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FLUX AND FIXATION IN CYCLIC PARTICLE SYSTEMS

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Start by randomly coloring each site of the one-dimensional integer lattice with any of N colors, labeled $0, 1, \dots, N-1$. Consider the following simple continuous time Markovian evolution. At exponential rate 1, the color $\xi(y)$ at any site y randomly chooses a neighboring site $x \in \{y-1, y+1\}$ and paints x with its color provided $\xi(y) - \xi(x) = 1 \bmod N$. Call this interacting process the *cyclic particle system on N colors*. We show that there is a qualitative change in behavior between the systems with $N \le 4$ and those with $N \ge 5$. Specifically, if $N \ge 5$ we show that the process *fixates*. That is, each site is painted a final color with probability 1. For $N \le 4$, on the other hand, we show that every site changes color at arbitrarily large times with probability 1.

1. Introduction. To date, the theory of interacting particle systems, beautifully described in Liggett's recent book [6], has focused on models with two possible states per site. This assumption greatly simplifies the ergodic theory, especially if the system enjoys additional nice properties such as reversibility or additivity. One class of multitype processes which has been studied is the collection of "linear systems" comprising the last chapter of [6]. Arguably the prototype for this class is the *stepping stone model*, sometimes called the "multitype voter model." See [2] and [3] for historical background with lots of references.

Recall the simple dynamics of the (continuous time) stepping stone model ζ_t on the d-dimensional integers Z^d. The possible values at each site are chosen from a finite or countably infinite set. Since this model has its origin in mathematical genetics, think of the values as representing competing allelic types. It is conceptually helpful (e.g., in simulation) to represent values as *colors*. If ζ is the current configuration, $\zeta(x)$ denoting the value at x and c a color, then the transition rule is

$$\zeta(x) \to c \text{ at exponential rate } \#\{y: |y-x| = 1, \zeta(y) = c\}.$$

Adopting a hybrid metaphor, we say that the color at a neighboring site y *eats* the color at x in this case. Thus all colors try to eat neighboring colors with the same appetite. In one and two dimensions the stepping stone model *clusters*, which means that regions of solid color grow over time. Color postcards which illustrate this clustering are available from the second author. Some theorems concerning the clustering are presented in [2] and [3].

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Our purpose here is to take a first tentative step into uncharted regions of the particle system realm. The long-range program is to explore multitype systems which are not additive and do not obey any simple duality equation. By flimsy analogy with differential equations, one expects a variety of new and more complex phenomena in such cases. Of course the mathematics also becomes much more difficult, but we hope to find a few problems which can be handled rigorously.

One such problem is the subject of this paper. The process ζ_t described above is actually a special example of the stepping stone evolutions considered by mathematical geneticists. The key feature of stepping stone dynamics, giving rise to what Wright [11] calls *isolation by distance*, is that interaction can only occur along the boundaries between distinct types. In this paper we consider a model ζ_t which also enjoys this key feature, but is otherwise extremely different from the color-exchangeable system ζ_t. As before, there are $N \geq 2$ colors, labeled $0, 1, \dots, N-1$. The transition rule is the same as for ζ_t, except that the color at neighboring site y can only eat the color at x if the former immediately follows the latter on a "color wheel":

$$(1)\qquad \begin{aligned} \xi(x) \to c \text{ at exponential rate } \#\{y\colon |y-x| = 1,\ \xi(y) = c,\\ \text{and } \xi(y) - \xi(x) = 1 \bmod N\}. \end{aligned}$$

We call this new process ξ_t the *cyclic particle system on N colors*. If $N = 2$ there is no constraint and the cyclic model agrees with the standard two-state voter model. Chapter 5 of [6] contains a survey of known results concerning this much studied process. If $N = 3$, the rule (1) is reminiscent of "Paper, Scissors, Stone," an age-old children's game (cf. [7], pages 25–28, 225, for the rules and some history). Variants which introduce additional hand signs correspond to models with $N \geq 4$, and to other models with more intricate appetite rules. In a biological context the $N = 3$ model is vaguely suggestive of Lotka–Volterra dynamics: $2 =$ predator, $1 =$ prey, $0 =$ vacant, where $0 \to 1$ represents migration of prey and $2 \to 0$ means starvation of predator. For $N \geq 4$ one can envision more complex "food cycles." The cyclic particle system makes sense in any dimension d, and the case of most biological interest is $d = 2$. Here we will restrict attention to one dimension, however, except for some concluding remarks. The reasons are technical. Our process is certainly not reversible and for $N \geq 3$ it is not additive or even attractive. So far, without the tools that go along with these properties, we are only able to get rigorous results for $d = 1$. The relative tractability of interactions on the line is a familiar theme in particle systems. A preliminary account of our results appeared in [1].

Throughout the remainder of the paper, then, unless explicitly stated otherwise, assume $d = 1$. Also, we will only start ξ_t from the product measure π which paints each site any of the N colors with equal probability $1/N$. Thereafter, colors eat one another when they can find the right food, without advantage to any color over any other. So what happens? Figures 1 and 2 show the result of (discrete-time) simulations on 400 sites up to $t = 250$.

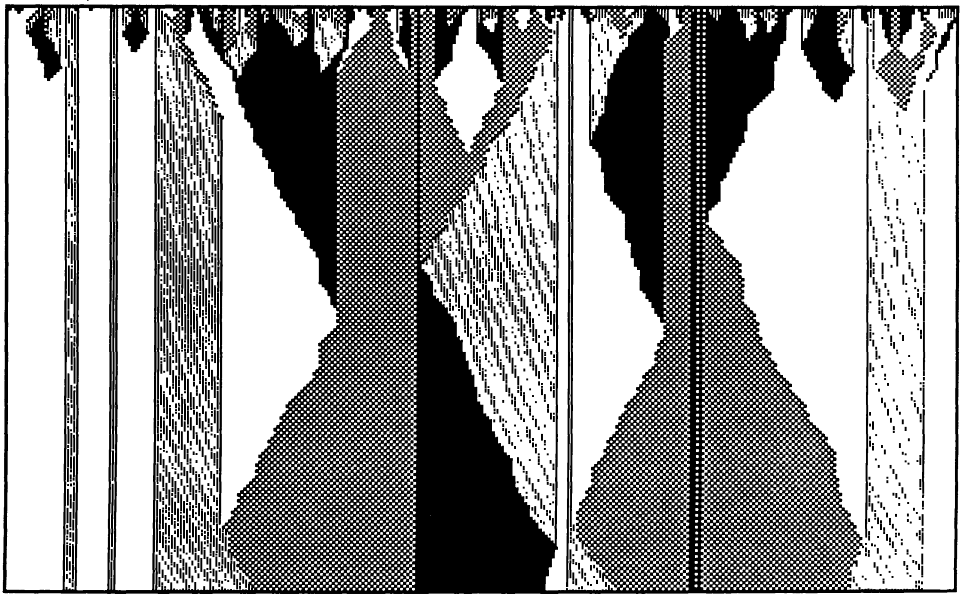

FIG. 1. *The cyclic particle system on Z with four colors.*

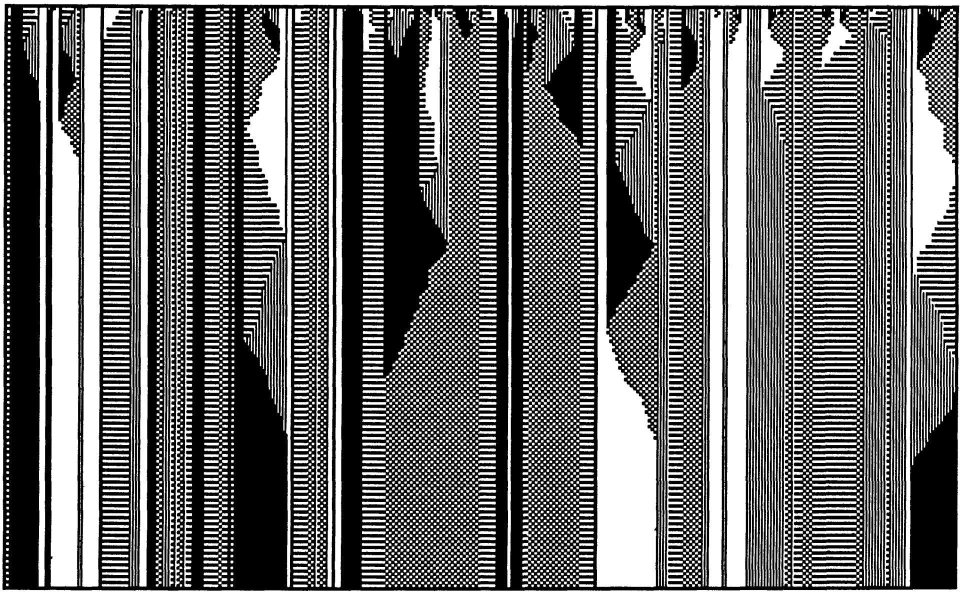

FIG. 2. *The cyclic particle system on Z with five colors.*

Color versions of these graphics are also contained in the particle postcard set which is available from the second author. Figure 1 illustrates the four color system, which appears to cluster as time (running down the page) evolves. We say that ξ_t *clusters* if

$$P(\xi_t(x) = \xi_t(y)) \to 1 \quad \text{as } t \to \infty \text{ for each } x, y. \tag{2}$$

