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EXPONENTIAL L_2 CONVERGENCE OF ATTRACTIVE REVERSIBLE NEAREST PARTICLE SYSTEMS[1]

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Nearest particle systems are continuous-time Markov processes on $\{0,1\}^Z$ in which particles die at rate 1 and are born at rates which depend on their distances to the nearest particles to the right and left. There is a natural parametrization of these systems with respect to which they exhibit a phase transition. When the process is attractive and reversible, the critical value λ_c above which a nontrivial invariant measure exists can be computed exactly. This invariant measure is the distribution ν of a stationary discrete time renewal process. Under a mild regularity assumption, we prove that the following three statements are equivalent: (a) The nearest particle system converges to equilibrium exponentially rapidly in $L_2(\nu)$. (b) The density of the interarrival times in the renewal process has exponentially decaying tails. (c) The nearest particle system is supercritical in the sense that $\lambda > \lambda_c$. Under an additional second-moment assumption, we prove that the critical exponent associated with the exponential convergence is 2. The proof of exponential convergence is based on an unusual comparison of the nearest particle system with an infinite system of independent birth and death chains. To carry out this comparison, a new representation is developed for a stationary renewal process with a log-convex renewal sequence in terms of a sequence of i.i.d. random variables.

1. Introduction. A nearest particle system is a certain type of one-dimensional interacting particle system which has a rich structure and which is more amenable to analysis than many other types are. It was introduced by Spitzer (1977) and takes its name from the nearest particle nature of the interaction. It is essentially a collection of $\{0,1\}$-valued Markov chains indexed by the integers in which the transition rates of any one chain depend on the distances from it to the nearest chains to the right and left, respectively, which are in state 1. Each chain loses the Markov property as a result of this interaction, but the system as a whole is a Markov process on the appropriate configuration space. In his article, Spitzer determined necessary and sufficient conditions for a nearest particle system to be reversible and showed that in the reversible case, the reversible invariant measure is the distribution ν of a stationary renewal process which is determined by the transition rates of the system.

Any time one has a Markov process with a finite invariant measure ν, a natural problem is to determine rates of convergence to equilibrium. It is of

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particular interest to determine when this convergence occurs exponentially rapidly. The precise form which the solution to such a problem takes depends on the nature of the Markov process. One can have exponential convergence in the uniform norm, or in the $L_2(v)$ norm, for example. Let $S(t)$ and Ω be the semigroup and generator of the process. We say that the process converges exponentially in the uniform norm if there are positive constants C and ε so that for each function f in a sufficiently rich class, there is a constant $B(f)$ so that

$$\sup_{\eta, \zeta} |S(t)f(\eta) - S(t)f(\zeta)| \leq CB(f)e^{-\varepsilon t}$$

for all $t \geq 0$. The process converges exponentially in the $L_2(v)$ norm if there is a positive ε so that for all $f \in L_2(v)$,

$$\left\| S(t)f - \int f\,dv \right\| \leq e^{-\varepsilon t} \left\| f - \int f\,dv \right\|,$$

where $\|\cdot\|$ denotes the $L_2(v)$ norm. The largest ε with this (latter) property will be called $\text{gap}(\Omega)$, for reasons explained in Section 2. Thus by definition, exponential L_2 convergence occurs if and only if $\text{gap}(\Omega) > 0$.

Exponential convergence in both the uniform and L_2 senses has been proved for various classes of stochastic Ising models by Holley and Stroock (1976), Holley (1984, 1985a, 1985b) and Aizenman and Holley (1987). Among the results proved in these articles is that exponential convergence holds in L_2 for all finite range one-dimensional stochastic Ising models and for all stochastic Ising models in higher dimensions under stronger hypotheses which imply that there is no phase transition for the corresponding Gibbs state. Under an additional attractiveness assumption, these results are strengthened to give exponential convergence in the uniform norm. Another result along these lines which is proved in these articles is that if the rate of convergence is faster than a certain polynomial rate depending on the dimension, then the rate must be exponential. For purposes of comparison with our results, note that these results for stochastic Ising models have been proved only for systems which: (a) have finite range interactions, (b) have uniformly positive transition rates and (c) do not exhibit phase transition.

The reversible nearest particle systems which we study in this article have none of these properties. Since our systems may have more than one invariant measure, convergence in the uniform norm will generally not hold. Thus in this article, we will treat the problem of exponential convergence to v in $L_2(v)$. Under mild regularity conditions, we will show that exponential convergence in this sense occurs for the nearest particle system if and only if the distribution of the interarrival times for the corresponding renewal process has exponentially decaying tails. With a natural parametrization of families of nearest particle systems, this translates into the statement that exponential convergence occurs if and only if the nearest particle system is supercritical. It is interesting that the proof of exponential convergence works all the way down to the critical value and not just for large values of the parameter as is often the case in this field. Our estimates will be sufficiently good that we will be able to identify the critical exponent associated with the exponential convergence under an additional second-moment assumption.

While these results are not altogether surprising, it is harder to prove them than might be expected. The primary reasons for this are that most nearest particle systems of interest have long-range interactions and transition rates which are not uniformly positive. The fact that the transition rates can be arbitrarily small makes it more difficult for the process to converge to equilibrium rapidly. The proof of exponential convergence in the supercritical case involves a comparison between the nearest particle system and an infinite collection of independent birth and death chains on the nonnegative integers. The key to this comparison is a new representation for a stationary renewal process in terms of a sequence of independent and identically distributed random variables.

We will begin by defining the processes we will study. Let $Y = \{0,1\}^Z$, where Z is the set of integers. An infinite nearest particle system is a continuous-time Markov process on

$$Y' = \left\{\eta \in Y: \sum_{x \geq 0} \eta(x) = \sum_{x \leq 0} \eta(x) = \infty\right\}.$$

Transitions are allowed at only one site at a time. Ones flip to zeros at rate 1, while a zero at site x flips to a one at rate $\beta(l, r)$, where l and r are the distances from x to the nearest sites to the left and right, respectively, at which there is a one. Here $\beta(l, r)$ is a nonnegative, bounded and symmetric function defined for $l, r = 1, 2, \ldots$. For a more precise description of this process and an account of most of what is known about it, see Sections 3, 4 and 5 of Chapter 7 of Liggett (1985).

We will assume throughout this article that the process is reversible and attractive. This is the case [see Theorems 4.2 and 4.7 of Chapter 7 of Liggett (1985)] if and only if $\beta(l, r)$ has the form

$$\beta(l, r) = \frac{\beta(l)\beta(r)}{\beta(l+r)}, \tag{1.1}$$

where $\beta(n)$ is a strictly positive probability density on the positive integers with finite mean M which is log-convex in the sense that

$$\frac{\beta(n)}{\beta(n-1)} \leq \frac{\beta(n+1)}{\beta(n)} \quad \text{for } n \geq 2. \tag{1.2}$$

This property implies that $\beta(l, r)$ is a decreasing function of l and r, and hence that the birth rate is an increasing function of the configuration.

One of the consequences of the attractiveness assumption (1.2) is that the process defined on Y' extends uniquely to a process on all of Y in such a way that it has the Feller property on Y [see Theorem 3.6 of Chapter 7 of Liggett (1985)]. The reversibility assumption (1.1) implies that the distribution v of the stationary renewal process corresponding to the density β is a reversible invariant measure for this process. This probability measure on Y' is defined by

$$v\{\eta: \eta(x_i) = 1 \text{ for } 1 \le i \le n \text{ and } \eta(x) = 0 \text{ for all other } x \text{ so that } x_1 < x < x_n\}$$
$$= M^{-1} \prod_{i=1}^{n-1} \beta(x_{i+1} - x_i)$$

for any integers $x_1 < \cdots < x_n$.

