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# HUNGARIAN CONSTRUCTIONS FROM THE NONASYMPTOTIC VIEWPOINT

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Let $x_1, \dots, x_n$ be independent random variables with uniform distribution over $[0,1]$, defined on a rich enough probability space $\Omega$. Denoting by $\hat{\mathbb{F}}_n$ the empirical distribution function associated with these observations and by $\alpha_n$ the empirical Brownian bridge $\alpha_n(t) = \sqrt{n}(\hat{\mathbb{F}}_n(t) - t)$, Komlós, Major and Tusnády (KMT) showed in 1975 that a Brownian bridge $\mathbb{B}^0$ (depending on $n$) may be constructed on $\Omega$ in such a way that the uniform deviation $\|\alpha_n - \mathbb{B}^0\|_\infty$ between $\alpha_n$ and $\mathbb{B}^0$ is of order of $\log(n)/\sqrt{n}$ in probability. In this paper, we prove that a Poisson bridge $\mathbb{L}_n^0$ may be constructed on $\Omega$ (note that this construction is not the usual one) in such a way that the uniform deviations between any two of the three processes $\alpha_n$, $\mathbb{L}_n^0$ and $\mathbb{B}^0$ are of order of $\log(n)/\sqrt{n}$ in probability. Moreover, we give explicit exponential bounds for the error terms, intended for asymptotic as well as nonasymptotic use.

**1. Introduction.** Let $x_1, \dots, x_n, \dots$ be independent random variables with uniform distribution over $[0,1]$ defined on a "rich enough" probability space $(\Omega, \mathscr{X}, P)$ in the following sense: There is a random variable, defined on $\Omega$, with uniform distribution over $[0,1]$, which is independent of the sequence $(x_j)_{j \geq 1}$.

Let $\hat{\mathbb{F}}_n$ be the empirical distribution function, defined by

$$\hat{\mathbb{F}}_n(t) = \frac{1}{n} \sum_{j=1}^{n} \mathbb{1}_{(x_j \leq t)}$$

for all $n \geq 1$ and $t$ in $[0,1]$, $\hat{\mathbb{F}}_0 \equiv 0$.

Throughout this paper, we shall consider the uniform behaviour of several "tied down" processes, so that the following notation will be useful.

NOTATION 1. For any function $f$: $[0,1] \to \mathbb{R}$, let $f^0$ be the function $t \to f(t) - tf(1)$, defined on $[0,1]$. Moreover we set $\|f\|_\infty = \sup_{t \in [0,1]} |f(t)|$.

As usual, whenever $\mathbb{B}$ is a Brownian motion on $[0,1]$, $\mathbb{B}^0$ is called a Brownian bridge. By analogy let $\sqrt{n}\,\hat{\mathbb{F}}_n^0$ be the empirical Brownian bridge. Finally, given a standard Poisson point process $\mathbb{L}$, we set $\mathbb{L}_n(t) = \mathbb{L}(nt)/\sqrt{n}$ for all $t \in [0,1]$. We call $\mathbb{L}_n^0$ a Poisson bridge with parameter $n$. As it is well known both processes $\sqrt{n}\,\hat{\mathbb{F}}_n^0$ and $\mathbb{L}_n^0$ converge in distribution to $\mathbb{B}^0$, in the Skorohod space $D[0,1]$. As a consequence of the following strong approximation theorem, we get that this convergence holds with rate $\log(n)n^{-1/2}$, in terms of the Lévy–Prohorov distance.

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THEOREM 1. *For each integer $n \geq 2$, there are a standard Poisson process $\mathbb{L}$ and a Brownian motion $\mathbb{B}$ (both depending on $n$), such that, whenever $\mathbb{U} \in \{\sqrt{n}\,\hat{\mathbb{F}}_n, \mathbb{L}_n\}$, we have for all positive $x$,*

$$P\big(\sqrt{n}\|\mathbb{U}^0 - \mathbb{B}^0\|_\infty > x + 12\log(n)\big) \leq 2\exp(-x/6). \tag{1.1}$$

COMMENT. In the above theorem, if we take $\mathbb{U} = \sqrt{n}\,\hat{\mathbb{F}}_n$, we get exactly Theorem 3 of KMT (1975) (with explicit constants). Note in particular that with the above construction, we have

$$\limsup_{n\to\infty}\big(\sqrt{n}/\log(n)\big)\big\|\sqrt{n}\,\hat{\mathbb{F}}_n^0 - \mathbb{B}^0\big\|_\infty \leq 12 \quad \text{a.s.}$$

As a byproduct of Theorem 1, we have for some Poisson process $\mathbb{L}$,

$$P\Big(\sqrt{n}\,\big\|\sqrt{n}\,\hat{\mathbb{F}}_n^0 - \mathbb{L}_n^0\big\|_\infty > x + 24\log(n)\Big) \leq 4\exp(-x/12). \tag{1.2}$$

This is a rather surprising result. In fact, one might think that the usual Kac construction of a Poisson bridge associated with $\hat{\mathbb{F}}$ would be convenient to approximate the empirical Brownian bridge as in (1.2) above. This is not true. More precisely, we show below that the uniform deviation between the Kac construction of a Poisson bridge and the empirical Brownian bridge is exactly of order of $n^{-1/4}$, in probability. Thus, (1.2) provides a new Poisson approximation for the empirical Brownian bridge. We first recall the Kac construction.

Let $p$ be a random variable which has the Poisson law with parameter $n$ [denoted from now on by $\mathscr{P}(n)$] and is independent of $(x_j)_{j\geq 1}$. Then, we set $\mathbb{L}_n = p\hat{\mathbb{F}}_p/\sqrt{n}$. $\mathbb{L}_n^0$ is well known [see, for instance, Shorack and Wellner (1986)] to be a Poisson bridge with parameter $n$, which we call the Kac Poisson bridge associated with $\hat{\mathbb{F}}$ [note that the Kac Poisson approximation of the empirical Brownian bridge may be successfully used to prove multidimensional analogues of Theorem 1; see Massart (1989)].

THEOREM 2. *Let $\mathbb{L}_n^0$ be the Kac Poisson bridge associated with $\hat{\mathbb{F}}$. Then the following quantity* (i) [*respectively,* (ii)] *is an upper* (*respectively, a lower*) *bound for $P(\|\sqrt{n}\,\hat{\mathbb{F}}_n^0 - \mathbb{L}_n^0\|_\infty > xn^{-1/4})$ for all $x > 0$ (respectively, for all $0 < x < 1/9$) and any integer $n \geq 2$:*

(i) $8\exp(-x)$,
(ii) $(1 - 9x)^2/50$.

## 2. Some technical lemmas.

*Exponential and maximal inequalities.* Inequality (2.1) below follows from the classical Cramér–Chernoff calculation and is essentially due to Bennett in the binomial case.

LEMMA 1. *Let $Z$ be a real valued random variable with Poisson or binomial distribution and expectation $m$. Then, for any positive $x$ and any sign $\varepsilon$, we have*

$$P(\varepsilon(Z - m) > x) \le \exp(-mh(\varepsilon x/m)), \tag{2.1}$$

*where $h(t) = (1 + t)\log(1 + t) - t$ for $t > -1$ and $h(t) = +\infty$ for $t \le -1$.*

The proof of Lemma 1 may be found in Shorack and Wellner (1986) (see inequalities 11.1.1 and 11.9.1). Then, following James and Shorack, a maximal inequality may be derived from Lemma 1 via a martingale argument [see Shorack and Wellner (1986) again, inequalities 11.1.2 and 14.5.7].

