T_1^{G_n} f\, d\mu - \int_{\partial G} \ln f\, d\mu \\ &\le C + \int_{\partial G} \ln T_1^{G_n} f\, d\mu. \end{aligned} \tag{2.2}$$

For any closed set $B \subseteq \mathbb{R}^d$ let $\tau_B := \inf\{t > 0: X_t \notin B\}$ where X_t is the process determined by L as given in the Introduction. Then since $G_n \downarrow \bar{G}$, $\tau_{G_n} \downarrow \tau_{\bar{G}}$ as $n \to \infty$. Together with the fact that $T_t^{G_n} f(x) = E_x f(X_t) 1_{\tau_{G_n} > t}$ this yields

$$T_t^{G_n} f(x) \downarrow E_x f(X_t) 1_{\tau_{\bar{G}} > t} \quad \text{as } n \to \infty,\ x \in \bar{G}. \tag{2.3}$$

For $x \in \partial G$ let K_x be an open truncated cone with vertex x and $K_x \subseteq \mathbb{R}^d \setminus G$. Such a cone can be chosen since ∂G is C^∞. Hence for $x \in \partial G$,

$$\begin{aligned} P_x(\tau_{\overline{G}} > 0) &\le P_x(\tau_{\mathbb{R}^d \setminus K_x} > 0) \\ &= 0 \end{aligned}$$

(see Dynkin [9, volume 2, Lemma 13.3, page 40]). Combined with (2.3) we get $T_t^{G_n} f(x) \downarrow 0$ as $n \to \infty$, $x \in \partial G$. Then since $\mu(\partial G) > 0$, monotone convergence in (2.2) yields $-\bar{I}(\mu) \le -\infty$ as desired. □

REMARK 2.2. If $L = \frac{1}{2}\Delta$, Lemma 2.1 and its proof still hold if we replace "∂G is C^∞" by the assumption that "G satisfies an exterior cone condition at every $x \in \partial G$."

LEMMA 2.3. *Let $G \subseteq \mathbb{R}^d$ be a bounded domain and define $l_G = \sup_{\mu(\overline{G})=1}[-\bar{I}(\mu)]$. Then for*

$$\mathscr{U}_G := \left\{ f \in C_+^\infty(\mathbb{R}^d) \colon \sup_G \frac{Lf}{f} \le 1 \right\}$$

we have $l_G = \inf_{f \in \mathscr{U}_G} \sup_G Lf/f$.

PROOF. For convenience, write $\mathscr{U} = \mathscr{U}_G$. From the work of Donsker and Varadhan [6] (see the first part of the proof of their Theorem 2.2 on page 599),

$$l_G = \sup_{\mu(\overline{G})=1} [-\bar{I}(\mu)] = \inf_{f \in C_+^\infty(\mathbb{R}^d)} \sup_{\overline{G}} \frac{Lf}{f}.$$

Hence it is clear that $l_G \le \inf_{f \in \mathscr{U}} \sup_{\overline{G}} Lf/f$. For the opposite inequality, let $\varepsilon \in (0,1)$ and choose $g \in C_+^\infty(\mathbb{R}^d)$ such that $\sup_{\overline{G}} Lg/g \le l_G + \varepsilon$. In particular, since $l_G \le 0$, $\sup_{\overline{G}} Lg/g \le \varepsilon < 1$. Hence $g \in \mathscr{U}$ and

$$\inf_{f \in \mathscr{U}} \sup_{\overline{G}} \frac{Lf}{f} \le \sup_{\overline{G}} \frac{Lg}{g} \le l_G + \varepsilon.$$

Letting $\varepsilon \downarrow 0$ gives the desired inequality and we are done. □

THEOREM 2.4. *Let $G \subseteq \mathbb{R}^d$ be a bounded domain with C^∞ boundary. If $\mathscr{K}_G := \{U \subseteq \mathbb{R}^d\colon U$ is open, ∂U is C^∞, and $U \subseteq \overline{U} \subseteq G\}$, then*

$$\sup_{U \in \mathscr{K}_G} \sup_{\mu(\overline{U})=1} [-\bar{I}(\mu)] = \sup_{\mu(\overline{G})=1} [-\bar{I}(\mu)]. \tag{2.4}$$

PROOF. Clearly LHS(2.4) $\le$ RHS(2.4). As for the opposite inequality, note that since $\bar{I}(\mu)$ is lower semicontinuous, the supremum on RHS(2.4) is actually taken on, say at μ_0, and $\bar{I}(\mu_0) < \infty$. Now $\mu_0(\bar{G}) = 1$ and by Lemma 2.1, $\mu_0(\partial G) = 0$.

Let $U \in \mathscr{K}_G$ with $\mu_0(U) > 0$ and consider any $f \in \mathscr{U}_U$. Writing $\mathscr{U} = \mathscr{U}_U$ and $\mu_U(B) := \mu_0(U)^{-1}\mu_0(U \cap B)$, we have

$$\begin{aligned}\sup_{\mu(\bar{G})=1} [-\bar{I}(\mu)] &= \bar{I}(\mu_0)\\ &\le \int_U \frac{Lf}{f} d\mu_0 + \int_{G\setminus U} \frac{Lf}{f} d\mu_0\\ &= \mu_0(U)\int_U \frac{Lf}{f} d\mu_U + \int_{G\setminus U} \frac{Lf}{f} d\mu_0\\ &\le \mu_0(U)\sup_{\bar{U}} \frac{Lf}{f} + \mu_0(G\setminus U) \qquad (\text{since } f \in \mathscr{U}).\end{aligned}$$

Taking the infimum over $f \in \mathscr{U}$ gives

$$\begin{aligned}\sup_{\mu(\bar{G})=1} [-\bar{I}(\mu)] &\le \mu_0(U)\inf_{f\in\mathscr{U}}\sup_{\bar{U}} \frac{Lf}{f} + \mu_0(G\setminus U)\\ &= \mu_0(U)\sup_{\mu(\bar{U})=1}[-\bar{I}(\mu)] + \mu_0(G\setminus U) \quad (\text{by Lemma 2.3})\\ &\le \mu_0(U)\sup_{V\in\mathscr{K}_G}\sup_{\mu(V)=1}[-I(\bar{\mu})] + \mu_0(G\setminus U).\end{aligned} \tag{2.5}$$

Since $\mu_0(\partial G) = 0$, $\mu_0(U) \uparrow \mu_0(G) = 1$ as $U \uparrow G$, so letting $U \to G$ in (2.5) gives RHS(2.4) $\le$ LHS(2.4) as desired. □

REMARK 2.5. 1. In the case of $L = \frac{1}{2}\Delta$, the conclusion of Theorem 2.4 is still true if we weaken the assumption that G has a C^∞ boundary to the assumption that G satisfies an exterior cone condition at every $x \in \partial G$ (cf. Remark 2.2).

