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BROWNIAN LOCAL TIME APPROXIMATED BY A WIENER SHEET

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Let $L(a, t)$ be the local time of a Wiener process. Our main result says that the process $L(a, t) - L(0, t)$ can be strongly approximated by a process obtained from a Wiener sheet $W(a, t)$ and a local time process $\hat{L}(0, t)$, independent of $W(\cdot, \cdot)$.

1. Introduction. Let $\{W(t); t \geq 0\}$ be a Wiener process with local time $L(a, t)$, that is, the two-parameter process $\{L(a, t); a \in R^1, t \geq 0\}$ satisfies

$$\int_A L(a, t)\, da = \lambda\{s: 0 \leq s \leq t, W(s) \in A\}$$

for any $t \geq 0$ and Borel set $A \in R^1$, where $\lambda(\cdot)$ is the Lebesgue measure. $L(0, t)$, the restriction of $L(a, t)$ for $a = 0$, will play a crucial role in the sequel. We will call both, the two-parameter process $L(a, t)$ as well as the one-parameter process $L(0, t)$, simply local time. Let

$$T_u = \inf\{t: t \geq 0, L(0, t) \geq u\},$$

and consider the process

$$\mathscr{L}(a, u) = L(a, T_u) - L(0, T_u) = L(a, T_u) - u.$$

It is well known that $\mathscr{L}(a, u)$ has a finite moment generating function in a neighbourhood of the origin,

(A.i) $$\mathbb{E}\mathscr{L}(a, u) = 0, \qquad \mathbb{E}\mathscr{L}^2(a, u) = 4au.$$

(A.ii) $\{\mathscr{L}(a, u); u \geq 0\}$ is a strictly stationary process of independent increments in u for any $a \in R^1$.

(A.iii) For any fixed $u > 0$, $\{L(a, T_u); a > 0\}$ is the square of a zero-dimensional Bessel process starting at u, that is, an R^1_+-valued diffusion process with generator $2xd^2/dx^2$ on $C^2(0, \infty)$. Moreover, $\{\mathscr{L}(a, u); a \geq 0\}$ for any fixed u has orthogonal increments and is a martingale in a.

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For these and further properties we refer to Ray (1963), Knight (1963), Walsh (1978) and Bass and Griffin (1985).

The Komlós, Major and Tusnády (1975) strong approximation theorem says that, on a rich enough probability space, a strictly stationary stochastic process $\{Z(u);\, u \geq 0\}$ of independent increments with $EZ(u) = 0$, $EZ^2(u) = u$ and a finite moment generating function in a neighbourhood of the origin can be approximated by a Wiener process $W(\cdot)$ such that $|Z(u) - W(u)| = O(\log u)$ a.s. $(u \to \infty)$. In short, we say that $Z(u)$ can be approximated by $W(u)$ with rate $O(\log u)$. Applying this theorem with $Z(u) = \frac{1}{2}a^{-1/2}\mathscr{L}(a, u)$, we obtain that the process $\{\frac{1}{2}a^{-1/2}\mathscr{L}(a, u);\, u \leq 0\}$ can be approximated by a Wiener process $\{W_a(u);\, u \geq 0\}$ for any fixed $a \neq 0$ with rate $O(\log u)$. We note that $W_a(u)$ has nothing to do with our original Wiener process $\{W(t);\, t \geq 0\}$ and, at this stage, we cannot say anything about the joint distribution of the process $\{W_a(t);\, t \geq 0\}$ in a.

Berkes and Philipp (1979) investigated the problem of approximating a finite family of pairwise orthogonal processes of independent increments by a family of independent Wiener processes. Their results imply that for any fixed $0 < a_1 < a_2 < \cdots < a_k < \infty$; $k = 1, 2, \ldots$, the vector-valued process

$$\left\{\frac{\mathscr{L}(a_1, t)}{2\sqrt{a_1}}, \frac{\mathscr{L}(a_2, t) - \mathscr{L}(a_1, t)}{2\sqrt{a_2 - a_1}}, \ldots, \frac{\mathscr{L}(a_k, t) - \mathscr{L}(a_{k-1}, t)}{2\sqrt{a_k - a_{k-1}}};\, t \geq 0\right\}$$

can be approximated by a vector-valued process

$$\{W_1(t), W_2(t), \ldots, W_k(t);\, t \geq 0\},$$

where $W_1, W_2, \ldots, W_k$ are independent Wiener processes, but the rate of approximation is worse than in the case of having only a single process of independent increments to be approximated. In any case, one can say equivalently that for any fixed $0 < a_1 < a_2 < \cdots < a_k < \infty$ there exists a Wiener sheet

$$\{W_{a_1, a_2, \ldots, a_k}(a, t);\, 0 \leq a < \infty, 0 \leq t < \infty\}$$

such that $W_{a_1, a_2, \ldots, a_k}(a_i, t)$ approximates the process $\frac{1}{2}\mathscr{L}(a_i, t)$, $i = 1, 2, \ldots, k$; $t \geq 0$. Clearly then, $W_{a_1, a_2, \ldots, a_k}(a, t)$ can be constructed as an arbitrary Wiener sheet satisfying $W_{a_1, a_2, \ldots, a_k}(a_j, t) = \sum_{l=1}^{j}(a_l - a_{l-1})^{1/2}W_l(t)$, $a_0 = 0$; $j = 1, 2, \ldots, k$. By a Wiener sheet we mean a two-parameter Gaussian process $\{W(a, u);\, 0 \leq a < \infty,\, 0 \leq u < \infty\}$ with covariance function $(a_1 \wedge a_2)(u_1 \wedge u_2)$ [cf., e.g., Section 1.11 in Csörgő and Révész (1981)].

This observation suggests the question: Can the process $\mathscr{L}(a, u)$ be strongly approximated by a single two-parameter Wiener process?

Since, by the law of interated logarithm, $L(a, u) = 0$ a.s. if $a \geq ((2 + \varepsilon)u \log\log u)^{1/2}$ and u is big enough, we have $\mathscr{L}(a, u) = -u$ for any u big enough. This clearly shows that the structure of $\mathscr{L}(a, u)$ is quite different from that of $W(a, u)$ whenever a is big. Hence we modify our question as follows. Can the process $\mathscr{L}(a, u)$ be strongly approximated by a Wiener sheet provided that u is big but a is not very big?

Our proposition of Section 2 gives a positive answer to this question by saying that one can find a Wiener sheet $W(a, u)$ for which $\mathscr{L}(a, u) - 2W(a, u)$ is small a.s. if u is big but a is not very big. Replacing u in this proposition by $L(0, t)$, we get

$$(1.1) \qquad (L(a,t) - L(0,t)) - 2W(a, L(0,t)) \quad \text{is small a.s.}$$

Since we do not know anything about the joint behaviour of $W(\cdot, \cdot)$ and $L(0, \cdot)$, it is hard to apply this result for the study of the process $L(a, t) - L(0, t)$. However, we also prove that the process $\{T_u;\, u \geq 0\}$ can be approximated by a process $\{\hat{T}_u;\, u \geq 0\}$ having the following two properties:

$$(\text{B.i}) \qquad \{\hat{T}_u;\, u \geq 0\} =_{\mathscr{D}} \{T_u;\, u \geq 0\}.$$

(B.ii) The processes $\{\hat{T}_u;\, u \geq 0\}$ and $\{W(a, u);\, a \geq 0, u \geq 0\}$ are independent.