A good example to keep in mind is that in which

$$\beta(l, r) = \lambda\left(\frac{1}{l} + \frac{1}{r}\right)^{\alpha}, \tag{1.3}$$

in which λ and α are positive parameters. This has the form (1.1) with $\beta(n) = \lambda n^{-\alpha}\rho^n$, provided that there exists a ρ which makes this into a density with finite mean. Such a ρ exists for all λ if $\alpha \le 1$, for all $\lambda > \lambda_c$ if $1 < \alpha \le 2$, and for all $\lambda \ge \lambda_c$ if $\alpha > 2$, where

$$\lambda_c = \left[\sum_{n=1}^{\infty} n^{-\alpha}\right]^{-1}. \tag{1.4}$$

Using the relative entropy technique, Liggett (1983) proved that in these cases, v is the only invariant measure for the system which is translation-invariant and concentrates on Y'. In all of the other cases, the only invariant measure on Y is the pointmass on the configuration which consists only of zeros. In the same article, Liggett proved that this pointmass is invariant if and only if $\alpha \ge 1$.

With this background, we can state our main result, which gives upper and lower bounds on gap(Ω). The lower bound is obtained in Corollary 5.21, while the upper bound is proved in Theorem 6.13. Let $\rho = \lim_n \beta(n+1)/\beta(n)$, which exists by (1.2) and satisfies $0 < \rho \le 1$ since β is positive and bounded. The density β will be called a moment sequence if there is a measure γ on $[0, \rho]$ so that

$$\beta(n) = \int_0^{\rho} z^n \, d\gamma \quad \text{for } n \ge 1.$$

THEOREM 1.5. (a) *Suppose that β is a moment sequence. Then*

$$\operatorname{gap}(\Omega) \ge \frac{\beta(1)}{4M\rho}(1 - \rho)^2.$$

(b) *Suppose that β satisfies*

$$\sum_{n=1}^{\infty} \frac{\beta^2(n)}{\beta(2n)} < \infty.$$

Then for any $1 \le \delta \le 2$,

$$\operatorname{gap}(\Omega) \le 4(1 - \rho)^{\delta} \sum_{n=1}^{\infty} n^{\delta}\beta(n)\rho^{-n}.$$

In example (1.3), the assumption of part (a) of Theorem 1.5 is always satisfied, and the assumption of part (b) is satisfied whenever $\alpha > 1$ (which corresponds to

$\lambda_c > 0$). That the assumption in (b) is satisfied is easy to check. To check the assumption in (a), see the proof of Proposition 4.9. In other examples, the assumption in (a) can be checked by using Theorem 2 of Section 7.3 of Feller (1971). The assumption of (b) is satisfied whenever $\beta(n)$ is regularly varying. It was shown at the end of Section 3 of Liggett (1983) that if $\rho = 1$, this assumption can fail only if $n^2\beta^2(n)/\beta(2n)$ oscillates between 0 and ∞ as $n \to \infty$. Thus Theorem 1.5 implies that under moderate regularity assumptions, $\text{gap}(\Omega) > 0$ if and only if $\rho < 1$.

The one example for which $\text{gap}(\Omega)$ can be computed exactly is the one in which $\beta(n) = (1 - \rho)\rho^{n-1}$, so that $\beta(l, r) = (1 - \rho)/\rho$. In this case, the coordinates of the process are independent two-state Markov chains, so that it is easy to use Theorem 2.6 and Proposition 3.3 to compute $\text{gap}(\Omega) = \rho^{-1}$. In this example, the lower bound in part (a) of Theorem 1.5 is $(1 - \rho)^2/4\rho$, while the upper bound in part (b) is infinite.

The first step in the proof of Theorem 1.5 is to identify $\text{gap}(\Omega)$ as the infimum of the quadratic form $-\int f\Omega f\,dv$ over all functions f in the domain of Ω which satisfy $\int f\,dv = 0$ and $\|f\| = 1$. This is done in Section 2. Part (b) of the theorem is then proved in Section 6 by evaluating this quadratic form for certain carefully chosen functions f.

The more interesting part of Theorem 1.5 is part (a), which gives exponential L_2 convergence when $\rho < 1$. The key to the proof of this part is a new representation for the renewal measure v in terms of a sequence of independent and identically distributed nonnegative integer-valued random variables. This representation is of independent interest. It is given in Section 4, along with other applications of the representation. Given this representation, it is natural to try to find a Markov process whose invariant measure is the distribution of this sequence of independent random variables, and whose rate of exponential convergence can be estimated relatively easily. The choice which works is a collection of independent and identically distributed birth and death chains on the nonnegative integers. In Section 3, we prove essentially that a birth and death chain converges exponentially in L_2 of its invariant measure if and only if its invariant measure has exponential tails. Since the rate of exponential convergence of a family of independent Markov processes is determined by the member of this family which converges most slowly (Theorem 2.6), the final step is to compare the quadratic form corresponding to the nearest particle system to the quadratic form corresponding to the family of independent birth and death chains. This comparison involves some renewal theory and relies heavily on the details of the representation. It is carried out in Section 5.

Before proceeding to the proofs of these results, it is of interest to restate Theorem 1.5 for a natural parametric family of nearest particle systems. This will allow us to discuss the critical exponent associated with the exponential convergence, which is defined by

$$\lim_{\lambda \downarrow \lambda_c} \frac{\log \text{gap}(\Omega_\lambda)}{\log(\lambda - \lambda_c)}.$$

In order to do so, let β be a probability density on the positive integers which satisfies $\beta(n+1)/\beta(n)\uparrow 1$ and has a finite first moment M. For $0 \le \rho \le 1$, define

$$\phi(\rho) = \sum_{n=1}^{\infty} \beta(n)\rho^n.$$

Consider the reversible attractive nearest particle system with birth rates

$$\beta_\lambda(l, r) = \lambda \frac{\beta(l)\beta(r)}{\beta(l+r)}. \tag{1.6}$$

Then $\lambda_c = 1$, and for $\lambda \ge 1$, $\beta_\lambda(l, r)$ has the form (1.1) with respect to the density $\beta_\lambda(n) = \lambda\beta(n)\rho^n$, where $\rho = \rho(\lambda)$ is the solution of $\lambda\phi(\rho) = 1$ [see Corollary 4.30 of Chapter 7 of Liggett (1985)]. Suppose in addition that $\beta(n)$ is a moment sequence which satisfies

$$\sum_{n=1}^{\infty} \frac{\beta^2(n)}{\beta(2n)} < \infty. \tag{1.7}$$

Then $\beta_\lambda(n)$ has the same two properties. If Ω_λ is the generator of the nearest particle system with parameter λ which has the birth rates given in (1.6), then Theorem 1.5 implies that $\text{gap}(\Omega_\lambda) > 0$ if and only if $\lambda > 1$. Furthermore, since

$$\lim_{\lambda \downarrow 1} \rho(\lambda) = 1$$

and

$$\phi'(\rho(\lambda)) \le \frac{1 - \lambda^{-1}}{1 - \rho(\lambda)} \le \phi'(1) = M$$

by the convexity of ϕ, Theorem 1.5 with $\delta = 2$ gives the bounds

$$\beta(1)M^{-3}/4 \le \liminf_{\lambda \downarrow 1} \frac{\text{gap}(\Omega_\lambda)}{(\lambda - 1)^2} \le \limsup_{\lambda \downarrow 1} \frac{\text{gap}(\Omega_\lambda)}{(\lambda - 1)^2} \le 4M^{-2} \sum_{n=1}^{\infty} n^2\beta(n).$$

It follows that the critical exponent associated with the exponential rate of convergence is 2 if $\beta(n)$ has a finite second moment.