2. When L is self-adjoint and ∂G is C^∞, it can be shown that the quantity $\sup_{\mu(\bar{G})=1}[-\bar{I}(\mu)]$ reduces to the classical variational formula for the principal eigenvalue λ_G of L on G (modulo sign, depending on the convention chosen). Hence the conclusion (2.4) in Theorem 2.4 is essentially a statement about the "continuous" dependence of the principal eigenvalue λ_G on the domain G. It is interesting to compare the proof of Theorem 2.4 in this case to a classical version of it that may be found in Courant and Hilbert [2, Theorem 11, page 423].

3. Upper bounds. In this section we prove the following theorem. We continue to use the notation and hypotheses of the Introduction.

THEOREM 3.1. *Let $G \subseteq \mathbb{R}^d$ be a bounded domain with C^∞ boundary, and let $h > 0$ be harmonic for L on G. Then for any open set $D \subseteq \bar{D} \subseteq G$,*

$$\limsup_{t\to\infty} t^{-1}\log \sup_{x\in D} P_x\left(\tau_G^h \ge t\right) \le \sup_{\mu(\bar{G})=1}\left[-\bar{I}(\mu)\right].$$

Here $\bar{I}(\mu)$ is defined in (2.1). *If* $\inf h > 0$, *we may replace D by G.*

For the proof, we need the following lemma.

LEMMA 3.2. *Let $G \subseteq \mathbb{R}^d$ be a bounded domain with C^∞ boundary. Then*

$$\limsup_{t\to\infty} t^{-1}\log \sup_{x,y\in G} p_G(t,x,y) \le \sup_{\mu(\overline{G})=1} \left[-\bar{I}(\mu)\right].$$

PROOF. By (1.1) choose $s > 0$ such that $\sup_{x,y\in G} p_G(s,x,y) \le 1$. Then for each $y \in G$, the function

$$u_y(t,x) := T_t^G \cdot 1(x) - p_G(t+s,x,y), \qquad (t,x) \in [0,\infty) \times \overline{G}$$

satisfies

$$\begin{aligned} &u_y \in C_b\big((0,\infty) \times \overline{G} \cup \{0\} \times G\big) \cap C^\infty((0,\infty) \times G),\\ &\left(\frac{\partial}{\partial t} - L\right) u_y = 0 \quad \text{on } (0,\infty) \times G,\\ &u_y(t,x) = 0 \quad \text{for } t > 0 \text{ and } x \in \partial G,\\ &u_y(0,x) = 1 - p_G(s,x,y) \ge 0 \quad \text{for } x \in G. \end{aligned} \tag{3.1}$$

Hence by an extended maximum principle (Il'in, Kalashnikov and Oleinik [11]; see Notes 1 and 2 after Theorem 11 on page 18), $u_y \ge 0$ on $(0,\infty) \times G$. Thus

$$\begin{aligned} \limsup_{t\to\infty} t^{-1}\log \sup_{x,y\in G} p_G(t,x,y) &\le \limsup_{t\to\infty} t^{-1}\log \sup_{x\in G} T_t^G \cdot 1(x)\\ &= \sup_{\mu(\overline{G})=1} \left[-\bar{I}(\mu)\right], \end{aligned}$$

where the last equality is due to Donsker and Varadhan [6, Theorem 2.2, page 598]. □

PROOF OF THEOREM 3.1. We have

$$\begin{aligned} \sup_{x\in D} P_x\left(\tau_G^h \ge t\right) &= \sup_{x\in D} h(x)^{-1} \int_G p_G(t,x,y) h(y)\,dy\\ &\le \left[\inf_D h\right]^{-1} \left[\sup_{x,y\in G} p_G(t,x,y)\right] \int_G h(y)\,dy. \end{aligned}$$

Since $\int_G h(y)\,dy < \infty$ by Lemma 1.2, the result is an immediate consequence of Lemma 3.2, and if $\inf h > 0$, we may replace D by G. □

4. Lower bounds. Let G be a bounded domain in $\mathbb{R}^d$ with C^∞ boundary. Let $G^* = G \cup \{\infty\}$ be the one point compactification of G. It is well known that G^* is metrizable. We will make free use of the notation of the Introduction.

Define for $t > 0$,

$$(4.1)\qquad p_{G^*}(t,x,y) := \begin{cases} p_G(t,x,y), & (x,y) \in G \times G, \\ 1 - p_G(t,x,G), & (x,y) \in G \times \{\infty\}, \\ 0, & (x,y) \in \{\infty\} \times G, \\ 1, & (x,y) \in \{\infty\} \times \{\infty\}, \end{cases}$$

where $p_G(t,x,G) := \int_G p_G(t,x,y)\,dy$. Let β denote the measure on the Borel sets $\mathscr{B}(G^*)$ of G^* defined by $\beta(A) = \lambda(A \cap G) + \delta_\infty(A)$, $A \in \mathscr{B}(G^*)$ (here λ is Lebesgue measure). Then it is not hard to show that $p_{G*}(t,x,y)$ is a transition density with respect to $d\beta(y)$ on G^* and gives rise to a semigroup $T_t^{G^*}$: $B(G^*) \to B(G^*)$ defined by

$$(4.2)\qquad T_t^{G^*}f(x) = \int_{G^*} p_{G^*}(t,x,y) f(y)\,d\beta(y), \qquad f \in B(G^*).$$

The next few lemmas will allow us to use the large deviation results of Donsker and Varadhan [7].

LEMMA 4.1. *The semigroup $T_t^{G^*}$ maps $B(G^*)$ into $C(G^*)$. In particular, $T_t^{G^*}$ is Feller.*

PROOF. First note [by (4.1) and (4.2)] for $f \in B(G^*)$

$$(4.3)\qquad T_t^{G^*}f(x) = \begin{cases} \displaystyle\int_G p_G(t,x,y) f(y)\,dy \\ \displaystyle\quad + \int_{\{\infty\}} [1 - p_G(t,x,G)]\, f(y)\,d\delta_\infty(y), & x \in G, \\ \displaystyle\int_G 0 \cdot f(y)\,dy \\ \displaystyle\quad + \int_{\{\infty\}} f(y)\,d\delta_\infty(y), & x \in \{\infty\}, \end{cases}$$

$$= \begin{cases} T_t^G f(x) + [1 - T_t^G \cdot 1(x)]\, f(\infty), & x \in G, \\ f(\infty), & x \in \{\infty\}. \end{cases}$$

Hence by (1.9), $T_t^{G^*}f \in C_b(G)$ and moreover,

$$T_t^{G^*}f(x) - T_t^{G^*}f(\infty) = T_t^G f(x) - f(\infty) T_t^G \cdot 1(x) \to 0 \quad \text{as } x \to \infty,\ x \in G.$$

Consequently, $T_t^{G^*}f \in C(G^*)$. □

Now define

$$(4.4)\qquad B_{00} := \Big\{ f \in C(G^*): \lim_{t \to 0} \sup_{x \in G^*} \int_{G^*} |f(y) - f(x)|\, p_{G^*}(t,x,y)\,d\beta(y) = 0 \Big\}.$$

LEMMA 4.2. $C_0(G) \subseteq B_{00}$.