Define the local time process $\hat{L}(0, \cdot)$ by

$$\hat{L}(0, \hat{T}_u) = u, \qquad u \geq 0.$$

By the continuity properties of $L(0, \cdot)$, we have [cf. Csáki, Csörgő, Földes and Révész (1983)]

$$\lim_{u \to \infty} \frac{|L(0, T_u) - L(0, \hat{T}_u)|}{|T_u - \hat{T}_u|^{1/2} \log u} = 0 \quad \text{a.s.}$$

Consequently, having defined the local time process $\hat{L}(0, \cdot)$ as the right-continuous (in fact, continuous) inverse of $\hat{T}_u$, we have

$$\begin{aligned} \hat{L}(0, \hat{T}_u) = L(0, T_u) &= L(0, \hat{T}_u) + L(0, T_u) - L(0, \hat{T}_u) \\ &= L(0, \hat{T}_u) + o(|T_u - \hat{T}_u|^{1/2} \log u) \quad \text{a.s.,} \end{aligned}$$

that is,

$$(1.2) \qquad \hat{L}(0, \hat{T}_u) - L(0, \hat{T}_u) = o(|T_u - \hat{T}_u|^{1/2} \log u) \quad \text{a.s.}$$

Thus, knowing (B.i) and (B.ii), we conclude that the process $\{\hat{L}(0, t);\, t \geq 0\}$ has the following properties:

$$(\text{C.i}) \qquad |\hat{L}(0,t) - L(0,t)| \quad \text{is small a.s., } t \to \infty,$$

$$(\text{C.ii}) \qquad \{\hat{L}(0,t);\, t \geq 0\} =_{\mathscr{D}} \{L(0,t);\, t \geq 0\},$$

(C.iii) $\{\hat{L}(0, t);\, t \geq 0\}$ and $\{W(a, u);\, a \geq 0, u \geq 0\}$ are independent.

Now (1.1), (C.i), (C.ii), (C.iii) and the continuity of $W(\cdot, \cdot)$ imply

$$(1.3) \qquad |(L(a,t) - L(0,t)) - 2W(a, \hat{L}(0,t))| \quad \text{is small a.s.,}$$

where $\hat{L}(0, t)$ satisfies (C.ii) and (C.iii).

The above sketched results are precisely formulated in Section 2. Section 3 is devoted to the proofs. In Section 4 some applications are given.

A weak convergence analog of our theorem was given by Yor (1983) as follows.

THEOREM A [Yor (1983)]. *As* $\lambda \to \infty$,

$$\left(\frac{1}{\lambda}W(\lambda^2 t), \frac{1}{\lambda}L(a, \lambda^2 t), \frac{1}{2\sqrt{\lambda}}\left(L(a, \lambda^2 t) - L(0, \lambda^2 t)\right)\right)$$
$$\to_{\mathscr{D}} \left(W(t), L(a, t)\right), W^*(a, L(0, t)),$$

where $W^*(a, u)$ *is a Wiener sheet independent of* $W(t)$ *and* $\to_{\mathscr{D}}$ *denotes weak convergence over the function space* $C(R_+^2; R^3)$.

This result also follows from our theorem. Here $C(R_+^2; R^3)$ is the space of all continuous functions from R_+^2 to R^3 endowed with the topology of compact uniform convergence.

2. Main results.

PROPOSITION. *There is a probability space with*

(i) *a standard Wiener process* $\{W(t); t \geq 0\}$, *its two-parameter local time process* $\{L(a, t); a \in R^1, t > 0\}$ *and the inverse process* T_u *of* $L(0, t)$ *defined by*

$$T_u = \inf\{t: t \geq 0, L(0, t) \geq u\},$$

(ii) *a two-time parameter Wiener process* $\{W(a, u); a \geq 0, u \geq 0\}$,
(iii) *a process* $\{\hat{T}_u; u \geq 0\}$ *with*

$$\{\hat{T}_u; u \geq 0\} =_{\mathscr{D}} \{T_u; u \geq 0\},$$

such that

$$(\alpha) \qquad \sup_{0 \leq a \leq a^* u^\delta} |L(a, T_u) - u - 2W(a, u)| = O(u^{(1+\delta)/2-\varepsilon}) \quad a.s., u \to \infty,$$

$$(\beta) \qquad |T_u - \hat{T}_u| = O(u^{15/8}) \quad a.s., u \to \infty,$$

and

(γ) $\{\hat{T}_u; u \geq 0\}$ *and* $\{W(a, u); a \geq 0, u \geq 0\}$ *are independent,*

where $a^* > 0$, $0 \leq \delta \leq 7/100$, $0 < \varepsilon < 1/72 - \delta/7$.

Applying the method sketched in the Introduction, we obtain the following theorem.

THEOREM. *There is a probability space with*

(i) *a standard Wiener process* $\{W(t); t \geq 0\}$ *and its two-parameter local time process* $\{L(a, t); a \in R^1, t \geq 0\}$,
(ii) *a two-time parameter Wiener process* $\{W(a, u); a \geq 0, u \geq 0\}$,
(iii) *a process* $\{\hat{L}(0, t); t \geq 0\}$ *with*

$$\{\hat{L}(0, t); t \geq 0\} =_{\mathscr{D}} \{L(0, t); t \geq 0\},$$

such that

$$(\alpha) \qquad \sup_{0 \le a \le a^* t^{\delta/2}} |L(a,t) - L(0,t) - 2W(a, \hat{L}(0,t))| = O\big(t^{((1+\delta)/4) - \varepsilon/2}\big) \quad a.s., \ t \to \infty,$$

$$(\beta) \qquad |\hat{L}(0,t) - L(0,t)| = O\big(t^{15/32} \log^2 t\big) \quad a.s., \ t \to \infty,$$

(γ) $\{\hat{L}(0,t); t \ge 0\}$ *and* $\{W(a,u); a \ge 0, u \ge 0\}$ *are independent,*

where $a^* > 0$, $0 \le \delta < 7/100$, $0 < \varepsilon < 1/72 - \delta/7$.

3. Proofs. At first we give a special case of a theorem of Berkes and Philipp (1979) for the sake of using it as a basic tool of construction in our proofs.

THEOREM B. *Let* $\{X_j; j = 1, 2, \dots\}$ *be a sequence of r.v.'s and let* $\{\mathscr{F}_j; j = 1, 2, \dots\}$ *be a nondecreasing sequence of σ-fields such that* X_j *is* $\mathscr{F}_j$*-measurable. Suppose that for some nonnegative* λ_j, δ_j *and* $D_j \ge 10^8$,

$$\mathbb{E}|\mathbb{E}(\exp iuX_j|\mathscr{F}_{j-1}) - e^{-u^2/2}| \le \lambda_j$$

for all u with $|u| \le D_j$ *and*

$$\mathbb{P}\{|N| \ge \tfrac{1}{4}D_j\} \le \delta_j,$$

where N is a normal $(0,1)$ *r.v. Then without changing its distribution we can redefine the sequence* $\{X_j; j \ge 1\}$ *together with a sequence* $\{Y_j; j \ge 1\}$ *of independent, standard normal r.v.'s such that*

$$\mathbb{P}\{|X_j - Y_j| \ge \eta_j\} \le \eta_j, \qquad j = 1, 2, \dots,$$

where

$$\eta_j = 16 D_j^{-1} \log D_j + 4\lambda_j^{1/2} D_j + \delta_j, \qquad j = 2, 3, \dots.$$

Now we formulate and prove our first of the six lemmas which will lead to the proofs of our proposition and theorem.

LEMMA 1. *Let* $a^* > 0$, $a_j = jr^{-\beta}$, $j = 0, 1, \dots, [a^* r^{\gamma+\beta}]$, *with some* $\beta > 0$, $\gamma \ge 0$. *Assume also that r is big enough. Then on a suitable probability space one can define a sequence of random vectors* $\{(X_j, Y_j); j = 1, 2, \dots, [a^* r^{\gamma+\beta}]\}$ *such that*

(i) $\{X_j; j = 1, 2, \dots, [a^* r^{\gamma+\beta}]\} =_{\mathscr{D}} \{X_j^*; j = 1, 2, \dots, [a^* r^{\gamma+\beta}]\}$,
(ii) $\{Y_j; j \ge 1\}$ *are independent, standard normal r.v.'s and*
(iii) $\mathbb{P}\{|X_j - Y_j| \ge \eta\} \le \eta$, $\quad j = 1, 2, \dots, [a^* r^{\gamma+\beta}]$, *where*

$$\eta = O\big((\log r) r^{(\gamma-1)/8}\big) \quad \text{and} \quad X_j^* = \frac{r^{(\beta-1)/2}}{2}\big(L(a_j, T_r) - L(a_{j-1}, T_r)\big).$$

PROOF. In order to apply Theorem A, we give an upper estimate of

$$E_j = \mathbb{E}|\mathbb{E}(\exp iuX_j^*|\mathscr{F}_{j-1}) - e^{-u^2/2}|,$$

where $\mathscr{F}_{j-1}$ is the smallest σ-algebra with respect to which the r.v.'s X_k^*, $k = 1, 2, \dots, j-1$, are measurable.