Figure 2 shows what happens when one more color is added: the five color process seems to "get stuck." Adopting more dignified terminology, we will say that ξ_t *fixates* if there is a limiting random configuration ξ_∞ (possibly deterministic) such that for each x,

$$P(\xi_t(x) = \xi_\infty(x) \text{ eventually in } t) = 1. \tag{3}$$

One can check that this is equivalent to (3) holding for one specific x. Finally, in contrast to (3), we say that ξ_t *fluctuates* provided that for each x,

$$P(\xi_t(x) \text{ changes value at arbitrarily large } t) = 1. \tag{4}$$

[Of course properties (2)–(4) could be investigated for processes with initial states other than π; we will not do so here.] The object of this paper is to establish rigorously a "phase transition" between four and five colors, as evidenced by properties (3) and (4). We will only address the question of clustering (2) at the end of the paper. Our principal result, then, is as follows.

THEOREM. *Let ξ_t denote the N-color cyclic particle system started from the uniform product distribution π over N colors. If $N \leq 4$, then ξ_t fluctuates* [(4) *holds*]. *However, if $N \geq 5$, then ξ_t fixates* [(3) *holds*].

For clarity, the remainder of the paper is divided into five parts. Section 2 describes a graphical representation of ξ_t, demonstrates fluctuation for $N \leq 4$ by simple zero-one law considerations and also contains a preparatory result. In Sections 3–5 we establish fixation. It turns out that the argument is quite simple if $N \geq 8$, a bit more involved for $N = 6$ and 7, and much more complicated when $N = 5$. In an attempt to distinguish the key ideas from technical details and bookkeeping, we present the analysis in three separate sections. Finally, Section 6 describes a number of interesting problems concerning one-dimensional cyclic systems and related generalized stepping stone models. The paper concludes with some tentative speculation about cyclic models in two dimensions.

2. Preliminaries. It is convenient to construct the cyclic particle system ξ_t by means of a *graphical representation*. This technique is widespread in the particle systems literature, so we describe it only briefly. See [6] (Chapter III, Section 6) for background and additional references.

On the space-time lattice $Z \times [0, \infty)$, put down a rate 1 Poisson stream of "arrows" (directed edges) connecting each site x to $x + 1$ and another independent stream from x to $x - 1$. These random arrows, together with increasing-time segments at fixed sites, give rise to *paths* which connect different space-time points. Label the locations of $Z \times \{0\}$ with random colors, according to product

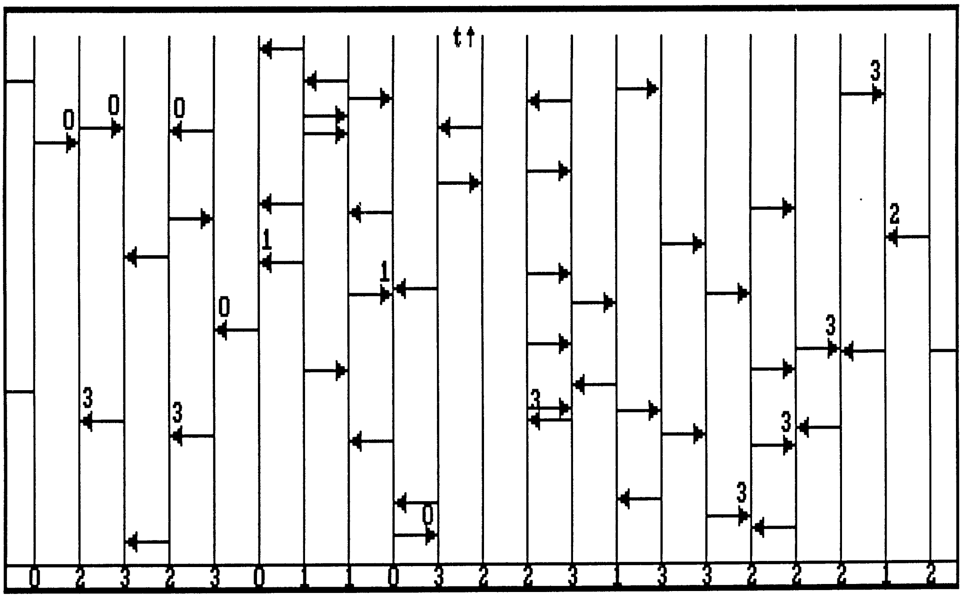

FIG. 3. *Graphical representation of the four color model on Z.*

measure π, and let ξ_0 be the resulting configuration. A change of state is said to occur at x at time t iff

(i) there is an arrow from (y, t) to (x, t) $(y = x \pm 1)$

and

(ii) $\xi_{t-}(y) - \xi_{t-}(x) = 1 \bmod N$.

Arrows which induce such a change of state will be called *active*. If one restricts the underlying probability space to a suitable set of measure one, then the dynamics of ξ_t are determined inductively. More details concerning similar constructions may be found in [5], for instance. Figure 3 illustrates how one realization of ξ_t (with $N = 4$) is generated by the underlying Poisson stream. Active arrows are labeled with the new state.

In order to show that ξ_t fluctuates for $N \leq 4$, we need little more than the graphical representation. Our argument is simple and "soft." Let Φ denote the set of all fixed configurations (absorbing states) for ξ_t. By standard Markov process reasoning, P-almost surely

$$\{\xi_t \text{ fixates}\} = \{\xi_\infty \in \Phi\},$$

where that right side means that $\xi_\infty(x)$ exists for each x, with $\xi_\infty \in \Phi$. Since π and the dynamics of ξ_t are invariant with respect to shifts in space and type (that is, translations in Z and translations mod N in the type space, respectively), it follows that Φ is invariant under these shifts. We apply this invariance to obtain the following result.

LEMMA 1. *Suppose that $A \subset \Phi$ is invariant with respect to spatial shifts and that a finite collection $A_0 = A, A_1, \dots, A_l$ $(l \geq 1)$ of type shifts of A forms a partition of Φ. Then*

$$P(\xi_t \text{ fluctuates}) = 1,$$

i.e., (4) *holds.*

PROOF. Let F_x denote the σ-algebra generated by $\xi_0(x)$ and all arrows (active or not) leading to x. We take $F = \sigma(F_x;\ -\infty < x < \infty)$ as the σ-algebra for our probability space and let I denote the sub-σ-algebra of events in F which are invariant with respect to spatial shifts. The F_x are P-independent, so I is P-trivial by a standard zero–one law. In particular, since $\{\xi_t \in A_0 \text{ eventually}\} \in I$,

$$P(\xi_t \in A_0 \text{ eventually}) = 0 \text{ or } 1.$$

By symmetry, the same holds with A_0 replaced by any other A_k $(1 \leq k \leq l)$. Since the A_k partition Φ, the only possibility is

$$P(\xi_t \in \Phi \text{ eventually}) = 0,$$

as desired. □

Fluctuation for $N \leq 4$ follows easily from Lemma 1. Simply check that Φ can be partitioned as

$$\begin{aligned} \Phi &= \{\xi_0\} \cup \{\xi_1\}, && N = 2,\\ &= \{\xi_0\} \cup \{\xi_1\} \cup \{\xi_2\}, && N = 3,\\ &= A_e \cup A_o, && N = 4, \end{aligned}$$

where $\xi_k(x) \equiv k$, $A_e = \{\xi\colon \xi(x) \in \{0,2\}\ \forall\ x\}$ and $A_o = \{\xi\colon \xi(x) \in \{1,3\}\ \forall\ x\}$.

