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THE SCALING LIMIT OF SELF-AVOIDING RANDOM WALK IN HIGH DIMENSIONS[1]

BY GORDON SLADE

McMaster University

The Brydges–Spencer lace expansion is used to prove that the scaling limit of the finite-dimensional distributions of self-avoiding random walk in the d-dimensional cubic lattice $\mathbb{Z}^d$ is Gaussian, if d is sufficiently large. It is also shown that the critical exponent γ for the number of self-avoiding walks is equal to 1, if d is sufficiently large.

1. Introduction. A T-step self-avoiding random walk in the d-dimensional cubic lattice $\mathbb{Z}^d$ is an ordered $(T+1)$-tuple $\omega = (\omega(0), \omega(1), \dots, \omega(T))$ with each $\omega(i) \in \mathbb{Z}^d$, $\omega(0) = 0$, $|\omega(i+1) - \omega(i)| = 1$ and $\omega(i) \neq \omega(j)$ for $i \neq j$. A probability measure is defined on the set of T-step self-avoiding walks by assigning equal probability to each walk. Although this model has a very simple definition, very little has been proved about it on a rigorous level. Two basic problems are to find the asymptotic behaviour as $T \to \infty$ of the number, c_T, of T-step self-avoiding walks and the expected value, R_T^2, of $\omega(T)^2$. The asymptotic behaviour of c_T and R_T^2 depends on the dimension d of the lattice, and up until now the rigorous results for these quantities are primarily for very high dimensions. A review of some general approaches to the problem for $d \geq 2$ is Freed (1981). The problem has been studied numerically in lower dimensions, see, for example, Madras and Sokal (1988).

In Kesten (1964) c_T is studied for high dimensions and it is shown that

$$\beta_d \equiv \lim_{T\to\infty} c_T^{1/T} = 2d - 1 - (2d)^{-1} + O(2d)^{-2}.$$

(The limit β_d is known to exist in all dimensions [Hammersley (1957)].) It is believed in fact that $c_T \sim \text{const.}\, \beta_d^T T^{\gamma-1}$, as $T \to \infty$, for some d-dependent exponent γ. We will show that for d sufficiently large, $c_T \sim \text{const.}\, \beta_d^T$ and hence $\gamma = 1$. In Slade (1987) it is shown that for d sufficiently large $R_T^2 \sim DT$ as $T \to \infty$, where the diffusion constant D is greater than 1. This last result reflects the Gaussian nature of self-avoiding walk in high dimensions. It is believed that for $d \geq 4$ the scaling limit of self-avoiding walk is Gaussian (with logarithmic corrections in $d = 4$). This view is supported by the work of Bovier, Felder and Fröhlich (1984), and for a related model, the loop-erased self-avoiding walk, by the work of Lawler (1980, 1986).

In this paper we prove that the scaling limit of the finite-dimensional distributions of self-avoiding walk is Gaussian, if d is sufficiently large. Our

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method is based on the lace expansion [Brydges and Spencer (1985)], which is related to the cluster expansions of statistical mechanics and constructive quantum field theory [Brydges (1986)]. The lace expansion was first used to study the weakly self-avoiding random walk for $d \geq 5$. In the weakly self-avoiding walk model, walks having self-intersections are not assigned probability 0, but rather a slightly smaller probability than walks which do not self-intersect. In Slade (1987) it was shown how the lace expansion could be used to study the strictly self-avoiding walk, obtaining convergence of the expansion by taking d to be large rather than by taking the probability penalty associated with self-intersections to be small as in Brydges and Spencer (1985).

To state our main result precisely, we first introduce some notation. For $t \in [0,1]$ and ω a self-avoiding walk, let

$$X_n(t,\omega) = n^{-1/2}\omega([nt]). \tag{1.1}$$

Expectation with respect to the uniform measure on the T-step self-avoiding walks will be denoted by $\langle \cdot \rangle_T$, and Wiener measure on $\mathbb{R}^d$ by dW. The measure dW is normalized so that $\int e^{ik\cdot B_t}\,dW = \exp[-Dk^2t/2d]$, where D is the diffusion constant referred to previously. Our main result is the following theorem.

THEOREM 1.1. *Let N be a positive integer, $0 < t_1 < t_2 < \cdots < t_N \leq 1$ and let f be a bounded, continuous, real-valued function on $\mathbb{R}^{dN}$. There is an integer $d_0 \geq 5$ such that for $d \geq d_0$,*

$$\lim_{n\to\infty} \langle f(X_n(t_1,\omega),\dots,X_n(t_N,\omega))\rangle_{[nt_N]} = \int f(B_{t_1},\dots,B_{t_N})\,dW.$$

In other words, the finite-dimensional distributions of scaled self-avoiding walk converge weakly to those of Brownian motion, in sufficiently high dimensions. It would be of interest to refine our methods to obtain Theorem 1.1 for $d \geq 4$, as well as to prove tightness and thereby show that self-avoiding walk converges in distribution to Brownian motion in high dimensions. It follows from Theorem 1.1 that for high dimensions, if self-avoiding walk does converge in distribution, it must be to Brownian motion.

No effort has been made here to obtain the best possible value of d_0. In Slade (1987), d_0 is certainly greater than 12, since otherwise one of the norms appearing in Theorem 4.1 there will be infinite. In addition, there are many places, both in Slade (1987) and this paper, where infinite series are dominated by an essentially geometric series whose ratio is proportional to an inverse power of d. By taking d large the infinite series can be controlled, but the value of d_0 becomes elusive, and possibly is larger in this paper than in Slade (1987). The problem of controlling infinite series appears to be a more significant difficulty in extending these methods to $d > 4$ than the divergent norms [which were avoided in Brydges and Spencer (1985)].

In the remainder of this section we introduce the main ideas involved in the proof of Theorem 1.1. Let

$$U_{st}(\omega) = \begin{cases} 0 & \text{if } \omega(s) \neq \omega(t), \\ -1 & \text{if } \omega(s) = \omega(t), \end{cases}$$

let $\tau \geq 0$, let

$$\mathscr{B}_\tau[a, b] = \{st: 0 < t - s \leq \tau, s, t \in [a, b] \cap \mathbb{Z}\}$$

and let

$$K_\tau[a, b] = \prod_{st \in \mathscr{B}_\tau[a, b]} (1 + U_{st}(\omega)). \tag{1.2}$$

Define

$$N_\tau(x, T) = (2d)^{-T} \sum_{\substack{\omega: 0 \to x \\ |\omega| = T}} K_\tau[0, T]. \tag{1.3}$$

The sum in (1.3) is over all *simple* random walks, i.e., nearest-neighbour walks with no self-avoidance constraint, which begin at the origin and end after T steps at $x \in \mathbb{Z}^d$. By convention, $N_\tau(x, 0) = \delta_{x,0}$. Then $(2d)^T N_\tau(x, T)$ is the number of distinct T-step walks from the origin to x which are self-avoiding on time intervals of length less than or equal to the memory τ. $\tau = 0$ corresponds to simple random walk while $\tau \geq T$ corresponds to self-avoiding random walk. The following Fourier and Laplace transforms of $N_\tau(x, T)$ will be distinguished from one another by their arguments. Define

$$N_\tau(k, T) = \sum_{x \in \mathbb{Z}^d} N_\tau(x, T) e^{ik \cdot x}, \qquad k \in [-\pi, \pi]^d, \tag{1.4}$$

and

$$N_\tau(k, z) = \sum_{T=0}^{\infty} N_\tau(k, T) z^T, \qquad z \in \mathbb{C}. \tag{1.5}$$

Then $c_T = (2d)^T N_T(k = 0, T)$.

