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FLUCTUATION RESULTS FOR THE WIENER SAUSAGE

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Some fluctuation results are proved for the volume of the Wiener sausage associated with a d-dimensional Brownian motion and a compact set of positive capacity. In high dimensions, the limiting distribution is normal, whereas, if $d = 2$, it is that of a renormalized local time of self-intersections of planar Brownian motion. For $d = 2$ or 3, these limit theorems are closely linked with the renormalization results for self-intersections of Brownian paths.

1. Introduction. Let $B = (B_t, t \geq 0)$ denote a Brownian motion with values in $\mathbb{R}^d$, $d \geq 2$, and let K be a compact set in $\mathbb{R}^d$. The Wiener sausage associated with B and the compact set K, on the time interval $[0; t]$, is defined by

$$S^K(0;t) = \bigcup_{s \leq t} (B_s + K)$$
$$= \left\{ y \in \mathbb{R}^d;\ y = B_s + a \text{ for some } s \leq t, a \in K \right\}.$$

Let m denote the Lebesgue measure on $\mathbb{R}^d$. If $d \geq 3$, Kesten, Spitzer and Whitman (see, e.g., [10], page 252) proved that

$$\lim_{t\to\infty} \frac{1}{t} m\big(S^K(0;t)\big) = C(K), \tag{1.a}$$

where $C(K)$ denotes the (Newtonian) capacity of K in $\mathbb{R}^d$, and convergence holds a.s. and in the L^p-norm ($p < \infty$). In the case $d = 2$, the following result holds, under the assumption that K has positive logarithmic capacity:

$$\lim_{t\to\infty} \frac{\log t}{t} m\big(S^K(0;t)\big) = 2\pi, \tag{1.b}$$

where convergence again holds a.s. and in the L^p-norm.

Note that the limit in (1.b) does not depend on the compact set K. Results (1.a) and (1.b) represent the law of large numbers for the volume of the Wiener sausage. Our goal here is to investigate the corresponding central limit theorems. We will also emphasize the fact that, for $d = 2$ or 3, these limit theorems are closely related to some renormalization results for the self-intersections of Brownian paths, which are essentially due to Varadhan [27] ($d = 2$) and Yor [29] ($d = 3$). Heuristically, this relationship can be explained by the trivial observation that, if there are many self-intersections, then the volume of the Wiener sausage will be small. In higher dimensions ($d \geq 4$), there are no self-intersections and a result similar to the usual central limit theorem for independent

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variables is expected to hold for the fluctuation of the Wiener sausage. We obtain the following results (see Theorems 2.1, 3.1 and 4.1 for more precise statements): If $d = 2$,

$$\text{(1.c)}\qquad \lim_{t\to\infty} \frac{(\log t)^2}{t}\left(m(S^K(0;t)) - 2\pi\frac{t}{\log t}\right) = 2\pi(1 + C - \log 2 + R(K)) - 4\pi^2\gamma,$$

where convergence holds in distribution, C is Euler's constant and $R(K)$ is the logarithm of the logarithmic capacity of K, and γ is the random variable formally defined by

$$\gamma = \iint_{(0\le s<t\le 1)} \left(\delta_{(0)}(B_s - B_t) - E\left[\delta_{(0)}(B_s - B_t)\right]\right) ds\, dt,$$

where $\delta_{(0)}$ denotes the Dirac measure at 0. If $d = 3$,

$$\text{(1.d)}\qquad \lim_{t\to\infty} (t\log t)^{-1/2}\left(m(S^K(0;t)) - C(K)t\right) = \frac{2^{-1/2}}{\pi}C(K)^2 N;$$

and, if $d \ge 4$,

$$\text{(1.e)}\qquad \lim_{t\to\infty} t^{-1/2}\left(m(S^K(0;t)) - C(K)t\right) = A(K)N,$$

where in both cases convergence holds in distribution, N denotes a standard normal variable and, for $d \ge 4$, $A(K)$ is some positive constant depending on K.

In lower dimensions ($d = 2$ or 3), it is interesting to compare our results with the asymptotic developments of $E(m(S^K(0;t)))$, which were obtained by Spitzer [25],

$$\text{(1.f)}\qquad E\left[m(S^K(0;t))\right] = 2\pi\frac{t}{\log t} + 2\pi(1 + C - \log 2 + R(K))\frac{t}{(\log t)^2} + o\left(\frac{t}{(\log t)^2}\right), \quad \text{if } d = 2,$$

$$\text{(1.g)}\qquad E\left[m(S^K(0;t))\right] = C(K)t + 4(2\pi)^{-3/2}C(K)^2 t^{1/2} + o(t^{1/2}), \quad \text{if } d = 3.$$

In particular, (1.f) may be considered as a special case of (1.c), once we have noted that the convergence in distribution in (1.c) can be turned into a convergence in the L^2-norm through a change of scale. In fact, we shall use (1.f) to prove (1.c). Note also that Spitzer's result (1.f) suggests the correct normalization factor for the fluctuation of the Wiener sausage if $d = 2$, but this is no longer true if $d = 3$.

Let us now state the renormalization results for self-intersections of Brownian paths from which we will deduce (1.c) and (1.d). It is well known that a Brownian path in $\mathbb{R}^d$ will intersect itself if and only if $d \le 3$. We now assume that $d = 2$ or 3. The local time of self-intersections of B, on the time interval $[0,1]$, is the Radon measure on $\mathcal{T} = \{(s,t);\ 0 \le s < t \le 1\}$ formally defined by

$$\alpha(A) = \iint_A \delta_{(0)}(B_s - B_t)\, ds\, dt,$$

where $\delta_{(0)}$ denotes the Dirac measure at 0 and A is any Borel subset of $\mathcal{T}$. We refer to Dynkin [4], Rosen [8] or Le Gall [13] for a precise definition of α. It turns out that $\alpha(\mathcal{T}) = \infty$, a.s. Heuristically, this corresponds to the fact that there are too many self-intersections near the diagonal $\{s = t\}$. If $\mathcal{T}_\varepsilon$ is the compact subset of $\mathcal{T}$ defined by $\mathcal{T}_\varepsilon = \{(s,t) \in \mathcal{T};\, t - s \geq \varepsilon\}$, it follows that

$$\lim_{\varepsilon \to 0} \alpha(\mathcal{T}_\varepsilon) = \infty, \quad \text{a.s.}$$

This limiting behavior is better understood if we compute

$$E[\alpha(\mathcal{T}_\varepsilon)] = \iint_{\mathcal{T}_\varepsilon} (2\pi(t-s))^{-d/2}\, ds\, dt = \frac{1}{2\pi}\left(\log\frac{1}{\varepsilon} - 1 + \varepsilon\right), \quad \text{if } d = 2,$$

$$= 2(2\pi)^{-3/2}(\varepsilon^{-1/2} - 2 + \varepsilon^{1/2}), \qquad \text{if } d = 3.$$

The renormalization results give some information on the asymptotic behavior of $\alpha(\mathcal{T}_\varepsilon) - E[\alpha(\mathcal{T}_\varepsilon)]$,

$$\text{(1.h)} \qquad \lim_{\varepsilon \to 0}\left(\alpha(\mathcal{T}_\varepsilon) - \frac{1}{2\pi}\log\frac{1}{\varepsilon}\right) = \gamma - \frac{1}{2\pi}, \quad \text{if } d = 2,$$

$$\text{(1.i)} \qquad \lim_{\varepsilon \to 0}\left(\log\frac{1}{\varepsilon}\right)^{-1/2}\left(\alpha(\mathcal{T}_\varepsilon) - 2(2\pi)^{-3/2}\varepsilon^{-1/2}\right) = \frac{2^{-1/2}}{\pi}N, \quad \text{if } d = 3,$$

where convergence holds in the L^2-norm for (1.h), in distribution for (1.i). The limit result (1.h) was obtained, in a slightly different context, by Varadhan [27], whereas (1.i) is due to Yor [29].

Different approaches to (1.h) have been proposed recently by Dynkin [4], Rosen [19], Yor [30] and Le Gall [13]. Note that (1.h) may be considered as a definition of the limit variable γ. The connection with our previous formal definition becomes obvious if we replace (1.h) by the equivalent statement

$$\text{(1.h}'\text{)} \qquad \lim_{\varepsilon \to 0}\left(\alpha(\mathcal{T}_\varepsilon) - E(\alpha(\mathcal{T}_\varepsilon))\right) = \gamma.$$

Let us now describe the relationship between (1.h) and (1.i) and our results concerning the fluctuation of the Wiener sausage in $\mathbb{R}^2$ or $\mathbb{R}^3$. First, we need to introduce the Wiener sausage of small radius ε associated with B and the compact set K on the time interval $[0;1]$,

$$S_\varepsilon^K = \bigcup_{s \leq 1}(B_s + \varepsilon K)$$

$$= \left\{y \in R^d;\, y = B_s + \varepsilon a \text{ for some } s \leq 1,\, a \in K\right\}.$$

An obvious change of scale shows that (1.c) and (1.d) are, respectively, equivalent to

$$\text{(1.c}'\text{)} \qquad \lim_{\varepsilon \to 0}\left(\log\frac{1}{\varepsilon}\right)^2\left(m(S_\varepsilon^K) - \pi\left(\log\frac{1}{\varepsilon}\right)^{-1}\right)$$

$$= \frac{\pi}{2}(1 + C - \log 2 + R(K)) - \pi^2\gamma, \quad \text{if } d = 2,$$

$$\text{(1.d}'\text{)} \qquad \lim_{\varepsilon \to 0}\varepsilon^{-2}\left(\log\frac{1}{\varepsilon}\right)^{-1/2}\left(m(S_\varepsilon^K) - \varepsilon C(K)\right) = \pi^{-1}C(K)^2 N, \quad \text{if } d = 3.$$

In order to study the asymptotic behavior of $m(S_\varepsilon^K)$, we will use the following device. We first write S_ε^K as the union of $S_\varepsilon^K(0; \frac{1}{2})$ and $S_\varepsilon^K(\frac{1}{2}; 1)$, where $S_\varepsilon^K(u; v)$ denotes the Wiener sausage of radius ε on the time interval $[u; v]$. This leads to the identity

$$\{m(S_\varepsilon^K)\} = \{m(S_\varepsilon^K(0; \tfrac{1}{2}))\} + \{m(S_\varepsilon^K(\tfrac{1}{2}; 1))\} - \{m(S_\varepsilon^K(0; \tfrac{1}{2}) \cap S_\varepsilon^K(\tfrac{1}{2}; 1))\},$$

where we use the notation $\{U\} = U - E[U]$ (this notation will be frequently used in the sequel). Then we divide $S_\varepsilon^K(0; \frac{1}{2})$, resp. $S_\varepsilon^K(\frac{1}{2}; 1)$, into two pieces according to the same device, but we do not modify the intersection term. After n steps, we obtain the formula

$$\text{(1.j)}\quad \begin{aligned}\{m(S_\varepsilon^K)\} &= \sum_{k=1}^{2^n} \left\{ m\left(S_\varepsilon^K\left(\frac{k-1}{2^n}; \frac{k}{2^n} \right)\right)\right\} \\ &- \sum_{p=1}^{n} \sum_{q=1}^{2^{p-1}} \left\{ m\left(S_\varepsilon^K\left(\frac{2q-2}{2^p}; \frac{2q-1}{2^p} \right) \cap S_\varepsilon^K\left(\frac{2q-1}{2^p}; \frac{2q}{2^p} \right)\right)\right\}.\end{aligned}$$