To conclude this section, we present a sufficient condition for fixation that will be applied subsequently to analyze cyclic particle systems with five or more colors. First the key notions of *active path* and *ancestral site* need to be introduced. A path from $(z, 0)$ to (x, t) in the graphical representation of ξ_t is said to be *active* if it is entirely comprised of (i) active arrows and (ii) fixed-site increasing-time segments which do not pass the head of any active arrow. Observe that there is always a unique location $z \in Z$ such that there is an active path from $(z, 0)$ to (x, t); we call z the *ancestor* of (x, t) and write $z = \alpha_t(x)$. (Conceptually, z is the initial location of the type that eats its way to x at time t.) Of course as time evolves, some ancestral sites may color sizable intervals, while others are eliminated. Offspring of a single ancestral site must constitute an interval because active paths cannot cross one another. The same principle dictates that by any time t when active paths from ancestors z and z' collide, all ancestors between z and z' must have been eliminated.

The key to our arguments is the Markov chain (ξ_t, α_t). More formally, the basic properties of this chain that we exploit are:

(i) $\alpha_t(x)$ is a nondecreasing function of x for each fixed t.
(ii) $\{\alpha_t(x);\ x \in Z\}$ is nonincreasing in t.

These properties are easily verified using the graphical representation.

Focusing on the origin, let us abbreviate $\alpha_t = \alpha_t(0)$ and define

$$T(z) = \min\{t: \alpha_t = z\} = \text{the first time that 0 has ancestor } z.$$

To show that ξ_t fixates, we will actually argue that the origin never has ancestors from a long distance away. A precise formulation of this condition and the argument for its sufficiency are the subject of our second lemma.

LEMMA 2. *ξ_t fixates if*

$$\lim_{n\to\infty} P(T(z) < \infty \text{ for some } z < -n) = 0. \tag{5}$$

PROOF. Define

$$\begin{aligned} \tau(j) &= \text{the time of the } j\text{th change of state at the origin,} \\ \alpha(j) = \alpha_{\tau(j)} &= \text{the ancestor responsible for the } j\text{th change at the origin,} \\ B &= \{\tau(j) < \infty \text{ for all } j\}, \\ G_n &= \{|\alpha(j)| \le n \text{ for all } j\}. \end{aligned}$$

First, we claim that

$$P(B \cap G_n) = 0 \quad \text{for each } n. \tag{6}$$

In fact, the claim follows from the previously mentioned fact that active paths cannot cross over one another: On the one hand, successive changes of state from the same side of 0 (i.e., all from $+1$ or all from -1) necessarily involve distinct ancestors. On the other hand, each alternation of sides (i.e., a change from $+1$ followed by a change from -1, or vice versa) eliminates all of the descendents of at least one ancestor. This reasoning would be incorrect for $N = 2$, since 0's and 1's can alternately push back and forth. But for $N > 2$, by the time $\alpha(j)$ induces an alternation of sides, $\alpha(j-2)$ has no descendents [these descendents having been trapped between descendents of $\alpha(j-1)$ and those of $\alpha(j)$]. Thus infinitely many changes of state at 0 necessarily involve infinitely many ancestors, and hence the event $B \cap G_n$ is actually impossible. Thus (6) holds. Since (5) and reflection symmetry imply that G_n occurs for some n with P_π-probability 1, applying (6) we conclude that

$$P(B) = P\Big(B \cap \Big(\bigcup_n G_n\Big)\Big) = P\Big(\bigcup_n (B \cap G_n)\Big) = 0.$$

But $B^c = \{\lim \xi_t(0) \text{ exists}\}$, so ξ_t fixates. □

3. Fixation for $N \ge 8$. In order to verify condition (5), we need to show that active paths have a hard time getting to the origin from far away. Thus we need to exploit the obstacles which block and often terminate such paths. To this end, we introduce an *edge process* $e_t(x)$ which keeps track of the distance on the N-torus between the type at x and the type at its left neighbor $x - 1$:

$$e_t(x) = \min\{\pm[\xi_t(x) - \xi_t(x-1)] \bmod N\}.$$

Say there is a *live edge at* x^- if $e_t(x) = 1$ and a *blockade at* x^- if $e_t(x) \geq 2$. The crucial observation to make is that live edges, which follow the motion of active paths, *cannot be created*; as time evolves they either move by nearest neighbor jumps or are annihilated by collision with a blockade. For example, the first case occurs if the states at $x = -1, 0, 1$ change from 2 1 1 to 2 2 1, whereas the second case occurs when 2 1 4 changes to 2 2 4. Of course the larger N, the greater the preponderance of blockades over live edges (and the greater their magnitude) and hence the less chance a live edge has of moving very far.

We tabulate the "credits" and "debits" due to edges of different values with the aid of a *comparison function* ϕ:

$$\begin{aligned} \phi(x) &= 0, && \text{if } e_0(x) = 0, \\ &= e_0(x) - 2, && \text{if } 1 \leq e_0(x) \leq N/2. \end{aligned}$$

If there is a blockade of magnitude $e_0(x)$ at x^- at time 0, then it is easy to check that at least $\phi(x)$ live edges arriving at x^- must be "sacrificed" before that location can possibly become live. (As in the scenario 2 1 3 → 2 2 3, no loss in live edges need occur when $e = 2$.) Our argument for $N \geq 8$ will be based on this type of worst-case bookkeeping.

Denote the events in (5) by H_n. On H_n, let ρ be the first time an active path from $(-\infty, -n) \times \{0\}$ reaches the origin, and let $m_- < -n$ be its initial position. Also, let $m_+ \geq 0$ be the rightmost (i.e., greatest) source of an active path which reaches the origin before time ρ. Now focus on the edges $e_0(x)$ at time 0 for locations x^- with x in the interval $I = (m_-, m_+]$. On H_n, each blockade $e_0(x)$, $x \in I$, must at some time before ρ be replaced by a live edge, and these live edges must also originate in I. (Again, this is a consequence of the fact that active paths cannot cross.)

Making use of our comparison function ϕ, we conclude that

$$\begin{aligned} H_n &\subset \left\{ \begin{array}{l} \text{there are at least as many live edges as the total amount of} \\ \text{"blockade mass" in interval } I \text{ at time } 0 \end{array} \right\} \\ &\subset \left\{ \sum_{x \in I} \phi(x) \leq 0 \right\} \\ &\subset \left\{ \sum_{-l}^{m} \phi(x) \leq 0 \text{ for some } l \geq n,\ m \geq 0 \right\}. \end{aligned} \tag{7}$$

The summands in this last event are (bounded) i.i.d., since ξ_0 is completely random on the N types. It is easy to compute $\mu_N = E[\phi(x)]$ and to check that $\mu_N > 0$ if $N \geq 8$. For instance,

$$\mu_8 = \tfrac{1}{8} \cdot 0 + \tfrac{1}{4}(-1) + \tfrac{1}{4} \cdot 0 + \tfrac{1}{4}(+1) + \tfrac{1}{8}(+2) = \tfrac{1}{4}.$$

A standard large deviations estimate shows that the probability of the event in (7) tends to 0 (exponentially fast) as $n \to \infty$. Hence $\lim P(H_n) = 0$, which proves fixation for eight or more colors.

4. Fixation for $N = 6, 7$. In order to show fixation with fewer than eight colors we need to improve our bookkeeping by taking advantage of some effects which are not worst-case. For instance, when a live edge collides with a blockade of size 2, the result may well be a blockade of size 3 rather than the new live edge assumed above, depending on the side from which the live edge arrives. By effectively exploiting this single improvement one can handle the cases $N = 6, 7$. There is a slight complication however when we incorporate this effect: Conditioning on which live edges first collide with certain blockade sites inevitably destroys independence between contributions to a credit/debit tally such as (7). Our remedy is a traditional one in probability, block averaging.