Let

$$D(k) = d^{-1} \sum_{j=1}^{d} \cos k^{(j)}, \tag{1.6}$$

the $k^{(j)}$ being the components of k. For simple random walk it is well known that

$$N_0(k, T) = D(k)^T. \tag{1.7}$$

Substituting (1.7) into (1.5) gives

$$N_0(k, z) = (1 - zD(k))^{-1}.$$

We define $\Pi_\tau(k, z)$ and $F_\tau(k, z)$ implicitly by

$$N_\tau(k, z) = (1 - zD(k) - \Pi_\tau(k, z))^{-1} = F_\tau(k, z)^{-1}. \tag{1.8}$$

The quantity $\Pi_\tau(k, z)$ is a measure of the difference between the self-avoiding walk and the simple walk. The lace expansion is an expansion of $\Pi_\tau(k, z)$ in a power series in z, which can be used to estimate $\Pi_\tau(k, z)$ and its derivatives. In Section 2 we give a self-contained derivation of the lace expansion. Related ideas are used in the work of Alkhimov (1984).

In Slade (1987) it was shown that for large d, $\Pi_\tau(k, z)$ is analytic in z for z in a disc centred at the origin with radius larger than $r_\tau(0)$, where $r_\tau(k)$ is the radius of convergence of (1.5). Using the Cauchy integral formula and the behaviour of $\Pi_\tau(k, z)$ for z near the pole of $N_\tau(k, z)$ at $r_\tau(k)$, it will be shown in Section 3 by the method of Brydges and Spencer (1985) that for large d,

$$\lim_{n\to\infty} \langle e^{ik\cdot X_n(t,\omega)}\rangle_{[nt]} = \lim_{n\to\infty} \frac{N_{nt}(n^{-1/2}k, [nt])}{N_{nt}(0, [nt])} = \exp[-Dk^2t/2d]. \tag{1.9}$$

By a standard result [Billingsley (1968), Theorem 7.6], weak convergence of probability measures on $\mathbb{R}^k$ is equivalent to pointwise convergence of characteristic functions, so (1.9) implies Theorem 1.1 in the case $N = 1$.

The general case of Theorem 1.1 is obtained in Section 4, by induction on N. Since weak convergence of probability measures is preserved by continuous functions, the conclusion of Theorem 1.1 can be replaced by

$$\begin{aligned}\lim_{n\to\infty} \langle f(X_n(t_1, \omega), X_n(t_2, \omega) - X_n(t_1, \omega), \ldots,\\ X_n(t_N, \omega) - X_n(t_{N-1}, \omega))\rangle_{[nt_N]}\\ = \int f(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_N} - B_{t_{N-1}})\, dW.\end{aligned} \tag{1.10}$$

Again using the fact that weak convergence of probability measures on $\mathbb{R}^k$ is equivalent to pointwise convergence of characteristic functions, it suffices to obtain (1.10) for the special case

$$f(y_1, \ldots, y_n) = \exp\left[i\sum_{j=1}^{N} k_j \cdot y_j\right], \tag{1.11}$$

where each y_j and each k_j is an element of $\mathbb{R}^d$.

Let $\mathbf{k} = (k_1, \ldots, k_N)$, and fix real numbers $a_0, a_1, \ldots, a_N$ with $0 = a_0 < a_1 < \cdots < a_N$. Let $\mathbf{k} \cdot \mathbf{y} = k_1 \cdot y_1 + \cdots + k_N \cdot y_N$ and let

$$\Delta\omega(\mathbf{a}) = (\omega([a_1]), \omega([a_2]) - \omega([a_1]), \ldots, \omega([a_N]) - \omega([a_{N-1}])).$$

Define

$$M_\tau(\mathbf{k}, \mathbf{a}) = (2d)^{-[a_N]} \sum_{\omega,\, |\omega|=[a_N]} e^{i\mathbf{k}\cdot\Delta\omega(\mathbf{a})} K_\tau[0, [a_N]]. \tag{1.12}$$

Then the expectation whose limit is being taken in (1.10), with f given by (1.11), is equal to

$$\frac{M_{nt_N}(n^{-1/2}\mathbf{k}, n\mathbf{t})}{N_{nt_N}(0, [nt_N])},$$

where $\mathbf{t} = (t_1, \ldots, t_N)$. Therefore it suffices to show that for large d,

$$\lim_{n\to\infty} \frac{M_{nt_N}(n^{-1/2}\mathbf{k}, n\mathbf{t})}{N_{nt_N}(0, [nt_N])} = \exp\left[-\frac{D}{2d}\sum_{j=1}^{N} k_j^2(t_j - t_{j-1})\right], \tag{1.13}$$

where $t_0 = 0$.

We obtain (1.13) in Section 4 by induction on N, with the case $N = 1$ being given by (1.9). Suppose that (1.13) holds when N is replaced by $N - 1$. To handle the induction step, we use an expansion for $M_\tau(\mathbf{k}, n\mathbf{t})$ which attempts to decouple the walk on the time intervals $[0, [nt_{N-1}]]$ and $[[nt_{N-1}], [nt_N]]$. In fact, in the first term in the expansion these two portions of the walk are independent of each other. Subsequent terms in the expansion involve three independent subwalks: a self-avoiding walk on an initial time interval $[0, I_1]$, a self-intersecting walk on an interval $I = [I_1, I_2]$ containing $[nt_{N-1}]$ and a self-avoiding walk on $[I_2, [nt_N]]$. It will be shown that the significant terms in the expansion are those for which $|I| = I_2 - I_1 \leq b_n$, where b_n is any sequence satisfying $b_n \to \infty$ and $b_n n^{-1/2} \to 0$, e.g., $b_n = n^{1/4}$. For these relatively short intervals I, the induction step can be taken.

The remainder of this paper is organized as follows. In Section 2 the lace expansion for $\Pi_\tau(k, z)$ is derived, and results from Slade (1987) concerning convergence of the expansion are recalled. A related expansion for $M_\tau(\mathbf{k}, \mathbf{a})$ is also derived. In Section 3 the induction proof is begun by proving (1.9). In Section 4 the expansion for $M_\tau(\mathbf{k}, \mathbf{a})$ is used to advance the induction and obtain (1.13), completing the proof of Theorem 1.1. Finally, in Section 5 it is shown that the critical exponent γ is equal to 1, if d is sufficiently large.

2. The lace expansion. In this section we derive the expansions for $\Pi_\tau(k, z)$ and $M_\tau(\mathbf{k}, \mathbf{a})$ referred to in Section 1, beginning with $\Pi_\tau(k, z)$. Substitution of (1.3) into (1.4), and the result into (1.5), yields

$$N_\tau(k, z) = 1 + \sum_{T=1}^{\infty} \left(\frac{z}{2d}\right)^T \sum_{\omega,\, |\omega| = T} e^{ik\cdot\omega(T)} K_\tau[0, T]. \tag{2.1}$$

Expanding the product (1.2) yields

$$K_\tau[0, T] = \sum_{B \subset \mathscr{B}_\tau[0, T]} \prod_{st \in B} U_{st}. \tag{2.2}$$

We want to perform a partial resummation of (2.2) to insert into (2.1), and to this end introduce several definitions.