Now the key observation is that we can choose n large so that the first term of the right-hand side of (1.j) is negligible in comparison with the second term. More precisely, this is true if n is large, independently of ε, when $d = 2$, if n is of the same order of magnitude as $\log(1/\varepsilon)$ when $d = 3$. Thus we have reduced the asymptotic behavior of $\{m(S_\varepsilon^K)\}$ to that of a sum of intersection terms. Because of the independence of Brownian increments, each of these terms can be interpreted as the volume of the intersection of two independent Wiener sausages. More precisely, take $t > 0$ and $h \le t$. Then $(B_s' \equiv B_{t-s} - B_t;\ 0 \le s \le h)$ and $(B_s'' \equiv B_{t+s} - B_t;\ 0 \le s \le h)$ are two independent Brownian motions starting from 0. Using obvious notation, we have

$$m(S_\varepsilon'^K(0; h) \cap S_\varepsilon''^K(0; h)) = m(S_\varepsilon^K(t-h; t) \cap S_\varepsilon^K(t; t+h)).$$

Moreover, the intersection local time of B' and B'', as defined in [7], is related to α by the equation

$$\int_0^h \int_0^h \delta_{(0)}(B_s' - B_t'')\, ds\, dt = \int_{t-h}^{t} \int_t^{t+h} \delta_{(0)}(B_u - B_v)\, du\, dv = \alpha([t-h; t) \times (t; t+h]).$$

Thus, at least when K is the unit ball, we can use Corollary 3.2 of [14] to obtain, for any p, q,

$$\text{(1.k)}\quad \begin{aligned}&\lim_{\varepsilon\to 0} \left(\log\frac{1}{\varepsilon}\right)^2 m\left(S_\varepsilon^K\left(\frac{2q-2}{2^p}; \frac{2q-1}{2^p} \right) \cap S_\varepsilon^K\left(\frac{2q-1}{2^p}; \frac{2q}{2^p} \right)\right) \\ &= \pi^2 \alpha\left(\left[\frac{2q-2}{2^p}; \frac{2q-1}{2^p} \right) \times \left(\frac{2q-1}{2^p}; \frac{2q}{2^q} \right] \right), \quad \text{if } d = 2,\end{aligned}$$

$$\text{(1.l)}\quad \begin{aligned}&\lim_{\varepsilon\to 0} \varepsilon^{-2} m\left(S_\varepsilon^K\left(\frac{2q-2}{2^p}; \frac{2q-1}{2^p} \right) \cap S_\varepsilon^K\left(\frac{2q-1}{2^p}; \frac{2q}{2^p} \right)\right) \\ &= C(K)^2 \alpha\left(\left[\frac{2q-2}{2^p}; \frac{2q-1}{2^p} \right) \times \left(\frac{2q-1}{2^p}; \frac{2q}{2^p} \right] \right), \quad \text{if } d = 3,\end{aligned}$$

where in both cases convergence holds in the L^2-norm. These results are proved in [14], in the special case when K is the unit ball. We shall see that they extend to the case of a general compact set K. Using (1.k), (1.l) and the remarks following (1.j), the asymptotic behavior of $\{m(S_\varepsilon^K)\}$ is reduced to that of $\alpha(\mathscr{T}_{(n)})$, where

$$\mathscr{T}_{(n)} = \bigcup_{p=1}^{n} \bigcup_{q=1}^{2^{p-1}} \left[\frac{2q-2}{2^p}; \frac{2q-1}{2^p}\right) \times \left(\frac{2q-1}{2^p}; \frac{2q}{2^p}\right], \tag{1.m}$$

and n is large independently of ε if $d = 2$, and $n \simeq \log(1/\varepsilon)$ if $d = 3$. The justification of this replacement is easy if $d = 2$, but for $d = 3$ we shall need a precise estimate of the rate of convergence in (1.l) (Corollary 3.4). This estimate makes it possible to improve some of the results in [14]. In particular, it shows that the convergence (1.l) holds a.s. in the special case when K is the unit ball.

Once we have justified the replacement of $\{m(S_\varepsilon^K)\}$ by $\{\alpha(\mathscr{T}_{(n)})\}$, we may use the "renormalization" results (1.h) and (1.i) to prove (1.c′) and (1.d′). In fact, it is not hard to see that $\{\alpha(\mathscr{T}_{(n)})\}$ has the same asymptotic behavior as $\{\alpha(\mathscr{T}_{2^{-n}})\}$.

The preceding method does not apply if $d \geq 4$ since the intersection local time does not exist. In this case, the idea of the proof is as follows. Starting again from (1.j), we note that we may choose n large, but not too large, so that the intersection terms of the second member are negligible in comparison with the term

$$\sum_{k=1}^{2^n} \left\{ m\left(S_\varepsilon^K\left(\frac{k-1}{2^n}; \frac{k}{2^n} \right)\right)\right\}.$$

But this term is nothing but the sum of 2^n independent identically distributed random variables with zero expectation. Thus we can use the usual central limit theorem for triangular arrays to study its asymptotic behavior. The only delicate point of the proof is the estimation of the variance of $m(S_\varepsilon^K)$.

Let us point out an important difference between results (1.c) and (1.d) on one hand, and (1.e) on the other hand. The limit random variables γ in (1.c) and N in (1.d) are the same for all compact sets K, in the sense that, if $K_1, \dots, K_p$ are p compact sets, we can state a limit theorem for the joint distribution of $\{m(S_{K_i}(0, t))\}$, $i = 1, \dots, p$, which involves only one limit variable, γ if $d = 2$ or N if $d = 3$. This remark is an obvious consequence of the relationship between (1.c) and (1.h) on one hand, and (1.d) and (1.i) on the other hand. Although our method does not provide such a result, we believe that, if $d \geq 4$, the limit variables corresponding to different compact sets are not the same. This observation is made plausible by the results obtained by Yor and Calais [31] on a slightly different problem (see the end of Section 4).

There is a close connection between our results and some limit theorems for the range of a random walk; it was noticed by Spitzer [25] that the discrete analog of the Wiener sausage is simply the range of a random walk. Thus it is worth comparing our results (1.c)–(1.e) with the analogous results for random walks, which were proved by Jain and Pruitt [11] ($d \geq 3$) and Le Gall [15] ($d = 2$). Let $X = (X_n, n \geq 0)$ denote a random walk with values in $\mathbb{Z}^d$. Assume that X has zero mean and finite second moments, and that X is adapted. Let R_n denote the number of distinct points visited by X before time n. Then

$$\text{(1.n)} \qquad \lim_{n\to\infty} \frac{(\log n)^2}{n}\{R_n\} = -C_X\gamma, \quad \text{if } d = 2,$$

$$\text{(1.o)} \qquad \lim_{n\to\infty} (n \log n)^{-1/2}\{R_n\} = C_X N, \quad \text{if } d = 3,$$

$$\text{(1.p)} \qquad \lim_{n\to\infty} n^{-1/2}\{R_n\} = C_X N \quad \text{if } d \geq 4,$$

where convergence holds in distribution, C_X is some constant depending on X and the limit variables γ and N have the same meaning as previously stated. In particular, the close connection between (1.d) and (1.i) provides some explanation for the extra factor $(\log n)^{-1/2}$ in (1.o).

Let us now mention various results about the Wiener sausage, which have been proved since the pioneering work of Kesten, Spitzer and Whitman. Some large deviations results for the volume of the Wiener sausage have been established by Donsker and Varadhan [1]; it is worth noting that similar results for the range of a random walk have also been proved by the same authors [2]. Asymptotic theorems for the Wiener sausage have often been motivated by physical problems (see, e.g., [12] and [23]). In particular, Kac and Luttinger [12] use both the law of large numbers for $m(S^K(0, t))$ and Donsker and Varadhan's result, which was only a conjecture at that time. In this connection, Eisele and Lang [6] have recently extended Donsker and Varadhan's result to the case of the Wiener sausage with drift. Sznitman [26] considers the Wiener sausage associated with an elliptic diffusion and proves analogs of (1.a) and (1.b). Sznitman's results suggest that an analog of (1.c′) and (1.d′) also holds for general elliptic diffusions. Note that the local time of self-intersections for smooth elliptic diffusions in $\mathbb{R}^2$ or $\mathbb{R}^3$ has been studied by Rosen [21]. In the Brownian case, Weinryb [28] obtains interesting extensions of (1.k) and (1.l) by replacing the Lebesgue measure m by an arbitrary measure satisfying some integrability conditions. Some extensions of the previous results may also hold for general Lévy processes; see Hawkes [9] for an extension of (1.a) and Shieh [22] for a construction of the intersection local time of Lévy processes.

In Section 2 we investigate the case $d = 2$. The arguments are similar to those given in [13] for the special case when K is the unit disk. However, details are provided for the sake of completeness. Section 3 is devoted to the case $d = 3$. The main ingredient is the technical estimate of Lemma 3.3. We also give an elementary proof of Yor's renormalization result (1.i) (Yor's original method was based on his Tanaka formula for the intersection local time.) Finally, Section 4 deals with the case $d \geq 4$ and contains the proof of (1.e).

2. The two-dimensional case. Throughout this section we assume that B is a two-dimensional Brownian motion starting from 0, and that K is a compact set of the plane of positive logarithmic capacity. We will use the same normalization of logarithmic capacity as Spitzer [25], so that the capacity of the unit disk is 1. We denote by $R(K)$ the logarithm of the logarithmic capacity of K. Our goal in this section is to prove the following theorem.

THEOREM 2.1. *Let S_ε^K denote the Wiener sausage of radius ε associated with K on the time interval $[0,1]$. Then*

$$\lim_{\varepsilon\to 0}\left(\log\frac{1}{\varepsilon}\right)\left(\left(\log\frac{1}{\varepsilon}\right)m(S_\varepsilon^K)-\pi\right)=\frac{\pi}{2}(1+C-\log 2+R(K))-\pi^2\gamma,$$

where convergence holds in the L^2-norm. Here C is Euler's constant and γ is the renormalized local time of self-intersections of B, formally defined by

$$\gamma=\iint_{0\le s<t\le 1}\left(\delta_{(0)}(B_s-B_t)-E\left[\delta_{(0)}(B_s-B_t)\right]\right)ds\,dt.$$

The simplest way to define the limit variable γ is to replace the Dirac measure by an approximating sequence $\delta_{(n)}$ and to take a limit as n tends to ∞. This approach was used by Varadhan [27]. One can prove (see [13]) that the limit does not depend on the approximating sequence. In the course of the proof of Theorem 2.1, we shall give an alternative construction of γ.