Since for fixed r $\{L(a, T_r); a \geq 0\}$ is a diffusion and

$$\mathbb{E}(\exp \eta L(a, T_r)) = \exp \frac{r\eta}{1 - 2a\eta} \tag{3.1}$$

[cf. Itô and McKean (1965), Problem 4, pages 73–74, or Bass and Griffin (1985)], we have

$$\begin{aligned}
&\mathbb{E}(\exp iuX_j^*|L(a_{j-1}, T_r) = z)\\
&= \mathbb{E}\left(\exp\left(\frac{iu}{2r^{(1-\beta)/2}}(L(a_1, T_z) - z)\right)\right)\\
&= \exp\left(-\frac{u^2 z}{2r}\frac{1}{1 - iu\sqrt{r^{-1-\beta}}}\right) = e^{-A}\exp(-iAu\sqrt{r^{-1-\beta}}),
\end{aligned}$$

where

$$A = \frac{u^2 z}{2r}\frac{1}{1 + u^2 r^{-1-\beta}}.$$

Since

$$|Re^{i\phi} - B|^2 = R^2 + B^2 - RB(e^{i\phi} + e^{-i\phi}),$$

we obtain

$$\begin{aligned}
&|e^{-A}\exp(-iAu\sqrt{r^{-1-\beta}}) - e^{-u^2/2}|^2\\
&= e^{-2A} + e^{-u^2} - \exp\left(-\frac{u^2}{2} - A - iAu\sqrt{r^{-1-\beta}}\right)\\
&\quad -\exp\left(-\frac{u^2}{2} - A + iAu\sqrt{r^{-1-\beta}}\right)\\
&= \exp\left(-\frac{u^2 z}{r}\frac{1}{1 + u^2 r^{-1-\beta}}\right)\\
&\quad -\exp\left(-\frac{u^2}{2} - \frac{u^2 z}{2r}\frac{1}{1 - iu\sqrt{r^{-1-\beta}}}\right)\\
&\quad -\exp\left(-\frac{u^2}{2} - \frac{u^2 z}{2r}\frac{1}{1 + iu\sqrt{r^{-1-\beta}}}\right) + e^{-u^2}.
\end{aligned}$$

Replacing z by $L(a_{j-1}, T_r) = L_{j-1}$ and applying (3.1), we get

$$E_j^2 \le \mathbb{E}\left|\exp\left(-\frac{u^2}{2r}\frac{L_{j-1}}{1-iu\sqrt{r^{-1-\beta}}}\right) - e^{-u^2/2}\right|^2$$

$$= \mathbb{E}\left(\exp\left(-\frac{u^2}{r}\frac{L_{j-1}}{1+u^2r^{-1-\beta}}\right)\right)$$

$$-\mathbb{E}\left(\exp\left(-\frac{u^2}{2}-\frac{u^2}{2r}\frac{L_{j-1}}{1-iu\sqrt{r^{-1-\beta}}}\right)\right)$$

$$-\mathbb{E}\left(\exp\left(-\frac{u^2}{2}-\frac{u^2}{2r}\frac{L_{j-1}}{1+iu\sqrt{r^{-1-\beta}}}\right)\right) + e^{-u^2}$$

$$= \exp\left(-\frac{ru^2}{r+u^2(a_j+a_{j-1})}\right)$$

$$-\exp\left(-\frac{u^2}{2}\left(1+\frac{1}{1+a_{j-1}u^2r^{-1}-iu\sqrt{r^{-1-\beta}}}\right)\right)$$

$$-\exp\left(-\frac{u^2}{2}\left(1+\frac{1}{1+a_{j-1}u^2r^{-1}+iu\sqrt{r^{-1-\beta}}}\right)\right) + e^{-u^2}$$

$$= \exp\left(-\frac{ru^2}{r+u^2(a_j+a_{j-1})}\right) + e^{-u^2} - Q,$$

where

$$Q = \exp\left(-\frac{u^2}{2}\left(1+\frac{1}{K-iL}\right)\right) + \exp\left(-\frac{u^2}{2}\left(1+\frac{1}{K+iL}\right)\right),$$

$$K = 1 + a_{j-1}u^2r^{-1} \quad \text{and} \quad L = u\sqrt{r^{-1-\beta}}.$$

Consequently,

$$Q = 2\exp\left(-\frac{u^2}{2}\left(1+\frac{K}{K^2+L^2}\right)\right)\cos\frac{u^2L}{2(K^2+L^2)}$$

$$= 2e^{-u^2}\exp\left(-\frac{u^2}{2}\left(\frac{K}{K^2+L^2}-1\right)\right)\cos\frac{u^2L}{2(K^2+L^2)}$$

and

$$K^2 + L^2 = 1 + \frac{j^2}{r^{2+2\beta}}u^4 - \frac{2j-1}{r^{2+2\beta}}u^4 + \frac{(2j-1)u^2}{r^{1+\beta}}$$
$$= 1 + a_{j-1}^2 u^4 r^{-2} + (a_j + a_{j-1})u^2 r^{-1}.$$

Hence $E_j^2 \le \mathrm{I} + \mathrm{II}$, where

$$\mathrm{I} = \exp\left(\frac{-ru^2}{r + u^2(a_j + a_{j-1})}\right) - e^{-u^2}$$

and

$$\mathrm{II} = 2\left(e^{-u^2} - e^{-u^2}\exp\left(-\frac{u^2}{2}\left(\frac{K}{K^2+L^2} - 1\right)\right)\cos\frac{u^2 L}{2(K^2+L^2)}\right).$$

Since

$$0 \le \exp\left(\frac{-u^2}{1+\varepsilon u^2}\right) - e^{-u^2} \le 2u^4\varepsilon e^{-u^2} \quad \text{if } u^4\varepsilon \le 1$$

and

$$\frac{a_{j-1} + a_j}{r} \le 2r^{\gamma-1}a^*,$$

we have

$$\mathrm{I} \le 4u^4 a^* r^{\gamma-1} e^{-u^2} \le C_1 r^{\gamma-1} a^* \quad \text{if } |u| \le \left(\frac{r^{1-\gamma}}{2a^*}\right)^{1/4},$$

where $C_1 = 4\max_u(u^4 e^{-u^2})$.

In order to estimate II we observe

$$\frac{K}{K^2+L^2} - 1 \le 0,$$

and conclude

$$\mathrm{II} \le 2e^{-u^2}\left(1 - \cos\frac{u^2 L}{2(K^2+L^2)}\right) \le e^{-u^2}\frac{u^4 L^2}{(K^2+L^2)^2} \le e^{-u^2}u^4 L^2$$
$$\le u^6 e^{-u^2} r^{-1-\beta} \le c_2 r^{\gamma-1}.$$

Hence we have

$$E_j \le \left(\mathbb{E}\left|\mathbb{E}\left(\exp iuX_j^* \mid \mathscr{F}_{j-1}\right) - e^{-u^2/2}\right|^2\right)^{1/2} \le \left(K_1 a^* r^{\gamma-1}\right)^{1/2},$$

provided that $|u| \le (r^{1-\gamma}/2a^*)^{1/4}$, where K_1 is an absolute constant.

Hence, by choosing

$$D_j = r^{(1-\gamma)/8}, \qquad \delta_j = \exp\left(-\tfrac{1}{2}r^{(1-\gamma)/4}\right), \qquad \lambda_j = \sqrt{K_1 a^*}\, r^{(\gamma-1)/2},$$

$$\eta_j = 2(1-\gamma)r^{(\gamma-1)/8}\log r + 4\sqrt[4]{K_1 a^*}\, r^{(\gamma-1)/8} + \exp\left(-\tfrac{1}{2}r^{(1-\gamma)/8}\right)$$

$$\le M(\log r) r^{(\gamma-1)/8}, \qquad j = 1,2,\dots,[a^* r^\gamma],$$

we can apply Theorem B and thus conclude Lemma 1. □

Lemma 1 can be easily reformulated by using Lemma A.1 of Berkes and Philipp (1979). First we state the latter and then the hinted at reformulation as Lemma 1*.