A pair $st \in \mathscr{B}_\tau[0, T]$ is called a *bond*. A set B of bonds is a *graph*. A *connected graph* G on $[a, b]$ is a graph consisting of bonds st with s and t in $[a, b]$, such that a and b are in bonds in G and for every $m \in (a, b)$ there is a bond $st \in G$ with $s < m < t$. A *lace* on $[a, b]$ is a connected graph on $[a, b]$ such that the removal of any one bond from the graph results in a graph on $[a, b]$ which is not connected. (Laces are called vines in the graph theory literature [Bondy and Locke (1981)].) The set of laces on $[a, b]$ having all bonds of length τ or less is denoted by $\mathscr{L}_\tau[a, b]$.

We now define a procedure which associates to every connected graph G on $[a, b]$ a corresponding lace $\mathscr{L}(G) \subset G$. $\mathscr{L}(G)$ has bonds $s_1t_1, s_2t_2, \ldots$, where $s_1 = a$, $t_1 = \max\{t\colon\ at \in G\}$, $t_{i+1} = \max\{t\colon\ st \in G,\ s < t_i\}$, $s_i = \min\{s\colon\ st_i \in G\}$. An example is illustrated in Figure 1. Given a lace

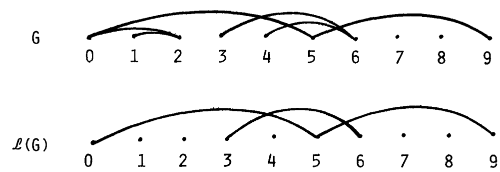

FIG. 1.

$L \in \mathscr{L}_\tau[a, b]$, a bond $st \in \mathscr{B}_\tau[a, b] \setminus L$ is said to be *compatible* with L if $\mathscr{L}(L \cup \{st\}) = L$. The set of bonds compatible with L is denoted $\mathscr{C}_\tau(L)$.

Returning to (2.2), we partition the set of graphs being summed over into two parts: those for which 0 is not in a bond in B and those for which it is. The sum over the former graphs is $K_\tau[1, T]$. If B does have a bond containing 0, let $a(B)$ be the largest value of a such that the set of bonds in B lying in the interval $[0, a]$ forms a connected graph. Then the sum over graphs having a bond containing 0 is

$$\sum_{a=2}^{T} \sum_{G \text{ on } [0, a]} \prod_{st \in G} U_{st} K_\tau[a, T],$$

where the second sum is over connected graphs G. The sum over a begins at $a = 2$ rather than $a = 1$ since $U_{01} = 0$. The sum over connected graphs can be further resummed as

$$\begin{aligned} \sum_{G \text{ on } [0, a]} \prod_{st \in G} U_{st} &= \sum_{L \in \mathscr{L}_\tau[0, a]} \sum_{G:\, \mathscr{L}(G) = L} \prod_{st \in L} U_{st} \prod_{s't' \in G \setminus L} U_{s't'} \\ &= \sum_{L \in \mathscr{L}_\tau[0, a]} \prod_{st \in L} U_{st} \prod_{s't' \in \mathscr{C}_\tau(L)} (1 + U_{s't'}) \equiv J_\tau[0, a]. \end{aligned} \tag{2.3}$$

Thus we have

$$K_\tau[0, T] = K_\tau[1, T] + \sum_{a=2}^{T} J_\tau[0, a] K_\tau[a, T]. \tag{2.4}$$

Substitution of (2.4) into (2.1) leads to

$$\begin{aligned} N_\tau(k, z) = 1 &+ \sum_{T=1}^{\infty} \left(\frac{z}{2d}\right)^T \sum_{|\omega| = T} e^{ik \cdot \omega(T)} K_\tau[1, T] \\ &+ \sum_{a=2}^{\infty} \sum_{T=a}^{\infty} \left(\frac{z}{2d}\right)^T \sum_{|\omega| = T} e^{ik \cdot \omega(T)} J_\tau[0, a] K_\tau[a, T]. \end{aligned} \tag{2.5}$$

In the second term on the right side of (2.5) the sum over ω can be factored into independent sums over walks on the intervals $[0, 1]$ and $[1, T]$. Performing the sum over walks on $[0, 1]$ leads to the value $zD(k)N_\tau(k, z)$ for the second term. Similarly the sum over ω in the third term on the right side of (2.5) can be factored into independent sums over walks on the intervals $[0, a]$ and $[a, T]$. A

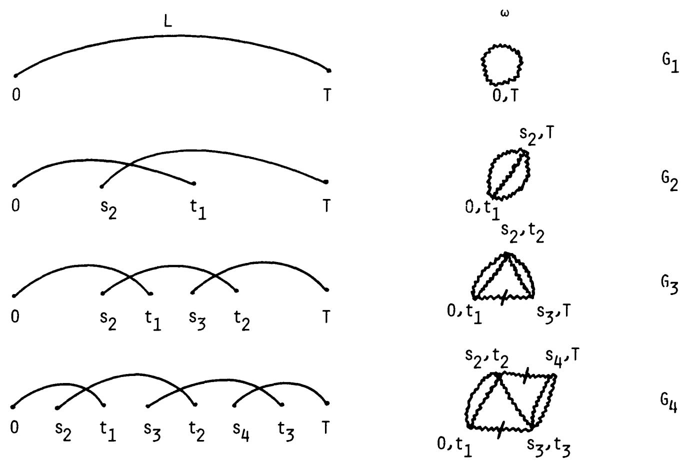

FIG. 2.

little algebra then leads to

$$N_\tau(k,z) = 1 + zD(k)N_\tau(k,z) + \sum_{a=2}^{\infty}\left(\frac{z}{2d}\right)^a \sum_{|\omega|=a} e^{ik\cdot\omega(a)} J_\tau[0,a]N_\tau(k,z). \tag{2.6}$$

Comparing (2.6) with (1.8) yields the lace expansion for $\Pi_\tau(k,z)$ [Brydges and Spencer (1985)]:

$$\Pi_\tau(k,z) = \sum_{T=2}^{\infty}\left(\frac{z}{2d}\right)^T \sum_{\omega,\,|\omega|=T} e^{ik\cdot\omega(T)} J_\tau[0,T]. \tag{2.7}$$

By symmetry the $e^{ik\cdot\omega(T)}$ in this formula could be replaced by $\cos(k\cdot\omega(T))$.

Inserting the expression (2.3) for $J_\tau[0,T]$ into (2.7) gives

$$\Pi_\tau(k,z) = \sum_{T=2}^{\infty}\left(\frac{z}{2d}\right)^T \sum_{\omega,\,|\omega|=T} e^{ik\cdot\omega(T)} \times \sum_{L\in\mathscr{L}_\tau[0,T]} \prod_{st\in L} U_{st} \prod_{s't'\in\mathscr{C}_\tau(L)} (1+U_{s't'}). \tag{2.8}$$

The factor $\prod_{st\in L}U_{st}$ is 0 unless ω has self-intersections as indicated in Figure 2. The generic walk whose topology is consistent with $\prod_{st\in L}U_{st}\neq 0$ for an N-bond lace L is denoted G_N. The walk G_N will be considered to consist of $2N-1$ subwalks on the intervals $[0,s_2],[s_2,t_1],[t_1,s_3],\dots$. Subwalks which may have length 0 are slashed in Figure 2. All unslashed subwalks consist of at least one step.