Before we proceed to the proof of Theorem 2.1, we will state a few preliminary results. Without loss of generality, we may and will assume that K is contained in the closed unit disk D. Our first lemma shows that, for any $y\neq 0$, $P[y\in S_\varepsilon^K]$ is of the same order of magnitude as $P[y\in S_\varepsilon^D]$. Let us recall the following result due to Spitzer [24]: For any $y\neq 0$, $t\ge 0$,

$$\lim_{\varepsilon\to 0}\left(\log\frac{1}{\varepsilon}\right)P\left[y\in S_\varepsilon^D(0,t)\right]=\pi\int_0^t p_s(0,y)\,ds, \tag{2.a}$$

where $p_s(0,y)=(2\pi s)^{-1}\exp(-|y|^2/2s)$ is the Gaussian density. Moreover, according to [14], there exists a constant c such that, for any $y\neq 0$, $\varepsilon\in(0;\frac{1}{2})$,

$$\left(\log\frac{1}{\varepsilon}\right)P\left[y\in S_\varepsilon^D\right]\le c\left(\left(\log\frac{1}{|y|}\right)\vee\exp\left(-\frac{|y|^2}{16}\right)\right). \tag{2.a$'$}$$

LEMMA 2.2. *For any $y\in\mathbb{R}^2$, $y\neq 0$,*

$$\lim_{\varepsilon\to 0}\left(\log\frac{1}{\varepsilon}\right)\left(P\left[y\in S_\varepsilon^D\right]-P\left[y\in S_\varepsilon^K\right]\right)=0.$$

REMARK. We could improve on the result of Lemma 2.2 by showing the convergence of

$$\left(\log\frac{1}{\varepsilon}\right)^2\left(P\left[y\in S_\varepsilon^D\right]-P\left[y\in S_\varepsilon^K\right]\right).$$

That kind of result would lead to a proof of Spitzer's result (1.d).

PROOF. For any $\varepsilon>0$, set

$$T_\varepsilon(y)=\inf\{t\ge 0,\ B_t\in y-\varepsilon D\},$$
$$T_\varepsilon^K(y)=\inf\{t\ge 0;\ B_t\in y-\varepsilon K\}$$

($y - \varepsilon K$ is the set $\{y - \varepsilon z;\ z \in K\}$). Note that $T_\varepsilon(y) \le T_\varepsilon^K(y)$ and that

$$P[y \in S_\varepsilon^D] - P[y \in S_\varepsilon^K] = P[T_\varepsilon(y) \le 1 < T_\varepsilon^K(y)].$$

Moreover,

$$P[T_\varepsilon(y) \le 1 < T_\varepsilon^K(y)] \le P[1 - \delta \le T_\varepsilon(y) \le 1] + P[T_\varepsilon(y) \le 1 - \delta < 1 \le T_\varepsilon^K(y)].$$

Now it is obvious from (2.a) that

$$\lim_{\delta \to 0}\left(\limsup_{\varepsilon \to 0}\left(\log\frac{1}{\varepsilon}\right)P[1 - \delta \le T_\varepsilon(y) \le 1]\right) = 0.$$

Thus we need only verify that, for any $\delta > 0$,

$$\lim_{\varepsilon \to 0}\left(\log\frac{1}{\varepsilon}\right)P[T_\varepsilon(y) \le 1 - \delta < 1 \le T_\varepsilon^K(y)] = 0. \tag{2.b}$$

We now apply the Markov property at time $T_\varepsilon(y)$ and denote by P_z the law of B starting from z. Using (2.a) again, we get, for some constant $C(y)$,

$$\begin{aligned}\left(\log\frac{1}{\varepsilon}\right)&P[T_\varepsilon(y) \le 1 - \delta < 1 \le T_\varepsilon^K(y)]\\ &\le C(y)\sup_{z \in y - \varepsilon D}\left(P_z[T_\varepsilon^K(y) > \delta]\right)\\ &\le C(y)\sup_{z \in D}\left(P_z\left[T_1^K(0) > \frac{\delta}{\varepsilon^2}\right]\right),\end{aligned} \tag{2.c}$$

where the last bound follows from an obvious change of scale. Now, since K has positive capacity, it is clear that

$$\lim_{t \to \infty}\left(\sup_{z \in D} P_z[T_1^K(0) > t]\right) = 0. \tag{2.d}$$

Putting (2.c) and (2.d) together, we get (2.b). □

Corollary 2.3. *Let B' denote another Brownian motion, independent of B, and let $S_\varepsilon'^K$ denote the corresponding Wiener sausage. Then*

$$\lim_{\varepsilon \to 0}\left(\log\frac{1}{\varepsilon}\right)^2 m(S_\varepsilon^K \cap S_\varepsilon'^K) = \pi^2\beta([0;1]^2),$$

where convergence holds in the L^2-norm and $\beta([0;1]^2)$ is the intersection local time of B and B' on $[0;1]^2$, formally defined by

$$\beta([0;1]^2) = \int_0^1\int_0^1 \delta_{(0)}(B_s - B_t')\,ds\,dt$$

(*see* [3] *or* [7] *for a precise definition of* β).

Proof. The result of the corollary is known to hold in the special case $K = D$ (see [14]). In order to extend it to a general K, it suffices to prove that

$$\lim_{\varepsilon \to 0}\left(\log\frac{1}{\varepsilon}\right)^2 E[m(S_\varepsilon^D \cap S_\varepsilon'^D) - m(S_\varepsilon^K \cap S_\varepsilon'^K)] = 0. \tag{2.e}$$

The proof of (2.e) is easy. We first write

$$E\left[m\left(S_\varepsilon^D \cap S_\varepsilon'^D\right) - m\left(S_\varepsilon^K \cap S_\varepsilon'^K\right)\right]$$
$$= \int dy \left(P\left[y \in S_\varepsilon^D\right] P\left[y \in S_\varepsilon'^D\right] - P\left[y \in S_\varepsilon^K\right] P\left[y \in S_\varepsilon'^K\right]\right).$$

Then we use Lemma 2.2 together with (2.a′), which allows us to apply the Lebesgue dominated convergence theorem. □

LEMMA 2.4. *There exists a constant C_1 such that, for any $0 < \varepsilon < \frac{1}{2}$,*

$$E\left[\left(m\left(S_\varepsilon^K\right) - E\left(m\left(S_\varepsilon^K\right)\right)\right)^2\right] \le C_1 \left(\log \frac{1}{\varepsilon}\right)^{-4}.$$

PROOF. Let

$$F(\varepsilon) = \left(\log \frac{1}{\varepsilon}\right)^2 E\left(\left(m\left(S_\varepsilon^K\right) - E\left(m\left(S_\varepsilon^K\right)\right)\right)^2\right)^{1/2}.$$

We aim to prove that F is bounded. We start from the trivial identity

$$m\left(S_\varepsilon^K\right) = m\left(S_\varepsilon^K(0; \tfrac{1}{2})\right) + m\left(S_\varepsilon^K(\tfrac{1}{2}; 1)\right) - m\left(S_\varepsilon^K(0; \tfrac{1}{2}) \cap S_\varepsilon^K(\tfrac{1}{2}; 1)\right). \tag{2.f}$$

Note that $m(S_\varepsilon^K(0; \frac{1}{2}))$ and $m(S_\varepsilon^K(\frac{1}{2}; 1))$ are independent random variables. Using (2.f) together with a suitable change of scale and Corollary 2.3, it follows that, for any $\rho > 2^{-1/2}$ and any ε sufficiently small,

$$F(\varepsilon) \le \rho F(2^{1/2}\varepsilon) + \bar{C}, \tag{2.g}$$

where $\bar{C}$ is some constant not depending on ε. Taking $\rho < 1$, we obtain that F is bounded when ε is small. □

PROOF OF THEOREM 2.1. We will use the notation $\{U\} = U - E[U]$. We start from formula (1.j) and we first notice that, for $n \ge 1$ and $\varepsilon \le 2^{-n}$,

$$E\left[\left\{\sum_{k=1}^{2^n} m\left(S_\varepsilon^K\left(\frac{k-1}{2^n}; \frac{k}{2^n}\right)\right)\right\}^2\right] = 2^n E\left[\left\{m\left(S_\varepsilon^K\left(0; \frac{1}{2^n}\right)\right)\right\}^2\right]$$
$$= 2^n \cdot 2^{-2n} E\left[\left\{m\left(S_{\varepsilon 2^{n/2}}^K(0; 1)\right)\right\}^2\right]$$
$$\le C_1 2^{-n} \left(\log \frac{1}{\varepsilon 2^{n/2}}\right)^{-4},$$

where the last bound follows from Lemma 2.4. Thus we see that, if n is large enough and if $\varepsilon \le 2^{-n}$, the L^2-norm of

$$\left(\log \frac{1}{\varepsilon}\right)^2 \left\{\sum_{k=1}^{2^n} m\left(S_\varepsilon^K\left(\frac{k-1}{2^n}; \frac{k}{2^n}\right)\right)\right\}$$

will be small. Then we fix n and apply Corollary 2.3, which implies that

$$\lim_{\varepsilon\to 0}\left(\log\frac{1}{\varepsilon}\right)^2\left\{\sum_{p=1}^{n}\sum_{q=1}^{2^{p-1}} m\left(S_\varepsilon^K\left(\frac{2q-2}{2^p};\frac{2q-1}{2^p}\right)\cap S_\varepsilon^K\left(\frac{2q-1}{2^p};\frac{2q}{2^p}\right)\right)\right\}$$
$$= \pi^2\{\alpha(\mathscr{T}_{(n)})\},$$

where convergence holds in the L^2-norm and α and $\mathscr{T}_{(n)}$ are defined in the Introduction. Then, if n is large, $\{\alpha(\mathscr{T}_{(n)})\}$ is close to γ in the L^2-norm. In fact, it is extremely easy to prove the L^2-convergence of the sequence $\{\alpha(\mathscr{T}_{(n)})\}$, and γ may be defined as the limit of this sequence (see [13]). The preceding remarks together with formula (1.j) imply that

$$\text{(2.h)}\qquad \lim_{\varepsilon\to 0}\left(\log\frac{1}{\varepsilon}\right)^2\{m(S_\varepsilon^K)\} = -\pi^2\gamma,$$

with convergence in the L^2-norm. On the other hand, Spitzer's result (1.f) and a suitable change of scale yield that

$$\text{(2.i)}\qquad \lim_{\varepsilon\to 0}\left(\log\frac{1}{\varepsilon}\right)E\left(\left(\log\frac{1}{\varepsilon}\right)m(S_\varepsilon^K) - \pi\right) = \frac{\pi}{2}(1 + C - \log 2 + R(K)).$$

Theorem 2.1 follows from (2.h) and (2.i). □

REMARKS. (i) Theorem 2.1 yields a simple proof of the almost sure convergence in (1.b). Taking $a_n = \exp(n^{3/4})$, we note that the series

$$\sum_n E\left(\left(\left(\frac{\log a_n}{a_n}\right)m(S^K(0;a_n)) - 2\pi\right)^2\right)$$

converges, and this is enough to get (1.b), since $m(S^K(0;t))$ increases with t, and $\lim(a_{n+1}/a_n) = 1$.