LEMMA A.1 [Berkes and Philipp (1979)]. *Let S_i, $i = 1,2,3$, be separable Banach spaces. Let F be a distribution on $S_1 \times S_2$ and let G be a distribution on $S_2 \times S_3$ such that the second marginal of F equals the first marginal of G. Then there exists a probability space and three random variables Z_i, $i = 1,2,3$, defined on it such that the joint distribution of Z_1 and Z_2 is F and the joint distribution of Z_2 and Z_3 is G.*

This lemma will be used repeatedly in the sequel without mention. It allows us to redefine sequences of random variables without changing their distributions and to embed them in a more extended probabilistic setup at the same time.

LEMMA 1*. *Let $a^* > 0$, $a_j = jr^{-\beta}$, $j = 0,1,2,\dots,[a^* r^{\gamma+\beta}]$, with some $\beta > 0$, $\gamma \ge 0$. Assume also that r is big enough. Then on a suitable probability space one can define two Wiener processes $\{W_1(t);\, t \ge 0\}$ and $\{W_2(t);\, t \ge 0\}$ such that*

$$\mathbb{P}\{|X_j^* - Y_j^*| \ge \eta\} \le \eta, \qquad j = 1,2,\dots,[a^* r^{\gamma+\beta}],$$

where $\eta \le (\log r) r^{(\gamma-1)/8}$,

$$X_j^* = \frac{1}{2\sqrt{r^{1-\beta}}}\left(L_1\left(a_j, T_r^{(1)}\right) - L_1\left(a_{j-1}, T_r^{(1)}\right)\right),$$

$$Y_j^* = r^{-\beta/2}\left(W_2(a_j) - W_2(a_{j-1})\right),$$

$L_1(\cdot,\cdot)$ is the local time of $W_1(\cdot)$ and $T_r^{(1)}$ is the inverse of the local time process $L_1(0,\cdot)$.

Our next lemma collects those properties of $L(a, T_r)$ which will be needed in the sequel.

LEMMA 2.

(i) *For $x > 0$ we have*

$$\mathbb{P}\left(\sup_{0 \le a \le A} L(a, T_1) > x\right) \le e^{1/(2A)} e^{-x/(4A)}.$$

(ii) *For* $0 < x < 2U$ *we have*

$$\mathbb{P}\Big(\sup_{u \le U} \sup_{a \le A} |L(a, T_u) - u| > x\Big) \le 8 \exp\left\{-\frac{x^2}{16AU}\right\}.$$

(iii) *For*

$$0 < x \le \min\left\{u, 2u\sqrt{\frac{h_2 - h_1}{h_1}}\right\}, \qquad 0 < h_1 < h_2,$$

we have

$$\mathbb{P}\Big(\sup_{h_1 \le z \le h_2} |L(z, T_u) - L(h_1, T_u)| > x\Big) \le 2 \exp\left\{-\frac{x^2}{16u(h_2 - h_1)}\right\}.$$

PROOF. (i) Since $\{L(a, T_1);\ a \ge 0\}$ is a martingale in a, by (3.1) we have

$$\mathbb{P}\Big(\sup_{0 \le a \le A} L(a, T_1) > x\Big) \le e^{-\lambda x} E\big(e^{\lambda L(A, T_1)}\big)$$
$$= \exp\left(-\lambda x + \frac{\lambda}{1 - 2A\lambda}\right)$$

for any $\lambda > 0$. Hence on choosing $\lambda = 1/(4A)$, we get the result in (i).

(ii) To prove this statement, we use the facts that $\{L(a, T_u);\ a \ge 0\}$ is a martingale in a, and that $\sup_{a \le A} |(L(a, T_u) - u|$ is a submartingale in u. Thus, for any $\lambda > 0$,

$$\mathbb{P}\Big(\sup_{u \le U} \sup_{a \le A} |L(a, T_u) - u| > x\Big)$$
$$\le e^{-\lambda x} \mathbb{E}\big(e^{\lambda \sup_{a \le A} |L(a, T_U) - U|}\big)$$
$$= e^{-\lambda x} \mathbb{E}\Big(\sup_{a \le A} \big(e^{(\lambda/2)|L(a, T_U) - U|}\big)^2\Big) \le 4 e^{-\lambda x} \mathbb{E}\big(e^{\lambda |L(A, T_U) - U|}\big)$$
$$\le 4 e^{-\lambda x} \big\{\mathbb{E}\big(e^{\lambda (L(A, T_U) - U)}\big) + \mathbb{E}\big(e^{-\lambda (L(A, T_U) - U)}\big)\big\}.$$

Based on (3.1) again, we get

$$\mathbb{P}\Big(\sup_{u \le U} \sup_{a \le A} |L(a, T_u) - u| > x\Big)$$
$$\le 4 e^{-\lambda x} \left(\exp\left\{-\lambda U + \frac{\lambda U}{1 - 2A\lambda}\right\} + \exp\left\{\lambda U - \frac{\lambda U}{1 + 2A\lambda}\right\}\right)$$
$$\le 8 \exp\left\{-\lambda x + \frac{2\lambda^2 UA}{1 - 2A\lambda}\right\} \le 8 \exp\{-\lambda x + 4\lambda^2 UA\},$$

where the last inequality holds if $\lambda \le 1/(4A)$. On choosing $\lambda = x/(8UA)$, we get (ii).

(iii) The proof of this statement is similar to that of the preceding one. Hence we give only a brief sketch of it. As in (ii) we get

$$\begin{aligned}\mathbb{P}\Big(\sup_{h_1\le z\le h_2}|L(z,T_u)-L(h_1,T_u)|>x\Big)\\ \le \exp\{-\lambda x\}\mathbb{E}(\exp\{\lambda L(h_2,T_u)-\lambda L(h_1,T_u)\})\\ +\exp\{-\lambda x\}\mathbb{E}(\exp\{\lambda L(h_1,T_u)-\lambda L(h_2,T_u)\}).\end{aligned}\tag{3.2}$$

In order to compute the above expectation we use the strong Markov property of $L(a,T_u)$. Namely, for any $y>0$ and $s>0$, we have

$$\mathbb{P}(L(h_2,T_u)-L(h_1,T_u)<y|L(h_1,T_u)=s)=\mathbb{P}(L(h_2-h_1,T_s)-s<y).$$

Consequently,

$$\begin{aligned}&\mathbb{E}(\exp\{\lambda(L(h_2,T_u)-L(h_1,T_u))\}|L(h_1,T_u)=s)\\ &=\mathbb{E}(\exp\{\lambda(L(h_2-h_1,T_s)-s)\})=\exp\left\{\frac{2\lambda^2 s(h_2-h_1)}{1-2(h_2-h_1)\lambda}\right\}.\end{aligned}$$

Hence

$$\begin{aligned}&\mathbb{E}(\exp\{\lambda(L(h_2,T_u)-L(h_1,T_u))\})\\ &=\mathbb{E}\left(\exp\left\{\frac{2\lambda^2(h_2-h_1)}{1-2(h_2-h_1)\lambda}L(h_1,T_u)\right\}\right)\\ &=\exp\left\{\frac{2\lambda^2(h_2-h_1)u}{1-2(h_2-h_1)\lambda-4\lambda^2h_1(h_2-h_1)}\right\},\end{aligned}$$

and similarly for the second term in (3.2). This implies that if

$$\lambda<\min\left\{\frac{1}{8(h_2-h_1)},\frac{1}{4\sqrt{h_1(h_2-h_1)}}\right\},$$

then

$$\mathbb{P}\Big(\sup_{h_1\le z\le h_2}|L(z,T_u)-L(h_1,T_u)|>x\Big)\le 2\exp\{-\lambda x+4\lambda^2(h_2-h_1)u\}.$$

By choosing $\lambda=x/(8u(h_2-h_1))$ we get (iii). □

Using the notation of Lemma 1* we prove the following lemma.

LEMMA 3. *For any*

$$\tau>\max\left\{\frac{9\gamma+4\beta+3}{8},\frac{1-\beta}{2}\right\}$$

we have

$$\mathscr{P} = \mathbb{P}\left\{ \sup_{0 \le a \le [a^* r^\gamma]} |L_1(a, T_r^{(1)}) - r - 2r^{1/2} W_2(a)| \ge r^\tau \right\} \le C^* r^A,$$

where $C^* > 0$ *is an absolute constant and* $A = (9\gamma - 1)/8 + \beta$.