PROOF. Let $f \in C_0(G)$ and $\varepsilon > 0$. Choose $g \in C_0^2(G)$ such that $0 \le g \le \sup|f|$ and $g(x) = \sup|f|$ for $x \in \operatorname{supp} f$. If $x \in K := \operatorname{supp} f$, then

$$\begin{aligned} g(x) - T_t^G g(x) &\ge \sup|f| - (\sup|f|)p_G(t,x,G) \\ &= (\sup|f|)\big(1 - T_t^G \cdot 1(x)\big) \\ &\ge 0. \end{aligned}$$

Hence for any $x \in G$,

$$|f(x)|\big(1 - T_t^G \cdot 1(x)\big) \le |g(x) - T_t^G g(x)|. \tag{4.5}$$

By uniform continuity of f, choose $\delta > 0$ so that

$$\sup_{\substack{x,y \in G \\ |x-y|<\delta}} |f(x) - f(y)| < \varepsilon. \tag{4.6}$$

Then for $B_\delta(x) := \{y \in \mathbb{R}^d: |x - y| < \delta\}$, by (4.6) and (1.1)

$$\begin{aligned} &\sup_{x\in G} \int_G |f(y) - f(x)| p_G(t,x,y)\,dy \\ &\quad = \sup_{x\in G}\left[\int_{G\cap B_\delta(x)} + \int_{G\cap B_\delta(x)^c}\right] \\ &\quad \le \varepsilon + 2(\sup|f|)Mt^{-d/2} \sup_{x\in G} \int_{G\cap B_\delta(x)^c} \exp(-\alpha|y-x|^2/t)\,dy \\ &\quad \le \varepsilon + M_1 t^{-d/2} e^{-\alpha\delta^2/t} \lambda(G). \end{aligned} \tag{4.7}$$

Since $f \in C_0(G)$, $f(\infty) = 0$, so by (4.1) and (4.3)

$$\begin{aligned} &\limsup_{t\to 0} \sup_{x\in G^*} \int_{G^*} |f(y) - f(x)| p_{G^*}(t,x,y)\,d\beta(y) \\ &\quad = \limsup_{t\to 0} \sup_{x\in G}\left\{\int_G |f(y) - f(x)| p_G(t,x,y)\,dy + |f(x)|\big[1 - T_t^G\cdot 1(x)\big]\right\} \\ &\quad \le \limsup_{t\to 0}\left\{\varepsilon + M_2 t^{-d/2} e^{-\alpha\delta^2/t} + \sup_{x\in G}|g(x) - T_t^G g(x)|\right\} \\ &\qquad\qquad [\text{by (4.7) and (4.5)}] \\ &\quad = \varepsilon \qquad\qquad [\text{by (1.12)}]. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $f \in B_{00}$. □

Define $\mathscr{M}(G^*)$ to be the set of probability measures on G^*, and give it the topology of weak convergence.

LEMMA 4.3. *Let $\alpha \in \mathscr{M}(G^*)$. Then every neighborhood N of α in $\mathscr{M}(G^*)$ contains a neighborhood of the form*

$$N_\alpha = \left\{\mu \in \mathscr{M}(G^*): \left|\int_{G^*} f_j d(\mu - \alpha)\right| < \varepsilon, 1 \le j \le k\right\},$$

where $\varepsilon > 0$ and $f_1, \dots, f_k \in B_{00}$.

PROOF. Let α and N be so given. It is no loss to assume that for some $\delta > 0$ and $h_1, \dots, h_p \in C(G^*)$,

$$N = \left\{\mu \in \mathscr{M}(G^*): \left|\int_{G^*} h_j d(\mu - \alpha)\right| < \delta, 1 \le j \le p\right\}.$$

Define $f_i(x) := h_i(x) - h_i(\infty)$ for $1 \le i \le p$. Then $f_1, \dots, f_p \in C(G^*) \cap \{f: f(\infty) = 0\}$ and hence we may choose $\tilde{f}_1, \dots, \tilde{f}_p \in C_0(G)$ satisfying

$$\sup_{G^*} |f_i - \tilde{f}_i| < \delta/3 \tag{4.8}$$

[here we take $\tilde{f}_i(\infty) = 0$]. By Lemma 4.2, $\tilde{f}_1, \dots, \tilde{f}_p \in B_{00}$, so to complete the proof it suffices to show that

$$N_\alpha := \left\{\mu: \left|\int_{G^*} \tilde{f}_i d(\mu - \alpha)\right| < \delta/3, 1 \le i \le p\right\} \subseteq N.$$

But that is easy: For $\mu \in N_\alpha$ and $1 \le i \le p$,

$$\begin{aligned}\left|\int h_i d(\mu - \alpha)\right| &= \left|\int [h_i(x) - h_i(\infty)]\, d(\mu - \alpha)(x)\right| \\ &= \left|\int f_i d(\mu - \alpha)\right| \\ &\le \left|\int (f_i - \tilde{f}_i) d(\mu - \alpha)\right| + \left|\int \tilde{f}_i d(\mu - \alpha)\right| \\ &\le 2\delta/3 + \delta/3 \qquad \text{[by (4.8) and that } \mu \in N_\alpha\text{]}.\end{aligned}$$

Hence $\mu \in N$ and we are done. □

LEMMA 4.4. *For any $\mu \in \mathscr{M}(G^*)$ with $\mu(G) = 1$ and $f \in B(G^*)$, there exist $f_n \in B_0 := \{f \in C(G^*): \lim_{t \to 0} \sup_{G^*} |T_t^{G^*} f - f| = 0\}$, $n \ge 1$, such that $\sup|f_n| \le \sup|f|$ and $f_n \to f$ a.s. (μ).*