(ii) Dynkin [5] has shown recently how to define a renormalized local time for k-multiple intersections, for any $k \geq 2$. Note that γ corresponds to the case $k = 2$. Dynkin's results can be used [17] to obtain higher asymptotic expansions for $m(S_\varepsilon^K(0;1))$, thus improving the statement of Theorem 2.1. The expansion at the order $(\log(1/\varepsilon))^{-k}$ involves the renormalized local times for p-multiple intersections, for $p = 1, 2, \ldots, k$ (Theorem 2.1 is the case $k = 2$).

3. The three-dimensional case. We now assume that B is a three-dimensional Brownian motion and that K is a compact subset of $\mathbb{R}^3$ with positive capacity $C(K)$. We use the same notation as in the Introduction. In particular, $\alpha(\cdot)$ denotes the local time of self-intersections of B, formally defined by

$$\alpha(F) = \int\int_F \delta_{(0)}(B_s - B_t)\,ds\,dt,$$

for any Borel subset F of $\mathscr{T} = \{(s,t);\ 0 \le s < t \le 1\}$. For any $\varepsilon > 0$, $\mathscr{T}_\varepsilon$ is the set $\{(s,t) \in \mathscr{T};\ t - s \ge \varepsilon\}$.

THEOREM 3.1. *There exists a constant C_2 such that, for any $\varepsilon \in (0, \frac{1}{2})$,*

$$E\Big(\big(\varepsilon^{-2}\{m(S_\varepsilon^K)\} - C(K)^2\{\alpha(\mathscr{T}_{\varepsilon^2})\}\big)^2\Big) \le C_2. \tag{3.a}$$

In particular,

$$\lim_{\varepsilon\to 0} \varepsilon^{-2}\left(\log\frac{1}{\varepsilon}\right)^{-1/2}\big(m(S_\varepsilon^K) - \varepsilon C(K)\big) = \frac{1}{\pi}C(K)^2 N, \tag{3.b}$$

where convergence holds in distribution and N denotes a standard normal variable.

Before proving Theorem 3.1, we need to establish a few technical estimates. Without loss of generality, we may and will assume that $B_0 = 0$ and that K is contained in the closed unit ball, which we denote by D. We shall sometimes use the probabilities P_y, $y \in \mathbb{R}^3$; under P_y, B is a Brownian motion starting from y. For $y \in \mathbb{R}^3$, $\varepsilon > 0$, we set

$$T_\varepsilon^K(y) = \inf\{t \ge 0;\ B_t \in y - \varepsilon K\},$$
$$L_\varepsilon^K(y) = \sup\{t > 0;\ B_t \in y - \varepsilon K\},$$

where $\sup \varnothing = 0$. Our estimates will involve the function ψ defined by

$$\psi(r) = r^{-1}1_{\{r<1\}} + \exp(-r^2/16), \qquad r > 0.$$

If $p_s(y,z)$ is now the three-dimensional Brownian transition density, we have, for some constant c,

$$\int_0^1 p_s(y,z)\,ds \le c\psi(|z-y|).$$

LEMMA 3.2. (i) *Assume that $|y| > \varepsilon > 0$. Then, for any $t \ge 0$,*

$$P\big[T_\varepsilon^D(y) \le t\big] = \frac{\varepsilon(|y| - \varepsilon)}{|y|}\int_0^t (2\pi s^3)^{-1/2}\exp\left(-\frac{(|y|-\varepsilon)^2}{2s}\right)ds. \tag{3.c}$$

Moreover, we may choose a constant C_3 such that, for any $y, z \in \mathbb{R}^3$, $\varepsilon, \varepsilon' \in (0;1)$,

$$\varepsilon^{-1}P\big[T_\varepsilon^D(y) \le 1\big] \le C_3\psi(|y|), \tag{3.d}$$

$$\begin{aligned}&(\varepsilon\varepsilon')^{-1}P\big[T_\varepsilon^D(y) \le 1;\ T_{\varepsilon'}^D(z) \le 1\big]\\ &\quad\le C_3\big(\psi(|y|) + \psi(|z|)\big)\psi\left(\frac{|z-y|}{2}\right).\end{aligned} \tag{3.d$'$}$$

(ii) *Let e_K denote the equilibrium measure of K. Then, for any $y \in \mathbb{R}^3$, $\varepsilon > 0$, $t \ge 0$,*

$$P\big[0 < L_\varepsilon^K(y) \le t\big] = \varepsilon\int e_K(dw)\int_0^t p_s(0, y - \varepsilon w)\,ds. \tag{3.e}$$

In particular, we may choose a constant C_4 *such that, for any* $y, z \in \mathbb{R}^3$, $\varepsilon, \varepsilon' \in (0;1)$ *with* $|y| \geq 2(\varepsilon + \varepsilon')$, $|z| \leq \varepsilon'$,

$$\text{(3.f)} \quad \left|\varepsilon^{-1} P_z\left[0 < L_\varepsilon^K(y) \leq t\right] - C(K)\int_0^t p_s(0, y)\,ds\right| \leq C_4(\varepsilon + \varepsilon')\psi\left(\frac{|y|}{2}\right)^2.$$

PROOF. Formulas (3.c) and (3.e) are well known (see, e.g., [10], pages 247–250, for the definition and main properties of the equilibrium measure). The bound (3.d) is easily derived from (3.c). In order to prove (3.d′), we note that it suffices to bound $P[T_\varepsilon^D(y) \leq T_{\varepsilon'}^D(z) \leq 1]$ and then we apply the Markov property at time $T_\varepsilon^D(y)$ and we use (3.d) twice. This method yields the desired result when $|y| > 2\varepsilon$, $|z - y| > 2(\varepsilon + \varepsilon')$. The remaining cases can be handled without difficulty.

We now proceed to the proof of (3.f). We start from (3.e) and note that e_K is supported by K and that $e_K(K) = C(K)$. Then (3.e) implies

$$\left|\varepsilon^{-1} P_z\left[0 < L_\varepsilon^K(y) \leq t\right] - C(K)\int_0^t p_s(0, y)\,ds\right|$$

$$\leq C(K)\int_0^t \sup_{|w| \leq \varepsilon + \varepsilon'} \left(|p_s(0, y + w) - p_s(0, y)|\right) ds.$$

The assumption $|y| \geq 2(\varepsilon + \varepsilon')$ implies that the following bound holds:

$$\sup_{|w| \leq \varepsilon + \varepsilon'} |p_s(0, y + w) - p_s(0, y)|$$

$$\leq (2\pi s)^{-3/2}\left(3(\varepsilon + \varepsilon')\frac{|y|}{s} \wedge 1\right)\exp\left(-\frac{|y|^2}{8s}\right).$$

By integrating with respect to ds, it follows that, for some constant c,

$$\left|\varepsilon^{-1} P_z\left[0 < L_\varepsilon^K(y) \leq t\right] - C(K)\int_0^t p_s(0, y)\,ds\right|$$

$$\leq c(\varepsilon + \varepsilon')|y|^{-2}\exp\left(-\frac{|y|^2}{16}\right),$$

which yields (3.f). □

LEMMA 3.3. *Let* K' *be another compact set in* $\mathbb{R}^3$, *also contained in the unit ball. There exists a constant* C_5 *such that, for any* $y, z \in \mathbb{R}^3$, $\varepsilon, \varepsilon' \in (0;1)$,

$$\left|P\left[T_\varepsilon^K(y) \leq T_{\varepsilon'}^{K'}(z) \leq 1\right] - \varepsilon\varepsilon' C(K)C(K')\int_0^1 p_s(0, y)\,ds\int_0^{1-s} p_t(y, z)\,dt\right|$$

$$\leq C_5\varepsilon\varepsilon'(\psi(|y|) + \psi(|z|))\psi\left(\frac{|z - y|}{2}\right)$$

$$\times\left(\varepsilon\psi(|y|) \wedge 1 + \varepsilon'\psi(|z|) \wedge 1 + (\varepsilon + \varepsilon')\psi\left(\frac{|z - y|}{2}\right) \wedge 1\right).$$

PROOF. We may restrict our attention to the case $|y| > 2\varepsilon$, $|z| > 2\varepsilon'$ $|z-y| > 2(\varepsilon+\varepsilon')$. Indeed, if one of these bounds is not satisfied, the statement of Lemma 3.3 is a trivial consequence of (3.d′) and the remark preceding Lemma 3.2.

Throughout the proof, c will denote a constant that does not depend on y, z, ε, ε', but may vary from place to place. The first step of the proof is to establish the bound

$$\begin{aligned} &\left|P\left[T_\varepsilon^K(y) \le T_{\varepsilon'}^{K'}(z) \le 1\right] - P\left[T_\varepsilon^K(y) \le 1;\, z \in S_{\varepsilon'}^{K'}\left(T_\varepsilon^K(y);1\right)\right]\right| \\ &\qquad \le c\varepsilon\varepsilon'^2\psi(|z|)\psi\left(\frac{|z-y|}{2}\right)^2 \end{aligned} \tag{3.g}$$

[note that the definition of $S_\varepsilon^K(a;b)$ makes sense even if a and b are random variables]. We have

$$\begin{aligned} &\left|P\left[T_\varepsilon^K(y) \le T_{\varepsilon'}^{K'}(z) \le 1\right] - P\left[T_\varepsilon^K(y) \le 1;\, z \in S_{\varepsilon'}^{K'}\left(T_\varepsilon^K(y);1\right)\right]\right| \\ &\quad = P\left[T_{\varepsilon'}^{K'}(z) < T_\varepsilon^K(y) \le 1;\, z \in S_{\varepsilon'}^{K'}\left(T_\varepsilon^K(y);1\right)\right] \\ &\quad \le P\left[T_{\varepsilon'}^{K'}(z) \le 1\right]\sup\left\{P_{z_{\varepsilon'}}\left[T_\varepsilon^K(y) \le 1\right];\, |z_{\varepsilon'} - z| \le \varepsilon'\right\} \\ &\qquad \times \sup\left\{P_{y_\varepsilon}\left[T_{\varepsilon'}^{K'}(z) \le 1\right];\, |y_\varepsilon - y| \le \varepsilon\right\} \\ &\quad \le C_3\varepsilon'\psi(|z|)C_3\varepsilon\psi\left(\frac{|z-y|}{2}\right)C_3\varepsilon'\psi\left(\frac{|z-y|}{2}\right), \end{aligned}$$

using (3.d), the fact that K, K' are contained in the unit ball and the assumptions $|z| > 2\varepsilon'$, $|z-y| > 2(\varepsilon+\varepsilon')$. Formula (3.g) follows immediately, with $c = (C_3)^3$.