LEMMA 2. *Let $\mathbb{L}$ be a standard Poisson process and $\mathbb{U} \in \{\sqrt{n}\,\hat{\mathbb{F}}_n, \mathbb{L}_n\}$. The following inequality holds for all $b \in ]0,1[$ and any positive $x$:*

$$P\Big(\sqrt{n} \sup_{t\in[0,b]} |\mathbb{U}^0(t)| > x\Big) \le 2\exp(-nb(1 - b)h(x/(nb))). \tag{2.2}$$

In the Brownian case, though a martingale argument could also lead to an analogue of inequality (2.2) [see Shorack and Wellner (1986), inequality 11.2.2], we shall use the following sharper result of independent interest which is due to Csáki [see Csáki (1974), Theorem 2.1].

LEMMA 3. *Let $\mathbb{B}$ be a Brownian motion. Defining $Q$: $\mathbb{R} \to [0,1]$ by*

$$Q: t \to \frac{1}{\sqrt{2\pi}} \int_t^\infty \exp(-x^2/2)\, dx,$$

*the following identity holds for all $b \in ]0,1]$ and any positive $x$:*

$$P\Big(\sup_{t\in[0,b]} \mathbb{B}^0(t) > x\Big) = Q\left(\frac{x}{\sqrt{b(1-b)}}\right) + \exp(-2x^2)Q\left(\frac{(1-2b)x}{\sqrt{b(1-b)}}\right). \tag{2.3}$$

REMARK 1. If we take $b = 1$ in the above identity, we get the classical result of Kolmogorov, $P(\sup_{t\in[0,1]}\mathbb{B}^0(t) > x) = \exp(-2x^2)$.

REMARK 2. As it is well known, $Q(u) \le (1/2)\exp(-u^2/2)$ for all positive $u$. Thus an immediate consequence of Lemma 3 is

$$P\Big(\sup_{t\in[0,b]} \mathbb{B}^0(t) > x\Big) \le \exp\left(-\frac{x^2}{2b(1-b)}\right), \tag{2.4}$$

where $0 < b \le 1/2$.

The original proof of Lemma 3 by Csáki being rather tortuous, we shall give a short and direct proof of Lemma 3 in the Appendix.

*Normal approximation of the symmetric binomial distribution.* Lemma 4 is due to Tusnády and was already used by Csörgő and Révész (1981) to prove a weakened version of KMT's theorem. Nevertheless, no proof of this result has been published yet as far as we know. So, since it is the key argument for getting Theorem 1, we shall prove it in detail in the Appendix.

NOTATION 2. Let $F$ be a distribution function. For all $t \in ]0,1]$, we set $F^{-1}(t) = \inf\{x\colon F(x) \geq t\}$.

LEMMA 4. *Let $\Phi = 1 - Q$, where $Q$ is defined as in Lemma 3. Let $Y$ be a random variable with distribution function $\Phi$. Denoting by $\Phi_n$ the distribution function of the binomial law $\mathscr{B}(n, 1/2)$, we set $B_n = \Phi_n^{-1} \circ \Phi(Y) - n/2$. Then the following inequalities hold:*

(i) $|B_n| \leq 1 + (\sqrt{n}/2)|Y|$,

(ii) $\left|B_n - (\sqrt{n}/2)Y\right| \leq 1 + Y^2/8$.

## 3. Proof of Theorem 1.

*A large deviation argument.* We first note that, whatever the joint distribution of $(\sqrt{n}\,\hat{\mathbb{F}}_n, \mathbb{L}_n, \mathbb{B})$ may be, the following large deviation inequality always holds.

LEMMA 5. *Let $\mathbb{L}$ be a standard Poisson process and $\mathbb{B}$ be a Brownian motion. Then, if $\mathbb{U} \in \{\sqrt{n}\,\hat{\mathbb{F}}_n, \mathbb{L}_n\}$, we have for all $x \geq 2n/5$,*

$$P\left(\sqrt{n}\|\mathbb{U}^0 - \mathbb{B}^0\|_\infty > x\right) \leq 8\exp(-x/6). \tag{3.1}$$

PROOF. Remark 1 and the Dvoretzky–Kiefer–Wolfowitz (DKW) inequality [see DKW (1956) for the original proof of this inequality and also Hu (1985) for an evaluation of the constant] yield for all positive $u$,

$$P\left(n\|\hat{\mathbb{F}}_n^0\|_\infty > u\right) \leq 6\exp(-2u^2/n), \tag{3.2}$$

$$P\left(\sqrt{n}\|\mathbb{B}^0\|_\infty > u\right) \leq 2\exp(-2u^2/n). \tag{3.3}$$

Thus, taking $\mathbb{U} = \sqrt{n}\,\hat{\mathbb{F}}_n$, (3.1) holds whenever $x \geq n/3$ [and so (3.1) is a fortiori valid when $x \geq 2n/5$].

Now, in order to bound $P(\sqrt{n}\|\mathbb{L}_n^0\|_\infty > u)$, we may always assume that $\mathbb{L}_n^0$ is the Kac Poisson bridge associated with $\hat{\mathbb{F}}$. So let $p$ be a random variable with Poisson law $\mathscr{P}(n)$ which is independent of $(x_j)_{j\geq 1}$. We have for any positive $\rho$,

$$\begin{aligned} P\left(\sqrt{n}\|\mathbb{L}_n^0\|_\infty > u\right) \leq\ & P(p \leq n)\max_{k\leq n} P\left(k\|\hat{\mathbb{F}}_k^0\|_\infty > u\right) \\ & + P(n < p \leq n + \rho u)\max_{k \leq n+\rho u} P\left(k\|\mathbb{F}_k^0\|_\infty > u\right) \\ & + P(p - n > \rho u). \end{aligned}$$

Now, using the (well known?) fact that $P(p \le n) \ge 1/2$, (2.1) and (3.2) yield for any positive $\rho$,

$$P\left(\sqrt{n}\|\mathbb{L}_n^0\|_\infty > u\right) \le 3\left(\exp(-2u^2/n) + \exp(-2u^2/(n+\rho u))\right) + \exp(-nh(\rho u/n)). \tag{3.4}$$

Thus, using (3.3) and (3.4), we get that the following inequality holds for all $\theta \in ]0,1[$ and any positive $\rho$, whenever $x \ge 2n/5$,

$$\begin{aligned} P&\left(\sqrt{n}\left(\|\mathbb{L}_n^0\|_\infty + \|\mathbb{B}^0\|_\infty\right) > x\right) \\ &\le 3\left(\exp(-4\theta^2 x/(2\rho\theta + 5)) + \exp(-4\theta^2 x/5)\right) \\ &\quad + \exp(-(5x/2)h(2\rho\theta/5)) + 2\exp\left(-4(1-\theta)^2 x/5\right). \end{aligned}$$

But we may assume that $x \ge 6\log(8)$, so taking $\theta = 0.54$ and $\rho = 1.8$, we easily get that $4\theta^2 x/5 - 4\theta^2 x/(2\rho\theta + 5) \ge 1.5$ and so

$$P\left(\sqrt{n}\left(\|\mathbb{L}_n^0\|_\infty + \|\mathbb{B}^0\|_\infty\right) > x\right) \le (3.67 + 1 + 2)\exp(-x/6),$$

completing the proof of (3.1). □

As a consequence of Lemma 5 above, we get that, given a fixed integer $n$, it is enough to prove inequality (1.1) for positive $x$ such that

$$6\log(2) \le x \le 2n/5 - 12\log(n). \tag{3.5}$$

[In particular this means $n \ge 164$ and $x + 6\log(n) \ge 34.75$.] What we have to do now is to describe the construction of the empirical, Poisson and Brownian processes.