An upper bound for $|\Pi_\tau(k,z)|$ can be obtained as follows. In (2.8) take absolute values inside all sums. In the product over $\mathscr{C}_\tau(L)$, omit all bonds $s't'$ for which s' and t' are not in the same subwalk. This has the effect of removing the interaction between distinct subwalks. Derivatives of $\Pi_\tau(k, z)$ with respect to z or k can be bounded similarly. It is shown in Brydges and Spencer (1985) and Slade (1987) how to obtain in this way the estimate:

$$|\partial_z^v \partial_k^u \Pi_\tau(k, z)| \le \delta_{u,0} \|\partial_z^v\big(zN_\tau^{(1)}(x, |z|)\big)\|_\infty$$
$$(2.9) \qquad + \sum_{N=2}^{\infty} \sum \prod_{\alpha=0}^{1} \prod_{G_N^{(\alpha)}} \|x^{u_i} \partial_z^{v_j} N_\tau^{(\alpha)}(x, |z|)\|_*.$$

Here

$$N_\tau^{(\alpha)}(x, |z|) = \sum_{T=\alpha}^{\tau} N_\tau(x, T)|z|^T,$$

$G_N^{(0)}$ and $G_N^{(1)}$ are, respectively, the subwalks in G_N which can and cannot have length 0, and $\prod_{\alpha=0}^{1}\prod_{G_N^{(\alpha)}}$ is the product over all subwalks in G_N. The product consists of $2N - 1$ factors, any one of which may be taken to be the x-space L^∞ norm. The other factors are x-space L^2 norms. The unlabelled sum is over ways of choosing nonnegative multiindices u_i such that $\Sigma u_i = u$ and nonnegative v_j such that $\Sigma v_j = v$.

Let $r_\tau(k)$ denote the radius of convergence of (1.5) and let $r_\tau = r_\tau(0)$. Then $r_\tau(k) \ge r_\tau \ge r_0 = 1$. Let

$$D_\tau(a) = \left\{z \in \mathbb{C}: |z| \le r_\tau\big(1 + a\tau^{-1}\ln \tau\big)\right\}.$$

Then by Theorem 4.3 of Slade (1987) we have the following result.

THEOREM 2.1. *There are a $d_0 \ge 5$ and a constant K such that for $d \ge d_0$, $|u| \le 2$ and all τ, $\partial_k^u \Pi_\tau(k, z)$ is analytic in $D_\tau(1)$, with $|\partial_z^v \Pi_\tau(k, z)| \le Kd^{-1}$, $v = 0, 1$, and $|\partial_k^u \Pi_\tau(k, z)| \le Kd^{-5/2}$, $|u| = 1, 2$.*

The proof of this theorem involves estimating the norms of $N_\tau^{(\alpha)}$ in (2.9). The convergence of (2.9) is due to the inverse powers of d occurring in each term in $N_\tau^{(1)}(x, |z|)$. Theorem 2.1 is stated in Slade (1987) only for $D_\tau(\frac{1}{2})$ but the proof gives analyticity in $D_\tau(1)$ without change.

The proof of (1.13) is based on an expansion for $M_\tau(\mathbf{k}, \mathbf{a})$ which we now derive. Any graph $B \subset \mathscr{B}_\tau[0, [a_N]]$ breaks up into connected components in a natural way. Given a graph $B \subset \mathscr{B}_\tau[0, [a_N]]$ and an integer m in the open interval $(0, [a_N])$, let $m_c(B)$ be the interval supporting the connected component of B which passes over m. If B has no bond st with $s < m < t$ we set $m_c(B) = [m, m]$. An example is illustrated in Figure 3. Then by (2.2) and (2.3),

$$\begin{aligned} K_\tau[0, [a_N]] &= \sum_{I \ni m} \sum_{B:\, m_c(B) = I} \prod_{st \in B} U_{st} \\ &= \sum_{I \ni m} K_\tau[0, I_1] \sum_{G \text{ on } I} \prod_{st \in G} U_{st} K_\tau[I_2, [a_N]] \\ &= \sum_{I \ni m} K_\tau[0, I_1] J_\tau[I_1, I_2] K_\tau[I_2, [a_N]]. \end{aligned} \tag{2.10}$$

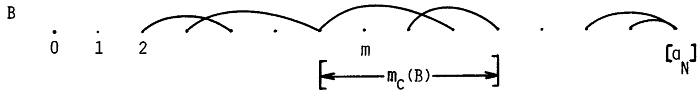

FIG. 3.

The sum over I in (2.10) is a sum over intervals $I = [I_1, I_2]$ with integer endpoints and $0 \le I_1 < m < I_2 \le [a_N]$ or $I_1 = I_2 = m$. (By convention, $J_\tau[m, m] = 1$.) Substituting (2.10) into (1.12) gives

$$M_\tau(\mathbf{k}, \mathbf{a}) = (2d)^{-[a_N]} \sum_{I \ni m} \sum_{\omega: |\omega| = [a_N]} e^{i\mathbf{k}\cdot\Delta\omega(\mathbf{a})} \times K_\tau[0, I_1] J_\tau[I_1, I_2] K_\tau[I_2, [a_N]]. \tag{2.11}$$

The expansion (2.11) will be used in Section 4 to prove (1.13).

3. The distribution of the endpoint. In this section we begin the induction proof of Theorem 1.1 by obtaining (1.9). We suppose throughout this section that $d \ge d_0$ for some sufficiently large d_0. Also, to simplify the notation the brackets denoting the integer part of a real number will be dropped. Thus we write nt in place of $[nt]$, for example; it should be clear from the context what is intended. We also use the notation $\kappa_n = n^{-1/2}k$.

To advance the induction, a slight generalization of (1.9) will be needed. Let g_n be any sequence such that $\lim_{n\to\infty} g_n = 0$, and let $\tilde{t} = t(1 + g_n)$. We will prove that

$$\lim_{n\to\infty} \frac{N_{nt}(\kappa_n, n\tilde{t})}{N_{nt}(0, n\tilde{t})} = \exp[-Dk^2t/2d]. \tag{3.1}$$

The proof uses the Cauchy integral formula to evaluate the numerator and denominator of the left side separately and follows the method of Brydges and Spencer (1985).

We begin by recalling from Corollary 4.2 of Slade (1987) that $|\partial_k^u \partial_z^v \Pi_\tau(k, z)|$ is bounded by a positive inverse power of d for $|u| \le 2$, $v \le 2$, $|u| + 2v \le 4$, uniformly in τ, k and $z \in D_\tau(0)$. It is straightforward to extend these bounds to $z \in D_\tau(b/\ln \tau)$ for any fixed constant b, possibly at the expense of increasing d_0. This follows from the fact that the bounds on Π_τ were obtained from bounds on norms of $N_\tau^{(\alpha)}(x, |z|) = \sum_{T=\alpha}^{\tau} N_\tau(x, T)|z|^T$, and the fact that for $z \in D_\tau(b/\ln \tau)$ and $T \le \tau$, $|z|^T \le r_\tau^T(1 + b\tau^{-1})^T \le r_\tau^T e^b$.

In Lemma 5.1 of Slade (1987) it is shown that there is a constant c such that $r_\tau(k) \le r_\tau(1 + cd^{-1}k^2)$, for $k^2 \le \text{const.}\, d\tau^{-1}\ln \tau$. Since $\kappa_n^2 = n^{-1}k^2 \le n^{-1}\pi^2 d$, $r_{nt}(\kappa_n) \le r_{nt}(1 + c\pi^2 n^{-1})$ and hence $r_{nt}(\kappa_n) \in D_{nt}(b/\ln(nt))$, for $b = c\pi^2 t$.