The second step is to prove

$$\begin{aligned} &\left|P\left[T_\varepsilon^K(y) \le 1;\, z \in S_{\varepsilon'}^{K'}\left(T_\varepsilon^K(y);1\right)\right] - P\left[T_\varepsilon^K(y) \le L_{\varepsilon'}^{K'}(z) \le 1\right]\right| \\ &\qquad \le c\varepsilon\varepsilon'^2\psi(|y|)\psi(|z-y|)^2. \end{aligned} \tag{3.h}$$

We have

$$\begin{aligned} &P\left[T_\varepsilon^K(y) \le 1;\, z \in S_{\varepsilon'}^{K'}\left(T_\varepsilon^K(y);1\right)\right] - P\left[T_\varepsilon^K(y) \le L_{\varepsilon'}^{K'}(z) \le 1\right] \\ &\quad = P\left[T_\varepsilon^K(y) \le 1;\, z \in S_{\varepsilon'}^{K'}\left(T_\varepsilon^K(y);1\right) \cap S_{\varepsilon'}^{K'}(1;\infty)\right] \\ &\quad \le P\left[T_\varepsilon^K(y) \le 1\right]\sup\left\{P_{y_\varepsilon}\left[T_{\varepsilon'}^{K'}(z) \le t \le L_{\varepsilon'}^{K'}(z)\right];\, |y_\varepsilon - y| \le \varepsilon,\, t \in [0;1]\right\} \\ &\quad \le C_3\varepsilon\psi(|y|)\sup\left\{P_{y_\varepsilon}\left[T_{\varepsilon'}^{D}(z) \le t \le L_{\varepsilon'}^{D}(z)\right];\, |y_\varepsilon - y| \le \varepsilon,\, t \in [0;1]\right\}. \end{aligned}$$

Then we use (3.c) and (3.e) to bound, for y_ε such that $|y_\varepsilon - y| \le \varepsilon$ and for $t \in [0;1]$,

$$\begin{aligned} &P_{y_\varepsilon}\left[T_{\varepsilon'}^{D}(z) \le t \le L_{\varepsilon'}^{D}(z)\right] \\ &\quad \le (2\pi)^{-1/2}\varepsilon'\int_0^t ds\, s^{-3/2}\exp\left(-\frac{|z-y|^2}{8s}\right)(2\pi)^{-1/2}\varepsilon'\int_{t-s}^\infty du\, u^{-3/2} \\ &\quad = 2(2\pi)^{-1}\varepsilon'^2\int_0^t ds\, s^{-3/2}(t-s)^{-1/2}\exp\left(-\frac{|z-y|^2}{8s}\right). \end{aligned}$$

Using this bound, it is an easy exercise to verify that

$$\sup\left\{P_{y_\varepsilon}\left[T_{\varepsilon'}^{D}(z) \le t \le L_{\varepsilon'}^{D}(z)\right];\ |y_\varepsilon - y| \le \varepsilon,\ t \in [0;1]\right\} \le c\varepsilon'^2\psi(|z-y|)^2,$$

from which (3.h) follows immediately.

The next step is to prove

$$\text{(3.i)}\qquad \begin{aligned}&\Big|P\left[T_\varepsilon^K(y) \le L_{\varepsilon'}^{K'}(z) \le 1\right]\\ &\quad -E\left[T_\varepsilon^K(y) \le 1;\ \varepsilon' C(K')\int_0^{1-T_\varepsilon^K(y)} p_s(y,z)\,ds\right]\Big|\\ &\le c\varepsilon\varepsilon'(\varepsilon+\varepsilon')\psi(|y|)\psi\left(\frac{|z-y|}{2}\right)^2.\end{aligned}$$

By applying the Markov property at time $T_\varepsilon^K(y)$ and using (3.f), we obtain

$$\begin{aligned}&\left|P\left[T_\varepsilon^K(y) \le L_{\varepsilon'}^{K'}(z) \le 1\right] - E\left[1_{(T_\varepsilon^K(y)\le 1)}\varepsilon' C(K')\int_0^{1-T_\varepsilon^K(y)} p_s(y,z)\,ds\right]\right|\\ &\le E\left[1_{(T_\varepsilon^K(y)\le 1)}\sup_{t\le 1}\left|P_{B_{T_\varepsilon^K(y)}}\left[0 < L_{\varepsilon'}^{K'}(z) \le t\right]\right.\right.\\ &\qquad\qquad \left.\left. -\varepsilon' C(K')\int_0^t p_s(y,z)\,ds\right|\right]\\ &\le C_4\varepsilon'(\varepsilon+\varepsilon')\psi\left(\frac{|z-y|}{2}\right)^2 P\left[T_\varepsilon^K(y) \le 1\right]\\ &\le C_3C_4\varepsilon\varepsilon'(\varepsilon+\varepsilon')\psi(|y|)\psi\left(\frac{|z-y|}{2}\right)^2.\end{aligned}$$

We now want to estimate

$$\begin{aligned}&E\left[1_{(T_\varepsilon^K(y)\le 1)}\varepsilon' C(K')\int_0^{1-T_\varepsilon^K(y)} p_s(y,z)\,ds\right]\\ &\qquad = \varepsilon' C(K')\int_0^1 ds\,p_s(y,z)P\left[T_\varepsilon^K(y) \le 1-s\right].\end{aligned}$$

First, we have

$$\text{(3.j)}\qquad \begin{aligned}&\left|\int_0^1 ds\,p_s(y,z)\left(P\left[T_\varepsilon^K(y) \le 1-s\right] - P\left[0 < L_\varepsilon^K(y) \le 1-s\right]\right)\right|\\ &\qquad \le c\varepsilon^2\psi(|y-z|)\psi(|y|)^2.\end{aligned}$$

The proof of (3.j) is very similar to that of (3.h) and will be left to the reader. Second, the arguments used in the proof of (3.i) easily imply that

$$\text{(3.k)}\qquad \begin{aligned}&\left|\int_0^1 ds\,p_s(y,z)\left(P\left[0 < L_\varepsilon^K(y) \le 1-s\right] - \varepsilon C(K)\int_0^{1-s} p_t(0,y)\,dt\right)\right|\\ &\qquad \le c\varepsilon^2\psi(|z-y|)\psi(|y|)^2.\end{aligned}$$

The statement of Lemma 3.3 now follows, by putting together (3.g)–(3.k). □

COROLLARY 3.4. *Let B' be another Brownian motion starting from 0, independent of B, and let $S_\varepsilon'^K$ denote the associated Wiener sausage. There exists a constant C_6 such that, for any $\varepsilon \in (0, \frac{1}{2})$,*

$$(3.1)\qquad E\Big[\big(\varepsilon^{-2}m(S_\varepsilon^K \cap S_\varepsilon'^K) - C(K)^2\beta([0;1]^2)\big)^2\Big] \le C_6\varepsilon \log\frac{1}{\varepsilon}.$$

Here $\beta([0;1]^2)$ denotes the intersection local time of B and B' on $[0;1]^2$,

$$\beta([0;1]^2) = \int_0^1\int_0^1 \delta_{(0)}(B_s - B_t')\,ds\,dt.$$

In the special case when K is star-shaped (i.e., $\varepsilon K \subset K$ for $\varepsilon < 1$), we have

$$(3.\text{m})\qquad \lim_{\varepsilon\to 0}\varepsilon^{-2}m(S_\varepsilon^K \cap S_\varepsilon'^K) = C(K)^2\beta([0;1]^2), \quad a.s.$$

PROOF. We first consider another compact set K', as in Lemma 3.3, and we establish the preliminary bound, for $\varepsilon, \varepsilon' \in (0, \frac{1}{2})$,

$$(3.\text{n})\qquad \begin{aligned}&\Big|E\Big[(\varepsilon\varepsilon')^{-2}m(S_\varepsilon^K \cap S_\varepsilon'^K)m(S_{\varepsilon'}^{K'} \cap S_{\varepsilon'}'^{K'}) - C(K)^2C(K')^2\beta([0;1]^2)^2\Big]\Big|\\ &\quad\le C_6'\Big(\varepsilon\log\frac{1}{\varepsilon} + \varepsilon'\log\frac{1}{\varepsilon'}\Big).\end{aligned}$$

Note that

$$(3.\text{o})\qquad \begin{aligned}E\big[m(S_\varepsilon^K \cap S_\varepsilon'^K)m(S_{\varepsilon'}^{K'} \cap S_{\varepsilon'}'^{K'})\big] &= \int dy\,dz\,P\big[T_\varepsilon^K(y) \le 1;\, T_{\varepsilon'}^{K'}(z) \le 1\big]^2\\ &= \int dy\,dz\,\big(P\big[T_\varepsilon^K(y) \le T_{\varepsilon'}^{K'}(z) \le 1\big]\\ &\qquad + P\big[T_{\varepsilon'}^{K'}(z) \le T_\varepsilon^K(y) \le 1\big]\big)^2.\end{aligned}$$

On the other hand (see, e.g., [15], Proposition 2.1),

$$(3.\text{p})\qquad \begin{aligned}E\big[\beta([0;1]^2)^2\big] = \int dy\,dz\Big(&\int_0^1 p_s(0,y)\,ds\int_0^{1-s}p_t(y,z)\,dt\\ &+ \int_0^1 p_s(0,z)\,ds\int_0^{1-s}p_t(z,y)\,dt\Big)^2.\end{aligned}$$

Lemma 3.3 gives us bounds on

$$\Big|(\varepsilon\varepsilon')^{-1}P\big[T_\varepsilon^K(y) \le T_{\varepsilon'}^{K'}(z) \le 1\big] - C(K)C(K')\int_0^1 p_s(0,y)\,ds\int_0^{1-s}p_t(y,z)\,dt\Big|.$$

Taking (3.o) and (3.p) into account and using both the bounds of Lemma 3.3 and (3.d′), we obtain (3.n). The point here is that $\psi(|y|)^2$ is integrable, but $\psi(|y|)^3$ is not.

The convergence (3.m) is known to hold in the L^2-norm when K is the unit ball D (see [14]). Thus we may take $K' = D$ and let ε' tend to 0 in (3.n) to get

$$(3.\text{q})\qquad (4\pi^2)\Big|E\Big[\varepsilon^{-2}m(S_\varepsilon^K \cap S_\varepsilon'^K)\beta([0;1]^2) - C(K)^2\beta([0;1]^2)^2\Big]\Big| \le C_6'\varepsilon\log\frac{1}{\varepsilon}.$$

Similarly, taking $K = K'$, $\varepsilon' = \varepsilon$, we have

$$\text{(3.r)} \qquad \left| E\left[\varepsilon^{-4} m\left(S_\varepsilon^K \cap S_\varepsilon'^K\right)^2 - C(K)^4 \beta\left([0;1]^2\right)^2\right] \right| \le C_6' \varepsilon \log \frac{1}{\varepsilon}.$$

Formula (3.l) follows immediately from these two bounds, with $C_6 = 3C_6'$.

In the special case when K is star-shaped, $m(S_\varepsilon^K)$ is a monotone function of ε, and (3.m) follows from (3.l) and an easy application of the Borel–Cantelli lemma.