Note added in revision. The role of the moment sequence assumption in part (a) of Theorem 1.5 is to guarantee the boundedness of

$$\frac{g(u+1) - g(\infty)}{g(u) - g(u+1)},$$

where g is the renewal sequence associated with the density β (see Proposition 4.9 and Theorem 5.7). Our work on the present article led us to consider the general problem of finding weaker conditions on β which are sufficient to imply this boundedness. The results of this investigation are reported in Liggett (1989). When combined with Theorem 5.7, they imply that $\text{gap}(\Omega) > 0$ provided that $\rho < 1$ and either $\beta(n + m + 1)$ is totally positive of order 3 [assumption (1.2) can be rephrased by saying that $\beta(n+m+1)$ is totally positive of order 2], or β satisfies (1.7).

2. Some general results. In this section, we will consider a general Markov process on a complete separable metric space which has an invariant probability measure v. Let $S(t)$ denote the semigroup of this process acting on $L_2(v)$, which is automatically a strongly continuous semigroup of positive contractions. Let Ω denote the generator of $S(t)$ and $D(\Omega)$ its domain. We will collect here some elementary results which we will need later connected with exponential convergence in $L_2(v)$ of such processes. One reason for doing so, is that these matters have usually been discussed in the context of reversibility, in which case the semigroup is self-adjoint, and one has additional tools at one's disposal such as the spectral theorem. We wish to point out that reversibility is not needed for these results.

In this section, $\|\cdot\|$ will denote the norm in $L_2(v)$. For $t \ge 0$, define

$$\sigma(t) = -\sup\left\{\log\|S(t)f\| : \|f\| = 1 \text{ and } \int f\,dv = 0\right\}. \tag{2.1}$$

By the contraction and semigroup properties, together with the invariance of v, we have

$$\|S(t+s)f\| \le e^{-\sigma(t)}\|S(s)f\| \le e^{-\sigma(t)-\sigma(s)}\|f\|$$

for $s, t \ge 0$ whenever $\int f\,dv = 0$. Taking the logarithm of both sides and then the appropriate supremum, it follows that

$$\sigma(t+s) \ge \sigma(t) + \sigma(s).$$

This superadditivity of σ implies that

$$\lim_{t \downarrow 0} \frac{\sigma(t)}{t} = \inf_{t>0} \frac{\sigma(t)}{t}. \tag{2.2}$$

We will denote the common value in (2.2) by gap(Ω). The reason for this notation is that in the reversible case, this quantity corresponds to the gap in the spectrum of Ω—that is, the largest ε so that there is no spectrum in the interval $(0, \varepsilon)$. It controls the exponential rate of convergence in $L_2(v)$ in the sense that it is the largest ε for which

$$\left\|S(t)f - \int f\,dv\right\| \le e^{-\varepsilon t}\left\|f - \int f\,dv\right\|$$

for all $f \in L_2(v)$ and all $t \ge 0$. We say that exponential L_2 convergence occurs if gap(Ω) > 0.

The definition of gap(Ω) is given in terms of the semigroup. It is usually more convenient to use the following expression for it in terms of the generator.

THEOREM 2.3.

$$\text{gap}(\Omega) = \inf\left\{-\int f\Omega f\colon f \in D(\Omega), \|f\| = 1 \textit{ and } \int f\,dv = 0\right\}. \tag{2.4}$$

PROOF. By (2.1) and (2.2),

$$\text{gap}(\Omega) = \inf_{t>0} \frac{1}{t} \inf\left\{-\log \|S(t)f\|\colon \|f\| = 1 \text{ and } \int f\,dv = 0\right\}. \tag{2.5}$$

Let A be the infimum on the right-hand side of (2.4), and B be the infimum on the right-hand side of (2.5). We must show that $A = B$. Suppose $f \in D(\Omega)$, $\|f\| = 1$ and $\int f\,dv = 0$. Then

$$\begin{aligned}\frac{d}{dt}\|S(t)f\|^2 &= \frac{d}{dt}\int [S(t)f]^2\,dv \\ &= 2\int [S(t)f][\Omega S(t)f]\,dv \\ &\le -2A\|S(t)f\|^2,\end{aligned}$$

by the definition of A applied to the function $S(t)f$, which is in $D(\Omega)$ and satisfies $\int S(t)f\,dv = 0$. Therefore,

$$\|S(t)f\|^2 \le e^{-2At}\|f\|^2,$$

from which it follows that $-\log\|S(t)f\| \ge At$, since $\|f\| = 1$. So $B \ge A$ by the definition of B, since $D(\Omega)$ is dense in $L_2(v)$. For the other inequality, take $f \in D(\Omega)$ such that $\|f\| = 1$ and $\int f\,dv = 0$. Then by the definition of the generator,

$$\int f\Omega f\,dv = \lim_{t\downarrow 0}\int f\frac{S(t)f - f}{t}\,dv = \lim_{t\downarrow 0}\frac{\int fS(t)f\,dv - 1}{t}.$$

By the definition of B, $\|S(t)f\| \le e^{-Bt}$, so it follows that $\int f\Omega f\,dv \le -B$. Using the definition of A, we see that $B \le A$ as required. □

For the next result, let $S(t)$ be the semigroup corresponding to a vector Markov process whose components are independent Markov processes with semigroups $S_k(t)$, generators Ω_k and invariant probability measures v_k, and let v be the product of the v_k's.

THEOREM 2.6. $\text{gap}(\Omega) = \inf_k \text{gap}(\Omega_k)$.

PROOF. To show that $\text{gap}(\Omega) \le \text{gap}(\Omega_k)$ for any k, simply note that the infimum in the expression given in (2.5) for $\text{gap}(\Omega)$ is smaller than what it would be if one considers only functions on the product space which depend only on the kth coordinate. For the other inequality, it is enough to consider the case in which there are only two components. Once this is done, an iteration argument gives the result for finitely many components. Then, a limiting argument gives it for infinitely many components, since functions which depend on finitely many coordinates are dense in $L_2(v)$. So, call the two coordinates x and y, number them 1 and 2 and let $\varepsilon = \min\{\text{gap}(\Omega_1), \text{gap}(\Omega_2)\}$. Take a function f on the product space which satisfies $\int f\,dv = 0$ and $\|f\| = 1$. Write

$$f(x, y) = h(x, y) + h_1(x) + h_2(y),$$

where $\int h(x, y)\,dv_1 = 0$ for a.e. y, $\int h(x, y)\,dv_2 = 0$ for a.e. x, $\int h_1(x)\,dv_1 = 0$ and $\int h_2(x)\,dv_2 = 0$. Then h, h_1 and h_2 are orthogonal in $L_2(v)$, so that

$$\|h\|^2 + \|h_1\|^2 + \|h_2\|^2 = 1. \tag{2.7}$$

Also, $S(t)h$, $S(t)h_1$ and $S(t)h_2$ are orthogonal in $L_2(v)$, so that

$$\|S(t)h\|^2 + \|S(t)h_1\|^2 + \|S(t)h_2\|^2 = \|S(t)f\|^2. \tag{2.8}$$

Since $S(t)h_i = S_i(t)h_i$, we have

$$\|S(t)h_i\| \le e^{-\varepsilon t}\|h_i\| \tag{2.9}$$

for $i = 1, 2$. On the other hand,

$$\|S(t)h\| = \|S_1(t)S_2(t)h\| \le e^{-\varepsilon t}\|S_2(t)h\| \le e^{-2\varepsilon t}\|h\|. \tag{2.10}$$

Combining (2.7)–(2.10), it follows that

$$\|S(t)f\|^2 \le e^{-4\varepsilon t}\|h\|^2 + e^{-2\varepsilon t}\|h_1\|^2 + e^{-2\varepsilon t}\|h_2\|^2 \le e^{-2\varepsilon t}.$$

Therefore, $\text{gap}(\Omega) \ge \varepsilon$ by (2.5). □

3. Markov chains on Z^+. In this section, we consider a positive recurrent continuous-time Markov chain on $Z^+ = \{0, 1, 2, \dots\}$ with no instantaneous states and with transition rates $q(x, y)$ for $y \ne x$. In order to avoid technical difficulties, we assume that the set G of functions on Z^+ which are constant off a finite set forms a core for the generator Ω of the chain and that

$$\Omega f(x) = \sum_y q(x, y)[f(y) - f(x)] \tag{3.1}$$

for $f \in G$. Let $\pi(x)$ be the stationary distribution for this chain and assume that

$$\sum_x \pi(x)\,|q(x, x)| < \infty \tag{3.2}$$

(which is equivalent to the positive recurrence of the embedded discrete time chain). Then $\Sigma_x \pi(x)q(x, y) = 0$ for each y. We wish to find useful upper and lower bounds for $\text{gap}(\Omega)$, and hence to find conditions which are separately necessary and sufficient for $\text{gap}(\Omega)$ to be strictly positive. First, we need to obtain a useful expression for $\text{gap}(\Omega)$. For this result, it is of course irrelevant that the state space of the chain is Z^+—any countable set will do.