Let $N$ and $\nu$ be the integers such that $\nu = 2^N \le n < 2\nu$. Using a classical measure theoretical argument [namely Lemma A1 of Berkes and Philipp (1979)], we know that in order to construct $n\hat{\mathbb{F}}_n$, $\sqrt{n}\,\mathbb{L}_n$ and $\sqrt{n}\,\mathbb{B}$ on the same probability space, it is enough to construct their respective increments $X_j$, $Y_j$ and $Z_j$ between time $(j-1)/\nu$ and $j/\nu$ for all $1 \le j \le \nu$. Next, a lemma of Skorohod (1976) ensures that the construction of the three processes may be performed on our rich enough initial probability space $\Omega$.

So, setting $\lambda = n\nu^{-1}$ and given a centered Gaussian random vector $Z$ of $\mathbb{R}^\nu$ with covariance matrix $\lambda I$, we have to define $X$ and $Y$ such that $X$ has the multinomial law $\mathscr{M}(n, \nu^{-1}, \dots, \nu^{-1})$ and $Y_1, \dots, Y_\nu$ are independent with common Poisson law $\mathscr{P}(\lambda)$.

*The dyadic scheme.* Throughout the proof, the intervals $]k, k']$ have to be interpreted as subsets of $\mathbb{N}$. $\mathbb{R}^\nu$ is given the canonical inner product $(\cdot\,|\,\cdot)$. The functions $\mathbb{1}_B$, $B \subset ]0,\nu]$ will be considered as vectors of $\mathbb{R}^\nu$. Given $I_{j,k} = ]k2^j, (k+1)2^j]$, we set $e_{j,k} = \mathbb{1}_{I_{j,k}}$. Then, let $\tilde{e}_{j,k} = e_{j-1,2k} - e_{j,k}/2$. It is easy to verify that the family $\mathscr{B} = \{\tilde{e}_{j,k}\colon 1 \le j \le N,\ 0 \le k < 2^{N-j}\} \cup \{e_{N,0}\}$ is an orthogonal basis of $\mathbb{R}^\nu$ with $(\tilde{e}_{j,k}|\tilde{e}_{j,k}) = 2^{j-2}$ and $(e_{N,0}|e_{N,0}) = \nu$. Moreover we set $W = (Z|e)$ and $\tilde{W} = (Z|\tilde{e})$.

Given any integer $m$, let $\Phi_m$ and $\Phi$ be defined as in Lemma 4 and $\mathscr{P}(m)$ still denotes (abusively) the distribution function of the Poisson law with parameter $m$. We finally set $Q_m = \Phi_m^{-1} \circ \Phi$ and next define either

(i) $U_{N,0} = n$ or
(ii) $U_{N,0} = \mathscr{P}(n)^{-1} \circ \Phi((2^{-N}/\lambda)^{1/2} W_{N,0})$ and then, inductively,

$$U_{j-1,2k} = Q_{U_{j,k}}\Big((2^{-j+2}/\lambda)^{1/2} \tilde{W}_{j,k}\Big) \quad \text{and} \quad U_{j-1,2k+1} = U_{j,k} - U_{j-1,2k}.$$

It is easy to prove by induction that the random vector $\{U_{j,k}: 0 \le k < 2^{N-j}\}$ has either the multinomial distribution $\mathscr{M}(n, 2^{-N+j}, \dots, 2^{-N+j})$ in case (i) or independent components with distribution $\mathscr{P}(n2^{-N+j})$ in case (ii), thus setting $X = (U_{0,0}, \dots, U_{0,\nu-1})$ in case (i) and $Y = (U_{0,0}, \dots, U_{0,\nu-1})$ in case (ii), the constructed $X$ and $Y$ have the desired distributions. Note moreover that either $U = (X|e)$ in case (i) or $U = (Y|e)$ in case (ii). Now, we set $\tilde{U} = (X|\tilde{e})$ in case (i) or $\tilde{U} = (Y|\tilde{e})$ in case (ii). It remains to control the quantity

$$P_0 = P\big(\sqrt{n}\,\|\mathbb{U}^0 - \mathbb{B}^0\|_\infty > x + 12\log(n)\big),$$

where $\mathbb{U} \in \{\sqrt{n}\,\hat{\mathbb{F}}_n, \mathbb{L}_n\}$.

*Regularization at the scale M.* Let $M$ be the least integer such that

$$\rho(x + 6\log(n)) \le \lambda 2^{M+1}, \tag{3.6}$$

where $\rho = 0.29$.

Next, we define, for any $t \in [0,1]$, $\Pi_M(t)$ to be the nearest point of $t$ on the grid$\{j2^{M-N}: 0 \le j \le 2^{N-M}\}$ (in case of ambiguity, take the smallest). Writing $D$ for the difference between either $X$ and $Z$ or $Y$ and $Z$ according to whether case (i) or (ii) is studied and setting $D[m] = \sum_{s=1}^{m} D_s$, we get

$$P_0 \le P_1 + P_2 + P(\Theta^c),$$

where

$$\Theta = \Big\{U_{j,k} \le \lambda(1+\varepsilon)2^j \text{ for all } M+1 < j \le N, 0 \le k < 2^{N-j}\Big\}$$
$$\cap \Big\{U_{j,k} \ge \lambda(1-\varepsilon)2^j \text{ for all } M < j \le N, 0 \le k < 2^{N-j}\Big\}, \quad \text{with } \varepsilon = 0.855,$$

$$P_1 = P\bigg(\sqrt{n}\,\big(\|\mathbb{U}^0 \circ (I - \Pi_M)\|_\infty + \|\mathbb{B}^0 \circ (I - \Pi_M)\|_\infty\big) > \frac{x}{2} + 3\log(n)\bigg)$$

and

$$P_2 = 2^{N-M} \max_{m \in 2^M \mathbb{N} \cap ]0,\nu]} P\bigg(\bigg(\bigg|D[m] - \frac{m}{\nu} D[\nu]\bigg| > \frac{x}{2} + 9\log(n)\bigg) \cap \Theta\bigg).$$

$P_1$ controls the small fluctuations of the processes $\mathbb{U}^0$ and $\mathbb{B}^0$. $P_2$ will be small because the construction of $X, Y, Z$ implies (precisely because of the fundamental Lemma 4) that the error $|D[m] - m/\nu D[\nu]|$ behaves like a $\chi^2(n)$ random variable on the event $\Theta$ which avoids too large deviations of $\mathbb{U}$.