□

PROOF OF (3.a). We start from formula (1.j) and take $n = n(\varepsilon)$ such that

$$\tfrac{1}{4} < \varepsilon 2^{n/2} < \tfrac{1}{2}.$$

We first bound

$$\text{(3.s)} \qquad \begin{aligned} &E\left[\left(\varepsilon^{-2} \sum_{k=1}^{2^n} \left\{ m\left(S_\varepsilon^K\left(\frac{k-1}{2^n}; \frac{k}{2^n}\right)\right)\right\}\right)^2\right]^{1/2} \\ &= 2^{n/2} E\left[\left(\varepsilon^{-2}\left\{ m\left(S_\varepsilon^K\left(0; \frac{1}{2^n}\right)\right)\right\}\right)^2\right]^{1/2} \\ &= \left(\varepsilon 2^{n/2}\right)^{-2} E\left[\left\{ m\left(S_{\varepsilon 2^{n/2}}^K(0;1)\right)\right\}^2\right]^{1/2} \le C_7. \end{aligned}$$

Then we bound

$$\begin{aligned} &E\left[\left(\varepsilon^{-2} \sum_{p=1}^{n} \sum_{q=1}^{2^{p-1}} \left\{ m\left(S_\varepsilon^K\left(\frac{2q-2}{2^p}; \frac{2q-1}{2^p}\right) \cap S_\varepsilon^K\left(\frac{2q-1}{2^p}; \frac{2q}{2^p}\right)\right)\right\}\right.\right. \\ &\qquad\qquad \left.\left. - C(K)^2 \left\{\alpha\left(\mathcal{T}_{(n)}\right)\right\}\right)^2\right]^{1/2} \\ &\le \sum_{p=1}^{n} \left(\sum_{q=1}^{2^{p-1}} E\left[\left(\varepsilon^{-2}\left\{ m\left(S_\varepsilon^K\left(\frac{2q-2}{2^p}; \frac{2q-1}{2^p}\right) \cap S_\varepsilon^K\left(\frac{2q-1}{2^p}; \frac{2q}{2^p}\right)\right)\right\}\right.\right.\right. \\ &\qquad \left.\left.\left. - C(K)^2 \left\{\alpha\left(\left[\frac{2q-2}{2^p}; \frac{2q-1}{2^p}\right) \times \left(\frac{2q-1}{2^p}; \frac{2q}{2^p}\right]\right)\right\}\right)^2\right]\right)^{1/2} \\ &\qquad\qquad\qquad\qquad \text{(independence of increments)} \\ &= \sum_{p=1}^{n} 2^{(p-1)/2} 2^{-3p/2} E\left[\left(\varepsilon^{-2}\left\{ m\left(S_{\varepsilon 2^{p/2}}^K(0;1) \cap S_{\varepsilon 2^{p/2}}^K(1;2)\right)\right\}\right.\right. \\ &\qquad\qquad \left.\left. - C(K)^2 2^p \left\{\alpha\left([0;1) \times (1;2]\right)\right\}\right)^2\right]^{1/2} \\ &\qquad\qquad\qquad\qquad \text{(change of scale)} \\ &\le 2^{-1/2} C_6^{1/2} \sum_{p=1}^{n} \left(\varepsilon 2^{p/2} \log \frac{1}{\varepsilon 2^{p/2}}\right)^{1/2} \quad \text{(Corollary 3.4)} \\ &\le C_8. \end{aligned}$$

Thus we have proved

$$E\Big[\big(\varepsilon^{-2}\{m(S_\varepsilon^K)\} - C(K)^2\{\alpha(\mathscr{T}_{(n)})\}\big)^2\Big] \le (C_7 + C_8)^2. \tag{3.t}$$

On the other hand, elementary calculations show that, with our choice of n,

$$E\Big[\big(\{\alpha(\mathscr{T}_{(n)})\} - \{\alpha(\mathscr{T}_{\varepsilon^2})\}\big)^2\Big] \le C_9. \tag{3.u}$$

Formula (3.a) follows from (3.t) and (3.u). □

PROOF OF (3.b). Keeping Spitzer's result (1.g) in mind, we see that (3.b) follows from (3.a) if we know that

$$\lim_{\varepsilon\to 0}\left(\log\frac{1}{\varepsilon}\right)^{-1/2}\{\alpha(\mathscr{T}_\varepsilon)\} = \frac{2^{-1/2}}{\pi}N, \tag{3.v}$$

with convergence in distribution. Formula (3.v) is proved by Yor [29] [in fact, Yor's result is more general than (3.v)]. For the sake of completeness, we will now sketch an elementary proof of (3.v). The first step of the proof is to estimate the variance of $\alpha(\mathscr{T}_\varepsilon)$. We show that

$$\lim_{\varepsilon\to 0}\left(\log\frac{1}{\varepsilon}\right)^{-1}E\left[\{\alpha(\mathscr{T}_\varepsilon)\}^2\right] = \frac{1}{2\pi^2}. \tag{3.w}$$

We first compute

$$\begin{aligned} E[\alpha(\mathscr{T}_\varepsilon)] &= \iint_{\mathscr{T}_\varepsilon}(2\pi(t-s))^{-3/2}\,ds\,dt \\ &= 2(2\pi)^{-3/2}\varepsilon^{-1/2} - 4(2\pi)^{-3/2} + 2(2\pi)^{-3/2}\varepsilon^{1/2}; \end{aligned}$$

then we write

$$E\left[\alpha(\mathscr{T}_\varepsilon)^2\right] = E\left[\iint_{\mathscr{T}_\varepsilon}\iint_{\mathscr{T}_\varepsilon} ds\,dt\,ds'\,dt'\,\delta_{(0)}(B_s - B_t)\delta_{(0)}(B_{s'} - B_{t'})\right].$$

We investigate separately the three cases $\{s < t < s' < t'\}$, $\{s < s' < t < t'\}$ and $(s < s' < t' < t\}$ and after some lengthy but straightforward calculations we find that

$$E\left[\alpha(\mathscr{T}_\varepsilon)^2\right] = (2\pi)^{-3}\left(4\varepsilon^{-1} - 16\varepsilon^{-1/2} + 4\pi\log\frac{1}{\varepsilon} + o\left(\log\frac{1}{\varepsilon}\right)\right),$$

from which (3.w) follows easily.

We now define

$$N = N(\varepsilon) = \left[\left(\log\frac{1}{\varepsilon}\right)^{1/2}\right],$$

where $[u]$ denotes the integer part of u. We note that

$$\alpha(\mathscr{T}_\varepsilon) = \sum_{i=1}^{N} X_i^\varepsilon + R^\varepsilon, \tag{3.x}$$

where

$$X_i^\varepsilon = \alpha\left(\left[\frac{i-1}{N};\frac{i}{N}\right)^2 \cap \mathcal{T}_\varepsilon\right), \qquad i = 1,\dots,N,$$

and the remainder R^ε satisfies

$$E[R^\varepsilon] \le \sum_{i=1}^{N} E\left[\alpha\left(\left[\frac{i-1}{N};\frac{i}{N}\right)\times\left(\frac{i}{N};1\right]\right)\right] \le C_{10}N^{1/2}.$$

Our choice of N implies that

$$\lim_{\varepsilon\to 0}\left(\log\frac{1}{\varepsilon}\right)^{-1/2} R^\varepsilon = 0, \quad \text{in the } L^1\text{-norm.} \tag{3.y}$$

On the other hand, the random variables $(X_i^\varepsilon,\ 1 \le i \le N)$ are independent and identically distributed and satisfy

$$\lim_{\varepsilon\to 0}\left(\log\frac{1}{\varepsilon}\right)^{-1/2} E\left[\{X_i^\varepsilon\}^2\right] = \lim_{\varepsilon\to 0}\left(\log\frac{1}{\varepsilon}\right)^{-1} E\left[\{\alpha(\mathcal{T}_{N\varepsilon})\}^2\right] = \frac{1}{2\pi^2},$$

where we have used (3.w). Formula (3.v) will follow from (3.x), (3.y) and an application of Lindeberg's theorem on triangular arrays to the family $((\log(1/\varepsilon))^{-1/2}\{X_i^\varepsilon\};\ 1 \le i \le N(\varepsilon))$. Of course, we need to verify Lindeberg's condition. It is clearly enough to prove that

$$E\left[\{\alpha(\mathcal{T}_\varepsilon)\}^4\right] \le C_{10}\left(\log\frac{1}{\varepsilon}\right)^2, \tag{3.z}$$

for some constant C_{10}, and for $0 < \varepsilon < \frac{1}{2}$. In order to prove (3.z), we set

$$\mathcal{T}_\varepsilon^1 = \mathcal{T}_\varepsilon \cap [0;\tfrac{1}{2}]^2, \qquad \mathcal{T}_\varepsilon^2 = \mathcal{T}_\varepsilon \cap [\tfrac{1}{2};1]^2.$$

The identity

$$\alpha(\mathcal{T}_\varepsilon) = \alpha(\mathcal{T}_\varepsilon^1) + \alpha(\mathcal{T}_\varepsilon^2) + \alpha([0;\tfrac{1}{2})\times(\tfrac{1}{2};1]\cap\mathcal{T}_\varepsilon)$$

implies that

$$E\left[\{\alpha(\mathcal{T}_\varepsilon)\}^4\right]^{1/4} \le E\left[\{\alpha(\mathcal{T}_\varepsilon^1)+\alpha(\mathcal{T}_\varepsilon^2)\}^4\right]^{1/4} + C_{11}.$$

On the other hand, using (3.w) again,

$$\begin{aligned} E\left[\{\alpha(\mathcal{T}_\varepsilon^1)+\alpha(\mathcal{T}_\varepsilon^2)\}^4\right] &= 2E\left[\{\alpha(\mathcal{T}_\varepsilon^1)\}^4\right] + 6E\left[\{\alpha(\mathcal{T}_\varepsilon^1)\}^2\right]^2 \\ &\le \tfrac{1}{2}E\left[\{\alpha(\mathcal{T}_{2\varepsilon})\}^4\right] + C_{12}\left(\log\frac{1}{\varepsilon}\right)^2. \end{aligned}$$

Hence, setting $h(\varepsilon) = (\log(1/\varepsilon))^2 E[\{\alpha(\mathcal{T}_\varepsilon)\}^4]$, we find that, for any $\rho > \frac{1}{2}$ and for ε sufficiently small,

$$h(\varepsilon) \le \left(\left(\rho h(2\varepsilon) + C_{12}\right)^{1/4} + \left(\log\frac{1}{\varepsilon}\right)^{-1/2} C_{11}\right)^4,$$

from which it follows immediately that $h(\varepsilon)$ is bounded when ε is small. This completes the proof of (3.z) and hence of (3.v). □

REMARKS. (i) The preceding arguments may be adapted to yield a simple proof of the following renormalization result for stable processes due to Rosen [20]. Let $X = (X_t; t \leq 0)$ denote a symmetric stable process in $\mathbb{R}^2$, with index $\nu > 1$. Then it is known that the path of X has double points. For any $\varepsilon > 0$, let $\delta_\varepsilon(x)$ be the transition density of X at time ε, so that δ_ε is an approximation of the Dirac measure $\delta_{(0)}$, and define

$$\alpha_\varepsilon(\mathscr{T}) = \int\int_{\mathscr{T}} \delta_\varepsilon(X_s - X_t)\, ds\, dt.$$

Rosen's result states that, if $\nu \leq \frac{4}{3}$,

$$\lim_{\varepsilon \to 0} \psi(\varepsilon)\{\alpha_\varepsilon(\mathscr{T})\} = C_\nu N,$$

where the convergence holds in distribution, C_ν is some constant depending on ν and $\psi(\varepsilon) = (\log(1/\varepsilon))^{-1/2}$ if $\nu = \frac{4}{3}$, $\varepsilon^{2/\nu - 3/2}$ if $\nu < \frac{4}{3}$. Rosen's proof is difficult and requires the estimation of all moments of $\{\alpha_\varepsilon(\mathscr{T})\}$. Using the same arguments as given previously for the proof of (3.v), one can reduce the proof of Rosen's result to an estimate of the second moment of $\{\alpha_\varepsilon(\mathscr{T})\}$.