PROPOSITION 3.3. *If $f \in G$, then*

$$\sum_x f(x)\Omega f(x)\pi(x) = -\tfrac{1}{2}\sum_{x,y} q(x, y)[f(y) - f(x)]^2\pi(x).$$

Therefore,

$$\operatorname{gap}(\Omega) = \tfrac{1}{2}\inf\Big\{ \sum_{x,y} q(x,y)[f(y)-f(x)]^2\pi(x):$$

$$f \in G, \sum_x f(x)\pi(x) = 0, \sum_x f^2(x)\pi(x) = 1\Big\}.$$

PROOF. For the first statement, square out the right-hand side and rearrange the terms, using (3.2) to justify the rearrangement. In doing so, recall that $\Sigma_y q(x, y) = 0$ for each x, and $\Sigma_x \pi(x)q(x, y) = 0$ for each y. The second statement follows from the first and Theorem 2.3, since G is a core for Ω. □

Since $\operatorname{gap}(\Omega)$ is an infimum, upper bounds for it are relatively easy to obtain by using special choices of the function f. For example, we have the following result.

THEOREM 3.4.

$$\operatorname{gap}(\Omega) \le \tfrac{1}{2}\inf_{n\ge 0} \frac{\Sigma_{x\le n<y}[\pi(x)q(x,y)+\pi(y)q(y,x)]}{\Sigma_{x\le n}\pi(x)\Sigma_{x>n}\pi(x)}.$$

PROOF. For a fixed $n \ge 0$, define $f \in G$ by $f(x) = ch(x) - d$, where h is the indicator function of $\{0, 1, \dots, n\}$ and c and d are chosen so that $\Sigma f(x)\pi(x) = 0$ and $\Sigma f^2(x)\pi(x) = 1$. Then

$$\sum_{x,y} q(x,y)[f(y)-f(x)]^2\pi(x) = c^2\sum_{x,y} q(x,y)[h(y)-h(x)]^2\pi(x)$$

$$= c^2 \sum_{x\le n<y}[\pi(x)q(x,y)+\pi(y)q(y,x)]$$

and

$$c^{-2} = \sum_{x\le n}\pi(x)\sum_{x>n}\pi(x).$$

Now apply Proposition 3.3. □

For the remainder of this section, which is devoted to obtaining lower bounds for $\operatorname{gap}(\Omega)$, we assume that the chain is a birth and death chain—that is, $q(x, y) = 0$ if $|x - y| > 1$. This simplifies matters considerably and is the case which is needed for our application to nearest particle systems in Section 5. Furthermore, since $\operatorname{gap}(\Omega)$ is an increasing function of the transition rates $q(x, y)$ by Proposition 3.3 (when considering different chains with the same stationary distribution), lower bounds for more general chains on Z^+ can often be obtained by comparing those chains to birth and death chains.

PROPOSITION 3.5. *Suppose that*

$$\sum_{y>u} \pi(y) \le c\pi(u)q(u, u+1)$$

and

$$\sum_{y>u} \gamma^{-y}\pi(y)q(y, y+1) \le d\pi(u)q(u, u+1)\gamma^{-u},$$

where c *and* d *are positive constants and* $0 < \gamma < 1$. *Then* $\mathrm{gap}(\Omega) \ge [cd + c(1-\gamma)^{-1}]^{-1}$.

PROOF. For any function f and any $0 \le x < y$, the Schwarz inequality gives

$$[f(y) - f(x)]^2 \le \sum_{u=x}^{y-1} [f(u+1) - f(u)]^2 \gamma^u \sum_{v=x}^{y-1} \gamma^{-v}.$$

If f has mean 0 and variance 1 relative to π, it then follows that

$$\begin{aligned} 1 &= \tfrac{1}{2} \sum_{x,y} \pi(x)\pi(y)[f(y) - f(x)]^2 = \sum_{x<y} \pi(x)\pi(y)[f(y) - f(x)]^2 \\ &\le \sum_u [f(u+1) - f(u)]^2 \gamma^u \Big\{ \sum_{v>u} \gamma^{-v} \sum_{y>v} \pi(y) \sum_{x \le u} \pi(x) \\ &\qquad + \sum_{v \le u} \gamma^{-v} \sum_{y>u} \pi(y) \sum_{x \le v} \pi(x) \Big\} \\ &\le c\big[d + (1-\gamma)^{-1}\big] \sum_u \pi(u)q(u, u+1)[f(u+1) - f(u)]^2, \end{aligned}$$

where we have used the fact that π is a probability measure, the hypotheses of the proposition and the inequality

$$\sum_{v \le u} \gamma^{-v} \le \gamma^{-u}(1-\gamma)^{-1}.$$

Now use Proposition 3.3 and the fact that $\pi(x)q(x, y)$ is symmetric (which is automatic for birth and death chains) to complete the proof. □

The following lemma is needed to simplify the statement of Proposition 3.5.

LEMMA 3.6. *Suppose that* $\alpha(u)$ *is a nonnegative function on* Z^+ *which satisfies*

$$\sum_{u>v} \alpha(u) \le b\alpha(v) \quad \text{for all } v \in Z^+.$$

Then

$$\sum_{u>v} \gamma^{-u}\alpha(u) \le b[\gamma - b(1-\gamma)]^{-1} \gamma^{-v}\alpha(v)$$

$$\text{for all } v \in Z^+ \text{ and all } \frac{b}{b+1} < \gamma < 1.$$

PROOF. It is enough to prove this statement when α has finite support, since the general result then follows by approximation. Let $A(v) = \sum_{u>v} \alpha(u)$. Then

$$\begin{aligned}\sum_{u>v} \gamma^{-u}\alpha(u) &= \sum_{u>v} \gamma^{-u}[A(u-1) - A(u)] \\ &= A(v)\gamma^{-v-1} + \sum_{u>v} A(u)\gamma^{-u}(\gamma^{-1} - 1) \\ &\le b\alpha(v)\gamma^{-v-1} + b(\gamma^{-1} - 1) \sum_{u>v} \alpha(u)\gamma^{-u}.\end{aligned}$$

Now multiply by γ and solve for the sum to complete the proof. □

THEOREM 3.7. *Suppose that*

$$\sum_{y>u} \pi(y) \le c\pi(u)q(u, u+1)$$

and

$$\sum_{y>u} \pi(y)q(y, y+1) \le b\pi(u)q(u, u+1),$$

where b and c are positive constants. Then

$$\text{gap}(\Omega) \ge \frac{1 - 2\sqrt{b^2 + b} + 2b}{c} \ge \frac{1}{2c(1 + 2b)}.$$

PROOF. Letting $\alpha(u) = \pi(u)q(u, u+1)$ in Lemma 3.6, we see that the hypotheses of Proposition 3.5 are satisfied with the same c and $d = b[\gamma - b(1-\gamma)]^{-1}$, provided that $b(b+1)^{-1} < \gamma < 1$. Elementary calculus shows that the choice of γ which provides the best lower bound for gap(Ω) satisfies $\gamma^2 = b(b+1)^{-1}$. The proof of the first inequality is completed by using this choice in Proposition 3.5. Now let $x = 4b(b+1)$, so that $1 + 2b = (1+x)^{1/2}$. The second inequality then becomes the arithmetic–geometric mean inequality applied to x and $x + 1$. □

Combining Theorems 3.4 and 3.7 gives the following corollary.