*Control of* $P(\Theta^c)$. Recall that $U_{j,k} = U_{j-1,2k} + U_{j-1,2k+1}$, so

$$\Theta^c \subset \bigcup_{0 \le k < 2^{N-M-2}} \left\{U_{M+2,k} > (1+\varepsilon)\lambda 2^{M+2}\right\} \cup \bigcup_{0 \le k < 2^{N-M-1}} \left\{U_{M+1,k} < (1-\varepsilon)\lambda 2^{M+1}\right\}.$$

Then, as a direct application of inequality (2.1), we get

$$P(\Theta^c) \le 2^{N-M-1}\Big(\exp\big(-\lambda 2^{M+2} h(\varepsilon)\big) + \exp\big(-\lambda 2^{M+1} h(-\varepsilon)\big)\Big).$$

Thus, using (3.6), we have

$$P(\Theta^c) \le \frac{6.9}{(x + 6\log(n))} \exp\left(\frac{-x}{6}\right).$$

*Control of* $P_1$. Since $\mathbb{U}^0$ and $\mathbb{B}^0$ have stationary increments, we may use Lemma 2 and Remark 2. Setting $b = 2^{M-N-1} \le 1/2$, this gives for all $\theta \in ]0,1[$,

$$(3.7)\qquad \begin{aligned} P_1 \le 2^{N-M+2}\Big(&\exp\big(-\theta^2(x + 6\log(n))^2/8nb\big) \\ &+\exp\big(-nb(1-b)h((1-\theta)(x + 6\log(n))/(2nb))\big)\Big). \end{aligned}$$

Now, because of (3.6), we have $2nb \le \rho(x + 6\log(n))$. Using furthermore (3.5), we get that $b \le \rho/5$, so, taking $\theta = 0.44$, (3.7) becomes

$$P_1 \le 2^{N-M+3}/n \exp(-x/6).$$

Using (3.6) again, we finally get

$$P_1 \le \frac{55.2}{(x + 6\log(n))} \exp(-x/6).$$

*Control of* $P_2$. We fix $m \in \mathbb{N}2^M \cap ]0, \nu]$. Let $k(j)$ be such that $m \in I_{j,k(j)}$ [then $k(M) = m2^{-M} - 1$ and $k(j) = [k(j-1)/2]$, where $[z]$ denotes the integral part of $z$]. For the sake of simplicity, any quantity of the type $t_{j,k(j)}$ which was introduced above, will be denoted by $t_j$ in what follows. For instance $\tilde{e}_j = \tilde{e}_{j,k(j)}$, etc. We shall use the following orthogonal expansions on $\mathscr{B}$:

$$(3.8)\qquad \mathbb{1}_{]0,m]} = \sum_{j>M} c_j \tilde{e}_j + \frac{m}{\nu} e_{N,0},$$

where $0 \le c_j \le 1$, $M < j \le N$.

The proof of (3.8) is easy when noting that $\mathbb{1}_{]0,m]}$ is orthogonal to $\tilde{e}_{j,k}$ when $k \ne k(j)$ and that $0 \le (\mathbb{1}_{]0,m]} | \tilde{e}_j) \le 2^{j-2} = (\tilde{e}_j | \tilde{e}_j)$. We have as well

$$(3.9)\qquad e_j = \sum_{s>j} (-1)^{k(s-1)} 2^{j+1-s} \tilde{e}_s + 2^{j-N} e_{N,0}.$$

So, setting $\tilde{\Delta} = (D|\tilde{e})$, we get from (3.8),

$$\left|D[m] - \frac{m}{\nu}D[\nu]\right| \le \sum_{j>M} |\tilde{\Delta}_j|.$$

Now, we set $\xi_j = \sqrt{2^{-j+2}/\lambda}\,\tilde{W}_j$, for all $M < j \le N$. Then, the $\xi$ are independent standard normal random variables. It follows from Lemma 4(ii) that

$$\left|\tilde{U}_j - \frac{\sqrt{U_j}}{2}\xi_j\right| \le 1 + \xi_j^2/8. \tag{3.10}$$

But because $|\sqrt{1+z} - 1| \le |z|/(1 + \sqrt{1 - |z|})$, the following inequality holds on $\Theta$ for all $j \ge M + 1$:

$$C_j = \left|\frac{\sqrt{U_j}}{2}\xi_j - \tilde{W}_j\right| \le \frac{|\xi_j|}{2(1 + \sqrt{1-\varepsilon})}\frac{|U_j - \lambda 2^j|}{\sqrt{\lambda}}2^{-j/2}.$$

Now, setting $\xi_{N+1} = (U_N - n)/\sqrt{n}\,\mathbb{1}_{\{|U_N - n| \le \varepsilon n\}}$, (3.9) and Lemma 4(i) yield

$$\begin{aligned}|U_j - \lambda 2^j| &\le 2\sum_{s=j+1}^{N} 2^{j-s}|\tilde{U}_s| + 2^{j-N}|U_N - n|\\ &\le 2 + \sqrt{\lambda(1+\varepsilon)}\sum_{s=j+1}^{N+1} 2^{j-s/2}|\xi_s|.\end{aligned}$$

Thus,

$$C_j \le 2^{-j/2}|\xi_j| + \frac{\sqrt{1+\varepsilon}}{2(1+\sqrt{1-\varepsilon})}\sum_{s=j+1}^{N+1} 2^{(j-s)/2}|\xi_j|\,|\xi_s|,$$

but $|\xi_j|\,|\xi_s| \le 1/2(\xi_j^2 + \xi_s^2)$ and $\sum_{j=M+1}^{s-1} 2^{j/2} = (1 + \sqrt{2})(2^{s/2} - 2^{(M+1)/2})$, so setting

$$\theta = \frac{(1+\sqrt{2})\sqrt{1+\varepsilon}}{2(1+\sqrt{1-\varepsilon})},$$

we have on $\Theta$

$$\begin{aligned}\sum_{j=M+1}^{N} C_j \le &\sum_{j=M+1}^{N} 2^{-j/2}|\xi_j|\left(1 - \frac{(1+\sqrt{2})}{8}2^{(M+1)/2}|\xi_j|\right)\\ &+\theta\sum_{j=M+1}^{N}\xi_j^2 + \frac{\theta}{2}\xi_{N+1}^2.\end{aligned}$$

Thus, via some easy calculations,

$$\sum_{j=M+1}^{N} C_j \le \sqrt{2}\,2^{-M} + \theta\sum_{j=M+1}^{N}\xi_j^2 + \frac{\theta}{2}\xi_{N+1}^2.$$

Now, because of (3.5) and (3.6) we have $M \ge 1$, so $M \ge \sqrt{2}\,2^{-M}$, hence, using

(3.10), we finally get on $\Theta$,

$$\sum_{j=M+1}^{N} |\tilde{\Delta}_j| \leq N + \left(\frac{1}{8} + \theta\right) \sum_{j=M+1}^{N} \xi_j^2 + \frac{\theta}{2}\xi_{N+1}^2. \tag{3.11}$$

In order to control the variable $\xi_{N+1}^2$, we need the following elementary lemma.

LEMMA 6. *Let $p$ be a random variable with Poisson distribution $\mathscr{P}(r)$ and define $\xi = (p - r)/\sqrt{r}\,\mathbb{1}_{\{|p-r| \leq \varepsilon r\}}$. Then $E(\exp(t\xi^2)) \leq 1 + 2t/(\psi(\varepsilon) - t)$ for all $0 < t < \psi(\varepsilon)$, where $\psi(\varepsilon) = \varepsilon^{-1}((1 + \varepsilon^{-1})\log(1 + \varepsilon) - 1)$.*

PROOF. As a direct application of Lemma 1, we get

$$E\big(\exp(t\xi^2)\big) = 1 + \int_0^{r\varepsilon^2} te^{tu}P\big(|p - r| > \sqrt{ru}\,\big)\,du$$

$$\leq 1 + 2\int_0^{r\varepsilon^2} te^{tu}\exp\Big(-rh\big(\sqrt{u/r}\,\big)\Big)\,du,$$

but $rh(\sqrt{u/r}\,) = u\psi(\sqrt{u/r}\,)$ and $\psi$ is a nonincreasing function, so

$$E\big(\exp(t\xi^2)\big) \leq 1 + 2\int_0^{\infty} t\exp(tu - \psi(\varepsilon)u)\,du,$$

which gives Lemma 6.