PROOF. Denote by μ_G the (Borel) measure on G induced by μ. Since $C_0(G)$ is dense in $L^1(G, d\mu_G)$, we may choose $f_n \in C_0(G)$ with $f_n \to f|_G$ in $L^1(G, d\mu_G)$ and $\sup_G |f_n| \le \sup_G |f|$. Extract a subsequence $f_{n_k} \to f|_G$ a.e. (μ_G). Since $f_{n_k} \in C_0(G) \subseteq B_{00} \subseteq B_0$ (by Lemma 4.2) and $\mu(G) = 1$, the lemma follows. □

Now $p_{G^*}(t, x, y)$ is the transition density (with respect to the measure β) of a Markov process Y_t with compact state space G^*. Define the random measure L_t

in $\mathcal{M}(G^*)$ by

$$L_t(A) = \frac{1}{t}\int_0^t I_A(Y_s)\,ds,$$

where A is a Borel subset of G^* and I_A is the indicator function of A. Thus $L_t(A)$ is just the proportion of time up to t spent by Y in A. Denoting by $\mathcal{P}_x$ the probability associated with $Y_0 = x$, we see L_t induces a probability measure $Q_{x,t}$ on $\mathcal{M}(G^*)$ defined by

$$Q_{x,t}(\mathcal{A}) = \mathcal{P}_x(L_t(\cdot) \in \mathcal{A}),$$

where $\mathcal{A}$ is a Borel subset of $\mathcal{M}(G^*)$.

Let $\mathcal{L}$ be the infinitesimal generator of the semigroup $T_t^{G^*}: C(G^*) \to C(G^*)$ and $\mathcal{D}^+(G^*)$ be the set of positive functions in the domain $\mathcal{D}(G)$ of $\mathcal{L}$. Then for $\mu \in \mathcal{M}(G^*)$, set

$$I(\mu) := -\inf_{f \in \mathcal{D}^+(G^*)} \int_{G^*} \frac{\mathcal{L}f}{f}(x)\,d\mu(x). \tag{4.9}$$

LEMMA 4.5. *Let $\mu \in \mathcal{M}(G^*)$ with $I(\mu) < \infty$ and* $\operatorname{supp}\mu \subseteq U$, *where U is open in G with C^∞ boundary and $U \subseteq \overline{U} \subseteq G$. If $x \in U$ and $E \subseteq U$ with $\beta(E) > 0$, then for $\sigma > 0$,*

$$\int_0^\infty e^{-\sigma t}\mathcal{P}_x(Y_s \in U, 0 \le s \le t; Y_t \in E)\,dt > 0. \tag{4.10}$$

PROOF. Since $E \subseteq U \subseteq \overline{U} \subseteq G$ and $x \in U$,

$$\begin{aligned}\text{LHS}(4.10) &= \int_0^\infty e^{-\sigma t}p_U(t, x, E)\,dt \qquad [\text{see } (4.3)]\\ &= \int_0^\infty e^{-\sigma t}\int_E p_U(t, x, y)\,dy\,dt.\end{aligned}$$

But $0 < \beta(E) = \lambda(E) = \int_E dy$ and $p_U(t, x, y) > 0$, so LHS(4.10) > 0 as desired. □

The following proposition gives a sufficient condition on $\mu \in \mathcal{M}(G^*)$ for the equality of $I(\mu)$ and $\bar{I}(\mu)$ defined in Section 2.

PROPOSITION 4.6. *Let $U \subseteq \mathbb{R}^d$ be open with C^∞ boundary and $U \subseteq \overline{U} \subseteq G$. Then for any $\mu \in \mathcal{M}(G^*)$ with* $\operatorname{supp}\mu \subseteq U$, $I(\mu) = \bar{I}(\mu)$.

PROOF. Let A be the infinitesimal generator of the semigroup $T_t^G: \hat{C}(G) \to \hat{C}(G)$ [cf. (1.9)] and denote the domain of A by $D(A)$. Let L be the differential operator (0.1) in the Introduction. Since $C_0^2(G) \subseteq D(A)$ and $Af = Lf$ for $f \in C_0^2(G)$ (see Dynkin [9, volume 1, Theorem 5.10, page 159; volume 2, Theorem 13.18, page 53], by (4.3) we see that $C_0^2(G) \subseteq \mathcal{D}(G^*)$ and $\mathcal{L}f = Lf$ for $f \in C_0^2(G)$. Then since $T_t^{G^*} \cdot 1 = 1$,

$$\begin{aligned}&C_0^2(G) + \mathbb{R} := \{f + C: f \in C_0^2(G), C \in \mathbb{R}\} \subseteq \mathcal{D}(G^*),\\ &\mathcal{L}(f + C) = Lf = \mathcal{L}f \quad \text{for } f + C \in C_0^2(G) + \mathbb{R}.\end{aligned} \tag{4.11}$$

Now consider any $f \in C_+^\infty(\mathbb{R}^d)$ and $\varepsilon > 0$. Extend $f|_{\bar{U}}$ by $\tilde{f} \in C_0^\infty(G)$ with $\tilde{f} \geq 0$. Then $\tilde{f} + \varepsilon \in \mathscr{D}^+(G^*)$. Since $\operatorname{supp}\mu \subseteq U$ we have

$$\begin{aligned}
\int \frac{Lf}{f}\,d\mu &= \int_U \frac{L\tilde{f}}{\tilde{f}}\,d\mu = \int_U \frac{\mathscr{L}(\tilde{f}+\varepsilon)}{\tilde{f}}\,d\mu && [\text{by } (4.11)]\\
&= \int_U \frac{\mathscr{L}(\tilde{f}+\varepsilon)}{\tilde{f}+\varepsilon}\,d\mu + \int_U \left[\frac{\mathscr{L}(\tilde{f}+\varepsilon)}{\tilde{f}} - \frac{\mathscr{L}(\tilde{f}+\varepsilon)}{\tilde{f}+\varepsilon}\right] d\mu\\
&= \int_U \frac{\mathscr{L}(\tilde{f}+\varepsilon)}{(\tilde{f}+\varepsilon)}\,d\mu + \varepsilon \int_U \frac{\mathscr{L}\tilde{f}}{\tilde{f}(\tilde{f}+\varepsilon)}\,d\mu\\
&\geq -I(\mu) - \varepsilon \int_U \frac{|\mathscr{L}\tilde{f}|}{\tilde{f}^2}\,d\mu && [\tilde{f}+\varepsilon \in \mathscr{D}^+(G^*)]\\
&= -I(\mu) - \varepsilon \int \frac{|Lf|}{f^2}\,d\mu && [(4.11)\text{ again}].
\end{aligned}$$

Letting $\varepsilon \downarrow 0$ and then taking the infimum over $f \in C_+^\infty(\mathbb{R}^d)$ yields $-\bar{I}(\mu) \geq -I(\mu)$.