COROLLARY 3.8. *Suppose that $q(u, u+1)$ is bounded away from 0 and ∞. Then a necessary and sufficient condition for* gap(Ω) *to be strictly positive is that π have exponentially decaying tails in the sense that*

$$\sup_{u \in Z^+} \frac{\sum_{y>u} \pi(y)}{\pi(u)} < \infty.$$

Versions of this result which apply to more general reversible Markov chains with bounded generators can be found in Lawler and Sokal (1989).

4. A representation for certain stationary renewal processes. We now give a convenient but somewhat unusual representation for a stationary renewal process with log-convex renewal sequence as a function of an i.i.d. sequence of random variables. Throughout this section, $\beta(n)$ will denote a strictly positive probability density on the positive integers which has a finite mean M. We define the corresponding renewal sequence $g(n)$ by $g(0) = 1$ and

$$g(n) = \sum_{k=1}^{n} \beta(k) g(n-k) \quad \text{for } n \geq 1,$$

which is known as the renewal equation. By the renewal theorem,

$$g(\infty) = \lim_{n \to \infty} g(n) = \frac{1}{M}. \tag{4.1}$$

We will assume that $g(n)$ is log-convex:

$$\frac{g(n)}{g(n-1)} \leq \frac{g(n+1)}{g(n)} \quad \text{for } n \geq 1. \tag{4.2}$$

This property has received considerable attention and is closely connected to renewal theory. A positive sequence satisfying $g(0) = 1$ and (4.2) is called a Kaluza sequence, and in fact, it is known that every bounded Kaluza sequence is a renewal sequence, possibly corresponding to a defective density. The density is defective if and only if g is summable [see Kaluza (1928) and Shanbhag (1977)]. de Bruijn and Erdös (1953) proved that a sufficient condition for (4.2) is that $\beta(n)$ be log-convex:

$$\frac{\beta(n)}{\beta(n-1)} \leq \frac{\beta(n+1)}{\beta(n)} \quad \text{for } n \geq 2. \tag{4.3}$$

Extensions of these implications to higher-order convexity properties of g and β can be found in Liggett (1989).

In order to describe the representation for the stationary renewal process, define π on Z^+ by $\pi(0) = g(1)$ and

$$\pi(n) = \frac{g(n+1)}{g(n)} - \frac{g(n)}{g(n-1)} \quad \text{for } n \geq 1. \tag{4.4}$$

This is a probability measure by (4.1) and (4.2).

Define a mapping $T: (Z^+)^Z \to \{0,1\}^Z$ by $T(x) = \eta$, where

$$\eta(n) = 1 \quad \Leftrightarrow \quad x(n+k) \leq k \quad \text{for all } k \geq 0. \tag{4.5}$$

Let $\{X(n), n \in Z\}$ be independent and identically distributed random variables with values in Z^+ and distribution given by $P\{X(n) = k\} = \pi(k)$. Let v be the probability measure on $\{0,1\}^Z$ which is the distribution of $T(X)$.

THEOREM 4.6. *Suppose that* (4.2) *holds. Then* v *is the stationary renewal measure corresponding to the density* β.

PROOF. Consider integers $n_1 < n_2 < \cdots < n_m$ and set $n_{m+1} = \infty$. Then by (4.5),

$$\begin{aligned}
&v\{\eta: \eta(n_1) = \eta(n_2) = \cdots = \eta(n_m) = 1\}\\
&\quad = P\{X(n_1) = 0, X(n_1+1) \le 1, \dots, X(n_2 - 1) \le n_2 - n_1 - 1,\\
&\qquad\quad X(n_2) = 0, X(n_2+1) \le 1, \dots, X(n_3 - 1) \le n_3 - n_2 - 1, \dots,\\
&\qquad\quad X(n_m) = 0, X(n_m + 1) \le 1, \dots\}\\
&\quad = \prod_{i=1}^{m} \prod_{j=0}^{n_{i+1}-n_i-1} \sum_{k=0}^{j} \pi(k)\\
&\quad = \prod_{i=1}^{m} g(n_{i+1} - n_i),
\end{aligned}$$

where the final equality follows from

$$\sum_{k=0}^{j} \pi(k) = \frac{g(j+1)}{g(j)},$$

which in turn follows from (4.4). But this is exactly the form which these probabilities would take if v were the stationary renewal measure corresponding to the density β. Since the sets of the form $\{\eta: \eta(n_1) = \eta(n_2) = \cdots = \eta(n_m) = 1\}$ are probability-determining, the proof is complete. □

Our main interest in this representation for a stationary renewal process satisfying (4.2) is due to its usefulness in studying nearest particle systems. However, it does have other applications. For example, it could be used to show that every Kaluza sequence $g(n)$ which is bounded above and below is the renewal sequence corresponding to a density with finite mean. To do so, simply define $\pi(n)$ as in (4.4) and show that the resulting probability measure v is a renewal measure.

Another application is related to the concept of association, which has played an important role in percolation, interacting particle systems, statistical mechanics and other areas [see, for example, Harris (1977), Newman (1983), Cox and Grimmett (1984) and Birkel (1988)]. The probability measure v is said to be associated if it has positive correlations in the sense that

$$\int FG\,dv \ge \int F\,dv \int G\,dv$$

for all bounded increasing functions F and G on $\{0,1\}^Z$.

COROLLARY 4.7. *Suppose that v is the distribution of a stationary renewal process whose renewal sequence satisfies* (4.2). *Then v is associated.*

PROOF. With the earlier notation, let μ be the distribution on $(Z^+)^Z$ of $\{X(n),\ n \in Z\}$. By Theorem 4.6,

$$\int F\,dv = \int F \circ T\,d\mu$$

for all bounded functions F on $\{0,1\}^Z$, where $\circ$ denotes composition. Since T is a decreasing function, $F \circ T$ is decreasing whenever F is increasing. Therefore, the association of v follows from the association of μ. But μ is associated because it is a product of probability measures on the linearly ordered set Z^+ [see page 78 of Liggett (1985), for example]. □

REMARK. Burton and Waymire (1986) proved a continuous-time version of Corollary 4.7 under the stronger assumption (4.3), using the FKG inequality. It seems unlikely that one could prove this result under (4.2) in that way.

We conclude this section with some estimates on the renewal sequence g which will be needed in the next section. The first gives an inequality which is related to various results concerning rates of convergence in the renewal theorem —see Stone (1965) and Grübel (1982), for example. These results say roughly that $g(n) - g(\infty)$ is bounded by the "tail of the tail" of the density β.

LEMMA 4.8. *Suppose that the density* β *satisfies* (4.3). *Then*

$$0 \le g(n) - g(n+1) \le \sum_{k=n+2}^{\infty} \beta(k)$$

for $n \ge 0$.