We may now finish the control of $P_2$. In fact, as it is well known, we have

$$E\Big(\exp\big(t\xi_j^2\big)\Big) = (1 - 2t)^{1/2} \quad \text{for } j \leq N. \tag{3.12}$$

But the variables $\xi$ are independent, so (3.11) yields via Lemma 6 and (3.12),

$$E\left(\exp\left(\tfrac{1}{3}\sum_{j=M+1}^{N} |\tilde{\Delta}_j|\right)\mathbb{1}_\Theta\right) \leq e^{N/3}\Big(\sqrt{(1 - 2(\theta + 1/8)/3)}\,\Big)^{M-N}$$

$$\times(1 + 2\theta/(6\psi(\varepsilon) - \theta)),$$

so, recalling that $\varepsilon = 0.855$, we have $\theta \leq 1.1907$ and $\psi(\varepsilon) \geq 0.3983$. Thus

$$E\left(\exp\left(\tfrac{1}{3}\sum_{j=M+1}^{N} |\tilde{\Delta}_j|\right)\mathbb{1}_\Theta\right) \leq 3\,2^{2N-(1.512)M}.$$

Then Chernoff's inequality yields, via (3.5) and (3.6),

$$P_2 \leq \left(\frac{10.68}{x + 6\log(n)}\right)^{2.512} \exp\left(-\frac{x}{6}\right).$$

Now, we recall that $x + 6\log(n) \geq 34.56$. Hence, collecting the above estimates, we get (1.1). □

## 4. Proof of Theorem 2.

*The upper bound*. Inequality (i) is a direct consequence of the DKW inequality [see (3.2)] and of Lemma 1. In fact, the distribution of $|n\hat{\mathbb{F}}_n^0 - \sqrt{n}\,\mathbb{L}_n^0|$, conditionally on $p = k$ is exactly equal to the distribution of $|\,|k-n|\mathbb{F}^0_{|k-n|}|$, so the DKW inequality leads to

$$P\big(\|\sqrt{n}\,\hat{\mathbb{F}}^0_n - \mathbb{L}^0_n\|_\infty > xn^{-1/4}|p = k\big) \le 6\exp\big(-2x^2\sqrt{n}/|k-n|\big). \tag{4.1}$$

Now, an upper bound for $P(\|\sqrt{n}\,\hat{\mathbb{F}}^0_n - \mathbb{L}^0_n\|_\infty > xn^{-1/4})$ is given by

$$P\big(|p-n| > 2x\sqrt{n}\big) + \sup_{|k-n|\le 2x\sqrt{n}} P\big(\|\sqrt{n}\,\hat{\mathbb{F}}^0_n - \mathbb{L}^0_n\|_\infty > xn^{-1/4}|p = k\big),$$

which means inequality (i) via 4.1 and Lemma 1.

*The lower bound.* We use the same method as Bretagnolle and Huber (1978) to prove their Proposition 2, page 336.

Let $\Delta = n\hat{\mathbb{F}}_n(1/2) - p\hat{\mathbb{F}}_p(1/2)$ and $\bar{\Delta} = |\Delta - (n-p)/2|$. We first work conditionally on $p$. Then $|\Delta|$ has the binomial distribution $\mathscr{B}(|p-n|, 1/2)$. So

$$E\big(\bar{\Delta}^2|p\big) = |p-n|/4 \tag{4.2}$$

and from Bretagnolle and Huber [(1978), page 339], we get

$$E\big(\bar{\Delta}|p\big) \ge \sqrt{|p-n|/28}\,. \tag{4.3}$$

Besides, the following elementary inequality holds for any $0 < a < b$:

$$P(\bar{\Delta} > a|p) \ge \frac{1}{b}\left(E(\bar{\Delta}|p) - a - \frac{1}{b}E(\bar{\Delta}^2|p)\right).$$

So, (4.2) and (4.3) yield

$$P(\bar{\Delta} > a|p) \ge \frac{1}{b}\left(\sqrt{\frac{|p-n|}{28}} - a - \frac{1}{4b}|p-n|\right).$$

Thus, taking the expectation in both sides of the above inequality, we get

$$P(\bar{\Delta} > a) \ge \frac{1}{b}\left(E\sqrt{\frac{|p-n|}{28}} - a - \frac{1}{4b}E(|p-n|)\right). \tag{4.4}$$

Now, following Bretagnolle and Huber again, we note that, since $\alpha \to \log((E(X^{1/\alpha}))^\alpha)$ is convex for any positive random variable $X$, we have

$$\frac{\sqrt{E(X^2)}}{E(X)} \le \frac{\sqrt{E(X^4)}}{E(X^2)}$$

and so, taking $X = \sqrt{|p-n|}$ and setting $\theta = \sqrt{E(|p-n|)}$, we get

$$\frac{\theta}{E\sqrt{|p-n|}} \le \frac{\sqrt{E(p-n)^2}}{\theta^2}, \tag{4.5}$$

However Stirling's formula implies

$$\theta^2 = 2nP(p = n-1) \ge \exp\left(-\frac{1}{12n}\right)\sqrt{\frac{2n}{\pi}}\,. \tag{4.6}$$

Since furthermore $E((p-n)^2)=n$, using (4.5) and (4.6), (4.4) becomes

$$P(\bar{\Delta}>a)\geq\frac{1}{b}\left(\exp\left(-\frac{1}{12n}\right)\sqrt{\frac{1}{(14\pi)}}\,\theta-a-\frac{\theta^2}{4b}\right),$$

so, choosing

$$a=9x\theta\exp(-1/12n)\sqrt{1/(14\pi)}\quad\text{and}\quad b=\left(\sqrt{14\pi}\,\theta/2(1-9x)\right)\exp(1/12n),$$

we get

$$P(\bar{\Delta}>a)\geq\frac{(1-9x)^2}{14\pi}\exp\left(-\frac{1}{6n}\right)\geq\frac{(1-9x)^2}{50}.$$

Next, (4.6) means that $a\geq xn^{1/4}$, yielding Theorem 2(ii) since of course $\|\sqrt{n}\,\hat{\mathbb{F}}_n^0-\mathbb{L}_n^0\|_\infty\geq\bar{\Delta}/\sqrt{n}$.

## APPENDIX

**1. Proof of Lemma 3.** We shall need the following property of the Brownian motion [see Karlin (1971), page 284].

LEMMA A1. *Let $\mathbb{B}$ be a standard Brownian motion. Then, for any real number and any positive $y$, the following identity holds almost surely*:

$$P\left(\sup_{t\in[0,1]}\mathbb{B}(t)-\alpha t>y|\mathbb{B}(1)\right)$$
$$=\exp\left(-2\left(y^2-(\mathbb{B}(1)-\alpha)^y\right)\right)\mathbb{1}_{\{\mathbb{B}(1)\leq y+\alpha\}}+\mathbb{1}_{\{\mathbb{B}(1)>y+\alpha\}}.$$

Now, we may pass to the proof of Lemma 3. Let $A_b(x)=\{\sup_{t\in[0,b]}\mathbb{B}^0(t)>x\}$. We set $\mathbb{W}(t)=b^{-1/2}\mathbb{B}(bt)$ for any positive $t$ and $\xi=\mathbb{B}(1)-\mathbb{B}(b)$. Then $\mathbb{W}$ is standard Brownian motion and $\xi$ is independent of $\mathbb{W}$ with normal distribution $\mathscr{N}(0,1-b)$. So, since

$$P(A_b(x)|\xi,\mathbb{W}(1))=P\left(\sup_{t\in[0,1]}\mathbb{W}(t)-\left(b\mathbb{W}(1)-\sqrt{b}\,\xi\right)t>\frac{x}{\sqrt{b}}\middle|\xi,\mathbb{W}(1)\right),$$

Lemma A1 yields

$$P(A_b(x)|\xi,\mathbb{W}(1))=\mathbb{1}_{\{U>1\}}+\exp\left(-\frac{2x^2}{b}(1-U)\right)\mathbb{1}_{\{U\leq1\}},$$

where $U=\sqrt{b}/x((1-b)\mathbb{W}(1)-\sqrt{b}\,\xi)$.