(ii) In the case when K is the unit ball D, it is possible to give a direct proof of (3.b), which does not involve the notion of intersection local time. Indeed, some tedious calculations show that

$$E\left[\left(m(S_\varepsilon^D)\right)^2\right] = 4\pi^2\varepsilon^2 + 8(2\pi)^{3/2}\varepsilon^3 + 16\pi^2\varepsilon^4\log\frac{1}{\varepsilon} + o\left(\varepsilon^4\log\frac{1}{\varepsilon}\right),$$

from which it follows that

$$\operatorname{var}\left(m(S_\varepsilon^D)\right) = 16\pi^2\varepsilon^4\log\frac{1}{\varepsilon} + o\left(\varepsilon^4\log\frac{1}{\varepsilon}\right)$$

and then the same arguments as in the preceding proof of (3.v) yield (3.b).

4. Results in higher dimensions. We now assume that B is a d-dimensional Brownian motion, with $d \geq 4$, and that K is a compact subset of $\mathbb{R}^d$. Then Spitzer's results (1.f) and (1.g) can be extended as follows (see Getoor [8]):

(4.a) $$E\left[m(S^K(0; t))\right] = C(K)t + \frac{C(K)^2}{4\pi^2}\log t + o(\log t), \quad \text{if } d = 4,$$

(4.b) $$E\left[m(S^K(0; t))\right] = C(K)t + \int dy\, P\left[T^K(y) < \infty\right]^2 + o(1), \quad \text{if } d \geq 5,$$

where we have used the same notation as in Section 3,

$$T^K(y) = \inf\{t; B_t \in y - K\}.$$

Since

$$E\left[m(S^K(0; t))\right] - C(K)t = E\left[m(S^K(0; t) \cap S^K(t; \infty))\right],$$

(4.a) and (4.b) can be interpreted in terms of intersections of independent Wiener sausages. Let $S'^K(0, t)$ denote the Wiener sausage associated with another Brownian motion B', independent of B but such that $B_0 = B'_0$, then

$$\text{(4.a}') \quad \lim_{t\to\infty} (\log t)^{-1} E\left[m\left(S^K(0;t) \cap S'^K(0;\infty)\right)\right] = \frac{C(K)^2}{4\pi^2}, \quad \text{if } d = 4,$$

$$\text{(4.b}') \quad \begin{aligned} &\lim_{t\to\infty} E\left[m\left(S^K(0;t) \cap S'^K(0;\infty)\right)\right] \\ &\quad = E\left[m\left(S^K(0;\infty) \cap S'^K(0;\infty)\right)\right] \\ &\quad = \int dy\, P\left[T^K(y) < \infty\right]^2 < \infty, \quad \text{if } d \geq 5. \end{aligned}$$

Note that (4.a′) can be strengthened by

$$\lim_{t\to\infty} (\log t)^{-1} m\left(S^K(0;t) \cap S'^K(0;\infty)\right) = \frac{C(K)^2}{4\pi^2} N^2,$$

where convergence holds in distribution and N denotes a standard normal variable. The latter result is proved in [16] in the special case when K is the unit ball, but the proof can be easily extended. If $d \geq 5$, $m(S^K(0;\infty) \cap S'^K(0;\infty))$ is a finite random variable in L^p for any $p < \infty$.

THEOREM 4.1. *There exists a nonnegative constant $A(K)$ such that*

$$\lim_{t\to\infty} t^{-1/2}\left(m\left(S^K(0;t)\right) - tC(K)\right) = A(K)N,$$

where convergence holds in distribution and N denotes a standard normal variable. Moreover, $A(K)$ is positive if and only if K has positive capacity.

The main ingredient of the proof of Theorem 4.1 will be the following lemma.

LEMMA 4.2. *For any compact subset K of $\mathbb{R}^d$, there exists a constant $A(K)$, positive if and only if K has positive capacity, such that*

$$\lim_{t\to\infty} t^{-1} \operatorname{var}\left(m\left(S^K(0;t)\right)\right) = A(K)^2.$$

PROOF. We may and will assume that $B_0 = 0$. We shall deal with a fixed compact set K, contained in the unit ball, and thus we drop the index K in the notation. For $0 \leq a \leq b$, set

$$\hat{S}(a;b) = S(a;b) - S(0;a).$$

Let n, N be two integers with $1 \le n \le N$. We start from the trivial formula

$$m(S(0;n)) + m(\hat{S}(n;N)) = m(S(0;N)).$$

By taking conditional expectation with respect to $\mathscr{F}_n = \sigma(B_s; s \le n)$ and then subtracting the expected values, it follows that

$$\{m(S(0;n))\} + \{E[m(\hat{S}(n;N))|\mathscr{F}_n]\} = \sum_{k=1}^{n} U_k^N, \tag{4.c}$$

where

$$U_k^N = E[m(S(0;N))|\mathscr{F}_k] - E[m(S(0;N))|\mathscr{F}_{k-1}].$$

The independence of increments clearly implies that

$$\{E[m(\hat{S}(n;N))|\mathscr{F}_n]\} = -\{E[m(S(0;n) \cap S(n;N))|\mathscr{F}_n]\}. \tag{4.d}$$

Similarly, for $k = 1, \dots, N$,

$$\begin{aligned} U_k^N &= E[m(\hat{S}(k-1;N))|\mathscr{F}_k] - E[m(\hat{S}(k-1;N))|\mathscr{F}_{k-1}] \\ &= m(\hat{S}(k-1;k)) - E[m(\hat{S}(N-1;N))|\mathscr{F}_{k-1}] \\ &\quad + E[m(S(0;k-1) \cap S(k-1;N-1))|\mathscr{F}_{k-1}] \\ &\quad - E[m(S(0;k) \cap S(k;N))|\mathscr{F}_k]. \end{aligned}$$

Let $\mathscr{F}_{a,b}$ denote the σ-field generated by the increments of B on $[a;b]$. Then

$$E[m(\hat{S}(N-1;N))|\mathscr{F}_{k-1}] =_{(d)} E[m(S(0;1) - S(1;N))|\mathscr{F}_{N-k+1,N}],$$

from which it follows that

$$\lim_{N\to\infty} E[m(\hat{S}(N-1;N))|\mathscr{F}_{k-1}] = E[m(S(0;1) - S(1;\infty))] = C(K),$$

in the L^1-norm. We finally obtain

$$\begin{aligned} \lim_{N\to\infty} U_k^N &= m(\hat{S}(k-1;k)) - C(K) \\ &\quad + E[m(S(0;k-1) \cap S(k-1;\infty))|\mathscr{F}_{k-1}] \\ &\quad - E[m(S(0;k) \cap S(k;\infty))|\mathscr{F}_k] \\ &= \{E[m(S(k-1;k) - S(k;\infty))|\mathscr{F}_k]\} \\ &\quad + E[m(S(0;k-1) \cap S(k-1;\infty))|\mathscr{F}_{k-1}] \\ &\quad - E[m(S(0;k-1) \cap S(k-1;\infty))|\mathscr{F}_k]. \end{aligned} \tag{4.e}$$

Letting N tend to infinity in (4.c) and taking into account (4.d) and (4.e), we obtain

$$\{m(S(0;n))\} = \{E[m(S(0;n) \cap S(n;\infty))|\mathscr{F}_n]\} + \sum_{k=1}^{n} Y_k, \tag{4.f}$$

where

$$Y_k = E[m(S(0;k-1)\cap S(k-1;\infty))|\mathscr{F}_{k-1}]$$
$$-E[m(S(0;k-1)\cap S(k-1;\infty))|\mathscr{F}_k]$$
$$+\{E[m(S(k-1;k)-S(k;\infty))|\mathscr{F}_k]\}.$$

It can easily be verified, for instance, by using the estimates in [14], that

$$E\left[E[m(S(0;n)\cap S(n;\infty))|\mathscr{F}_n]^2\right] \le E\left[m(S(0;n)\cap S(n;\infty))^2\right]$$

(4.g)
$$\le \text{constant}(\log n)^2, \quad \text{if } d=4,$$
$$\le \text{constant}, \quad \text{if } d\ge 5.$$

On the other hand, Y_k is $\mathscr{F}_k$-measurable and, for $k<l$,

$$E[Y_l|\mathscr{F}_k]=0.$$

This implies that

(4.h)
$$E\left[\left(\sum_{k=1}^{n} Y_k\right)^2\right] = \sum_{k=1}^{n} E\left[(Y_k)^2\right].$$

Lemma 4.2 now follows from (4.f)–(4.h) and the following lemma.

LEMMA. *There exists a nonnegative constant $A(K)$ such that*

$$\lim_{k\to\infty} E\left[(Y_k)^2\right] = A(K)^2.$$

Moreover, $A(K)$ is positive if, and only if, K has positive capacity.

PROOF. It will be convenient to assume that B_t is defined for any $t\in\mathbb{R}$, by setting $B_t = B'_{-t}$ for $t<0$, where B' is another Brownian motion independent of B starting from 0. The sausage $S(a;b)$ is thus defined for any a, $b\in[-\infty;\infty]$, $a\le b$. Moreover, we may assume that B is defined on the canonical space $\Omega = C(\mathbb{R},\mathbb{R}^d)$ of continuous functions from $\mathbb{R}$ to $\mathbb{R}^d$, so that, for $\omega\in\Omega$ and $t\in\mathbb{R}$,

$$B_t(\omega)=\omega(t).$$

The shift θ on Ω is then defined by setting

$$\theta\omega(t)=\omega(1+t)-\omega(1), \qquad t\in\mathbb{R}.$$

Note that θ preserves the probability P on Ω. For $t\in\mathbb{R}$, we denote by $\mathscr{G}_t$ the σ-field generated by $(B_s;\ -\infty<s\le t)$. Then

(4.i)
$$Y_k = Z_{k-1}\circ\theta^{k-1},$$

where

$$Z_k = E[m(S(-k;0)\cap S(0;\infty))|\mathscr{G}_0] - E[m(S(-k;0)\cap S(0;\infty))|\mathscr{G}_1]$$
$$+\{E[m(S(0;1)-S(1;\infty))|\mathscr{G}_1]\}.$$

We shall prove the existence of a random variable Z such that

$$\lim_{k\to\infty} Z_k = Z, \tag{4.j}$$

in the L^2-norm. The first assertion of the lemma follows immediately from (4.j), with $A(K)^2 = E[Z^2]$.