For the opposite inequality, consider any $g \in \mathscr{D}^+(G^*)$ and $\varepsilon > 0$. By (1.10) and (4.3), for each $t > 0$, $T_t^{G^*}g \in C^\infty(G)$ and $LT_t^{G^*}g = (d/dt)T_t^{G^*}g = \mathscr{L}T_t^{G^*}g = T_t^{G^*}\mathscr{L}g$ on G. Then since $T_t^{G^*}g \geq 0$ we may extend $(T_t^{G^*}g + \varepsilon)|_{\bar{U}}$ by some $\tilde{g} \in C_+^\infty(\mathbb{R}^d)$. Thus

$$\begin{aligned}
-\bar{I}(\mu) &\leq \int \frac{L\tilde{g}}{\tilde{g}}\,d\mu = \int_U \frac{L\tilde{g}}{\tilde{g}}\,d\mu\\
&= \int_U \frac{LT_t^{G^*}g}{T_t^{G^*}g + \varepsilon}\,d\mu \qquad (4.12)\\
&= \int_U \frac{T_t^{G^*}\mathscr{L}g}{T_t^{G^*}g + \varepsilon}\,d\mu.
\end{aligned}$$

Now $g, \mathscr{L}g \in C(G^*) \subseteq C_b(G)$; hence, by (1.11) and (4.3),

$$\lim_{t\to 0} T_t^{G^*}(\mathscr{L}g)(x) = \mathscr{L}g(x), \qquad x \in G,$$

$$\lim_{t\to 0} T_t^{G^*}g(x) = g(x), \qquad x \in G.$$

Using dominated convergence in (4.12) with $t \to 0$ yields

$$-\bar{I}(\mu) \leq \int_U \frac{\mathscr{L}g}{g+\varepsilon}\,d\mu = \int \frac{\mathscr{L}g}{g+\varepsilon}\,d\mu.$$

Letting $\varepsilon \to 0$ and then taking the infimum over $g \in \mathscr{D}^+(G^*)$ gives $-\bar{I}(\mu) \leq -I(\mu)$ as desired. □

Now we are ready to apply the results of Donsker and Varadhan to obtain the main result of this section.

Theorem 4.7. *Let $G \subseteq \mathbb{R}^d$ be a bounded domain with C^∞ boundary and suppose the differential operator L is given by* (0.1) *in the Introduction. Let $h > 0$ be harmonic for L on G. Then for τ_G^h and P_x as defined in the Introduction, for any $x \in G$,*

$$\liminf_{t\to\infty} \frac{1}{t} \log P_x\left(\tau_G^h > t\right) \geq \sup_{\mu(\overline{G})=1} \left[-\bar{I}(\mu)\right].$$

Remark 4.8. In the case when $L = \frac{1}{2}\Delta$, rather than require ∂G to be C^∞, we may assume G satisfies an exterior cone condition at every $x \in \partial G$. The conclusion of Theorem 4.7 holds and the same proof works (cf. Remarks 1.1, 2.2 and 2.5).

Proof of Theorem 4.7. Let $U \subseteq \mathbb{R}^d$ be open with C^∞ boundary and $U \subseteq \overline{U} \subseteq G$. Extend the coefficients of $L_h|_{\overline{U}}$ to be in $C^\infty(\mathbb{R}^d)$ such that the resulting differential operator $\tilde{L}$ on $C^2(\mathbb{R}^d)$ satisfies the same hypotheses as L. Thus the analogues of Lemmas 4.1–4.5 and Proposition 4.6 for $\tilde{L}$ continue to hold. For convenience we will write $\tilde{Q}_{x,t}$, $\tilde{I}(\mu)$, $\bar{\tilde{I}}(\mu)$, $\tilde{X}_t$, etc., for $\tilde{L}$ analogues of $Q_{x,t}$, $I(\mu)$, $\bar{I}(\mu)$, X_t, etc. Note that $\tilde{X}_t$ and Z_t^h (as in the Introduction) have the same law on $\overline{U}$.

Consider any $\mu \in \mathscr{M}_U(G^*) := \{\mu \in \mathscr{M}(G^*) : \operatorname{supp}\mu \subseteq U\}$ with $\tilde{I}(\mu) < \infty$, and let N be any neighborhood of μ in $\mathscr{M}(G^*)$. By Theorem 8.1 of Donsker and Varadhan [7, page 446],

$$\lim_{t\to\infty} \inf \frac{1}{t} \log \tilde{Q}_{x,t}\left(N \cap \mathscr{M}_U(G^*)\right) \geq -\tilde{I}(\mu) \tag{4.13}$$

provided their hypotheses H_1–H_4 hold. By our Lemmas 4.1 and 4.2 for $\tilde{L}$ analogues hypotheses H_1 and H_2 hold. Hypotheses H_3 and H_4 are only concerned with the measure μ appearing in the statment of Theorem 8.1 of Donsker and Varadhan. In the present context, $\mu \in \mathscr{M}_U(G^*)$, and hence by our Lemmas 4.4 and 4.5, the hypotheses H_3 and H_4 hold for this particular μ. Hence (4.13) is indeed valid.

Letting $N = \mathscr{M}(G^*)$, and taking the supremum over $\mu \in \mathscr{M}_U(G^*)$, we get from (4.13),

$$\liminf_{t\to\infty} \frac{1}{t} \log \tilde{Q}_{x,t}\left(\mathscr{M}_U(G^*)\right) \geq \sup_{\mu\in\mathscr{M}_U(G^*)} \left[-\tilde{I}(\mu)\right]. \tag{4.14}$$

But for $U \subseteq \overline{U} \subseteq G$: $\eta := \inf\{t > 0 : Z_t^h \notin U\}$ and $\tilde{\tau}_U := \inf\{t > 0 : \tilde{X}_t \notin U\}$, we have for $x \in U$,

$$\begin{aligned} P_x\left(\tau_G^h > t\right) \geq P_x(\eta > t) &= \tilde{P}_x(\tilde{\tau}_U > t) \\ &= \tilde{\mathscr{P}}_x\left(\tilde{L}_t(\cdot) \in \mathscr{M}_U(G^*)\right) \\ &= \tilde{Q}_{x,t}\left(\mathscr{M}_U(G^*)\right). \end{aligned}$$