Now, $U$ has the normal distribution $\mathscr{N}(0,b(1-b)/x^2)$, so

$$P(A_b(x))=P(U\geq1)+\exp(-2x^2/b)E\left(\mathbb{1}_{\{U\leq1\}}\exp(2x^2U/b)\right).$$

Thus,

$$P(A_b(x)) = Q\left(\frac{x}{\sqrt{b(1-b)}}\right)$$
$$+\exp\left(-\frac{2x^2}{b}\right)\frac{x}{\sqrt{2\pi b(1-b)}}\int_{-\infty}^{1}\exp\left(\frac{2x^2u}{b}-\frac{x^2u^2}{2b(1-b)}\right)du.$$

Setting $v = xu/\sqrt{b(1-b)} - 2x\sqrt{(1-b)/b}$ in the last integral, we easily get (2.3).

**2. Proof of Lemma 4.** We follow the book by Csörgő and Révész [(1981), pages 133 and 134]. It suffices to prove Lemma 4.4.1 (see page 133), which we prefer to write from the right side. Precisely,

TUSNÁDY'S LEMMA. *Let $Y$ be a standard normal random variable and $B_n$ be a binomial random variable with law $\mathscr{B}(n, \frac{1}{2})$. Then the following inequalities hold for any integer $j$, with $0 \le j \le n$:*

(i) $P\{B_n \ge (n+j)/2\} \ge P\{\sqrt{n}\,Y/2 \ge n(1-\sqrt{1-j/n})\}$,
(ii) $P\{B_n \ge (n+j)/2\} \le P\{\sqrt{n}\,Y/2 \ge (j-2)/2\}$.

We set

$$p_{nj} = P(B_n = (n+j)/2) = \binom{n}{(n+j)/2}2^{-n}.$$

In what follows, $n+j$ is always even and $0 \le j \le n$.

Using Stirling's formula, we can expand $p_{nj}$,

$$p_{nj} = CS(x_j, n)\sqrt{2/\pi n}\exp\left(-nh(x_j)/2 - (1/2)\log\left(1-x_j^2\right)\right) \quad \text{as } j < n, \tag{A.1}$$

where $j = nx_j$, $h(x) = (1+x)\log(1+x) + (1-x)\log(1-x)$, and the correction term is defined via

$$CS(x,n) = (1+\beta/12n)(1+\beta'/[6n(1-x)])^{-1}(1+\beta''/[6n(1+x)])^{-1}$$

with $1 \le \beta \le \beta'' \le \beta' \le 1^+ = 12(\sqrt{(e/2\pi)} - 1) \le 1.01325$. (Obviously this is an incorrect statement, since $\beta, \beta', \beta''$ depend on $j$ and $n$, but we use in what follows only the inequality $1 \le \beta \le \beta'' \le \beta' \le 1^+$.) As $CS(x, n)$ is monotonic wrt $\beta', \beta'', \beta$, we have

$$CS(x,n) \ge (1+1/12n)\left\{1 + 1^+/[3n(1-x^2)] + (1^+)^2/\left[36n^2(1-x^2)^2\right]\right\}^{-1}. \tag{A.2.a}$$

Moreover, $CS(x, n)$ is monotonic wrt $x^2$ when $\beta' = \beta''$, monotonic wrt $\beta$ when $\beta = \beta' = \beta''$ and $x = 0$ and finally, as

$$\log\left[(1+y)(1+2y)^{-2}\right] \le -3y + 7y^2/2 \quad \text{for } y \ge 0,$$

setting $12ny = 1$, we get

$$\text{(A.2.b)} \qquad \log CS(x, n) \le -1/4n + 7/288n^2 \quad \text{as } 0 < x < 1.$$

(A.3) PROOF OF (i) WHEN $j^2 \ge 2n$ (OR $x_j \ge \sqrt{2/n}$). Using the classical bound $P\{Y \ge t\} \le (\sqrt{2\pi}\, t)^{-1}\exp(-t^2/2)$, we prove easily (i) when $j = n$ or $j = n - 2$ and $n \ge 5$. Usual tables give the result when $j = n$ or $j = n - 2$ and $n < 5$. As, for $n \le 7$, $(n - 4) < 2n$, we may assume now $2n \le j^2 \le (n - 4)^2$ and $n \ge 8$. Let $f_n(x)$ be $\sqrt{n/2\pi(1 - x)}\, \exp(-2n(1 - \sqrt{1 - x})^2)$. It suffices to prove that

$$p_{nj} \ge \int_{x_j}^{x_{j+2}} f_n(x)\, dx.$$

Between $x_2$ ($n$ even) or $x_3$ ($n$ odd) and $x_{n-4}$, $f_n$ is monotonic as $n \ge 5$, and it remains to prove that when $\sqrt{2/n} \le x \le 1 + 4/n$ and $n \ge 8$,

$$CS(x, n)(1 + x)^{-1/2}\exp\left[n\left(4(1 - \sqrt{1 - x})^2 - h(x)\right)/2\right] \ge 1.$$

Let $J(x)$ be defined as $J(x) = 4(1 - \sqrt{1 - x})^2 - h(x)$. Using bounds (A.1) and (A.2.a) it turns to prove that

$$(1 + 1/12n)\exp(nJ(x)/2) \ge \sqrt{1 + x}\left(1 + 1^+/3n(1 - x^2) + 1^{+2}/36n^2(1 - x^2)^2\right).$$

The two sides are increasing wrt $x$.

When $x \le 1 - 4/n$, $n \ge 8$, the right-hand side is less than 1.48. For $x \ge 0.55$ and $n \ge 8$, the left-hand side is greater than 1.57, the result is thus proved when $0.55 \le x$.