PROOF. The proof is by coupling. Construct Bernoulli random variables $\eta(n)$ for $n \ge 1$ and $\zeta(n)$ for $n \ge 0$ so that $\eta(1) = \zeta(0) = 1$, $P\{\zeta(1) = 1\} = \beta(1)$, $\zeta(n) \le \eta(n)$ for $n \ge 1$, and so that conditional on $\eta(n)$ and $\zeta(n)$ for $n \le m-1$, $\eta(m)$ and $\zeta(m)$ satisfy $\zeta(m) \le \eta(m)$ and have the appropriate conditional probabilities,

$$P\{\eta(m) = 1 | \eta(1),\dots,\eta(m-1),\zeta(0),\dots,\zeta(m-1)\}$$
$$= p(m - \max\{k < m: \eta(k) = 1\})$$

and

$$P\{\zeta(m) = 1 | \eta(1),\dots,\eta(m-1),\zeta(0),\dots,\zeta(m-1)\}$$
$$= p(m - \max\{k < m: \zeta(k) = 1\}),$$

where

$$p(k) = \frac{\beta(k)}{\sum_{j \ge k} \beta(j)} \quad \text{for } k \ge 1.$$

This is possible because $p(k)$ decreases in k as a consequence of (4.3). By construction, the distributions of $\{\eta(n): n \ge 1\}$ and $\{\zeta(n): n \ge 0\}$ are those of the renewal process conditioned on $\eta(1) = 1$ and $\zeta(0) = 1$, respectively. Therefore,

$$g(n) = P\{\eta(n+1) = 1\} = P\{\zeta(n) = 1\} \quad \text{for } n \geq 0,$$

and since $\zeta(n+1) \leq \eta(n+1)$,

$$\begin{aligned} g(n) - g(n+1) &= P\{\eta(n+1) = 1, \zeta(n+1) = 0\} \\ &\leq P\{\zeta(1) = \cdots = \zeta(n+1) = 0\} \\ &= \sum_{k=n+2}^{\infty} \beta(k). \end{aligned}$$

The inequality follows from the fact that if $\zeta(m) = \eta(m) = 1$, then $\zeta(n) = \eta(n)$ for all $n \geq m$. □

One consequence of Lemma 4.8 is that $g(n)$ converges to $g(\infty)$ exponentially rapidly if $\beta(n)$ has exponential tails. We will need a stronger form of exponential convergence, which we prove below under the additional assumption that $\beta(n)$ is a moment sequence.

Proposition 4.9. *Suppose that the density β is a moment sequence*:

$$\beta(n) = \int_0^\rho z^n \, d\gamma \quad \textit{for } n \geq 1,$$

where $0 < \rho < 1$ and γ is a measure on $[0, \rho]$. Then

$$g(k+1) - g(\infty) \leq \frac{\rho}{1-\rho}[g(k) - g(k+1)].$$

In particular, this is the case if β is of the form

$$\beta(n) = K(\alpha, \rho)\frac{\rho^n}{n^\alpha}, \tag{4.10}$$

where $0 < \rho < 1$, $\alpha > 0$ and $K(\alpha, \rho)$ is a normalizing constant.

Proof. Since β is a moment sequence, so is g:

$$g(n) = \int_0^1 z^n \, d\psi \quad \text{for } n \geq 1,$$

where ψ is a measure on $[0,1]$ [see Kaluza (1928), Horn (1970) or Hansen and Steutel (1988) for this and related facts]. Writing

$$g(n) - g(n+1) = \int_0^1 z^n (1-z) \, d\psi,$$

we see by Lemma 4.8 that $\psi((\rho,1)) = 0$. Furthermore, $g(\infty) = \psi(\{1\})$, so that

$$g(k+1) - g(\infty) = \int_0^\rho z^{k+1}\,d\psi \le \frac{\rho}{1-\rho}\int_0^\rho z^k(1-z)\,d\psi$$
$$= \frac{\rho}{1-\rho}[g(k) - g(k+1)].$$

The final statement follows from the fact that any density of the form (4.10) is a moment sequence. This can be checked by applying Theorem 2 of Section 7.3 of Feller (1971), for example, or by using the identity

$$\Gamma(\alpha) = n^\alpha \rho^{-n}\int_0^\rho z^{n-1}\left(\log\frac{\rho}{z}\right)^{\alpha-1} dz,$$

which follows from the usual definition of the gamma function Γ by a change of variables. □

5. Nearest particle systems: The lower bound. In this section, we will use the results of the previous two sections to obtain a lower bound for gap(Ω), and hence a sufficient condition for exponential L_2 convergence of the nearest particle system. Throughout this section, we will assume that the system is reversible and attractive. The first step is to obtain a useful expression for gap(Ω). Some technicalities must be dealt with in doing so, because the birth rates have a long-range dependence. In particular, we need to recall the construction of attractive nearest particle systems which was given in Chapter 7 of Liggett (1985).

For $m \le n$, let

$$Z_{m,n} = \{m, m+1, \dots, n\}, \qquad Y_{m,n} = \{0,1\}^{Z_{m,n}},$$

and let $C_{m,n}$ be the set of all functions on $Y_{m,n}$. Consider the continuous-time Markov chain on $Y_{m,n}$ in which a one flips to a zero at rate 1 and a zero flips to a one at rate $\beta(l,r) = \beta(l)\beta(r)/\beta(l+r)$, where l and r are the distances to the nearest ones to the left and right, respectively, and β is a log-convex density on $\{1,2,\dots\}$ with finite mean M. In order that this be well defined, we adopt the convention that there are fixed ones at $m-1$ and at $n+1$ which are used in determining the nearest one to the left or right of a site if there are no ones in $Z_{m,n}$ to the left or right of that site. Let $S_{m,n}(t)$ and $\Omega_{m,n}$ be the semigroup and generator corresponding to this Markov chain:

$$\Omega_{m,n} f(\eta) = \sum_{m \le x \le n,\, \eta(x)=1} [f(\eta_x) - f(\eta)]$$
$$+ \sum_{m \le x \le n,\, \eta(x)=0} \beta(l_x(\eta), r_x(\eta))[f(\eta_x) - f(\eta)],$$

where $\eta_x(x) = 1 - \eta(x)$ and $\eta_x(y) = \eta(y)$ for $y \ne x$, and $l_x(\eta)$ and $r_x(\eta)$ are the distances from x to the nearest ones in η to the left and right, respectively.

As in the previous section, let v be the distribution of the stationary renewal process on $Y = \{0,1\}^Z$ with interarrival distribution β, and let $v_{m,n}$ be the probability measure on $Y_{m,n}$ obtained by conditioning v on the event $\{\eta: \eta(m-1) = \eta(n+1) = 1\}$. Then it is easy to check that $v_{m,n}$ is a reversible invariant measure for the chain with generator $\Omega_{m,n}$. Note that

$$\int_{\{\eta(x)=0\}} [f(\eta_x) - f(\eta)]^2 \beta(l_x(\eta), r_x(\eta))\, dv_{m,n}$$
$$= \int_{\{\eta(x)=1\}} [f(\eta_x) - f(\eta)]^2\, dv_{m,n}$$
$$= \int [f(\eta^x) - f(\eta)]^2\, dv_{m,n},$$

where $\eta^x(x) = 0$ and $\eta^x(y) = \eta(y)$ for $y \neq x$. Therefore, Proposition 3.3 can be applied to this chain to obtain

$$\text{(5.1)}\qquad \operatorname{gap}(\Omega_{m,n}) = \inf\left\{ \sum_{u=m}^{n} \int [f(\eta^u) - f(\eta)]^2\, dv_{m,n}: \; f \in C_{m,n}, \int f\, dv_{m,n} = 0, \int f^2\, dv_{m,n} = 1 \right\}.$$

The semigroup $S(t)$ of the nearest particle system is defined in the following way in terms of the chains on $Y_{m,n}$ described above. Let

$$Y' = \left\{ \eta \in Y: \sum_{x \geq 0} \eta(x) = \sum_{x \leq 0} \eta(x) = \infty \right\}.$$

If $f \in C(Y)$ depends on finitely many coordinates, then

$$\text{(5.2)}\qquad S(t)f = \lim_{m \to -\infty,\, n \to \infty} S_{m,n}(t)f,$$

where the convergence is uniform on compact subsets of Y'. By Theorem 3.6 of Chapter 7 of Liggett (1985), $S(t)f$ extends continuously to all of Y, and this defines a Feller semigroup on $C(Y)$. Let Ω be the generator of this semigroup. It should be kept in mind that the set of functions which depend on finitely many coordinates does not in general form a core for Ω—see page 334 of Liggett (1985).