Let $\sigma(x) = \inf\{t\colon e_t(x) \neq e_0(x)\}$ be the time of the first edge change at x^-. For fixed $L < \infty$, introduce the *cylinder event*:

$$C_x = \left\{ \begin{matrix} \text{any path (active or not) to } (x, \sigma(x)) \text{ stays within distance } L \\ \text{of } x \text{ between time 0 and time } \sigma(x) \end{matrix} \right\}.$$

To obtain (5), it is easiest to argue by contradiction. So we assume that ξ_t does not fixate and hence that $\sigma(x) < \infty$ with probability 1. Then given any $\varepsilon > 0$, we can choose L large enough that $P(C_x) \equiv P(C_0) \geq 1 - \varepsilon$.

As in the previous section, let H_n denote the event in (5). Divide the integer lattice Z into contiguous blocks of sites, with M sites per block, where M will be much larger than L, but still small compared to n. Define a *modified comparison function* ϕ' on sites x by

(8a) $\qquad \phi'(x) = 2, \qquad$ if $e_0(x) = 2$, $e_{\sigma(x)}(x) = 3$, x is at least distance L from the edge of the M-block to which it belongs and C_x occurs,

(8b) $\qquad \phi'(x) = \phi(x), \qquad$ otherwise.

This modification of ϕ allows us to exploit the fact that under the conditions of (8a), two live edges need to be sacrificed at x^- in order for that location to become alive.

One can mimic the argument of the previous section, but with the difference that one now uses the improved comparison function ϕ'. Repeating the reasoning down through (7), we conclude that

$$H_n \subset \left\{ \sum_{-l}^{m} \phi'(x) \leq 0 \text{ for some } l \geq n,\ m \geq 0 \right\}. \tag{9}$$

By construction, the summands of (9) which belong to different M-blocks are *independent*. (They use disjoint regions of the Poisson diagram.) So it suffices to show that the expected contribution from any single M-block is strictly positive. The same large deviations reasoning as in Section 3 then yields (5). We will compute this contribution for $N = 6$; the case $N = 7$ is easier.

If there is an initial blockade of size 2 at x^-, then until time $\sigma(x)$ the dynamics of ξ_t on the half-lines $(-\infty, x - 1]$ and $[x, \infty)$ are independent. By symmetry, the blockade has probability $\frac{1}{2}$ of becoming a size 3 blockade and probability $\frac{1}{2}$ of becoming a live edge; the probability that C_x also occurs is at least $\frac{1}{2} - \varepsilon$ for a site x further than distance L from the boundary of its

M-block. Of course, for $N = 6$, on the average $\frac{1}{3}$ of these sites have $e_0(x) = 2$. Making use of such sites, we see that the contribution within a single M-block from terms $\phi'(x)$ which satisfy the conditions of (8a) has expectation at least

$$(M - 2L)\tfrac{1}{3}\left(\tfrac{1}{2} - \varepsilon\right)(+2).$$

Moreover, the contribution from $\phi'(x)$ as in (8b) is at least

$$M\left[\tfrac{1}{6}\cdot 0 + \tfrac{1}{3}(-1) + \tfrac{1}{6}\cdot 0 + \tfrac{1}{6}(+1)\right].$$

For small ε and M sufficiently large, the total is clearly positive.

5. Fixation for $N = 5$. Our method for establishing fixation, which was detailed above, can be summarized as follows. We specify *effects* E_k which occur by (random) time ρ, which take place with density p_k on the line, which eliminate at least r_k live edges if they happen, and are defined so that distinct effects eliminate distinct edges. If $\Sigma r_k p_k$ exceeds the initial density of live edges, then the blocking argument above implies fixation.

The proof when $N = 5$ proceeds along the same lines, but seems to require minor modifications and some rather tedious bookkeeping. First of all, we need to take advantage of effects which involve up to three changes of state per site. This involves appropriate redefinition of the cutoff constant L and events C_x to ensure that the overwhelming majority of sites in an M-block undergo at least three changes of state before influence is felt from outside the M-block. [Now C_x should also require that paths originating at x stay within distance L of x until time $\sigma_3(x)$ = time of the third edge change at x^-, in order to maintain independence of the M-block summands.] By choosing L and M sufficiently large, one can establish fixation provided that $\Sigma r_k p_k$ exceeds the initial density of live edges, even when the E_k involve up to three (or, in fact, any finite number of) changes per site. We omit further details on this point.

Second, we will exploit contributions from nine different types of effects in order to overcome the initial density of live edges, which equals $\frac{2}{5}$. Below, we specify representatives F_k of each different effect, tabulate corresponding *rewards* r_k and give lower bounds on the probability π_k that F_k occurs at a given location x^-. Each effect F_k will be represented by an icon and a more formal description, both of which assume that $\xi_0(x-1) = 0$. A *symmetry index* s_k will indicate how many distinct situations are represented by F_k. In the one case where $\xi_0(x-1) = \xi_0(x) = 0$, the symmetry index equals 5, due to rotations in the type space. In all other cases the symmetry index is 10, due to type rotations and reflection about x^- on the line. Under this scheme, the average number of live edges eliminated per site is given by $\Sigma r_k s_k \pi_k$.

We leave it to the reader to check the crucial fact that there is no "double counting" in our tally. After detailing the tally sheet and checking that it does indeed establish fixation, we will then proceed to demonstrate the lower bounds on π_k.

The Effects F_k.

F_1: $\quad \tilde{1} \quad 0 \quad 1 \longleftarrow 2, \qquad r_1 = 2,\ s_1 = 10,\ \pi_1 \geq 0.00316.$

This icon means that $\xi_0(x-2) \neq 1$, $\xi_0(x-1) = 0$, $\xi_0(x) = 1$ and that a 2 from the right eats the 1 at x before any change occurs at $x-1$. The effect eliminates two live edges, hence $r_1 = 2$.

F_2: $\quad 1 \dashrightarrow 0 \quad 1 \quad \tilde{2}, \qquad r_2 = 2,\ s_2 = 10,\ \pi_2 \geq 0.00316.$

This icon means that $\xi_0(x-1) = 0$, $\xi_0(x) = 1$, $\xi_0(x+1) \neq 2$ and that a 1 from the left eats the 0 at $x-1$ before any change occurs at x. The effect eliminates two live edges.

F_3: $\quad 1 \longrightarrow 0 \vee 1 \longleftarrow 2, \qquad r_3 = 2,\ s_3 = 10,\ \pi_3 \geq 0.00093.$

This icon means that $\xi_0(x-2) = 1$, $\xi_0(x-1) = 0$, $\xi_0(x) = 1$, $\xi_0(x+1) = 2$ and that either the 1 at $x-2$ eats the 0 at $x-1$ or the 2 at $x+1$ eats the 1 at x before the 1 at x eats the 0 at $x-1$. The effect eliminates two live edges.

$$\begin{array}{l} \qquad\quad \dashleftarrow\!\!\dashleftarrow 3 \\ \tilde{1} \quad 0 \quad 1 \longleftarrow 2, \end{array}$$

F_4: $\quad r_4 = 1,\ s_4 = 10,\ \pi_4 \geq 0.00135.$

This icon means that $\xi_0(x-2) \neq 1$, $\xi_0(x-1) = 0$, $\xi_0(x) = 1$, $\xi_0(x+1) = 2$, the 2 at $x+1$ eats the 1 at x and then a 3 from the right arrives at x before there is any change at $x-1$. The effect eliminates one live edge not counted in F_1.

F_5: $\quad 1 \longrightarrow 0 \quad 0 \longleftarrow 1, \qquad r_5 = 2,\ s_5 = 5,\ \pi_5 \geq 0.00057.$

This icon means that $\xi_0(x-2) = 1$, $\xi_0(x-1) = 0$, $\xi_0(x) = 0$, $\xi_0(x+1) = 1$, the 1 at $x-2$ eats the 0 at $x-1$ and the 1 at $x+1$ eats the 0 at x (in either order). The effect eliminates two live edges.