We expand $J$ and get $J(x) \ge x^3/2 + 7x^4/48$. Thus, as $nx^2 \ge 2$ and $n \ge 8$, $nJ(x)/2 \ge x/2 + 7/24n$. Let $K(x)$ be $\exp(x/2)/\sqrt{1 + x}$. $x \mapsto x^{-2}(K(x) - 1)$ is decreasing wrt $x$ in $[0, 1]$ and thus $K(x) \ge 1 + 0.3799/n$ when $\sqrt{2/n} \le x \le 0.55$. It remains to prove that when $x \le 0.55$,

$$(1 + 1/12n)(1 + 0.3799/n)\exp(7/24n)$$
$$\ge \left(1 + 1^+/3n(1 - x^2) + (1^+)^2/36n^2(1 - x^2)^2\right).$$

At $x = 0.55$, the right-hand side is less than $1 + 0.543/n$, thus the result is also proved for $\sqrt{2/n} \le x \le 0.55$ as $0.543 \le 1/12 + 0.3799 + 7/24$. □

(A.4) PROOF OF (i) WHEN $j < \sqrt{2n} - 2$. Using the fact that when $j = 0$ the two sides of (i) are equal if we subtract $p_{n0}/2$ to the left-hand side when $n$ is even, we just have to prove the following "reversed form" in which $p_{n0}/2$ is 0 when $n$ is odd. For $n \ge 1$ and $0 \le j < \sqrt{2n} - 2$,

$$\text{(A.5)} \quad p_{n0}/2 + \sum_{0 < l \le j} p_{nl} \le P\left\{\sqrt{n}\, Y/2 \in \left[0, n\left(1 - \sqrt{1 - (j + 2)/n}\right)\right]\right\}.$$

We construct a family of intervals $I_0, I_1, \ldots, I_k$, such that, when $j < \sqrt{2n} - 2$,

(a) $p_{nj} \le P\{\sqrt{n}\, Y/2 \in I_k\}$ with $j = 2k + 1$ and $j \ge 1$, $n$ odd,
(b) $p_{nj} \le P\{\sqrt{n}\, Y/2 \in I_k\}$ with $j = 2k$ and $j \ge 2$, $n$ even,

(c) $p_{n0}/2 \le p\{\sqrt{n}\,Y/2 \in I_0\}$ for $j = 0$, $n$ even,

(d) the intervals are adjacent, and $I_0 \cup I_1 \cup \cdots \cup I_k \subset [0, n(1 - \sqrt{1 - x_{j+2}})]$.

We set

$$\text{(A.6)}\quad \begin{aligned} \delta_{k+1} &= (k+1)/n + k(k+1/2)(k+1)/n^{3/2}, \quad k \ge 0,\\ \Delta_{k+1} &= \delta_{k+1} + k + 1/2 = \delta_{k+1} + (j+1)/2, \quad k \ge 0,\ n \text{ even},\\ \Delta_{k+1} &= \delta_{k+1} + k + 1 = \delta_{k+1} + (j+1)/2, \quad k \ge 0,\ n \text{ odd},\\ I_k &= [\Delta_k, \Delta_{k+1}] \quad \text{with } \Delta_0 = 0. \end{aligned}$$

We notice that the assumption $0 \le j < \sqrt{2n-2}$ implies $n \ge 4$ (if even) and $n \ge 5$ (if odd).

**Proof of** (d). We just have to prove that $\Delta_{k+1} \le n(1 - \sqrt{1 - x_{j+2}})$.

As $j + 2 \le \sqrt{2n}$, $\delta_{k+1} \le (k+1)/n + k(k+1/2)/n\sqrt{2} \le nx^2/4\sqrt{2} + (1 - 1/\sqrt{2})/n$ with $x = (j+2)/n$. As $\Delta_{k+1} = nx/2 - 1/2 + \delta_{k+1}$, and as $1 - \sqrt{1-x}) \ge x/2 + x^2/8$, it remains to prove that

$$1/2 + nx^2(1/8 - 1/4\sqrt{2}) - (1 - 1/\sqrt{2})/n \ge 0.$$

This is true when $nx^2 \le 2$, and (d) is proved. □

**Proof of** (a)–(c). First we prove that

$$\text{(A.7)}\quad \begin{aligned} p_{nj} &\le \sqrt{2/\pi n}\exp\big[-1/4n + 7/288n^2 - (n-1)j^2/2n^2 \\ &\qquad + (j/n)^{2n}/2n(1 - j^2/n^2)\big],\\ p_{nj} &\le \sqrt{2/\pi n}\exp\big[-0.249/n + 7/288n^2 - (n-1)j^2/2n^2\big] \\ &\qquad \text{when } j < \sqrt{2n} - 2. \end{aligned}$$

In view of (A.1) and (A.2), it suffices to prove, with $x = j/n$, that

$$-\big[nh(x) + \log(1 - x^2) - (n-1)x^2\big]/2 \le x^{2n}/4n(1 - x^2).$$

The left-hand side is expanded as $\sum_{l \ge 2} x^{2l}(1 - n/(2l-1))/2l = A + B$, where $A = \sum_{2 \le l \le n-1}$ and $B = \sum_{l \ge n}$.

$$d^2A/dx^2 = \sum_{2 \le l \le (n+1)/2} (2l - n - 1)(x^{2l-2} - x^{2n-2l}), \quad \textit{negative} \text{ as } 0 \le x \le 1.$$

As $dA/dx = 0$ for $x = 0$, $A$ is negative; finally, $2nB$ is bounded by $x^{2n}/(1 - x^2)$, less than $4 \times 10^{-6}$ if $n \ge 4$ and $x \le [\sqrt{2n} - 2]/n$. □

Second, it is an easy exercise to show that

$$\text{(A.8)}\quad P\{Y \in [a, b]\} \ge \sqrt{1/2\pi}\,(b - a)\exp\big[-a^2/4 - b^2/4\big]\varphi(a, b),$$

where $\varphi(a, b) = \sinh[(b^2 - a^2)/4]/[(b^2 - a^2)/4]$, always greater than 1 (this property will often be used in what follows to "neglect" $\varphi$). For the intervals $I_k$, we get

$$(\text{A.9})\qquad \begin{aligned} &P\{\sqrt{n}\,Y/2 \in I_k\} \\ &\quad\geq \sqrt{2/\pi n}\,\varphi_k \exp\left[-\left(\Delta_{k+1}^2 + \Delta_k^2\right)/n + \log(\Delta_{k+1} - \Delta_k)\right], \\ &\qquad \varphi_k = \varphi\left(2\Delta_{k+1}/\sqrt{n}, 2\Delta_k/\sqrt{n}\right). \end{aligned}$$

We have to prove that the ratio between bounds (A.9) and (A.7) is greater than 1 [or greater than 1/2 in case (c)].

*We first study the case where* $k = 0$. If $n$ is *even*, (c) is equivalent (neglecting $\varphi$) to the positivity of $0.249/n - 7/288n^2 - 1/4n - 1/n^2 - 1/n^3 + \log(1 + 2/n)$. As $\log(1 + u) \geq u - u^2/2$, it suffices to prove that

$$(E)_n = 1.999/n - 3/n^2 - 7/288n^2 - 1/n^3 \geq 0.$$

As $n \geq 4$, $n(E)_n \geq 1.18$.

If $n$ is *odd*, using the same arguments, we have to prove that

$$(E)_n = 0.749/n - 3/n^2 - 7/288n^2 - 1/n^3 \geq 0.$$

This time, as $n \geq 5$, $n(E)_n \geq 0.104$.