THEOREM 5.3. *With Ω defined above,*

$$\operatorname{gap}(\Omega) \geq \inf_{m \leq n} \inf\left\{ \sum_{u=m}^{n} \int [f(\eta^u) - f(\eta)]^2\, dv_{m,n}: \; f \in C_{m,n}, \int f\, dv_{m,n} = 0, \int f^2\, dv_{m,n} = 1 \right\}.$$

PROOF. Let $\varepsilon = \inf_{m \leq n} \operatorname{gap}(\Omega_{m,n})$, which is the right-hand side above by (5.1). Therefore, by (2.5),

$$\text{(5.4)}\qquad \int [S_{m,n}(t)f]^2\, dv_{m,n} \leq e^{-2\varepsilon t} \int f^2\, dv_{m,n}$$

for all $f \in C_{m,n}$ such that $\int f\,dv_{m,n} = 0$, and for all $m \le n$. We want to pass to the limit in this inequality. First note that the renewal theorem implies that $v_{m,n}$ converges weakly to v as m and $n \to -\infty$ and ∞, respectively. Therefore by Skorohod's theorem, one can construct η with distribution v and $\eta_{m,n}$ with distribution $v_{m,n}$ on a common probability space so that $\eta_{m,n}$ converges to η almost surely. Using (5.2), it then follows from (5.4) that

$$\int [S(t)f]^2\,dv \le e^{-2\varepsilon t}\int f^2\,dv$$

for all f which depend on finitely many coordinates and satisfy $\int f\,dv = 0$. Therefore, $\text{gap}(\Omega) \ge \varepsilon$ as required. □

We come now to the part of the argument in which we compare the nearest particle system with a system of independent birth and death chains. Define the mapping T as in (4.5). Note that T is not one-to-one, so that it is not easy to use it to go back and forth between functions or measures on $(Z^+)^Z$ and functions or measures on $\{0,1\}^Z$. Nevertheless, the following lemma will make it possible to use T for this purpose. For $x \in (Z^+)^Z$ and $m \in Z$, define $x_m \in (Z^+)^Z$ by $x_m(n) = x(n)$ for $n \ne m$, and $x_m(m) = x(m) + 1$. The lemma asserts that while $T(x_m)$ is not determined by $T(x)$ alone, the only additional information about x which is needed is the value of $x(m)$.

LEMMA 5.5. *If* $\eta = T(x)$, *then* $T(x_m) = \eta^{m-x(m)}$.

PROOF. Let $\zeta = T(x_m)$. By the definition of T, $\zeta(n) = 1 \Leftrightarrow x_m(n+k) \le k$ for all $k \ge 0 \Leftrightarrow x(n+k) \le k$ for all $k \ge 0$, $n + k \ne m$, and $x(m) + 1 \le m - n$ if $m \ge n$. Therefore, $\zeta(n) = \eta(n)$ unless $x(m) = m - n$, in which case $\zeta(n) = 0$ [regardless of the value of $\eta(n)$]. It follows that $\zeta = \eta^{m-x(m)}$. □

As in Section 4, let $X = \{X(k), k \in Z\}$ be independent and identically distributed random variables with density π given in (4.4), and let μ be its distribution on $(Z^+)^Z$. Recall that we observed in Section 4 that (4.2) follows from the log-convexity of $\beta(n)$, which we are assuming here. We will need the following bound on the conditional distribution of X given $T(X)$. In this bound, g is the renewal sequence defined in Section 4.

LEMMA 5.6. *If* $0 \le j \le l$ *and* $\zeta \in \{0,1\}^Z$ *satisfies* $\zeta(i-l) = \zeta(i+r) = 1$ *and* $\zeta(k) = 0$ *for all* $i - l < k < i + r$, *where* $l \ge 0$ *and* $r \ge 1$, *then*

$$P(X(i) = j \mid T(X) = \zeta) \le \frac{\pi(j)}{\beta(r+j)\pi(0)} \sum_{k=0}^{j} \beta(r+k)g(j-k).$$

PROOF. We will need to isolate the dependence of $T(X)$ on $X(i)$ from its dependence on the other coordinates of X. To do so, write

$$\{T(X)(i-l)=1\} = \{X(i) \le l\} \cap E(i-l)$$

and for $i-l<u\le i-j$,

$$\{T(X)(u)=0\} = \{X(i) > i-u\} \cup E(u),$$

where the events $E(i-l),\dots,E(i-j)$ are independent of $X(i)$. Note that $\{T(X)(u)=0\} \supset \{X(i)=j\}$ if $i-j<u\le i$, and that $E(u)=\{T(X)(u)=0\}$ is independent of $X(i)$ if $i<u\le i+r$. Therefore,

$$\begin{aligned}
&P(T(X)(u)=\zeta(u) \text{ for all } i-l\le u\le i+r \mid X(i)=j)\\
&\quad = P(T(X)(u)=\zeta(u) \text{ for all } i-l\le u\le i-j\\
&\qquad\qquad \text{and all } i<u\le i+r \mid X(i)=j)\\
&\quad = P(E(i-l)\cap\cdots\cap E(i-j)\cap E(i+1)\cap\cdots\cap E(i+r))\\
&\quad \le \frac{P(T(X)(u)=\zeta(u) \text{ for all } i-l\le u\le i-j \text{ and all } i<u\le i+r)}{P(X(i)\le l)}.
\end{aligned}$$

Therefore, using Theorem 4.6, we can compute

$$\begin{aligned}
&P(X(i)=j \mid T(X)=\zeta)\\
&\quad = P(X(i)=j \mid T(X)(u)=\zeta(u) \text{ for all } i-l\le u\le i+r)\\
&\quad = \frac{P(T(X)(u)=\zeta(u) \text{ for all } i-l\le u\le i+r \mid X(i)=j)\pi(j)}{v\{\eta\colon \eta(u)=\zeta(u) \text{ for all } i-l\le u\le i+r\}}\\
&\quad \le \frac{\pi(j)P(T(X)(u)=\zeta(u) \text{ for all } i-l\le u\le i-j \text{ and all } i<u\le i+r)}{v\{\eta\colon \eta(0)=1\}\beta(l+r)P(X(i)\le l)}\\
&\quad \le \frac{\pi(j)v\{\eta\colon \eta(u)=\zeta(u) \text{ for all } i-l\le u\le i-j \text{ and all } i<u\le i+r\}}{v\{\eta\colon \eta(0)=1\}\beta(l+r)\pi(0)}.
\end{aligned}$$

Since the density β is log-convex, the right-hand side above is a decreasing function of l. So, replacing the l by j, and then using the fact that v is a renewal measure, gives the required inequality. □

THEOREM 5.7. *Let M be as in* (4.1), *and define ρ by*

$$\rho = \lim_{n\to\infty} \frac{\beta(n+1)}{\beta(n)},$$

which exists since β is log-convex. Then

$$\operatorname{gap}(\Omega) \ge \frac{(1-\rho)\beta(1)}{2M\rho}\left[1 + 2\sup_{u\ge 0}\frac{g(u+1)-g(\infty)}{g(u)-g(u+1)}\right]^{-1}.$$

Proof. Let $\mu_{m,n}$ be the measure obtained from μ by conditioning on the set

$$\begin{aligned}\{x \in (Z^+)^Z: T(x)(m-1) = T(x)(n+1) = 1\}\\ = \{x \in (Z^+)^Z: x(m+k-1) \le k \text{ and } x(n+k+1) \le k \text{ for all } k \ge 0\}.\end{aligned}$$

Then the measure on $\{0,1\}^Z$ induced by $\mu_{m,n}$ under the mapping T is just $v_{m,n}$. Let f be a fixed function in $C_{m,n}$ which satisfies $\int f\,dv_{m,n} = 0$ and $\int f^2\,dv_{m,n} = 1$, and let F be the function on $(Z^+)^Z$ obtained by composing f with T. Then

$$\int F\,d\mu_{m,n} = 0 \quad\text{and}\quad \int F^2\,d\mu_{m,n} = 1. \tag{5.8}$$