PROOF.

$$\mathscr{P} = \mathbb{P}\left\{ \sup_{1 \le j \le [a^* r^{\gamma+\beta}]} \sup_{a_{j-1} \le a \le a_j} |L_1(a, T_r^{(1)}) - r - 2r^{1/2} W_2(a)| \ge r^\tau \right\}$$

$$= \mathbb{P}\left\{ \sup_{1 \le j \le [a^* r^{\gamma+\beta}]} \sup_{a_{j-1} \le a \le a_j} \left| \sum_{l=1}^{j-1} \left(L_1(a_l, T_r^{(1)}) - L_1(a_{l-1}, T_r^{(1)}) \right) + L_1(a, T_r^{(1)}) - L_1(a_{j-1}, T_r^{(1)}) - 2r^{1/2} \left(\sum_{l=1}^{j-1} (W_2(a_l) - W_2(a_{l-1})) + W_2(a) - W_2(a_{j-1}) \right) \right| \ge r^\tau \right\}$$

$$\le \sum_{j=1}^{[a^* r^{\gamma+\beta}]} \mathbb{P}\left\{ |L_1(a_j, T_r^{(1)}) - L_1(a_{j-1}, T_r^{(1)}) - 2r^{1/2}(W_2(a_j) - W_2(a_{j-1}))| \ge \frac{r^\tau}{3[a^* r^{\gamma+\beta}]} \right\}$$

$$+ \mathbb{P}\left\{ \sup_{1 \le j \le [a^* r^{\gamma+\beta}]} \sup_{a_{j-1} \le a \le a_j} |L_1(a, T_r^{(1)}) - L_1(a_{j-1}, T_r^{(2)})| \ge \frac{r^\tau}{3} \right\}$$

$$+ \mathbb{P}\left\{ \sup_{1 \le j \le [a^* r^{\gamma+\beta}]} \sup_{a_{j-1} \le a \le a_j} |W_2(a) - W_2(a_{j-1})| \ge \frac{r^\tau}{6\sqrt{r}} \right\} = \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

Choose such a τ for which we have

$$(\log r) r^{(\gamma-1)/8} < \frac{r^\tau}{6[a^* r^{\gamma+\beta}]\sqrt{r^{1-\beta}}},$$

that is, $\tau > (9\gamma + 4\beta + 3)/8$. Then by Lemma 1* we obtain

$$\mathrm{I} \le [a^* r^{\gamma+\beta}](\log r) r^{(\gamma-1)/8} \le C(\log r) r^A,$$

and by Lemma 2(iii)

$$\mathrm{II} \le 2[a^* r^{\gamma+\beta}] \exp\left(-\frac{r^{2\tau}}{144 r^{1-\beta}} \right)$$

if $\tau < 1 - (\beta + \gamma)/2$. Obviously, we have to choose $\tau > (1 - \beta)/2$ in order to get an exponent tending to ∞.

A similar estimate can be obtained for III. Hence we have Lemma 3. □

Our next aim is to prove that the local time process $L_1(a, T_r^{(1)})$, resp. the inverse process $T_r^{(1)}$ of $L_1(0, \cdot)$, can be replaced by another local time process $L(a, T_r)$, resp. the inverse process T_r of $L(0, \cdot)$, such that

(i) the statement of Lemma 3 remains valid with $L(a, T_r)$, resp. T_r, instead of $L_1(a, T_r^{(1)})$, resp. $T_r^{(1)}$,
(ii) T_r will be independent of $W_2(\cdot)$.

In order to construct the processes $L(a, T_r)$ and T_r, consider a Wiener process $\{W_3(t);\, t \geq 0\}$, independent of $W_1(\cdot)$ and $W_2(\cdot)$. Let $L_3(\cdot, \cdot)$ be the local time of $W_3(\cdot)$ and $T_r^{(3)}$ be the inverse process of $L_3(0, \cdot)$.

Consider the increments $T_i^{(3)} - T_{i-1}^{(3)}$, $i = 1, 2, \dots, r$, and call these increments large if $T_i^{(3)} - T_{i-1}^{(3)} > r^{\theta}$, where $0 < \theta < 2$ will be specified later. Otherwise these increments will be called small. Similarly, we define the large and small increments of the process $T_r^{(1)}$. Assume that the number of the large increments of $T_r^{(3)}$, resp. those of $T_r^{(1)}$, are $\mu^{(3)}$ and $\mu^{(1)}$, resp. Denote the sequences of the small increments by

$$T_{j_1}^{(3)} - T_{j_1-1}^{(3)}, T_{j_2}^{(3)} - T_{j_2-1}^{(3)}, \dots, T_{j_{r-\mu^{(3)}}}^{(3)} - T_{j_{r-\mu^{(3)}}-1}^{(3)},$$
$$1 \leq j_1 < j_2 < \cdots < j_{r-\mu^{(3)}} < r,$$

and similarly by

$$T_{i_1}^{(1)} - T_{i_1-1}^{(1)}, T_{i_2}^{(1)} - T_{i_2-1}^{(1)}, \dots, T_{i_{r-\mu^{(1)}}}^{(1)} - T_{i_{r-\mu^{(1)}}-1}^{(1)},$$
$$1 \leq i_1 < i_2 < \cdots < i_{r-\mu^{(1)}} < r.$$

Now the constructions of T_r and $L(a, T_r)$ are as follows. The large increments of $T_r^{(3)}$ and the corresponding increments of $L^{(3)}(a, T_r^{(3)})$ should remain unchanged but the small increments should be changed by the corresponding increments of $T_r^{(1)}$, resp., by those of $L^{(1)}(a, T_r^{(1)})$, that is,

$$T_u = \begin{cases} T_u^{(3)}, & \text{if } T_1^{(3)} - T_0^{(3)} \text{ is large,} \\ T_{i_1+u}^{(1)} - T_{i_1}^{(1)}, & \text{if } T_1^{(3)} - T_0^{(3)} \text{ is small,} \end{cases}$$

where $0 \leq u \leq 1$ and i_1 is the smallest integer for which $T_{i+1}^{(1)} - T_i^{(1)}$ is small,

$$T_{1+u} - T_1 = \begin{cases} T_{1+u}^{(3)} - T_1^{(3)}, & \text{if } T_2^{(3)} - T_1^{(3)} \text{ is large,} \\ T_{i_2+u}^{(1)} - T_{i_2}^{(1)}, & \text{if } T_2^{(3)} - T_1^{(3)} \text{ is small,} \end{cases}$$

where again $0 \leq u \leq 1$, and i_2 is the smallest integer not used before for which $T_{i_2+u}^{(1)} - T_{i_2}^{(1)}$ is small. In the case of $\mu^{(3)} < \mu^{(1)}$ one cannot perform all the indicated changes. Hence we stop after the first $r - \mu^{(1)}$ changes and leave the remaining short increments and local time increments unchanged. We denote the arising processes by T_s, $L(a, T_s)$, $s \leq r$. Observe that in T_r the number and ordering of the large and small increments are the same as in $T_r^{(3)}$. Clearly,

$$\begin{aligned} &\{(T_s, L(a, T_s));\, a \geq 0, r \geq s \geq 0\} \\ &=_{\mathscr{D}} \{(T_s^{(3)}, L^{(3)}(a, T_s^{(3)}));\, a \geq 0, r \geq s \geq 0\} \\ &=_{\mathscr{D}} \{(T_s^{(1)}, L^{(1)}(a, T_s^{(1)}));\, a \geq 0, r \geq s \geq 0\}. \end{aligned}$$

We prove that

$$T_r \text{ is close to } T_r^{(3)} \quad \text{(Lemma 4)}$$

and

$$L(a, T_r) - r \text{ is close to } L^{(1)}(a, T_r^{(1)}) - r \quad \text{(Lemma 5)}.$$

Consequently, by Lemma 3 the latter is also close to $2r^{1/2}W_2(a)$. The independence of $T_r^{(3)}$ and $W_2(a)$ is clear from the above construction.