*Second, we study the case where* $k \geq 1$. We observe that $j < \sqrt{2n} - 2$ implies $n \geq 10$ (if even) or $n \geq 13$ (if odd). As $k \geq 1$, $\Delta_k$ and $\Delta_{k+1}$ follow the regular form (A.6). Thus, if we set $s = \delta_{k+1} + \delta_k$, $d = \delta_{k+1} - \delta_k$, we get

$$(\text{A.10})\qquad \Delta_{k+1} + \Delta_k = j + s, \qquad \Delta_{k+1} - \Delta_k = 1 + d,$$

$$(\text{A.11})\qquad s = (2k+1)/n + (2k^3 + k)/n^{3/2}, \qquad d = 1/n + 3k^2/n^{3/2},$$

$$(\text{A.12})\qquad \begin{aligned} &\log \varphi_k \geq \log\left(1 + j^2/6n^2\right) \\ &\left(\sinh(u)/u \geq 1 + u^2/6,\ \varphi_k = \sinh(u)/u \right. \\ &\qquad\qquad \left.\text{with } u = \left(\Delta_{k+1}^2 - \Delta_k^2\right)/n \geq j/n\right). \end{aligned}$$

First we observe that

$$(\text{A.13})\qquad d \leq 3/2\sqrt{n}, \qquad \log \varphi_k \geq 0.98 j^2/6n^2, \qquad \log(\Delta_{k+1} - \Delta_k) \geq \lambda d,$$

where $\lambda = 0.9$ when $n$ is even and $n \geq 20$, $\lambda = 0.93$ when $n$ is odd and $n \geq 25$, $\lambda = 0.913$ when $k = 1$ and $n \geq 10$. For the first inequality, we use $2k \leq \sqrt{2n} - 2$, $n \geq 10$; for the third one, in the general case, $2k$ is less than $k_n = \sqrt{2n} - 2$ ($n$ even) or $\sqrt{2n} - 3$ ($n$ odd), and $d_n$ defined as $n \to 1/n + 3k_n^2/n^{3/2}$ is decreasing in both cases. Next, we compute the infimum of $\log(1 + d_n)/d_n$ in both cases. We check the particular case when $k = 1$, $n \geq 10$ exactly as above. The proof of the second one is similar. □

Second, we prove that

$$(\text{A.14})\qquad s \leq 1/n + k/\sqrt{n}.$$

Obvious for $n \geq 14$, using the bound $2k \leq \sqrt{2n} - 2$. If $10 \leq n \leq 14$, $k = 1$ and a direct computation proves the general result. □

After some computation, proving that the ratio between bounds (A.9) and (A.7) is greater than 1 turns to proving that

$$\text{(A.15)}\qquad \begin{aligned} &-\left[js/n + d/n + s^2/2n + d^2/2n + 1/2n + 7/288n^2 - j^2/2n^2\right] \\ &\quad + \left[0.249/n + \log(1+d) + \log \varphi_k\right] \geq 0. \end{aligned}$$

PROOF OF (A.15). (a) We assume first that $n \geq 20$, if even, or $n \geq 25$, if odd. Using bound (A.13) for $d^2/2n$ and $\log(1 + d) + \log \varphi_k$ and bound (A.14) for $js/n$ we get the following lower bound, denoted by $A$, for the left-hand side of (A.15):

$$\text{(A.16)}\qquad A = \alpha\left[k^2/n^{3/2}\right] - 2\beta\left[k/n^{5/4}\right] + \gamma[1/n].$$

When $n$ is even, $j = 2k$ (recall that $\lambda = 0.9$) and

$$\alpha = 0.7 - \{2.5 - 4(0.98)/6\}/n^{1/2} - 3/n,$$

$$\beta = n^{-3/4} + n^{-5/4}/2,$$

$$\gamma = 0.649 - \{17/8 + 7/288\}/n - 1/2n^2.$$

It remains to prove that $\alpha\gamma - \beta^2 \geq 0$ for $n \geq 20$, the infimum is achieved at $n = 20$, where its value is greater than 0.06.

When $n$ is odd, $j = 2k + 1$ (recall that $\lambda = 0.93$). We get

$$\alpha = 0.79 - \{2.5 - 4(0.98)/6\}/n^{1/2} - 3/n,$$

$$\beta = 1/2n^{1/4} + 2(1 - 0.98/6)/n^{3/4} + 1/2n^{5/4},$$

$$\gamma = 0.679 - (3.625 + 7/288 - 0.98/6)/n - 1/2n^2.$$

The infimum of $\alpha\gamma - \beta^2$ is achieved at $n = 25$, where its value is greater than 0.01.

(b) When $k = 1$, we use exact values for $s$ and $d$ given in (A.11) and the second and third inequalities of (A.13) reducing all powers of $n$ greater than 3/2 to the power 3/2, using $n \geq 10$ if even or $n \geq 13$ if odd, and $\lambda = 0.913$. We get that (A.15) is implied by

$$0.662/n - 0.29/n^{3/2} \geq 0, \quad n \text{ even},$$

or

$$0.662/n - 1.72/n^{3/2} \geq 0, \quad n \text{ odd},$$

obviously fulfilled.

(c) The proof of (A.15) is finished when noting that $n$ less than 20 and even, or less than 25 and odd imply that $k \leq 1$. □

The proof (A.4) is now finished. □

END OF THE PROOF OF (i). Let $j^-$ be the greatest integer such that $n + j$ is even, $j^- < \sqrt{2n} - 2$, $j^+$ the least integer such that $n + j$ is even, $j^+ \geq \sqrt{2n}$,

$j_0 = (j^- + j^+)/2$. We have already proved (i) when $j > j_0$ and $j < j_0$. Now, if

$$p_{nj_0} \geq \int_{x_{j_0}}^{x_{j_0}+2} f_n(x)\,dx$$

[see the proof of (A.3) above], we can apply the scheme (A.3) up to $j_0$. If not, as $I_0 \cup I_1 \cup \cdots \cup I_{j_0-1} \leq [0, n(1-\sqrt{1-x_{j_0}})]$, then $I_0 \cup I_1 \cup \cdots \cup I_{j_0-1} \cup [n(1-\sqrt{1-x_{j_0}}), n(1-\sqrt{1-x_{j_0+2}})] \subset [0, n(1-\sqrt{1-x_{j_0+2}})]$ and we can use the "reversed form" (A.4) up to $j_0$. □

PROOF OF (ii). For $n$ odd, (ii) is obvious when $j = 1$. Thus we may assume $j \geq 3$. For $n$ even, (ii) is obvious when $j = 2$. See below the case when $j = 0$, $n$ even. In what follows, we may assume $j \geq 3$.

For $j = n$, the left-hand side's value is $2^{-n}$, and the result is proved when $n \leq 4$, using tables. For $j = n$ and $n > 4$ we use the classical bound

$$P\{Y \geq t\} \geq \left(\sqrt{2\pi}\right)^{-1}\left((t^2-1)/t^3\right)\exp(-t^2/2)$$

and the result is easily proved. We set as usual $x = j/n$. Then, it suffices to prove that $p_{nj} \leq P\{\sqrt{n}\,Y/2 \in [(j-2)/2, j/2]\}$. Using (A.7) and (A.8), where we neglect $\varphi$, it turns to prove

$$-\frac{1}{4n} + \frac{7}{288n^2} - \frac{(n-1)x^2}{2} + \frac{x^{2n}}{2n(1-x^2)} \leq -\frac{n\left[(x-2/n)^2 + x^2\right]}{4},$$

$$\text{with } \frac{3}{n} \leq x \leq 1 - \frac{2}{n}.$$

We remark that $2n(1-x^2) \geq 4$. Thus we have to prove that

$$x - x^2/2 - x^{2n}/4 \geq 3/4n + 7/288n^2$$

when $x \geq 3/n$ and $n \geq 4$, and that is easy to check (the left-hand side being concave, we have to test the inequality at the two extremities).

Last, we have to prove the result when $n$ is even and $j = 2$. Using the symmetry of laws, we have to prove that $p_{n0} \leq P\{|\sqrt{n}\,Y/2| \leq 1\}$, obvious when $n = 2$. If $n \geq 4$, we bound $p_{n0}$ by $\sqrt{2/\pi n}$, the right-hand term being greater than $2\sqrt{2/\pi n}\exp(-2/n)$ by (A.8). Finally, $\exp(-2/n) \geq 1/2$ when $n \geq 4$. □

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