Hence using the elementary equality

$$(4.15)\qquad \inf_{f\in C_+^\infty(\mathbb{R}^d)}\int\frac{Lf}{f}\,d\mu=\inf_{f\in C_+^\infty(\mathbb{R}^d)}\int\frac{L_h f}{f}\,d\mu=\inf_{f\in C_+^\infty(\mathbb{R}^d)}\int\frac{\tilde{L}f}{f}\,d\mu$$

for any $\mu\in\mathscr{M}_U(G^*)$, we get

$$\begin{aligned}
\liminf_{t\to\infty}\frac{1}{t}\log P_x\big(\tau_G^h>t\big) &\geq \liminf_{t\to\infty}\frac{1}{t}\log\tilde{Q}_{x,t}\big(\mathscr{M}_U(G^*)\big)\\
&\geq \sup_{\mu\in\mathscr{M}_U(G^*)}\big[-\tilde{I}(\mu)\big] && [\text{by (4.14)}]\\
&= \sup_{\mu\in\mathscr{M}_U(G^*)}\big[-\bar{\tilde{I}}(\mu)\big] && (\text{by Proposition 4.6})\\
&= \sup_{\mu\in\mathscr{M}_U(G^*)}\big[-\bar{I}(\mu)\big] && [\text{by (4.15)}]\\
&= \sup\big\{-\bar{I}(\mu):\mu\in\mathscr{M}(\mathbb{R}^d),\operatorname{supp}\mu\subseteq U\big\}, && x\in U.
\end{aligned}$$

Taking the supremum over all open $U\subseteq\overline{U}\subseteq G$ with C^∞ boundary and then applying Theorem 2.4, we get the conclusion of the theorem. □

5. Asymptotics. We continue to use the notation of the Introduction. The main result is the following theorem.

THEOREM 5.1. *Let G be a bounded domain in $\mathbb{R}^d$ with C^∞ boundary. Let $h>0$ be harmonic for L on G. Then for any open set $D\subseteq\mathbb{R}^d$ with $D\subseteq\overline{D}\subseteq G$,*

$$(5.1)\qquad \begin{aligned}\lim_{t\to\infty}\frac{1}{t}\log\sup_{x\in D}P_x\big(\tau_G^h>t\big) &= \sup_{\mu(\overline{G})=1}\big[-\bar{I}(\mu)\big]\\
&= \lim_{t\to\infty}\frac{1}{t}\log\sup_{x\in D}P_x\big(\tau_G>t\big).\end{aligned}$$

If $\inf h>0$, *we may replace D by G.*

REMARK 5.2. In the case when L is self-adjoint, it can be shown that the sup-inf in (5.1) reduces to the classical variational formula for the first eigenvalue of L on G with Dirichlet boundary conditions.

PROOF OF THEOREM 5.1. All the hard work has been done. The first equality in (5.1) is an immediate consequence of Theorems 3.1 and 4.7. As for the second equality, observe $P_x(\tau_G>t)=P_x(\tau_G^1>t)$, and hence another application of Theorems 3.1 and 4.7 with $h=1$ does the trick. □

REMARK 5.3. It is clear from the preceding that we may replace $\sup_{x\in D}P_x(\tau_G>t)$ by $\sup_{x\in G}P_x(\tau_G>t)$ in (5.1).

6. Brownian motion. In this section we restrict our attention to Brownian motion, so $L=\frac{1}{2}\Delta$. Our main result is Theorem 6.4 which generalizes (0.4) in the

Introduction. Let $G \subseteq \mathbb{R}^d$ be a bounded domain satisfying an exterior cone condition at every boundary point. For each $t > 0$ and $\eta \in G$ we may condition the Brownian motion killed at ∂G (written $\{X_s^G: s \geq 0\}$ as in the Introduction) by the event $\{X_t^G = \eta\}$. This gives rise to a new sample continuous and time inhomogenous Markov process $\{X_\eta^t(s): s \in [0, t)\}$ with nonstationary transition density (from $\zeta_1 \in G$ at time s_1, to $\zeta_2 \in G$ at time s_2 with $0 \leq s_1 < s_2 < t$):

$$p_\eta^t(s_1, \zeta_1; s_2, \zeta_2) = \frac{p_G(s_2 - s_1, \zeta_1, \zeta_2) p_G(t - s_2, \zeta_2, \eta)}{p_G(t - s_1, \zeta_1, \eta)} \tag{6.1}$$

(see Doob [8, Section 2.VI.14, pages 567–568]). With the help of the tied down process $X_\eta^t(\cdot)$ we obtain the following upper bound.

THEOREM 6.1. *With G as before, suppose for any open $D \subseteq \overline{D} \subseteq G$ there exist positive numbers δ and T such that*

$$M := \sup\left\{ \int_G p_\eta^t(0, \zeta_1; s, \zeta_2)^{1+\delta}\, d\zeta_2 : \zeta_1 \in D, t > 2T, \right. \tag{6.2}$$
$$\left. \eta \in G, T < s < t - T \right\} < \infty.$$

If h is positive and harmonic for $\frac{1}{2}\Delta$ on G and $h \in L^\rho(G, dx)$ for some $\rho \in (0, 1]$, then

$$\limsup_{t \to \infty} \frac{1}{t} \log \sup_{x \in D} P_x\left(\tau_G^h > t\right) \leq \sup_{\mu(\overline{G}) = 1} \left[-\bar{I}(\mu)\right]. \tag{6.3}$$

REMARK 6.2. 1. If G is a Lipschitz domain, then the condition $h \in L^\rho(G, dx)$ for some $\rho > 0$ is automatically satisfied (see [5], especially Theorem 2.4 and the proof of Lemma 3.2).

2. By (6.1),

$$p_\eta^t(0, \zeta_1; s, \zeta_2) = \frac{p_G(s, \zeta_1, \zeta_2) p_G(t - s, \zeta_2, \eta)}{p_G(t, \zeta_1, \eta)}.$$

The trouble is for t large and/or η near ∂G the denominator $p_G(t, \zeta_1, \eta)$ is small; however, for s small and η near ∂G the factor $p_G(t - s, \zeta_2, \eta)$ in the numerator is also small. Also, if $t - s$ is small and t is large, both $p_G(s, \zeta_1, \zeta_2)$ in the numerator and $p_G(t, \zeta_1, \eta)$ in the denominator are small. So we hope when the denominator is small, the numerator is small enough to "cancel" the denominator in some sense. In fact, hypotheses (6.2) is just a way of making this idea precise: It says for t large with s and $t - s$ bounded away from zero, the denominator does not go to zero too much faster than the numerator. Also note it is possible to show that in the case when ∂G is C^∞, we have for some $T > 0$,

$$\sup\left\{ p_\eta^t(0, \zeta_1; s, \zeta_2): t > 2T, \zeta_1 \in D, \zeta_2 \in G, T < s < t - T, \eta \in G\right\} < \infty.$$

To prove Theorem 6.1 we need the following proposition.