LEMMA 4. *For any $\mathscr{K} > 0$ we have*

$$\mathbb{P}\left\{\sup_{0 \le s \le r} |T_s - T_s^{(3)}| \ge r^{\mathscr{K}}\right\} \le Cr^{1+\theta/2-\mathscr{K}}.$$

PROOF. Observe that

$$\sup_{s \le r} |T_s - T_s^{(3)}| \le \sum_{i=1}^{r} \left(T_i^{(3)} - T_{i-1}^{(3)}\right)\mathbb{1}\left[T_i^{(3)} - T_{i-1}^{(3)} < r^{\theta}\right] + \sum_{i=1}^{r} \left(T_i^{(1)} - T_{i-1}^{(1)}\right)\mathbb{1}\left[T_i^{(1)} - T_{i-1}^{(1)} < r^{\theta}\right].$$

Clearly,

$$T_i^{(3)} - T_{i-1}^{(3)} =_{\mathscr{D}} T_i^{(1)} - T_{i-1}^{(1)} =_{\mathscr{D}} T_1, \qquad i = 1, 2, \ldots, r,$$

and

$$\mathbb{P}(T_1 < u) = 2\left(1 - \phi(u^{-1/2})\right).$$

Consequently,

$$\mathbb{E}\left(T_1 \mathbb{1}\left[T_1 < r^{\theta}\right]\right) \le Cr^{\theta/2}$$

and

$$\mathbb{E}\left(\sup_{0 \le s \le r} |T_s - T_s^{(3)}|\right) \le Cr^{1+\theta/2}.$$

Thus the lemma follows by the Markov inequality. □

LEMMA 5. *For $\gamma < \tau$ we have*

$$\mathbb{P}\left\{\sup_{0 \le a \le a^* r^{\gamma}} |L^{(1)}(a, T_r^{(1)}) - L(a, T_r)| > r^{\tau}\right\} \le 5r \exp\{-Cr^{B}\}$$

with $B = (\tau - \gamma)/2$.

PROOF. It is easy to see that

$$\begin{aligned}\sup_{0 \le a \le a^* r^{\gamma}} &|L^{(1)}(a, T_r^{(1)}) - L(a, T_r)| \\ &\le \left(\mu^{(1)} + \mu^{(3)}\right)\max\left(\sup_{1 \le i \le r}\ \sup_{0 \le a \le a^* r^{\gamma}} |L^{(3)}(a, T_i^{(3)}) - L^{(3)}(a, T_{i-1}^{(3)})|,\right. \\ &\qquad \left.\sup_{1 \le i \le r}\ \sup_{0 \le a \le a^* r^{\gamma}} |L^{(1)}(a, T_i^{(1)}) - L^{(1)}(a, T_{i-1}^{(1)})|\right).\end{aligned}$$

Moreover, it is also clear that:

(i) $\mu^{(1)} + \mu^{(3)}$ is a binomial r.v. with parameters $2r$ and $p = \mathbb{P}(T_1 > r^{\theta})$.
(ii) $\mathbb{P}(T_1 > r) = 2\phi(r^{-\theta/2}) - 1 \leq r^{-\theta/2}$, $\theta > 0$.

Hence

$$\begin{aligned}
&\mathbb{P}\big(\mu^{(1)} + \mu^{(3)} \geq 4r^{1-\theta/2}\big) \\
&\quad \leq \mathbb{E}\big(\exp(\mu^{(1)} + \mu^{(3)})\exp(-4r^{1-\theta/2})\big) \\
&\quad = (ep + 1 - p)^{2r}\exp(-4r^{1-\theta/2}) \leq \exp\{-Cr^{1-\theta/2}\}, \qquad 0 < \theta < 2.
\end{aligned}$$

Now Lemma 2(i) gives

$$\mathbb{P}\Big\{\sup_{0 \leq a \leq a^* r^{\gamma}} |L^{(3)}(a, T_i^{(3)}) - L^{(3)}(a, T_{i-1}^{(3)})| > r^{\eta}\Big\} \leq C\exp\Big(-\frac{r^{\eta-\gamma}}{4a^*}\Big)$$

if $\eta > \gamma$. Consequently,

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{0 \leq a \leq a^* r^{\gamma}} |L^{(1)}(a, T_r^{(1)}) - L(a, T_r)| \geq r^{\tau}\Big\} \\
&\quad \leq \exp\{-Cr^{1-\theta/2}\} \\
&\qquad + 2\mathbb{P}\Big\{\sup_{1 \leq i \leq r}\ \sup_{0 \leq a \leq a^* r^{\gamma}} |L^{(3)}(a, T_i^{(3)}) - L^{(3)}(a, T_{i-1}^{(3)})| \geq \tfrac{1}{8}r^{\tau-1+\theta/2}\Big\} \\
&\quad \leq C\exp\{-Cr^{\tau-1+\theta/2-\gamma}\} + \exp\{-Cr^{1-\theta/2}\}.
\end{aligned}$$

Choosing

$$\tau - 1 + \theta/2 - \gamma = 1 - \theta/2,$$

that is,

$$\theta = 2 - (\tau - \gamma),$$

we obtain Lemma 5. □

Combining Lemmas 3, 4 and 5, we have

LEMMA 6. *For any*

$$\tau > \max\Big\{\gamma, \frac{1-\beta}{2}, \frac{9\gamma + 4\beta + 3}{8}\Big\} = \max\Big\{\frac{1-\beta}{2}, \frac{9\gamma + 4\beta + 3}{8}\Big\}$$

we have

$$\mathbb{P}\Big\{\sup_{0 \leq a \leq [a^* r^{\gamma}]} |L(a, T_r) - r - 2r^{1/2}W_2(a)| \geq r^{\tau}\Big\} \leq Cr^{A},$$

where $A = (9\gamma - 1)/8 + \beta$. *Furthermore, the inverse local time process* T_r *can be approximated by an inverse local time process* $T_r^{(3)}$, *which is independent of*

$W_2(\cdot)$, *such that for any* $\mathscr{K} > 0$,

$$\mathbb{P}\left\{\sup_{0\le s\le r} |T_s - T_2^{(3)}| \ge r^{\mathscr{K}}\right\} \le Cr^{\overline{B}},$$

where

$$\overline{B} = 2 - \frac{\tau - \gamma}{2} - \mathscr{K}.$$

Consider the sequence $0 = A_0 < A_1 < A_2 < \cdots$ defined by

$$A_k - A_{k-1} = k^{\alpha}, \qquad k = 1, 2, \dots, \dots; \alpha > 0.$$

The required Wiener processes as well as local time and Wiener sheet will be constructed bit by bit via Lemma 6, separately on the intervals $[A_{k-1}, A_k]$. In fact for each $k = 1, 2, \dots$ one can construct a local time process $\{{}_kL(a, T_s); 0 \le s \le A_k - A_{k-1}\}$, a Wiener process $\{{}_kW_2(t); t \ge 0\}$ and an inverse local time process $\{{}_kT_s^{(3)}; 0 \le s \le A_k - A_{k-1}\}$ such that the statements of Lemma 6 [$T_s = {}_kT_s$, the inverse of ${}_kL(0, \cdot)$] hold true, and the processes $\{{}_kW_2(\cdot), {}_kL(\cdot, \cdot), {}_kT_s^{(3)}\}$, $k = 1, 2, \dots$, are independent.

Now we proceed to construct a Wiener sheet. We let

$$W(a, A_k) = \sum_{i=1}^{k} (A_i - A_{i-1})^{1/2}\, {}_iW_2(a)$$

and define $W(a, t)$ for $A_k \le t \le A_{k+1}$ by joining $W(a, A_k)$ and $W(a, A_{k+1})$ by a suitable independent Wiener sheet [cf. Csörgő and Révész (1981), Section 1.11].

Similarly, we define

$$T_u = \sum_{i=1}^{k-1} {}_iT_{A_i - A_{i-1}} + {}_kT_{u - A_{k-1}} \quad \text{if } A_{k-1} \le u \le A_k,$$

$$T_u^{(3)} = \sum_{i=1}^{k-1} {}_iT^{(3)}_{A_i - A_{i-1}} + {}_kT^{(3)}_{u - A_{k-1}} \quad \text{if } A_{k-1} \le u \le A_k$$

and

$$L(a, T_u) = \sum_{i=1}^{k-1} {}_iL\left(a, {}_iT_{A_i - A_{i-1}}\right) + {}_kL\left(a, {}_kT_{u - A_{k-1}}\right), \qquad A_{k-1} \le u \le A_k, a \ge 0.$$

The independence of $L(a, T_u)$ and $T_u^{(3)}$ is clear from the above construction. By Lemma A.1 we can also construct Wiener processes $W(t)$ and $W^{(3)}(t)$ with local times $L(a, t)$ and $L^{(3)}(a, t)$, respectively, such that T_u is the inverse of $L(0, t)$, $T_u^{(3)}$ is the inverse of $L^{(3)}(0, t)$ and $W^{(3)}(t)$ and $W(a, u)$ are independent.