If $d \geq 5$, (4.j) is obvious, and

$$Z = E\left[m(S(-\infty;0) \cap S(0;\infty))|\mathscr{G}_0\right] - E\left[m(S(-\infty;0) \cap S(0;\infty))|\mathscr{G}_1\right] + \left\{E\left[m(S(0;1) - S(1;\infty))|\mathscr{G}_1\right]\right\}.$$

We now consider the case $d = 4$. The point here is that

$$m(S(-\infty;0) \cap S(0;\infty)) = \infty, \quad \text{a.s.},$$

and thus we cannot define Z as previously stated. However, we have

$$\begin{aligned} Z_k = &\left\{E\left[m(S(0;1) - S(1;\infty))|\mathscr{G}_1\right]\right\} \\ &-E\left[m((S(0;1) - S(1;\infty)) \cap S(-k;0))|\mathscr{G}_1\right] \\ &+E\left[m(S(-k;0) \cap S(0;\infty))|\mathscr{G}_0\right] \\ &-E\left[m(S(-k;0) \cap S(1;\infty))|\mathscr{G}_1\right], \end{aligned}$$

and

$$\begin{aligned} &E\left[m(S(-k;0) \cap S(0;\infty))|\mathscr{G}_0\right] - E\left[m(S(-k;0) \cap S(1;\infty))|\mathscr{G}_1\right] \\ &\quad = \int dy\, 1_{S(-k;0)}(y)(g(y) - g(y - B_1)), \end{aligned}$$

where we have set $g(y) = P[y \in S(0;\infty)]$. The explicit formula

$$g(y) = c_d \int e_K(dw)|y - w|^{2-d}$$

(e_K is the equilibrium measure of K) can be used to derive the bound, valid for $x,\ y \in \mathbb{R}^d$,

$$|g(y) - g(y - x)| \leq c(g(y) + g(y - x))\left(\frac{|x| + 1}{|y|} \wedge 1\right). \tag{4.k}$$

Then

$$\begin{aligned} &E\left[\int dy\, 1_{S(-\infty;0)}(y)|g(y) - g(y - B_1)|\right] \\ &\quad \leq c\int\int dx\, dy\, p_1(0,x)g(y)(g(y) + g(y - x))\left(\frac{|x| + 1}{|y|} \wedge 1\right) < \infty. \end{aligned}$$

Similarly, straightforward calculations using (4.k) show that

$$\begin{aligned} &E\left[\int\int dy\, dz\, 1_{S(-\infty;0)}(y)\, 1_{S(-\infty;0)}(z)|g(y) - g(y - B_1)||g(z) - g(z - B_1)|\right] \\ &\quad < \infty. \end{aligned}$$

Hence

$$\int dy\, 1_{S(-\infty;0)}(y)(g(y) - g(y - B_1))$$

is well defined as a random variable in $L^2(\Omega)$. Using dominated convergence, it follows that

$$\lim_{k\to\infty} Z_k = Z,$$

in the L^2-norm, and

$$\begin{aligned} Z = &\{E[m(S(0;1) - S(1;\infty))|\mathscr{G}_1]\} \\ &-E[m((S(0;1) - S(1;\infty)) \cap S(-\infty;0))|\mathscr{G}_1] \\ &+\int dy\, 1_{S(-\infty;0)}(y)(g(y) - g(y - B_1)). \end{aligned}$$

This completes the proof of the first assertion of the lemma. Note that the preceding arguments apply to any $d \geq 4$.

It remains to prove that $Z \neq 0$ if K has positive capacity (the converse is obvious). Coming back to (4.f) and using (4.i), we have

$$\{m(S(0;n))\} = \sum_{k=0}^{n-1} Z \circ \theta^k + H_n,$$

where

$$H_n = \{E[m(S(0;n) \cap S(n;\infty))|\mathscr{G}_n]\} + \sum_{k=0}^{n-1} (Z_k - Z) \circ \theta^k.$$

In particular,

$$m(S(0;n)) \geq nC(K) + \sum_{k=0}^{n-1} Z \circ \theta^k + H_n.$$

We shall establish the existence of a constant $\bar{c}$ (depending on K but not on n) such that, for every $n \geq 1$,

$$P[m(S(0;n)) \leq \bar{c};\ |H_n| \leq \bar{c}] > 0. \tag{4.l}$$

If we assume $Z = 0$, the inequality

$$m(S(0;n)) \geq nC(K) + H_n,$$

together with (4.l) forces $C(K) = 0$. Thus, in order to complete the proof of the lemma, it is enough to prove (4.l).

We first need to obtain a simpler expression of H_n. We have

$$\begin{aligned} (Z - Z_k)\circ\theta^k = \int dy\, 1_{S(-\infty;0)-S(0;k)}(y)\big(&E[1_{S(k;\infty)}(y)|\mathscr{G}_k] \\ &-E[1_{S(k;\infty)}(y)|\mathscr{G}_{k+1}]\big). \end{aligned}$$

After some calculations, we get

$$
\begin{aligned}
&\sum_{k=0}^{n-1} (Z - Z_k)\circ\theta^k \\
&\quad = \int dy \Big(1_{S(-\infty;0)}(y) E\big[1_{S(-\infty;0)}(y)|\mathscr{G}_0\big] \\
&\qquad\qquad - 1_{S(-\infty;0)-S(0;n-1)}(y) E\big[1_{S(n-1;\infty)}(y)|\mathscr{G}_n\big]\Big) \\
&\qquad - m(S(-\infty;0)\cap S(0;n-1)) \\
\text{(4.m)}\quad &\quad = \int dy\, 1_{S(-\infty;0)}(y)(g(y) - g(y-B_n)) \\
&\qquad + \int dy\, 1_{S(-\infty;0)\cap S(0;n-1)}(y) g(y-B_n) \\
&\qquad - \int dy\, 1_{S(-\infty;0)-S(0;n-1)}(y) E\big[1_{S(n-1;n)-S(n;\infty)}(y)|\mathscr{G}_n\big] \\
&\qquad - m(S(-\infty;0)\cap S(0;n-1)).
\end{aligned}
$$

The bound (4.k) shows that

$$
\begin{aligned}
&\left|\int dy\, 1_{S(-\infty;0)}(y)(g(y) - g(y-B_n))\right| \\
&\qquad \le c\int dy\, 1_{S(-\infty;0)}(y)(g(y)+g(y-B_n))\left(\frac{|B_n|+1}{|y|}\wedge 1\right).
\end{aligned}
$$

We may choose a constant c' such that, for any $x \in \mathbb{R}^d$ with $|x| \le 1$,

$$
\int dy\, g(y)(g(y)+g(y-x))\left(\frac{|x|+1}{|y|}\wedge 1\right) \le c'.
$$

It follows that, for every $n \ge 1$,

$$
P\left[\sup(|B_s|;\, s\le n)\le 1;\ \left|\int dy\, 1_{S(-\infty;0)}(y)(g(y)-g(y-B_n))\right| \le c(c'+1)\right] > 0.
$$

On the other hand, the explicit formula for H_n, which follows from (4.m), shows that, for some constant $\bar{c}$,

$$
\begin{aligned}
&\left(\{\sup(|B_s|;\, s\le n)\le 1\}\cap\left\{\left|\int dy\, 1_{S(-\infty;0)}(y)(g(y)-g(y-B_n))\right|\le c(c'+1)\right\}\right) \\
&\qquad \subset (\{m(S(0;n))\le\bar{c}\}\cap\{|H_n|\le\bar{c}\}).
\end{aligned}
$$

This completes the proof of (4.l) and of the lemma. □

PROOF OF THEOREM 4.1. For t large enough, set $n = n(t) = [\log t]$. Then

$$
m(S^K(0;t)) = \sum_{i=1}^{n} X^i(t) + R(t),
$$

where

$$X^i(t) = m\left(S^K\left(\frac{i-1}{n}t; \frac{i}{n}t\right)\right), \qquad 1 \le i \le n,$$

and the remainder $R(t)$ satisfies

$$\text{(4.n)} \quad E\left[|R(t)|\right] = E\left[\sum_{i=1}^{n-1} m\left(S^K\left(\frac{i-1}{n}t; \frac{i}{n}t\right) \cap S^K\left(\frac{i}{n}t; t\right)\right)\right] \le c(\log t)^2,$$

for some constant c (we use the fact that

$$E\left[m(S^K(0; t) \cap S'^K(0, t))\right] \le c, \qquad \text{if } d \ge 5,$$
$$\le c \log t, \quad \text{if } d = 4).$$

The random variables $X^i(t)$, $i = 1, \dots, n$, are independent and identically distributed and, by Lemma 4.2,

$$\lim_{t \to \infty} \frac{\log t}{t} E\left[(X^i(t))^2\right] = A(K)^2.$$

Theorem 4.1 follows from (4.n) and an application of Lindeberg's theorem on triangular arrays to the family $(\{X^i(t)\}, 1 \le i \le n(t))$. In order to verify Lindeberg's condition, it is enough to establish the following bound:

$$\text{(4.o)} \qquad E\left[\{m(S^K(0; t))\}^4\right] \le \bar{c}t^2,$$

for some constant $\bar{c}$ and for t large enough. Note that (4.o) is equivalent to

$$\text{(4.o}'\text{)} \qquad E\left[\{m(S_\varepsilon^K)\}^4\right] \le \bar{c}\varepsilon^{4(d-1)},$$

for ε small enough. The bound (4.o′) is easily established using the same method as in the proof of (3.z): We divide the sausage S_ε^K into two pieces, namely $S_\varepsilon^K(0; 1/2)$ and $S_\varepsilon^K(1/2; 1)$ and we use the fact that the fourth moment of $m(S_\varepsilon^K(0; 1/2) \cap S_\varepsilon^K(1/2; 1))$ is bounded by a constant times $\varepsilon^{4d}(\log(1/\varepsilon))^4$ (cf. [16] for the case $d = 4$; if $d \ge 5$, ε^{4d} clearly suffices). □

REMARK. Yor and Calais [31] have studied the asymptotic behavior of the double integrals

$$\text{(4.p)} \qquad I_t(f) = \int_0^t ds \int_0^s du \, \{f(B_s - B_u)\},$$

where $f: \mathbb{R}^d \to \mathbb{R}$ is a continuous function with compact support. The fluctuation results for the integrals (4.p) are very close to those obtained previously for the Wiener sausage. Let us consider k functions $f_1, \dots, f_k$. Then

$$\text{(4.q)} \qquad \lim_{t \to \infty} t^{-1}(I_t(f_i); 1 \le i \le k) = (\bar{f}_i\gamma; 1 \le i \le k), \quad \text{if } d = 2,$$

$$\text{(4.r)} \qquad \lim_{t \to \infty} (t \log t)^{-1/2}(I_t(f_i); 1 \le i \le k) = \frac{2^{-1/2}}{\pi}(\bar{f}_i N; 1 \le i \le k), \quad \text{if } d = 3,$$

where convergence holds in distribution, $\bar{f_i} = \int f_i(x)\,dx$ and γ and N have the same meaning as before (see [13] and [29] for proofs of these results). On the other hand, if $d \geq 4$, the following limit result holds:

$$\lim_{t\to\infty} t^{-1/2}(I_t(f_i); 1 \leq i \leq k) = (N_i; 1 \leq i \leq k), \tag{4.s}$$

where $(N_i; 1 \leq i \leq k)$ is a Gaussian vector whose covariance matrix depends on the f_i's and is described in [31]. Note that, in contrast to (4.q) and (4.r), the limit variable N_i in (4.s) "depends on" the function f_i. These results suggest that similar properties hold for the fluctuation of the Wiener sausage and motivated our previous remarks in the Introduction.

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