F_6: $\quad 0 \quad 2 \dashleftarrow 3, \qquad r_6 = 1,\ s_6 = 10,\ \pi_6 = 0.02000.$

This icon means that $\xi_0(x-1) = 0$, $\xi_0(x) = 2$ and that a 3 from the right eats the 2 at x before a change occurs at $x-1$. The effect eliminates one live edge.

$$\begin{array}{l} 1 \dashrightarrow \\ \qquad\quad 0 \quad 2 \longleftarrow 3, \end{array}$$

F_7: $\quad r_7 = 1,\ s_7 = 10,\ \pi_7 \geq 0.00288.$

This icon means that $\xi_0(x-1) = 0$, $\xi_0(x) = 2$, $\xi_0(x+1) = 3$, that the 3 at $x+1$ eats the 2 at x and then a 1 from the left eats the 0 at $x-1$ before any other change at x. The effect eliminates one live edge not counted in F_6.

$$\begin{array}{l} 1 \dashrightarrow \\ \qquad\quad 0 \quad 2 \overset{\leftarrow}{} 2 \longleftarrow 3, \end{array}$$

F_8: $\quad r_8 = 1,\ s_8 = 10,\ \pi_8 \geq 0.00030.$

This icon means that $\xi_0(x-1) = 0$, $\xi_0(x) = 2$, $\xi_0(x+1) = 2$, $\xi_0(x+2) = 3$, that the 3 at $x+2$ eats the 2's at $x+1$ and x and then a 1 from the left eats the 0 at $x-1$ before any other change at x. The effect eliminates one live edge not counted in F_6.

F_9: $\quad 1 \longrightarrow 0 \quad 2 \overset{\leftarrow}{} 3, \qquad r_9 = 2,\ s_9 = 10,\ \pi_9 \geq 0.00028.$

This icon means that $\xi_0(x-2) = 1$, $\xi_0(x-1) = 0$, $\xi_0(x) = 2$, $\xi_0(x+1) = 3$, that the 1 at $x-2$ eats the 0 at $x-1$ and then the 3 at $x+1$ eats the 2 at x before any other change at $x-1$. The effect eliminates two live edges.

Computing our claimed lower bound on the mean number of live edges eliminated per site, we find that

$$\Sigma r_k s_k \pi_k \geq 0.4016 > \tfrac{2}{5},$$

as desired. So the remainder of this section is devoted to our techniques for estimating the π_k.

The above effects only take place when there is no interference from extraneous sites. Such interference can certainly take place, so we need to show that it does so with small probability. The next lemma gives estimates for interference from the right when it is competing against (independent) exponential clocks with rate $\lambda = 1, 2, 3$. Of course the same estimates will control interference from the left, by symmetry. We formulate the lemma in terms of a cyclic model ξ_t^+ on the half-line $[x, \infty)$, with $\xi_0^+(x) = 1$, with the colors at sites $y > x$ initially Bernoulli and with the same dynamics as ξ_t except that arrows from sites in $(-\infty, x-1]$ are suppressed. The 1 at x initially is for notational convenience; the same bounds clearly apply to configurations which are shifted in the type space. Let $T^{(\lambda)}$ ($\lambda = 1, 2, 3$) be independent exponential random variables with respective rates λ. Let $U^{(\lambda)}$ ($\lambda = 1, 2$) be the sum of two independent exponentials with rate λ.

Lemma 3. *With ξ_t^+, $T^{(\lambda)}$ and $U^{(\lambda)}$ as above, the following estimates hold.*

(A) $P(\xi_t^+(x) = 2$ *for some* $t \leq T^{(1)}) \leq 0.14$.
(B1) $P(\xi_t^+(x) = 2$ *for some* $t \leq T^{(2)}) \leq 0.077$.
(B2) $P(\xi_0^+(x+1) \neq 2,\ \xi_t^+(x) = 2$ *for some* $t \leq T^{(2)}) \leq 0.01$.
(B3) $P($*at least two changes of state at* x *before* $T^{(2)}) \leq 0.003$.
(C) $P(\xi_t^+(x) = 2$ *for some* $t \leq T^{(3)}) \leq 0.06$.
(D) $P(\xi_t^+(x) = 2$ *for some* $t \leq U^{(1)}) \leq 0.31$.
(E) $P(\xi_t^+(x) = 2$ *for some* $t \leq U^{(2)}) \leq 0.15$.

Proof of Lemma 3. (A), (B1), (B2), (C): We estimate five separate contributions, according to the different possible initial values of $\xi_0^+(x+1)$ (with probability $\frac{1}{5}$ each).

(a) $\xi_0^+(x+1) = 2$. The 1 at x cannot become a 2 unless the first arrow from $x+1$ to x occurs before $T^{(\lambda)}$. Suppose that $\xi_0^+(x+2) = 3$, that an arrow occurs from $x+2$ to $x+1$ before $T^{(\lambda)}$ and before any arrow from $x+3$ to $x+2$ or from $x+1$ to x. Then there is an arrow from $x+1$ to x before $T^{(\lambda)}$ and before any arrow from $x+2$ to $x+1$, and subsequently $T^{(\lambda)}$ occurs before the next arrow from $x+1$ to x. In this case the 1 does not change before $T^{(\lambda)}$ even though the first arrow from $x+1$ to x precedes $T^{(\lambda)}$. We obtain

$$P\big(\xi_t^+(x) = 2 \text{ for some } t \leq T^{(\lambda)},\ \xi_0^+(x+1) = 2\big)$$
$$\leq \frac{1}{5}\left(\frac{1}{\lambda+1}\right) - \frac{1}{5}\cdot\frac{1}{5}\left(\frac{1}{\lambda+3}\right)\left(\frac{1}{\lambda+2}\right)\left(\frac{\lambda}{\lambda+1}\right).$$

(The small improvement from the second term simplifies bookkeeping when $\lambda = 1$.)

(b) $\xi_0^+(x+1)=1$. In order for a 2 to arrive at x, a 2 must first arrive at $x+2$ (or be there initially) and then at least two more transitions must occur before $T^{(\lambda)}$. Thus this case contributes at most

$$\frac{1}{5}\cdot P\big(2 \text{ at } x+2 \text{ before } T^{(\lambda)} | \xi_0^+(x+1)=1\big)\left(\frac{1}{\lambda+1}\right)^2.$$

We rewrite this last probability as

$$\tfrac{1}{5}\sum_{j=0}^{4} P\big(2 \text{ at } x+2 \text{ before } T^{(\lambda)} | \xi_0^+(x+1)=1,\ \xi_0^+(x+2)=2-j \,(\mathrm{mod}\, 5)\big).$$

We claim that the jth term in the above sum is at most $(1/(\lambda+1))^j$. This is clear for $j=0,1$. For $j=2$, the 0 at $x+2$ can become a 1 at rate 2 if there is a 1 at $x+3$ (since there is also a 1 at $x+1$), but then at least two rate-1 transitions are needed to change the 1 at $x+2$ into a 2. If there is not a 1 at $x+3$, then at least two rate-1 transitions are needed. So we get the desired bound. Each additional increase in j requires at least one more transition and the claim follows. We conclude that the contribution from this case is at most

$$\frac{1}{5}\cdot\frac{1}{5}\sum_{j=0}^{4}\left(\frac{1}{\lambda+1}\right)^j\left(\frac{1}{\lambda+1}\right)^2 \le (0.04)\left(\frac{\lambda+1}{\lambda}\right)\left(\frac{1}{\lambda+1}\right)^2 = (0.04)\frac{1}{\lambda(\lambda+1)}.$$

Throughout the remainder of the proof and in estimates for the π_k to follow, we will use the weight factor of $(\lambda+1)/\lambda$ given above to bound similar sums of conditional probabilities.