Now we prove the statements (α) and (β) of the proposition.

Replace r in Lemma 6 by $k^{\alpha} = A_k - A_{k-1}$ and, in order to obtain a convergent series for the sake of applying the Borel–Cantelli lemma, choose α such that $\alpha A = (9\gamma\alpha - \alpha)/8 + \alpha\beta < -1$, $\alpha\overline{B} = \alpha(2 - (\tau - \gamma)/2 - \mathscr{K})\alpha < -1$. Then

Lemma 6 and the Borel–Cantelli lemma imply

$$\sup_{0\le a\le[a^*k^{\alpha\gamma}]} |{}_kL(a, {}_kT_{A_k-A_{k-1}}) - (A_k - A_{k-1}) - 2(A_k - A_{k-1})^{1/2}{}_kW_2(a)|$$
$$= O(k^{\alpha\tau}) \quad \text{a.s.}$$

and

$$\sup_{0\le s\le k^\alpha} |{}_kT_s - {}_kT_s^{(3)}| = O(k^{\alpha\mathscr{K}}) \quad \text{a.s.}$$

Let now k be a given positive integer and define k_0 by $k_0 = [k^{9/110}]$. Then

$$\sup_{0\le a\le a^*k_0^{\alpha\gamma}} |L(a, T_{A_k}) - A_k - 2W(a, A_k)|$$
$$\le \sup_{0\le a\le a^*k_0^{\alpha\gamma}} |L(a, T_{A_{k_0}}) - A_{k_0}| + 2\sup_{0\le a\le a^*k_0^{\alpha\gamma}} |W(a, A_{k_0})|$$
$$+ \sum_{i=k_0+1}^{k} \sup_{0\le a\le a^*k_0^{\alpha\gamma}} |{}_iL(a, {}_iT_{A_i-A_{i-1}}) - (A_i - A_{i-1})$$
$$-2(A_i - A_{i-1})^{1/2}{}_iW_2(a)|$$
$$= \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

By (iii) of Lemma 2

$$\mathrm{I} = O\big((\log k)k_0^{\alpha\gamma/2}A_{k_0}^{1/2}\big),$$

and routine calculations yield

$$\mathrm{II} = O\big((\log k)k_0^{\alpha\gamma/2}A_{k_0}^{1/2}\big).$$

Hence our above inequalities imply

$$\sup_{0\le a\le a^*k_0^{\alpha\gamma}} |L(a, T_{A_k}) - A_k - 2W(a, A_k)|$$
$$= O\big((\log k)k_o^{\alpha\gamma/2}A_{k_0}^{1/2}\big) + \sum_{i=k_0+1}^{k} O(i^{\alpha\tau})$$
$$= O\big((\log k)k^{(9\alpha\gamma/20)+9(\alpha+1)/20}\big) + O(k^{\alpha\tau+1})$$

and

$$\sup_{0\le s\le A_k} |T_s - T_s^{(3)}| = O\left(\sum_{i=1}^{k} i^{\alpha\mathscr{K}}\right) = O(k^{\alpha\mathscr{K}+1}).$$

We have to choose the parameters involved here such that we should have

$$\tau > \max\left\{\frac{1-\beta}{2}, \frac{9\gamma+4\beta+3}{8}\right\},$$
$$A = \frac{9\gamma-1}{8} + \beta, \qquad \alpha A < -1,$$
$$\overline{B} = 2 - \frac{\tau-\gamma}{2} - \mathscr{K}, \qquad \alpha\overline{B} < -1 \text{ and } \mathscr{K} < 2.$$

Some possible choices are

$$\alpha = 35, \qquad \beta = \frac{1}{11} - \frac{9\gamma}{8}, \qquad \tau = \frac{1}{2} - \frac{1}{24} + \frac{9\gamma}{16} \quad \text{and} \quad \mathscr{K} = \frac{15}{8}.$$

Consequently, we have

$$\sup_{0 \le a \le a^* r^{9\alpha\gamma/10}} |L(a, T_{A_k}) - A_k - 2W(a, A_k)| = O(k^Q) \quad \text{a.s.},$$

where

$$Q > \max\left\{\alpha\tau + 1, \tfrac{9}{20}(\alpha\gamma + \alpha + 1)\right\},$$

as well as

$$\sup_{0 \le s \le k^{\alpha+1}} |T_s - T_s^{(3)}| = O\left(k^{(15\alpha/8)+1}\right).$$

Let $A_{k-1} \le u < A_k$. Then by standard methods and (ii) of Lemma 2

$$\begin{aligned}
&\sup_{0 \le a \le a^* u^\delta} |L(a, T_u) - u - 2W(a, u)| \\
&\quad \le \sup_{0 \le a \le a^*(k-1)^{(\alpha+1)\delta}} |L(a, T_{A_{k-1}}) - A_{k-1} - 2W(a, A_{k-1})| \\
&\qquad + \sup_{0 \le a \le a^*(k-1)^{(\alpha+1)\delta}} \sup_{A_{k-1} \le u < A_k} |L(a, T_u) - L(a, T_{A_{k-1}}) - (u - A_{k-1})| \\
&\qquad + 2 \sup_{0 \le a \le a^*(k-1)^{(\alpha+1)\delta}} \sup_{A_{k-1} \le u < A_k} |W(a, u) - W(a, A_{k-1})| \\
&\quad \le O(k^Q) + O\left(k^{[\alpha + (\alpha+1)\delta]/2} \log k\right) \\
&\quad = O\left(u^{Q/(\alpha+1)}\right) + O\left(u^{[\alpha + (\alpha+1)\delta]/(2(\alpha+1))} \log u\right),
\end{aligned}$$

provided that $(\alpha + 1)\delta \le 9\alpha\gamma/10$ and

$$\sup_{0 \le s \le u} |T_s - T_s^{(3)}| = O\left(u^{(15\alpha+1)/(8(\alpha+1))}\right) = O\left(u^{15/8}\right).$$

By choosing $\gamma = 8\delta/7$, a simple calculation shows that our proposition holds with $\hat{T}_u = T_u^{(3)}$.

We can now formalize the proof of the theorem as follows. Let $\hat{L}(a, t)$ be the local time of the Wiener process $W^{(3)}(t)$, that is, $\hat{L}(a, t) = L^{(3)}(a, t)$. Then (1.2) and ($\beta$) of the proposition imply that almost surely

$$\begin{aligned}
\hat{L}(0, t) - L(0, t) &\le \hat{L}\left(0, \hat{T}_{\hat{L}(0,t)}\right) - L\left(0, \hat{T}_{\hat{L}(0,t)}\right) \\
&= O\left(\left(\hat{L}(0, t)\right)^{15/16} \log \hat{L}(0, t)\right) \\
&= O\left(t^{15/32} \log^2 t\right).
\end{aligned}$$

Here we have used the facts that $\hat{T}_{\hat{L}(0,t)} \le t$ and that $\hat{L}(0, t) = O(t^{1/2} \log t)$ a.s. Similarly, one gets

$$L(0, t) - \hat{L}(0, t) = O\left(t^{15/32} \log^2 t\right) \quad \text{a.s.},$$

and hence (β) of the theorem.

To verify (α) of the theorem, we let $u = L(0, t)$ in (α) of the proposition. Then we have almost surely

$$\sup_{0 \le a \le a^*(L(0,t))^\delta} |L(a, T_{L(0,t)}) - L(0,t) - 2W(a, L(0,t))|$$
$$= O\big((L(0,t))^{(1+\delta)/2-\varepsilon}\big) = O\big(t^{(1+\delta)/4-\varepsilon_1/2}\big),$$

where $\varepsilon_1 < \varepsilon$. This implies also

$$\sup_{0 \le a \le a^* t^{\delta_1/2}} |L(a, T_{L(0,t)}) - L(0,t) - 2W(a, L(0,t))| \tag{3.3}$$
$$= O\big(t^{(1+\delta)/4-\varepsilon_1/2}\big) \quad \text{a.s.},$$

where $\delta_1 < \delta$.