PROPOSITION 6.3. *Let $r, \gamma \in (0,1)$. For some positive numbers δ and C_1 we have*

$$\frac{x}{1+ax} \le C_1 \frac{x^r a^{r-1}}{1+a^{\gamma+1}x^r} \quad \text{for } x \ge 0 \text{ and } a \in (0,\delta).$$

PROOF. Let $a > 0$ and define

$$f(x) := f_a(x) := \left[\frac{x}{1+ax}\right]\left[\frac{x^r}{1+a^{\gamma+1}x^r}\right]^{-1}, \qquad x > 0.$$

We need to find $C_1 > 0$ and $\delta > 0$ depending only on γ and r such that $f(x) \le C_1 a^{r-1}$ for any $x > 0$ and $a \in (0,\delta)$.

First observe we may write

$$f(x) = (x^{-r} + a^{1+\gamma})/(x^{-1} + a), \qquad x > 0.$$

Then

$$\begin{aligned} f'(x) &= [(1-r)/a + a^\gamma x^r - rx]\,x^{-r-2}a/(x^{-1}+a)^2 \\ &= [(1-r)/a + g(x)]\,x^{-r-2}a/(x^{-1}+a)^2, \end{aligned} \tag{6.4}$$

where $g(x) := a^\gamma x^r - rx$. Since $r < 1$, it is routine to show g is strictly increasing on $(0, a^{\gamma/(1-r)})$ and strictly decreasing on $(a^{\gamma/(1-r)}, \infty)$. Moreover, $g(a^{\gamma/(1-r)}r^{-1/(1-r)}) = 0 = g(0)$. Hence there is a unique $x_0 > a^{\gamma/(1-r)}r^{-1/(1-r)}$ $(> a^{\gamma/(1-r)})$ such that $g(x_0) = -(1-r)/a < 0$. Now $x \in (0, x_0) \Rightarrow g(x) > g(x_0) \Rightarrow f'(x) > 0$ and also $x \in (x_0, \infty) \Rightarrow g(x) < g(x_0) \Rightarrow f'(x) < 0$. Thus

$$f(x) \le f(x_0), \qquad x > 0. \tag{6.5}$$

We wish to obtain upper and lower bounds on x_0. First let $0 < \beta_1 < (1-r)/r$. Then

$$\begin{aligned} g(\beta_1 a^{-1}) &= \beta_1^r a^{\gamma-r} - r\beta_1 a^{-1} = a^{-1}(\beta_1^r a^{\gamma+1-r} - r\beta_1) \\ &> a^{-1}(-(1-r)), \end{aligned}$$

provided $a > 0$ is small, say $a < \delta_1 := \delta_1(r, \beta_1, \gamma)$. Thus $\beta_1 a^{-1} < x_0$ for $a < \delta_1$. Similarly, if $\beta_2 > (1-r)/r$, then for some $\delta_2 := \delta_2(r, \beta_2, \gamma) > 0$, $\beta_2 a^{-1} > x_0$ for $a < \delta_2$. As a result, for $a < \min(\delta_1, \delta_2)$, $\beta_1 a^{-1} < x_0 < \beta_2 a^{-1}$, and so we may choose $\beta \in (\beta_1, \beta_2)$ satisfying $x_0 = \beta a^{-1}$. Hence

$$\begin{aligned} f(x_0) &= (x_0^{-r} + a^{1+\gamma})/(x_0^{-1} + a) \\ &= (\beta^{-r}a^r + a^{1+\gamma})/(\beta^{-1}a + a) \\ &= (\beta^{-r}a^{r-1} + a^\gamma)/(\beta^{-1} + 1) \\ &\le (\beta_1^{-r}a^{r-1} + a^\gamma)/(\beta_2^{-1} + 1) \\ &\le C_1 a^{r-1} \end{aligned}$$

for a sufficiently small, say $a < \delta$ (remember, $r < 1$). From (6.5) the desired conclusion follows. □

PROOF OF THEOREM 6.1. Let T, δ, D and ρ be as in the hypotheses. Let $U \subseteq \overline{U} \subseteq G$ be an open set with C^∞ boundary where $\overline{D} \subseteq U$. Extend $h|_{\overline{U}}$ by some $\tilde{h} \in C_+^\infty(\mathbb{R}^d)$. Consider any $\varepsilon > 0$ and $f \in C_+^\infty(\mathbb{R}^d)$ with $\Delta f \le 0$ on G. Then with $g := f/\tilde{h} + \varepsilon$ and $V = -\Delta(\tilde{h}g)/2\tilde{h}g$, the Feynman–Kac formula (cf. Stroock and Varadhan [16, Problem 4.6.7, page 114]) yields

$$
\begin{aligned}
(\tilde{h}g)(x) &= E_x(\tilde{h}g)(X_t)\exp\left\{\int_0^t V(X_s)\,ds\right\} \\
&\ge \varepsilon E_x h(X_t)1_{\tau_U > t}\exp\left\{-\int_0^t [\Delta f/2(f+\varepsilon h)](X_s)1_{\tau_U > s}\,ds\right\}.
\end{aligned}
\tag{6.6}
$$

Next, choose $r \in (0,1)$ so that

$$\rho = r(1+\delta)/\delta, \tag{6.7}$$

where ρ and δ are from the hypotheses of the theorem. By Proposition 6.3, given $\gamma \in (0,1)$, for any $x \in G$ and ε small,

$$
\begin{aligned}
&h(x)/(\inf f + \varepsilon h(x)) \\
&\quad = (\inf f)^{-1}h(x)/(1 + (\varepsilon/\inf f)h(x)) \\
&\quad \le (\inf f)^{-1}C_1 h(x)^r(\varepsilon/\inf f)^{r-1}/\left(1 + [\varepsilon/\inf f]^{\gamma+1}h(x)^r\right) \\
&\quad \le C_2\varepsilon^{r-1}h(x)^r,
\end{aligned}
\tag{6.8}
$$

where C_2 is independent of ε, x and the open set U. Since $\Delta f|_G \le 0$, on G we have, for ε small,

$$
\begin{aligned}
\Delta f/2(f+\varepsilon h) &= \Delta f/2f - \varepsilon h\,\Delta f/2f(f+\varepsilon h) \\
&= \Delta f/2f + \varepsilon h|\Delta f|/2f(f+\varepsilon h) \\
&\le \sup_G \Delta f/2f + \left[\sup|\Delta f|/2f\right]\varepsilon h/(\inf f + \varepsilon h) \\
&\le \sup_G \Delta f/2f + C_3\varepsilon^r h^r \qquad [\text{by } (6.8)],
\end{aligned}
\tag{6.9}
$$

where C_3 is independent of x, ε and U. Thus for $t > 2T$ (T as in hypotheses) and ε small,