(c) $\xi_0^+(x+1)=0$. The 1 at x eats the 0 at $x+1$ before $T^{(\lambda)}$ (and before any change at $x+1$ from the right) with probability at most $1/(\lambda+1)$. At that point the situation reduces to (b). It is also possible for a 1 at $x+2$ to eat the 0 at $x+1$; three more transitions must then occur before $T^{(\lambda)}$. The total contribution is at most

$$\left(\frac{1}{\lambda+1}\right)\frac{1}{5}\cdot\frac{1}{5}\left(\frac{\lambda+1}{\lambda}\right)\left(\frac{1}{\lambda+1}\right)^2+\frac{1}{5}\cdot\frac{1}{5}\left(\frac{\lambda+1}{\lambda}\right)\left(\frac{1}{\lambda+1}\right)\left(\frac{1}{\lambda+1}\right)^3$$
$$=(0.04)\frac{\lambda+2}{\lambda(\lambda+1)^3}.$$

(d) $\xi_0^+(x+1)=4$. A 0 must first appear at $x+2$ and eat the 4 at $x+1$; we estimate the chance of this as in (b). Then five additional changes must take place (one at x, two at $x+1$, two at $x+2$), in any of four possible ways. The total contribution is at most

$$\frac{1}{5}\cdot\frac{1}{5}\left(\frac{\lambda+1}{\lambda}\right)\left(\frac{1}{\lambda+1}\right)4\left(\frac{1}{\lambda+1}\right)^5=(0.16)\frac{1}{\lambda(\lambda+1)^5}.$$

(e) $\xi_0^+(x+1)=3$. A 4 must appear at $x+2$ and eat the 3 at $x+1$. Then seven additional changes must occur (one at x, three at $x+1$, three at $x+2$), in any of four possible ways. The contribution is at most

$$\frac{1}{5}\cdot\frac{1}{5}\left(\frac{\lambda+1}{\lambda}\right)\left(\frac{1}{\lambda+1}\right)4\left(\frac{1}{\lambda+1}\right)^7=(0.16)\frac{1}{\lambda(\lambda+1)^7}.$$

Add together the contributions from (a)–(e) to obtain the desired bounds (A), (B1), (B2), (C). [For (B2), omit the contribution from (a).]

(B3): We estimate three pieces, depending on the initial state.

(f) $\xi_0^+(x+1) \neq 2$. First the 1 at x must change to a 2 and then two more changes (at $x+1$, then at x) must occur before $T^{(2)}$. By (B2), the probability of the first event is at most 0.01; the contribution is therefore at most $(0.01)(\frac{1}{3})^2$.

(g) $\xi_0^+(x+1) = 2$, $\xi_0^+(x+2) \neq 3$. The 2 at $x+1$ must at some point change to a 3 and then a change must occur at x. Applying (B2) to the first event, we see that the contribution is at most $\frac{1}{5}(0.01)(\frac{1}{3})$.

(h) $\xi_0^+(x+1) = 2$, $\xi_0^+(x+2) = 3$. The first change on $[x, x+2]$ may occur at any of these three sites. The first change is at x (and before $T^{(2)}$) with probability at most $\frac{1}{4}$. In this case two more changes are needed before $T^{(2)}$. The first change is at $x+1$ with probability at most $\frac{1}{4}$ and then at least nine more changes must occur, in one of four ways. And the first change is at $x+2$ with probability at most $\frac{1}{4}$, after which at least five more changes must occur. The total contribution is at most

$$\frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{4}\left[\left(\frac{1}{3}\right)^2 + 4\left(\frac{1}{3}\right)^9 + \left(\frac{1}{3}\right)^5\right].$$

Add together the contributions from (f)–(h) to verify the claim.

(D), (E): Our method is entirely analogous to that for (A), (B), (C), but the arithmetic is a bit messier. The probability of j arrows before $U^{(\lambda)}$ is less than

$$u^\lambda(j) = \left(\frac{1}{\lambda+1}\right)^j + j\left(\frac{\lambda}{\lambda+1}\right)\left(\frac{1}{\lambda+1}\right)^j.$$

The weight factor is now $w^\lambda = \Sigma_0^4 u^\lambda(j)$. Below, we write contributions (a′)–(e′) corresponding to the previous contributions (a)–(e). The probability of "j additional changes before $U^{(2)}$" is given by $u^\lambda(j)$.

(a′) $\frac{1}{5}u^\lambda(1)$;
(b′) $\frac{1}{5} \cdot \frac{1}{5}w^\lambda u^\lambda(2)$;
(c′) $\frac{1}{5} \cdot \frac{1}{5}[u^\lambda(1)w^\lambda u^\lambda(2) + w^\lambda u^\lambda(4)]$;
(d′) $\frac{1}{5} \cdot \frac{1}{5}w^\lambda 4u^\lambda(6)$;
(e′) $\frac{1}{5} \cdot \frac{1}{5}w^\lambda 4u^\lambda(8)$.

The sum of these contributions is less than 0.31 if $\lambda = 1$ and less than 0.15 if $\lambda = 2$, as desired. □

We now proceed to derive the claimed lower bounds on the probabilities π_k of the effects F_k, one by one. As in the proof of fixation for $N = 6, 7$, we will assume that ξ_t fluctuates with probability 1, in order to ensure that desired changes of state occur eventually. Thus our argument is essentially a proof by contradiction. We will use the abbreviation P_* to denote probabilities where conditioning is implicit, but clear from the context. For instance, the conditioning on the right side of the first inequality below is with respect to $\{\xi_0(x-1) = 0,\ \xi_0(x) = 1\}$.

The estimate for π_1. $\tilde{1}\ 0\ 1 \longleftarrow 2$. This effect makes a major contribution, so we exercise some care in its estimation. Let $T^{(2)}$ = the time of the first arrow from x to $x-1$ *or* from $x+1$ to x, T_- = the time of the first change at $x-1$ from the left and T_+ = the time of the first change at $x+1$ from the right. We have

$$\begin{aligned}
P(F_1) \geq{}& \tfrac{1}{5}\cdot\tfrac{1}{5}\big[P_*\big(\xi_0(x-2)\neq 1,\ \xi_0(x+1)=2,\\
&\qquad \text{arrow from } x+1 \text{ to } x \text{ at } T^{(2)},\ T^{(2)}<T_+,\ T^{(2)}<T_-\big)\\
&\quad +P_*\big(\xi_0(x-2)\neq 1,\ \xi_0(x+1)=1,\\
&\qquad \text{arrow from } x+1 \text{ to } x \text{ at } T^{(2)},\\
&\qquad \text{exactly one change at } x+1 \text{ before } T^{(2)},\ T^{(2)}<T_-\big)\big]\\
\geq{}& \tfrac{1}{5}\cdot\tfrac{1}{5}\cdot\tfrac{1}{5}\cdot\tfrac{1}{2}\big[\big(P_*\big(\xi_0(x-2)\neq 1,\ T^{(2)}<T_-\big)\\
&\quad -P_*\big(\xi_0(x-2)\neq 1, \text{ exactly one change at } x+1\\
&\qquad\qquad \text{before } T^{(2)},\ T^{(2)}<T_-\big)\\
&\quad -P_*\big(\xi_0(x-2)\neq 1,\ \geq 2 \text{ changes at } x+1\\
&\qquad\qquad \text{before } T^{(2)},\ T^{(2)}<T_-\big)\big)\\
&\quad +P_*\big(\xi_0(x-2)\neq 1, \text{ exactly one change at } x+1\\
&\qquad\qquad \text{before } T^{(2)},\ T^{(2)}<T_-\big)\big]\\
={}& \tfrac{1}{5}\cdot\tfrac{1}{5}\cdot\tfrac{1}{5}\cdot\tfrac{1}{2}\big[P_*\big(\xi_0(x-2)\neq 1\big)-P_*\big(\xi_0(x-2)\neq 1,\ T^{(2)}\geq T_-\big)\\
&\quad -P_*\big(\xi_0(x-2)\neq 1,\ \geq 2 \text{ changes at } x+1\\
&\qquad\qquad \text{before } T^{(2)},\ T^{(2)}<T_-\big)\big]\\
\geq{}& \tfrac{1}{5}\cdot\tfrac{1}{5}\cdot\tfrac{1}{5}\cdot\tfrac{1}{2}\big[\tfrac{4}{5}-(0.01)-(0.003)\big]=0.003148,
\end{aligned}$$

this last inequality following from Lemma 3(B2), (B3). To beef up the estimate a bit, we argue as follows. Suppose that there is a 1 at $x+1$ and a 2 at $x+2$ initially. Suppose further that an arrow occurs from $x+1$ to x before any other of the types $\{x-3 \text{ to } x-2,\ x \text{ to } x-1,\ x+2 \text{ to } x+1,\ x+3 \text{ to } x+2\}$. Then successive arrows occur from $x+2$ to $x+1$ and from $x+1$ to x before any other of the types $\{x-3 \text{ to } x-2,\ x \text{ to } x-1,\ x+3 \text{ to } x+2\}$. Then F_1 occurs after time $T^{(2)}$ and so the event is disjoint from that of the preceding estimate. The chance of this event (assuming the initial data of F_1) is

$$\left(\tfrac{4}{5}\cdot\tfrac{1}{5}\cdot\tfrac{1}{5}\cdot\tfrac{1}{5}\cdot\tfrac{1}{5}\right)\cdot\left(\tfrac{1}{5}\cdot\tfrac{1}{4}\cdot\tfrac{1}{4}\right)=0.000016.$$

Combining pieces, we obtain the claimed bound of 0.00316.