It follows from

$$L(a,t) \le L\big(a, T_{L(0,t)+1}\big)$$

and

$$\sup_{a \le a^* u^\delta} \big(L(a, T_{u+1}) - L(a, T_u)\big) = O(u^{2\delta}) \quad \text{a.s.},$$

which can be easily seen from the proof of Lemma 5 [see (i) of Lemma 2], that

$$\sup_{a \le a^* t^{\delta_1/2}} \big(L(a,t) - L(a, T_{L(0,t)})\big) = O\big(t^\delta \log t\big) \tag{3.4}$$
$$= O\big(t^{(1+\delta)/4-\varepsilon_1/2}\big) \quad \text{a.s.}$$

Furthermore, by standard methods one gets

$$\sup_{a \le a^* t^{\delta_1/2}} |W(a, L(0,t)) - W(a, \hat{L}(0,t))|$$
$$= O\big(t^{\delta_1/4} |L(0,t) - \hat{L}(0,t)|^{1/2} \log t\big) \tag{3.5}$$
$$= O\big(t^{\delta_1/4 + 15/64} \log^3 t\big) = O\big(t^{(1+\delta)/2-\varepsilon_1/4}\big) \quad \text{a.s.}$$

Now (3.3), (3.4) and (3.5) imply (α) of the theorem, while (γ) is obvious by the construction. This also completes the proof of the theorem.

4. Applications. At first we list a few simple properties of the process $(2W(a, \hat{L}(0,t)), t^{-1/4}\hat{L}(0,t))$ which are mostly known or can be obtained by standard methods of proof. Namely, we have

$$\frac{W(a, \hat{L}(0,t))}{\sqrt{a\hat{L}(0,t)}} =_{\mathscr{D}} N_1 \quad \text{for any } a > 0, \tag{4.1}$$

$$\frac{\hat{L}(0,t)}{\sqrt{t}} =_{\mathscr{D}} |N_2|, \tag{4.2}$$

$$\frac{W(a, \hat{L}(0,t))}{a^{1/2} t^{1/4}} =_{\mathscr{D}} N_1 |N_2|^{1/2} \quad \text{for any } a > 0, \tag{4.3}$$

where N_1, N_2 are independent normal (0, 1) r.v.'s.

Also, for any $a > 0$, the set of the limit points of

$$U_t = \frac{W(a, \hat{L}(0,t))}{\sqrt{2a\hat{L}(0,t)\log\log t}} \tag{4.4}$$

is the interval $[-1, 1]$ a.s. The set of the limit points of

$$V_t = \frac{\hat{L}(0,t)}{\sqrt{2t\log\log t}}$$

is the interval $[0, 1]$ a.s. The set of the limit points of

$$(U_t, V_t)$$

is the semicircle $\{(x, y)\colon y \geq 0, x^2 + y^2 \leq 1\}$. The set of the limit points of

$$U_t V_t^{1/2} = \frac{W(a, \hat{L}(0,t))}{2^{3/4}a^{1/2}t^{1/4}(\log\log t)^{3/4}}$$

is the interval $[0, 2^{1/2}3^{-3/4}]$ a.s. for any $a > 0$, that is,

$$\limsup_{t\to\infty} \frac{W(a, \hat{L}(0,t))}{a^{1/2}t^{1/4}(\log\log t)^{3/4}} = 2^{5/4}3^{-3/4}. \tag{4.5}$$

Similarly, one can obtain

$$\begin{aligned} &\limsup_{t\to\infty} \sup_{0<a\leq a^*t^\delta} \frac{W(a, \hat{L}(0,t))}{\sqrt{2a^*\hat{L}(0,t)t^\delta\log\log t}} \\ &= \limsup_{t\to\infty} \sup_{0<a\leq a^*t^\delta} \left(\frac{27}{32}\right)^{1/4} \frac{W(a, \hat{L}(0,t))}{(a^*t^\delta)^{1/2}t^{1/4}(\log\log t)^{3/4}} = 1 \quad \text{a.s.} \end{aligned} \tag{4.6}$$

for any $a^* > 0$ and $\delta \geq 0$.

Consequently, by our theorem we obtain

$$\frac{L(a,t) - L(0,t)}{2\sqrt{aL(0,t)}} \to_{\mathscr{D}} N_1, \qquad t \to \infty, \text{ for any } a > 0, \tag{4.1*}$$

$$t^{-1/2}L(0,t) =_{\mathscr{D}} |N_2|, \tag{4.2*}$$

$$\frac{L(a,t) - L(0,t)}{2a^{1/2}t^{1/4}} \to_{\mathscr{D}} N_1|N_2|^{1/2}, \qquad t \to \infty, \text{ for any } a > 0, \tag{4.3*}$$

as well as

$$\limsup_{t\to\infty} \frac{L(a,t)-L(0,t)}{2\sqrt{2aL(0,t)\log\log t}} = 1 \quad \text{a.s. for any } a>0, \tag{4.4*}$$

$$\limsup_{t\to\infty} \frac{L(a,t)-L(0,t)}{a^{1/2}t^{1/4}(\log\log t)^{3/4}} = \tfrac{4}{3}\cdot 6^{1/4} \quad \text{a.s. for any } a>0, \tag{4.5*}$$

and

$$\begin{aligned}&\limsup_{t\to\infty} \sup_{0<a\le a^*t^\delta} \frac{L(a,t)-L(0,t)}{2\sqrt{2a^*t^\delta L(0,t)\log\log t}} \\ &\quad = \limsup_{t\to\infty} \sup_{0<a\le a^*t^\delta} \tfrac{3}{4}\cdot 6^{-1/4}\frac{L(a,t)-L(0,t)}{(a^*t^\delta)^{1/2}t^{1/4}(\log\log t)^{3/4}} = 1 \quad \text{a.s.}\end{aligned} \tag{4.6*}$$

for any $a^* > 0$ and $0 \le \delta < 7/200$.

We note that (4.1*), (4.2*) and (4.3*) are also consequences of Theorem A, while (4.4*), (4.5*) and (4.6*) cannot be obtained from any weak invariance principle like Theorem A. For a direct proof of (4.4*) and (4.5*) we refer to Csáki and Földes (1988).

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REFERENCES

BASS, R. F. and GRIFFIN, P. S. (1985). The most visited site of Brownian motion and simple random walk. *Z. Wahrsch. verw. Gebiete* **70** 417–436.

BERKES, I. and PHILIPP, W. (1979). Approximation theorems for independent and weakly dependent random vectors. *Ann. Probab.* **7** 29–54.

CSÁKI, E. and FÖLDES, A. (1988). On the local time process standardized by the local time at zero. *Acta Math. Hungar.* **52** 175–186.

CSÁKI, E., CSÖRGŐ, M., FÖLDES, A. and RÉVÉSZ, P. (1983). How big are the increments of the local time of a Wiener process? *Ann. Probab.* **11** 593–608.

CSÖRGŐ, M. and RÉVÉSZ, P. (1981). *Strong Approximations in Probability and Statistics*. Academic, New York.

ITÔ, K. and MCKEAN, H. P., JR. (1965). *Diffusion Processes and Their Sample Paths*. Springer, Berlin.

KNIGHT, F. B. (1963). Random walks and a sojourn density process of Brownian motion. *Trans. Amer. Math. Soc.* **109** 56–86.

KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1975). An approximation of partial sums of independent rv's and the sample df, I. *Z. Wahrsch. verw. Gebiete* **32** 111–131.

RAY, D. B. (1963). Sojourn times of diffusion processes. *Illinois J. Math.* **7** 615–630.

WALSH, J. B. (1978). Downcrossings and the Markov property. *Astérisque* **52–53** 89–115. Soc. Math. France, Paris.

YOR, M. (1983). Le drap Brownian comme limite en loi de temps locaux linéaires. *Séminaire de Probabilités XVII, 1981 / 82. Lecture Notes in Math.* **986** 89–105. Springer, New York.

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