Differentiating the equation $F_{nt}(\kappa_n, r_{nt}(\kappa_n)) = 0$ twice with respect to k yields

$$\partial_{ij}^2 F_{nt} + \partial_{iz}^2 F_{nt}\, \partial_j r_{nt} + \partial_{jz}^2 F_{nt}\, \partial_i r_{nt} + \partial_z F_{nt}\, \partial_{ij}^2 r_{nt} + \partial_z^2 F_{nt}\, \partial_i r_{nt}\, \partial_j r_{nt} = 0. \tag{3.2}$$

This equation can be used to obtain a formula for $\partial_{ij}^2 r_{nt}(\kappa_n)$. Since $r_{nt}(k)$ is an even function of k, and since by the previous remarks all of the derivatives of F_{nt} appearing in (3.2) are bounded, it follows that the second, third and fifth terms in (3.2) all vanish in the limit $n \to \infty$.

The proof of (3.1) uses the following lemma.

LEMMA 3.1. *$\lim_{n\to\infty}[r_{nt}(\kappa_n)/r_{nt}(0)]^{nt} = \exp[Dk^2t/2d]$, if d is sufficiently large.*

PROOF. By Taylor's theorem,

$$r_{nt}(\kappa_n) = r_{nt} + \frac{1}{2n}\sum_{i,\,j=1}^{d} \partial_{ij}^2 r_{nt}(\kappa) k^{(i)} k^{(j)},$$

for some κ on the line segment joining the origin to κ_n. It suffices to show that

$$\lim_{n\to\infty} r_{nt}^{-1}\,\partial_{ij}^2 r_{nt}(\kappa) = \delta_{ij} D d^{-1}. \tag{3.3}$$

In view of the remark following (3.2),

$$\lim_{n\to\infty} \partial_{ij}^2 r_{nt}(\kappa) = -\lim_{n\to\infty} \frac{\partial_{ij}^2 F_{nt}(\kappa, r_{nt}(\kappa))}{\partial_z F_{nt}(\kappa, r_{nt}(\kappa))}.$$

A straightforward argument using Taylor's theorem shows that $\partial_{ij}^2 F_{nt}(\kappa, r_{nt}(\kappa))$ and $\partial_z F_{nt}(\kappa, r_{nt}(\kappa))$ have the same limits as $\partial_{ij}^2 F_{nt}(0, r_{nt})$ and $\partial_z F_{nt}(0, r_{nt})$, and hence

$$\lim_{n\to\infty} \partial_{ij}^2 r_{nt}(\kappa) = \lim_{n\to\infty} \partial_{ij}^2 r_{nt}(0).$$

By (1.3)–(1.5) and (1.8) $F_\tau(k, z)$ is invariant under replacement of $k^{(j)}$ by $-k^{(j)}$, and hence $\partial_{ij}^2 r_{nt}(0) = 0$ if $i \neq j$. By symmetry and the fact [proved in Section 5 of Slade (1987)] that $D = \lim_{t\to\infty} r_\tau^{-1} \nabla_k^2 r_\tau(0)$, the lemma follows. □

The following theorem yields (1.9) and is general enough to allow the induction step to be taken.

THEOREM 3.2. *Let h_n be any nonnegative sequence such that $\lim_{n\to\infty} h_n = 0$ and $g = (g_n)$ be any sequence with $|g_n| \le h_n$ for all n. Let $\tilde{t} = t(1 + g_n)$. Then for $d \ge d_0$,*

$$\lim_{n\to\infty} \frac{N_{nt}(\kappa_n, n\tilde{t})}{N_{nt}(0, n\tilde{t})} = \exp[-Dk^2t/2d] \quad \textit{uniformly in } g.$$

PROOF. By Lemma 5.1 of Slade (1987), for $k^2 \le \text{const.}\, d\tau^{-1}\ln\tau$, $N_\tau(k, z)$ is analytic in $z \in D_\tau(1)$ except at the simple pole $r_\tau(k) \in D_\tau(\frac{1}{4})$. [In Lemma 5.1 the result is stated only for $z \in D_\tau(\frac{1}{2})$ but the same proof in fact gives the result for $z \in D_\tau(1)$.] Choose n so large that $\kappa_n^2 \le \text{const.}\, d(nt)^{-1}\ln(nt)$. Let C be the circle of radius $\frac{1}{2}$ centred at the origin of the complex plane, with counterclockwise orientation. Then by the Cauchy integral formula and the residue theorem,

$$
\begin{aligned}
N_{nt}(\kappa_n, n\tilde{t}) &= \frac{1}{2\pi i}\oint_C N_{nt}(\kappa_n, z) z^{-n\tilde{t}-1}\,dz \\
&= \frac{r_{nt}(\kappa_n)^{-n\tilde{t}-1}}{-\partial_z F_{nt}(\kappa_n, r_{nt}(\kappa_n))} \qquad (3.4)\\
&\quad\times\left[1 - \frac{\partial_z F_{nt}(\kappa_n, r_{nt}(\kappa_n))}{2\pi i}\oint_{\partial D_{nt}(1)} N_{nt}(\kappa_n, z)\left[\frac{r_{nt}(\kappa_n)}{z}\right]^{n\tilde{t}+1} dz\right].
\end{aligned}
$$

To estimate the second term in the square brackets on the right side of (3.4), we first note that $\partial_z F_{nt}(\kappa_n, r_{nt}(\kappa_n))$ is bounded by Theorem 2.1. Taking absolute values inside the integral, the factor $|r_{nt}(\kappa_n)/z|^{n\tilde{t}+1}$ is equal to

$$|r_{nt}(\kappa_n)/r_{nt}|^{n\tilde{t}+1}|1 + (nt)^{-1}\ln(nt)|^{-n\tilde{t}-1}.$$

By Lemma 3.1 and the fact that $|g_n| \le h_n$, given $\varepsilon > 0$ this can be bounded above by $O(n^{-1+\varepsilon})$ uniformly in g. Since $N_{nt}(\kappa_n, z)$ has a simple pole at $r_{nt}(\kappa_n)$, the integral of $|N_{nt}(\kappa_n, z)|$ around $\partial D_{nt}(1)$ diverges like ln|distance from $r_{nt}(\kappa_n)$ to $\partial D_{nt}(1)| \sim \ln n$. Therefore

$$N_{nt}(\kappa_n, n\tilde{t}) = \frac{r_{nt}(\kappa_n)^{-n\tilde{t}-1}}{-\partial_z F_{nt}(\kappa_n, r_{nt}(\kappa_n))}\left[1 + O(n^{-1+\varepsilon})\right]. \qquad (3.5)$$

It can be shown that $\partial_z F_{nt}(\kappa_n, r_{nt}(\kappa_n))$ and $\partial_z F_{nt}(0, r_{nt})$ have the same limit as $n \to \infty$, and hence the theorem follows from (3.5) and Lemma 3.1. □

4. The finite-dimensional distributions. In this section we complete the proof of Theorem 1.1 by obtaining (1.13). The proof is by induction on N, with the case $N = 1$ having been obtained in Theorem 3.2. Throughout this section the assumption $d \ge d_0$ is implicit, for some sufficiently large d_0. We use the notation $\boldsymbol{\kappa}_n = n^{-1/2}\mathbf{k}$, and omit brackets denoting the integer part of a real number.