The estimate for π_2. $1 \dashrightarrow 0\ 1\ \tilde{2}$. $P(F_1)=P(F_2)$, so we use the same lower bound as above.

The estimate for π_3. $1 \longrightarrow 0\ {}^{\vee}1 \longleftarrow 2$. Lemma 3(C) handles this situation. Interference from outside $[x-2,\ x+1]$ can only occur if an effect comes in from either the left or the right before $T^{(3)}$ = the time of the first arrow from

$x - 2$ to $x - 1$ *or* from x to $x - 1$ *or* from $x + 1$ to x. Let $T_- =$ the time of the first change at $x - 2$ from the left and $T_+ =$ the time of the first change at $x + 1$ from the right. Then

$$\begin{aligned} P(F_3) &\geq \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} P_*\big(\text{arrow from } x-2 \text{ to } x-1 \\ &\qquad \text{or from } x+1 \text{ to } x \text{ at time } T^{(3)}, T^{(3)} < T_-, T^{(3)} < T_+\big) \\ &\geq \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{2}{3}\big(P_*\big(T^{(3)} < T_-, T^{(3)} < T_+\big)\big) \\ &\geq \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{2}{3}\big(1 - 2P_*\big(T^{(3)} \geq T_+\big)\big) \\ &\geq \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{2}{3}\big(1 - 2(0.06)\big) > 0.00093. \end{aligned}$$

The estimate for π_4. $\tilde{1}\ 0\ 1 \longleftarrow 2 \quad \overset{\longleftarrow\text{-----}\ 3}{}$. Lemma 3(B1) and symmetry (assuming fluctuation) will be applied. $T^{(2)}$ will denote the time of the first arrow from x to $x - 1$ *or* from $x + 1$ to x. Let S_+ be the time of the first change at $x + 1$ from the right and let T_+ be the time of the first arrow from $x + 1$ to x after S_+. Similarly, T_- will be the time of the first arrow from $x - 2$ to $x - 1$ after the first change at $x - 2$ from the left. Then

$$\begin{aligned} P(F_4) &\geq \tfrac{4}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} P_*\big(\text{arrow from } x+1 \text{ to } x \text{ at } T^{(2)}, T^{(2)} < T_+ < T_-\big) \\ &\geq \tfrac{4}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5}\big[P_*\big(\text{arrow from } x+1 \text{ to } x \text{ at } T^{(2)}, T_+ < T_-\big) \\ &\qquad - P_*\big(\text{arrow from } x+1 \text{ to } x \text{ at } T^{(2)}, S_+ \leq T^{(2)}\big)\big]. \end{aligned}$$

The two events of the second bracketed probability are clearly independent and the two events of the first bracketed probability are positively correlated. Positive correlation follows from the observation that $T^{(2)}$ is the time of the first arrow in a rate-2 stream which is put down either from x to $x - 1$ or from $x + 1$ to x according to a fair coin toss. Thus the conditional law of T_+, given that the first arrow is from $x + 1$ to x, is stochastically smaller than the conditional law of T_+, given that the first arrow is from x to $x - 1$. (In a suitable coupling, the influence from the right has an extra arrow to use.) We conclude that

$$P(F_4) \geq \tfrac{4}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{2}\big(\tfrac{1}{2} - (0.077)\big) > 0.00135.$$

The estimate for π_5. $1 \longrightarrow 0\ 0 \longleftarrow 1$. Introduce

$T^{(2)} =$ time of first arrow from $x - 2$ to $x - 1$ *or* from $x + 1$ to x,
$G_- = \{$arrow from $x - 2$ to $x - 1$ at $T^{(2)}\}$,
$G_+ = \{$arrow from $x + 1$ to x at $T^{(2)}\}$,
$U^{(2)} =$ the time of the first arrow to x after $T^{(2)}$, on G_-,
$=$ the time of the first arrow to $x - 1$ after $T^{(2)}$, on G_+,
$H_- = \{$arrow from $x - 2$ to $x - 1$ at $U^{(2)}\}$,
$H_+ = \{$arrow from $x + 1$ to x at $U^{(2)}\}$,
$T_- =$ the time of the first change at $x - 2$ from the left,
$T_+ =$ the time of the first change at $x + 1$ from the right,
$U_- =$ the time of the first change at $x - 1$ after $T^{(2)}$,
$U_+ =$ the time of the first change at x after $T^{(2)}$.

We have

$$P(F_5) \geq \left(\tfrac{1}{5}\right)^4 \left[P_*\left(G_-, H_+, T^{(2)} < T_-, U^{(2)} < T_+, U^{(2)} < U_-\right) + P_*\left(G_+, H_-, T^{(2)} < T_+, U^{(2)} < T_-, U^{(2)} < U_+\right)\right].$$

These two probabilities are equal. Since $H_\pm$ are independent of the other events and since both have probability $\frac{1}{2}$, this is at least

$$\left(\tfrac{1}{5}\right)^4 \left[\tfrac{1}{2} - P_*\left(G_-, T_- \leq T^{(2)}\right) - P_*\left(G_-, T_+ \leq U^{(2)}\right) - P_*\left(G_-, U_- \leq U^{(2)}\right)\right].$$

Lemma 3(B1), (E) gives the lower bound

$$\left(\tfrac{1}{5}\right)^4 \tfrac{1}{2}\left[1 - (0.077) - (0.15) - 2P_*\left(G_-, U_- \leq U^{(2)}\right)\right].$$

On G_-, the event $\{U_- \leq U^{(2)}\}$ requires two arrows in $[T^{(2)}, U^{(2)}]$ if $\xi_{T^{(2)}}(x-3) = 2$ and at least three otherwise. The weight factor corresponding to $T^{(2)}$ is $\frac{3}{2}$, so the chance of a 2 at $x - 3$ at time $T^{(2)}$ is at most $\frac{3}{2} \cdot \frac{1}{5}$. The time between $T^{(2)}$ and $U^{(2)}$ is governed by a rate-2 exponential. Thus

$$P_*\left(G_-, U_- \leq U^{(2)}\right) \leq \tfrac{1}{2}\left[(0.3)\left(\tfrac{1}{3}\right)^2 + (0.7)\left(\tfrac{1}{3}\right)^3\right] < 0.03.$$

Substitute this estimate to obtain the claimed bound of 0.00057.