$$
\begin{aligned}
&\int_0^t [\Delta f/2(f+\varepsilon h)](X_s)1_{\tau_U > s}\,ds \\
&\quad \le \int_T^{t-T} [\Delta f/2(f+\varepsilon h)](X_s)1_{\tau_U > s}\,ds \\
&\quad \le (t-2T)\sup_G \Delta f/2f + C_3\varepsilon^r \int_T^{t-T} h(X_s)^r 1_{\tau_U > s}\,ds,
\end{aligned}
$$

where we have used $\Delta f|_G \le 0$ in the first inequality and (6.9) in the second. Using this in (6.6) gives, for $t \ge 2T$ and $x \in D \subseteq U$,

$$
\begin{aligned}
\sup_D(f+\varepsilon h) &= \sup_D \tilde{h}g \ge \tilde{h}g(x) \\
&\ge \varepsilon E_x h(X_t)1_{\tau_U > t}\exp\left\{-C_3\varepsilon^r\int_T^{t-T} h(X_s)^r 1_{\tau_U > s}\,ds\right\} \\
&\quad \times \exp\left\{-(t-2T)\sup_G \Delta f/2f\right\}.
\end{aligned}
$$

Letting $U \uparrow G$, after some rearrangement we get for $t \geq 2T$ and $x \in D$,

$$\begin{aligned} &\varepsilon^{-1}\Big[\sup_D (f + \varepsilon h)\Big]\exp\Big\{(t - 2T)\sup_G \Delta f/2f\Big\} \\ (6.10)\qquad &\geq E_x h(X_t)1_{\tau_G > t}\exp\Big\{-C_3\varepsilon^r \int_T^{t-T} h(X_s)^r 1_{\tau_G > s}\, ds\Big\} \\ &= E_x h(X_t^G)\exp\Big\{-C_3\varepsilon^r \int_T^{t-T} h(X_s^G)^r\, ds\Big\} \end{aligned}$$

(here X_t^G is Brownian motion killed at ∂G).

Now we use the tied down process $X_\eta^t(\cdot)$ to study the right-hand side of (6.10). We have by the conditional Jensen inequality

$$\begin{aligned} \mathrm{RHS}(6.10) &= E_x\Big[h(X_t^G)E_x\Big(\exp\Big\{-C_3\varepsilon^r\int_T^{t-T} h(X_s^G)^r\, ds\Big\}\Big| X_t^G\Big)\Big] \\ (6.11)\qquad &\geq E_x\Big[h(X_t^G)\exp\Big\{-C_3\varepsilon^r\int_T^{t-T} E_x\big[h(X_s^G)^r | X_t^G\big]\, ds\Big\}\Big]. \end{aligned}$$

But for $T < s < t - T$, $\eta \in G$ and $x \in D$,

$$\begin{aligned} &E_x\big(h(X_s^G)^r | X_t^G = \eta\big) \\ &\quad= \int_G h(y)^r p_\eta^t(0, x; s, y)\, dy \\ &\quad\leq \Big[\int_G h(y)^{r(1+\delta)/\delta}\, dy\Big]^{\delta/(1+\delta)}\Big[\int_G p_\eta^t(0, x; s, y)^{1+\delta}\, dy\Big]^{1/(1+\delta)} \\ &\quad\leq M^{1/(1+\delta)}\Big[\int_G h(y)^\rho\, dy\Big]^{\delta/(1+\delta)} \qquad \text{[by (6.2) and (6.7)]} \\ &\quad= M_1 < \infty \qquad [\text{since } h \in L^\rho(G, dx)]. \end{aligned}$$

Combined with (6.10) and (6.11), this yields

$$\begin{aligned} &\varepsilon^{-1}\Big[\sup_D (f + \varepsilon h)\Big]\exp\Big\{(t - 2T)\sup_G \Delta f/2f\Big\} \\ &\quad\geq E_x h(X_t^G)\exp\{-M_1C_3\varepsilon^r(t - 2T)\} \\ &\quad= E_x h(X_t)1_{\tau_G > t}\exp\{-M_1C_3\varepsilon^r(t - 2T)\}, \qquad \varepsilon \text{ small}, x \in D. \end{aligned}$$

Hence for ε small,

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t}\log \sup_{x\in D} P_x(\tau_G^h > t) \\ &\quad= \limsup_{t\to\infty} \frac{1}{t}\log \sup_{x\in D}\Big[h(x)^{-1}E_x h(X_t)1_{\tau_G > t}\Big] \\ &\quad\leq \limsup_{t\to\infty} \frac{1}{t}\Big\{\log\Big[\inf_D h\Big]^{-1} + \log\Big[\varepsilon^{-1}\sup_D (f + \varepsilon h)\Big] \\ &\qquad\qquad + (t - 2T)\sup_G \Delta f/2f + M_1C_3\varepsilon^r(t - 2T)\Big\} \\ &\quad= \sup_G \Delta f/2f + M_1C_3\varepsilon^r. \end{aligned}$$

Letting $\varepsilon \to 0$ and then taking the inf over $f \in C_+^\infty(\mathbb{R}^d)$ with $\Delta f \leq 0$ on G gives

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in D} P_x\left(\tau_G^h > t\right) \tag{6.12}$$
$$\leq \inf\left\{\sup_G \Delta f/2f\colon f \in C_+^\infty(\mathbb{R}^d),\, \Delta f \leq 0 \text{ on } G\right\}.$$

But for $L = \frac{1}{2}\Delta$, we have $l_G := \sup_{\mu(\overline{G})=1}[-\bar{I}(\mu)] < 0$, so an argument similar to that in the proof of Lemma 2.3 shows

$$l_G = \inf\left\{\sup_G \Delta f/2f\colon f \in C_+^\infty(\mathbb{R}^d),\, \Delta f \leq 0 \text{ on } G\right\}.$$

Using this in (6.12) yields (6.3), as desired. □

Notice by Remark 4.8, Theorem 4.7 is valid in the present context. Hence just as in the proof of Theorem 5.1, we may use Theorems 4.7 and 6.1 to obtain

THEOREM 6.4. *Let $G \subseteq \mathbb{R}^d$ be a bounded domain whose boundary satisfies an exterior cone condition at every point. Suppose the hypothesis* (6.2) *holds for any open $D \subseteq \overline{D} \subseteq G$. If $h > 0$ is harmonic for $\frac{1}{2}\Delta$ on G and $h \in L^p(G, dx)$ for some $p > 0$, then*

$$\lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in D} P_x\left(\tau_G^h > t\right) = \sup_{\mu(\overline{G})=1} \left[-\bar{I}(\mu)\right]$$
$$= \lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in D} P_x\left(\tau_G > t\right).$$

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