As for the case $N = 1$, some flexibility in the number of steps in the walk is needed to perform the induction step. Let B be any fixed positive constant, let $g = (g_n)$ be any sequences such that $-Bn^{-1/2} \le g_n \le 0$, let $t_0 = 0$, and let $\tilde{\mathbf{t}} = (t_1, t_2, \dots, t_{N-1}, \tilde{t}_N)$, where $\tilde{t}_N = t_N(1 + g_n)$. We prove the following slight generalization of (1.13):

$$\lim_{n\to\infty}\frac{M_{nt_N}(\boldsymbol{\kappa}_n, n\tilde{\mathbf{t}})}{N_{nt_N}(0, n\tilde{t}_N)} = \exp\left[-\frac{D}{2d}\sum_{j=1}^{N} k_j^2(t_j - t_{j-1})\right] \quad \text{uniformly in } g. \qquad (4.1)$$

Our induction hypothesis is that (4.1) holds when $N \ge 2$ is replaced by $N - 1$. We shall prove that this implies (4.1).

By (2.11) with $m = nt_{N-1}$,

$$
\begin{aligned}
M_{nt_N}(\boldsymbol{\kappa}_n, n\tilde{\mathbf{t}}) &= (2d)^{-n\tilde{t}_N}\sum_{I \ni nt_{N-1}}\ \sum_{\omega,\, |\omega| = n\tilde{t}_N} e^{i\boldsymbol{\kappa}_n\cdot\Delta\omega(n\tilde{\mathbf{t}})} \qquad (4.2)\\
&\qquad\times K_{nt_N}[0, I_1]J_{nt_N}[I_1, I_2]K_{nt_N}[I_2, n\tilde{t}_N].
\end{aligned}
$$

In (4.2) we factor the walk ω into three independent subwalks on the time intervals $[0, I_1]$, $I = [I_1, I_2]$ and $[I_2, n\tilde{t}_N]$. The first and third of these are forced to be self-avoiding by the factors of K, while the factor J forces the second to intersect itself. Denoting the components of κ_n by $\kappa^{(1)}, \ldots, \kappa^{(N)}$, the exponential in (4.2) can be factored as

$$\begin{aligned}(4.3)\quad &\exp\left[i\sum_{j=1}^{N-2}\kappa^{(j)}\cdot\left(\omega(nt_j)-\omega(nt_{j-1})\right)+i\kappa^{(N-1)}\cdot\left(\omega(I_1)-\omega(nt_{N-2})\right)\right]\\ &\times\exp\left[i\kappa^{(N-1)}\cdot\left(\omega(nt_{N-1})-\omega(I_1)\right)+i\kappa^{(N)}\cdot\left(\omega(I_2)-\omega(nt_{N-1})\right)\right]\\ &\times\exp\left[i\kappa^{(N)}\cdot\left(\omega(n\tilde{t}_N)-\omega(I_2)\right)\right].\end{aligned}$$

It will be shown that the dominant contribution to (4.2) comes from intervals I with $|I| \le b_n$, where b_n is any fixed sequence satisfying $\lim_{n\to\infty} b_n = \infty$ and $\lim_{n\to\infty} b_n n^{-1/2} = 0$, e.g., $b_n = n^{1/4}$. For such I and for sufficiently large n,

$$0 < nt_1 < \cdots < nt_{N-2} < I_1 \le nt_{N-1} \le I_2 < n\tilde{t}_N.$$

Substituting (4.3) into (4.2) and summing over only those I with $|I| \le b_n$ gives

$$\begin{aligned}(4.4)\quad \Sigma_{\mathbf{k}}^{\le} \equiv &\sum_{\substack{I\ni nt_{N-1}\\ |I|\le b_n}} M_{nt_N}\left(\kappa^{(1)},\ldots,\kappa^{(N-1)}; nt_1,\ldots,nt_{N-2}, I_1\right)(2d)^{-|I|}\\ &\times\sum_{\omega,\,|\omega|=|I|} E(\omega, I) J_{nt_N}[0,|I|] N_{nt_N}\left(\kappa^{(N)}, n\tilde{t}_N - I_2\right).\end{aligned}$$

In (4.4) the first and third subwalks referred to in the last paragraph have been summed over, the time scale for the second subwalk has been shifted to begin at 0 and

$$E(\omega, I) = \exp\left[i\kappa^{(N-1)}\cdot\omega(nt_{N-1} - I_1) + i\kappa^{(N)}\cdot\left(\omega(|I|) - \omega(nt_{N-1} - I_1)\right)\right].$$

Now $\kappa^{(j)} = n^{-1/2}k_j$, and for $|I| \le b_n$, $|\omega(nt_{N-1} - I_1)| \le b_n$ and $|\omega(|I|)| \le b_n$. It follows that

$$(4.5)\qquad E(\omega, I) = 1 + f(\omega, I),$$

where $|f(\omega, I)| \le O(b_n n^{-1/2})$ uniformly in ω and $|I| \le b_n$. Also, for $|I| \le b_n$, $I_1 = nt_{N-1}(1 - O(n^{-1}b_n)) \le nt_{N-1}$, and so by the induction hypothesis

$$\begin{aligned}&M_{nt_N}\left(\kappa^{(1)},\ldots,\kappa^{(N-1)}; nt_1,\ldots,nt_{N-2}, I_1\right)\\ &= M_{nt_{N-1}}\left(\kappa^{(1)},\ldots,\kappa^{(N-1)}; nt_1,\ldots,nt_{N-2}, I_1\right)\\ (4.6)\quad &= N_{nt_{N-1}}(0, I_1)\left[\exp\left[-\frac{D}{2d}\sum_{j=1}^{N-1}k_j^2(t_j - t_{j-1})\right] + F(I)\right]\\ &= N_{nt_N}(0, I_1)\left[\exp\left[-\frac{D}{2d}\sum_{j=1}^{N-1}k_j^2(t_j - t_{j-1})\right] + F(I)\right],\end{aligned}$$

where $|F(I)| \le o(1)$ uniformly in $|I| \le b_n$. Similarly, for $|I| \le b_n$, $n\tilde{t}_N - I_2 =$

$n(t_N - t_{N-1})(1 - |O(g_n)| - O(n^{-1}b_n)) \le n(t_N - t_{N-1})$, and so by Theorem 3.2 it follows that

$$
\begin{aligned}
N_{nt_N}\left(\kappa^{(N)}, n\tilde{t}_N - I_2\right) &= N_{n(t_N - t_{N-1})}\left(\kappa^{(N)}, n\tilde{t}_N - I_2\right) \\
&= N_{n(t_N - t_{N-1})}(0, n\tilde{t}_N - I_2) \\
&\quad \times \left(\exp\left[-\frac{D}{2d}k_N^2(t_N - t_{N-1})\right] + h(I)\right) \\
&= N_{nt_N}(0, n\tilde{t}_N - I_2)\left(\exp\left[-\frac{D}{2d}k_N^2(t_N - t_{N-1})\right] + h(I)\right),
\end{aligned} \tag{4.7}
$$

where $|h(I)| \le o(1)$ uniformly in $|I| \le b_n$.