The estimate for π_6. $0\ 2 \longleftarrow 3$. By assuming that ξ_t fluctuates, so that either the 0 at $x - 1$ or the 2 at x is eaten eventually, we can compute the probability of this event exactly. As in the proof for $N = 6, 7$, symmetry gives

$$\pi_6 = \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{2} = 0.02.$$

The estimate for π_7. $\overset{1 \longrightarrow}{0\ 2 \longleftarrow 3}$. We use Lemma 3(A) and symmetry (again assuming fluctuation). Let $T^{(1)}$ = the time of the first arrow from $x + 1$ to x, T_- = the time of the first change at $x - 1$ from the left and T_+ = the time of the first change at $x + 1$ from the right. It follows that

$$\begin{aligned} P(F_7) &\geq \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} P_*\left(T^{(1)} < T_-, T^{(1)} < T_+, T_- < T_+\right) \\ &= \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{2} P_*\left(T^{(1)} < T_-, T^{(1)} < T_+\right) \\ &\geq \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{2}(1 - 2(0.14)) = 0.00288. \end{aligned}$$

The estimate for π_8. $\overset{1 \longrightarrow}{0\ 2 \overset{\leftarrow}{}\ 2 \longleftarrow 3}$. We use Lemma 3(D) and symmetry. Let $U^{(1)}$ = the time of the first arrow from $x + 1$ to x after the first arrow from $x + 2$ to $x + 1$, T_- = the time of the first change at $x - 1$ from the left and T_+ = the time of the first change at $x + 2$ from the right. It follows that

$$\begin{aligned} P(F_8) &\geq \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} P_*\left(U^{(1)} < T_-, U^{(1)} < T_+, T_- < T_+\right) \\ &= \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{2} P_*\left(U^{(1)} < T_-, U^{(1)} < T_+\right) \\ &\geq \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{5} \cdot \tfrac{1}{2}(1 - 2(0.31)) > 0.00030. \end{aligned}$$

The estimate for π_9. $1 \longrightarrow 0\ 2 \leftarrow 3$. Argue just as for the G_- half of F_5 to get half the bound obtained for $P(F_5)$,

$$P(F_9) \geq (0.5)(0.00057) > 0.00028.$$

REMARK. If the clearance in our proof (0.4%) seems too close for comfort, we note that improvements can be made. For instance, a simple correlation inequality shows that the events in the second to last lines of the computations for F_7 and F_8 are positively correlated. Thus, extra terms of $(0.14)^2$ and $(0.31)^2$ can be added within the respective large parentheses of the final lines in those computations. This adds another 0.0015 to our tally. Also, a few additional effects could be listed, making contributions of the same order of magnitude. But we do not know an easier proof of fixation for $N = 5$ and doubt that we have missed a "big piece."

6. Related results and future directions. We conclude with an informal discussion of several intriguing questions raised by our work. We hope to address some of these problems in the future.

First of all, note that our model ξ_t is closely related to a discrete time *deterministic* cellular automaton [10]. Namely, suppose that at each time $n \in \{0, 1, \dots\}$ the color $\xi_n(y)$ at site y paints every neighbor x such that

$$\xi_n(y) - \xi_n(x) = 1 \bmod N.$$

This rule is interesting if $N \geq 3$. Once we specify a random initial configuration of N colors on the integers, the entire trajectory of ξ_n is prescribed. Suppose that, as in the theorem, the initial measure is π. Then the conclusions of the theorem hold for $N = 3, 4$ and for $N \geq 6$. In fact our proof goes through in these cases without significant change once the graphical representation is tailored to the discrete space-time lattice. Thus we obtain one-dimensional cellular automata with novel ergodic behavior, due to the rich structure of the invariant set Φ. With substantial additional effort, our method can be adapted to prove fixation for the five-color deterministic dynamic; details may be found in [4].

As noted in the Introduction, for three- and four-color models we have sidestepped the issue of *clustering* (2). In the two-color case (the voter model), clustering occurs at rate $t^{1/2}$ and is well understood. See Section V.3 of [6], for instance. Suitable rescaling leads to annihilating Brownian edges in the limit. One can investigate the corresponding "clustering rates" when $N = 3$ and 4. A natural first step is to study the same problem for the corresponding deterministic systems of the previous paragraph. Here the dynamics are particularly simple and some exact connections with random walks occur [4].

Another tack is to allow more general "appetite rules," while still maintaining the simple nearest neighbor spatial interaction. If all N colors eat at the same rate, but each color is only allowed to eat certain others, then there is one model for each directed graph on N sites. Exchangeable stepping stone dynamics correspond to the complete graph, whereas this paper treats the cyclic graph. More generally, if appetites are allowed to differ, then there is a model for every

Q-matrix of rates q_{ij} at which color i eats color j. All such systems preserve the key feature that change only occurs along boundaries between color clusters; they therefore constitute a natural class of models for spatial competition.

Ideally one would like to categorize all possible types of ergodic behavior for such generalized stepping stone models, along the lines of the classification of cellular automata [10], iterates of maps or other dynamical systems. Four possibilities are:

(i) Some color takes over the world.
(ii) The system converges to a nontrivial equilibrium.
(iii) There is some set of two or more colors which form continually expanding clusters.
(iv) The system fixates.

Are there other scenarios? Who wins in class (i) situations? (The answer is sometimes paradoxical.) What are sufficient conditions to ensure that a given model belongs to each of the four classes? In the case of clustering, when is there a "natural scale"? One-dimensional generalized stepping stone models are sufficiently tractable that satisfactory answers to at least some of these intriguing questions should be forthcoming.

Analogous models in two dimensions are much more interesting and much more meaningful to mathematical biologists/geneticists. We conclude the discussion with a few remarks concerning N-color cyclic particle system dynamics on the two-dimensional integers. ξ_t evolves as in one dimension, except that types try to eat one of the four nearest neighbors selected at random. Essentially the same argument as in one dimension shows that the three- and four-color models fluctuate in two (or more) dimensions. We have simulated a discrete-time version of the four-color two-dimensional model on a high-performance Cellular Automaton Machine (CAM) [8], developed by Tom Toffoli at the MIT Laboratory for Computer Science. A still frame is shown in Figure 4, but this in no way does justice to the intriguing dynamic which one sees with the aid of the CAM. Started from a random state, the simulation suggests quite convincingly that ξ_t *does not cluster*. Rather, the simulation rapidly stabilizes to a "bubbling equilibrium" in which waves of color grow for a while, then "fold over" and are "swallowed up." Connected clusters containing several hundred cells of the same color are common. If we start from large squares of the four colors, then the simulation quickly settles down to the same steady state. And if we start with a small patch of randomness surrounded by a solid sea of one color, then cyclic waves spread out from the random source. This last effect is nicely illustrated in [8]. Charles Bennett has pointed out a resemblance to Zhabotinsky's oscillating chemical reactions [12]. The cover article [9] of the June 1974 *Scientific American* gives a colorful account of those dynamics.

Preliminary two-dimensional simulations suggest that fixation sets in somewhere between 8 and 16 colors. (One must exercise caution, however. If the initial state consists of a large enough solid square of one color surrounded by a sea of random colors, then simulations also suggest that fixation is questionable for even more colors.) We are presently unable to prove fixation for any N when $d = 2$; the unusual self-organizing wave patterns make rigorous analysis difficult.

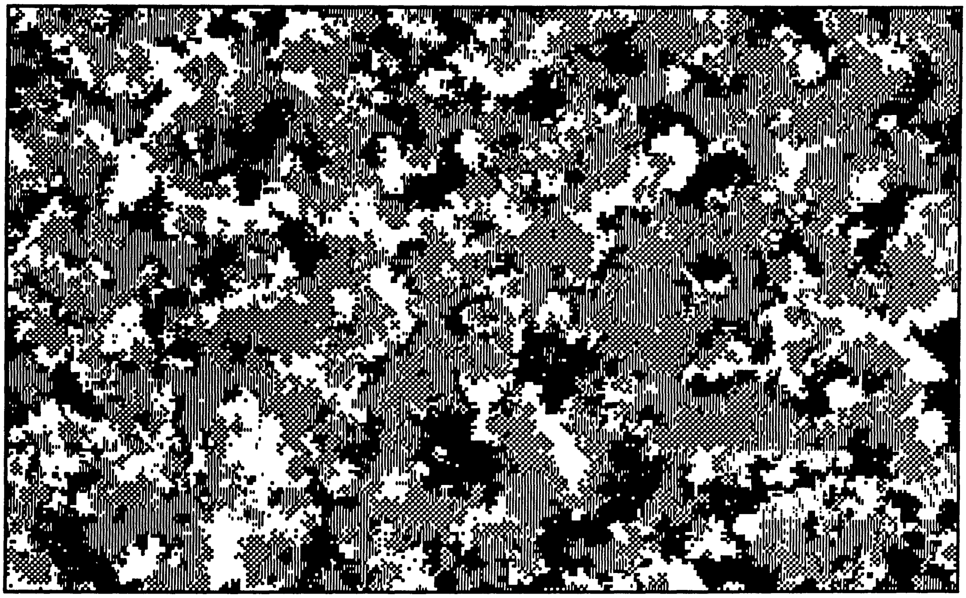

FIG. 4. *The cyclic particle system on Z^2 with four colors.*

We are beginning to investigate some simpler two-dimensional models, for which one can prove fixation once the number of colors is sufficiently large.

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