Substituting (4.5)–(4.7) into (4.4) leads to

$$
\Sigma_{\mathbf{k}}^{\le} = \exp\left[-\frac{D}{2d}\sum_{j=1}^{N} k_j^2(t_j - t_{j-1})\right]\Sigma_{\mathbf{0}}^{\le} + A, \tag{4.8}
$$

where

$$
\begin{aligned}
|A| &\le o(1) \sum_{\substack{I \ni nt_{N-1} \\ |I| \le b_n}} N_{nt_N}(0, I_1)(2d)^{-|I|} \\
&\quad \times \sum_{|\omega| = |I|} |J_{nt_N}[0, |I|]| N_{nt_N}(0, n\tilde{t}_N - I_2) \\
&\equiv o(1)\Sigma_{\text{a.v.}}^{\le}.
\end{aligned} \tag{4.9}
$$

The subscript a.v. in (4.9) stands for absolute value. Using the notation $M_{nt_N}(\kappa, n\tilde{\mathbf{t}}) \equiv \Sigma_{\mathbf{k}}^{\le} + \Sigma_{\mathbf{k}}^{>}$ (the first term involves the sum over $|I| \le b_n$ and the second involves $|I| > b_n$), it follows from (4.8) that

$$
\begin{aligned}
\frac{M_{nt_N}(\kappa, n\tilde{\mathbf{t}})}{N_{nt_N}(0, n\tilde{t}_N)} &= \frac{\Sigma_{\mathbf{k}}^{\le} + \Sigma_{\mathbf{k}}^{>}}{N_{nt_N}(0, n\tilde{t}_N)} \\
&= \frac{\exp\left[-(D/2d)\Sigma_{j=1}^{N} k_j^2(t_j - t_{j-1})\right]\Sigma_{\mathbf{0}}^{\le}}{N_{nt_N}(0, n\tilde{t}_N)} \\
&\quad + \frac{A}{N_{nt_N}(0, n\tilde{t}_N)} + \frac{\Sigma_{\mathbf{k}}^{>}}{N_{nt_N}(0, n\tilde{t}_N)} \\
&\equiv \exp\left[-\frac{D}{2d}\sum_{j=1}^{N} k_j^2(t_j - t_{j-1})\right]T_1 + T_2 + T_3.
\end{aligned} \tag{4.10}
$$

To complete the proof of (4.1), it suffices to show that as $n \to \infty$, $T_1 \to 1$, $T_2 \to 0$ and $T_3 \to 0$, uniformly in g. Since

$$T_1 = 1 - \frac{\Sigma_0^>}{N_{nt_N}(0, n\tilde{t}_N)},$$

$$|T_2| \le \frac{o(1)\Sigma_{\text{a.v.}}^{\le}}{N_{nt_N}(0, n\tilde{t}_N)}$$

and

$$|T_3| \le \frac{\Sigma_{\text{a.v.}}^{>}}{N_{nt_N}(0, n\tilde{t}_N)},$$

where $\Sigma_{\text{a.v.}}^{>}$ is defined as in (4.9) with the sum over $|I| \le b_n$ replaced by the sum over $|I| > b_n$, it suffices to prove the following lemma.

LEMMA 4.1. (a) $\lim_{n\to\infty} N_{nt_N}(0, n\tilde{t}_N)^{-1}\Sigma_{\text{a.v.}}^{>} = 0$ *uniformly in* g.
(b) $N_{nt_N}(0, n\tilde{t}_N)^{-1}\Sigma_{\text{a.v.}}^{\le}$ *is bounded uniformly in* n *and* g.

PROOF. (a) By (3.5), $r_{nt_N}^{n\tilde{t}_N+1}N_{nt_N}(0, n\tilde{t}_N)$ is bounded above and bounded below away from 0, so it suffices to show that

$$\lim_{n\to\infty} r_{nt_N}^{n\tilde{t}_N+1}\Sigma_{\text{a.v.}}^{>} = 0. \tag{4.11}$$

It is shown in Slade (1987) that $r_\infty = \lim_{\tau\to\infty} r_\tau$ exists and that $r_\infty - r_\tau \le O(\tau^{-1})$. This means that

$$\left(\frac{r_{nt_N}}{r_\infty}\right)^{n\tilde{t}_N} = \left(1 - \frac{r_\infty - r_{nt_N}}{r_\infty}\right)^{n\tilde{t}_N}$$

remains bounded above and bounded below away from 0 as $n \to \infty$, and hence (4.11) follows from

$$\lim_{n\to\infty} r_\infty^{n\tilde{t}_N+1}\Sigma_{\text{a.v.}}^{>} = 0. \tag{4.12}$$

To prove (4.12), we first note that by definition of $\Sigma_{\text{a.v.}}^{>}$,

$$\begin{aligned} &r_\infty^{n\tilde{t}_N+1}\Sigma_{\text{a.v.}}^{>} \\ &\quad = r_\infty^{n\tilde{t}_N+1} \sum_{\substack{I\ni nt_{N-1}\\ |I|>b_n}} N_{I_1}(0, I_1)(2d)^{-|I|} \\ &\qquad \times \sum_{\omega,\,|\omega|=|I|} |J_{nt_N}[0, |I|]| N_{n\tilde{t}_N-I_2}(0, n\tilde{t}_N - I_2). \end{aligned} \tag{4.13}$$

By (3.5), for some constant C,

$$\begin{aligned} &r_\infty^{n\tilde{t}_N+1}\Sigma_{\text{a.v.}}^{>} \\ &\quad \le r_\infty^{n\tilde{t}_N+1} C \sum_{\substack{I\ni nt_{N-1}\\ |I|>b_n}} r_{I_1}^{-I_1-1} r_{n\tilde{t}_N-I_2}^{-n\tilde{t}_N+I_2-1}(2d)^{-|I|} \sum_{\omega,\,|\omega|=|I|} |J_{nt_N}[0, |I|]|. \end{aligned} \tag{4.14}$$

Arguing as before to replace r_{I_1} and $r_{n\tilde{t}_N - I_2}$ by r_∞ gives

$$(4.15)\qquad \begin{aligned} r_\infty^{n\tilde{t}_N+1}\Sigma^>_{\text{a.v.}} &\le C_1 \sum_{\substack{I \ni nt_{N-1}\\ |I|>b_n}} (2d)^{-|I|} r_\infty^{|I|-1} \sum_{\omega,\,|\omega|=|I|} |J_{nt_N}[0,|I|]| \\ &\le C_1 \sum_{T=b_n+1}^{\infty} (2d)^{-T} T r_\infty^{T-1} \sum_{\omega,\,|\omega|=T} |J_{nt_N}[0,T]|. \end{aligned}$$

From (2.7) it can be seen that the sum on the right side of (4.15) is the tail of the lace expansion for $\partial_z \Pi_{nt_N}(k, r_\infty)$, with absolute values taken inside the sums. The lace expansion for $\partial_z \Pi_{nt_N}(k, r_\infty)$, with absolute values taken inside the sums, converges uniformly in n by Theorem 2.1 since $r_\infty = r_{nt_N} + O(n^{-1}) \in D_{nt_N}(1)$. (In the proof of Theorem 2.1 the lace expansion was shown to be absolutely convergent.) We now show that the right side of (4.15) goes to 0 as $n \to \infty$.

The lace expansion is estimated by first writing the sum over laces in (2.8) as a sum over the number N of bonds in a lace, followed by a sum over all laces which consist of exactly N bonds. Denoting by $\mathscr{L}_\tau^N[a,b]$ the set of N-bond laces on $[a,b]$ whose bonds are of length τ or less, for any positive integer M we have

$$(4.16)\qquad \begin{aligned} &\sum_{T=b_n+1}^{\infty} (2d)^{-T} r_\infty^{T-1} T \sum_{|\omega|=T} |J_\tau[0,T]| \\ &\quad\le \sum_{N=1}^{M} \sum_{T=b_n+1}^{\infty} (2d)^{-T} r_\infty^{T-1} T \\ &\qquad \times \sum_{|\omega|=T} \sum_{L \in \mathscr{L}_\tau^N[0,T]} \Big| \prod_{st \in L} U_{st} \Big| \prod_{s't' \in \mathscr{C}_\tau(L)} (1 + U_{s't'}) \\ &\qquad + \sum_{N=M+1}^{\infty} \sum_{T=1}^{\infty} (2d)^{-T} r_\infty^{T-1} T \\ &\qquad \times \sum_{|\omega|=T} \sum_{L \in \mathscr{L}_\tau^N[0,T]} \Big| \prod_{st \in L} U_{st} \Big| \prod_{s't' \in \mathscr{C}_\tau(L)} (1 + U_{s't'}). \end{aligned}$$

Given an $\varepsilon > 0$, the second term on the right side of (4.16) can be made less than $\varepsilon/2$ by choosing M sufficiently large, independent of τ, since it is bounded above [as in (2.9)] by

$$\|\partial_z N_\tau^{(1)}(x, r_\infty)\|_\infty \sum_{N=M+1}^{\infty} (2N-1) \|N_\tau^{(1)}(x, r_\infty)\|_2^N \|N_\tau^{(0)}(x, r_\infty)\|_2^{N-2},$$

which goes to 0 as $M \to \infty$ uniformly in τ, by Corollary 4.2 of Slade (1987) (extended as remarked in the third paragraph of Section 3).

It remains to show that the first term on the right side of (4.16) can be made less than $\varepsilon/2$ by choosing n large, independent of τ. This term is given by a sum of M diagrams, consisting of $2N-1$ subwalks, $1 \le N \le M$. Each subwalk is bounded by a norm of $\partial_z^v N_\tau^{(\alpha)}(x, r_\infty)$, $v = 0, 1$, as in (2.9). At least one subwalk in each diagram consists of at least $b_n(2M-1)^{-1}$ steps. At the expense of an additional z-derivative, a factor of $b_n^{-1}(2M-1)$ can be extracted from the norm bounding such a long subwalk, yielding an overall factor of b_n^{-1} [cf. the proof that $r_\infty - r_T \le O(T^{-2})$ in Section 5]. The quantity multiplying b_n^{-1} involves norms of $\partial_z^v N_\tau^{(\alpha)}(x, r_\infty)$, $\alpha = 0, 1$; $v = 0, 1, 2$, and is bounded uniformly in τ, again by Corollary 4.2 of Slade (1987). Thus the first term on the right side of (4.16) can be made less than $\varepsilon/2$ by taking n large, independently of τ.

It follows that the right side of (4.15) goes to 0 at $n \to \infty$.

(b) The proof of part (b) is essentially the same as the beginning of the proof of part (a). The only difference is that in (4.13)–(4.15) the sums over $|I| > b_n$ and $T > b_n$ are replaced by sums over $|I| \le b_n$ and $T \le b_n$. The sum over $T \le b_n$ can then be replaced by the sum over $T < \infty$, and the result follows from convergence of the sum uniformly in n. □

5. The critical exponent γ. In this section we show that $\gamma = 1$ if d is sufficiently large. By (3.5), for large d,

$$
\begin{aligned}
c_T &= (2d)^T N_T(k = 0, T) \\
&= (2d/r_T)^T \left[-r_T\, \partial_z F_T(0, r_T)\right]^{-1}\left[1 + O(T^{-1+\varepsilon})\right].
\end{aligned} \tag{5.1}
$$

Now $r_\infty = \lim_{T\to\infty} r_T$ exists. Similarly it can be shown that

$$\lim_{T\to\infty}\left[-\partial_z F_T(0, r_T)\right] = 1 + \lim_{T\to\infty} \partial_z \Pi_T(0, r_T)$$

exists. In fact, for two distinct memories $\sigma < \tau$,

$$
\begin{aligned}
&\partial_z \Pi_\tau(0, r_\tau) - \partial_z \Pi_\sigma(0, r_\sigma) \\
&\quad= \partial_z \Pi_\tau(0, r_\tau) - \partial_z \Pi_\tau(0, r_\sigma) + \partial_z \Pi_\tau(0, r_\sigma) - \partial_z \Pi_\sigma(0, r_\sigma) \\
&\quad= \partial_z^2 \Pi_\tau(0, r^*)(r_\tau - r_\sigma) + \partial_z \delta\Pi(0, r_\sigma),
\end{aligned} \tag{5.2}
$$

where $r^* \in (r_\sigma, r_\tau)$ and $\delta\Pi = \Pi_\tau - \Pi_\sigma$. By Corollary 4.2 of Slade (1987), $|\partial_z^2 \Pi_\tau(0, r^*)|$ is bounded uniformly in σ and τ, and hence the first term on the right side of (5.2) is $O(\sigma^{-1})$ uniformly in τ. The second term is also $O(\sigma^{-1})$ uniformly in τ, by an argument virtually identical to that used in Section 5 of Slade (1987) to show that $|\delta\Pi(0, r_\sigma)| \le O(\sigma^{-1})$. Therefore

$$c_T \sim \left[-r_\infty\, \partial_z F_\infty(0, r_\infty)\right]^{-1}(2d/r_T)^T. \tag{5.3}$$

In (5.3) we would like to replace r_T by r_∞. This replacement is justified if $\lim_{T\to\infty}(r_\infty/r_T)^T = 1$. Since r_T increases as T increases and since $r_T \ge 1$,

$$1 \le \left(\frac{r_\infty}{r_T}\right)^T = \left(1 + \frac{r_\infty - r_T}{r_T}\right)^T \le (1 + r_\infty - r_T)^T.$$

Hence the r_T in (5.3) can be replaced by r_∞ if $\lim_{T\to\infty}(1 + r_\infty - r_T)^T = 1$. It is not enough here that $r_\infty - r_T \le O(T^{-1})$, but this bound can be improved to $r_\infty - r_T \le O(T^{-2})$ as follows. The $O(T^{-1})$ decay was obtained in Slade (1987) from the fact that for $\sigma < \tau$, $r_\tau - r_\sigma$ is bounded above by a constant multiplied by $\|\Sigma_{T=\sigma/6}^{\infty} N_\sigma(x, T) r_\sigma^T\|_\infty$. By inserting a factor of $T(T-1)36\sigma^{-2}$ in the sum over T, this x-space L^∞ norm can be bounded above by a constant times $\sigma^{-2}\|\partial_z^2 N_\sigma(x, r_\sigma)\|_\infty$. For d sufficiently large $\|\partial_z^2 N_\sigma(x, r_\sigma)\|_\infty$ is bounded uniformly in σ by Corollary 4.2 of Slade (1987) and hence $r_\infty - r_T \le O(T^{-2})$. It follows that $\lim_{T\to\infty}(1 + r_\infty - r_T)^T = 1$ and therefore

$$c_T \sim \left[-r_\infty\, \partial_z F_\infty(0, r_\infty)\right]^{-1}(2d/r_\infty)^T. \tag{5.4}$$

Comparing (5.4) with the definition of γ given in the Introduction leads to $\gamma = 1$, for d sufficiently large. The connective constant $\beta_d = 2dr_\infty^{-1}$ can be computed using $r_T = 1 - \Pi_T(0, r_T)$ and (2.8) to write $2dr_\infty^{-1}$ as a series involving inverse powers of d. In this way it is possible to recover the result of Kesten (1964) mentioned in the Introduction.

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Note added in proof. Since this work was completed, it has been shown in Slade (1988) that for d sufficiently large $\{X_n\}$ is tight, and hence the scaled self-avoiding random walk converges in distribution to Brownian motion. In addition, Lawler (1987) has used the convergent lace expansion to construct the infinite self-avoiding walk in high dimensions.

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DEPARTMENT OF MATHEMATICS
AND STATISTICS
MCMASTER UNIVERSITY
HAMILTON, ONTARIO
CANADA L8S 4K1