Consider the continuous-time Markov process on $\{x \in (Z^+)^Z: x(m+k-1) \le k$ and $x(n+k+1) \le k$ for all $k \ge 0\}$ in which the coordinates evolve independently according to birth and death chains. The transition rates $q(i,j)$ for the kth coordinate chain are determined by

$$\pi(j)q(j,j+1) = \pi(j+1)q(j+1,j) = g(j) - g(j+1) \tag{5.9}$$

for j and $j+1$ in its state space: $0 \le j \le k-m$ if $m \le k \le n$, and $0 \le j \le k-n-2$ if $k \ge n+2$. [Note that by (4.1) and (4.2), $g(j)$ is decreasing in j, and if $g(j) = g(j+1)$ for some j, then $g(i) = g(i+1)$ for all $i \ge j$.] Let Γ_k be the generator of the kth coordinate chain and Γ be the generator of the product Markov process on $(Z^+)^Z$. Then (5.8) implies that

$$\operatorname{gap}(\Gamma) \le \sum_k \int [F(x_k) - F(x)]^2 q(x(k), x(k)+1)\mu_{m,n}(dx). \tag{5.10}$$

(Apply Proposition 3.3 to the chain in which only the coordinates between m and n move.) Summing over the possible values of $x(k)$ and using Lemma 5.5 and the definition of F, the right-hand side of (5.10) can be written as

$$\sum_{j,k} q(j,j+1)\int_{\{x(k)=j\}} \left[f\left(\left(T(x)\right)^{k-j}\right) - f(T(x))\right]^2 \mu_{m,n}(dx). \tag{5.11}$$

For $m \le i \le n$, let $Q_i(\eta)$ be the conditional expectation given $T(x) = \eta$ of

$$\sum_j q(j,j+1)1_{\{x(i+j)=j\}},$$

where x is distributed according to $\mu_{m,n}$. Writing $k = i + j$ and conditioning on the value of η, we see that the expression in (5.11) can be rewritten so that (5.10) yields

$$\operatorname{gap}(\Gamma) \le \int \sum_i [f(\eta^i) - f(\eta)]^2 Q_i(\eta) v_{m,n}(d\eta). \tag{5.12}$$

We next need to obtain a uniform upper bound on $Q_i(\eta)$. Take η so that $\eta(m-1) = \eta(n+1) = 1$ and define $u = \min\{j > i: \eta(j) = 1\}$. Then by Lemma 5.6,

$$Q_i(\eta) = \sum_{j=0}^{u-i-1} q(j, j+1) P(X(i+j) = j | T(X) = \eta)$$

$$\leq \sum_{j=0}^{u-i-1} \frac{q(j, j+1)\pi(j)}{\beta(u-i)\pi(0)} \sum_{k=0}^{j} \beta(u-i-j+k) g(j-k) \tag{5.13}$$

$$\leq \sum_{j=0}^{u-i-1} \frac{g(j) - g(j+1)}{\beta(u-i)\pi(0)} \sum_{k=u-i-j}^{\infty} \beta(k),$$

where in the last step, we have used (5.9) and the fact that g is bounded by 1. We now compute

$$\sum_{j=0}^{u-1} [g(j) - g(j+1)] \sum_{k=u-j}^{\infty} \beta(k)$$

$$= \sum_{j=0}^{u-1} g(j) \sum_{k=u-j}^{\infty} \beta(k) - \sum_{j=1}^{u} g(j) \sum_{k=u-j+1}^{\infty} \beta(k) \tag{5.14}$$

$$= \sum_{j=1}^{u-1} g(j)\beta(u-j) + \sum_{k=u}^{\infty} \beta(k) - g(u) = \sum_{k=u+1}^{\infty} \beta(k)$$

by the renewal equation (see Section 4). Since β is log-convex, $\beta(k+1) \leq \rho\beta(k)$, so that

$$\sum_{k=v}^{\infty} \beta(k) \leq \beta(v) \sum_{k=v}^{\infty} \rho^{k-v} = \frac{1}{1-\rho}\beta(v). \tag{5.15}$$

Combining (5.13), (5.14) and (5.15) gives

$$Q_i(\eta) \leq \frac{\rho}{\pi(0)(1-\rho)}. \tag{5.16}$$

Substituting this bound for $Q_i(\eta)$ in (5.12) gives

$$\text{gap}(\Gamma) \leq \frac{\rho}{(1-\rho)\pi(0)} \int \sum_i [f(\eta^i) - f(\eta)]^2 \, d\nu_{m,n}. \tag{5.17}$$

Since this is true for all f satisfying $\int f \, d\nu_{m,n} = 0$ and $\int f^2 \, d\nu_{m,n} = 1$ and for all $m \leq n$, it follows from Theorem 5.3 that

$$(1-\rho)\pi(0)\text{gap}(\Gamma) \leq \rho \, \text{gap}(\Omega). \tag{5.18}$$

By Theorem 2.6,

$$\text{gap}(\Gamma) = \inf_k \text{gap}(\Gamma_k). \tag{5.19}$$

By (4.1), (4.4) and (5.9),

$$\sum_{y>u} \pi(y) = 1 - \frac{g(u+1)}{g(u)} \le M[g(u) - g(u+1)] = M\pi(u)q(u, u+1)$$

for $u \ge 0$. Using (5.9) again, we see that

$$\sum_{y>u} \pi(y)q(y, y+1) = g(u+1) - g(\infty).$$

Therefore, the hypotheses of Theorem 3.7 are satisfied with $c = M$ and

$$b = \sup_{u \ge 0} \frac{g(u+1) - g(\infty)}{g(u) - g(u+1)},$$

provided that $b < \infty$. Noting that the coordinate chains with generators Γ_k have stationary distributions which are constant multiples of π on their respective state spaces, it follows from Theorem 3.7 that

$$\text{gap}(\Gamma_k) \ge \frac{1}{2M(1+2b)} \tag{5.20}$$

with that choice of b. Since $\pi(0) = g(1) = \beta(1)$, (5.18), (5.19) and (5.20) combine to give the statement of the theorem. □

COROLLARY 5.21. *Suppose that the density β is a moment sequence, and define ρ as in Theorem* 5.7. *Then*

$$\text{gap}(\Omega) \ge \frac{\beta(1)}{4M\rho}(1-\rho)^2 \ge \frac{(1-\rho)^4}{4\rho}.$$

PROOF. The first inequality follows from Theorem 5.7 and Proposition 4.9. From the moment sequence expression for β, it is easily seen that

$$\beta(1) \ge M(1-\rho)^2,$$

which gives the second inequality. □

6. Nearest particle systems: The upper bound. In this section, we will obtain an upper bound for gap(Ω), and hence a necessary condition for exponential L_2 convergence of the nearest particle system. We continue to assume that the system is reversible and attractive and that υ is the renewal measure corresponding to the positive log-convex probability density β with finite mean M. Let D be the set of all functions on Y which depend on finitely many coordinates.

THEOREM 6.1.

$$\operatorname{gap}(\Omega) \le \inf\left\{ \sum_u \int [f(\eta^u) - f(\eta)]^2 \, d\nu : f \in D, \int f \, d\nu = 0, \int f^2 \, d\nu = 1 \right\}. \tag{6.2}$$

PROOF. By Theorem 3.5 of Chapter 7 of Liggett (1985), $D(\Omega) \supset D$. Therefore, by Theorem 2.3, we need only show that for $f \in D$,

$$-\int f \Omega f \, d\nu = \sum_u \int [f(\eta^u) - f(\eta)]^2 \, d\nu.$$

The analogous identity for the finite approximation to the nearest particle system was shown at the beginning of Section 5 using Proposition 3.3. To prove it for the infinite system, simply pass to the limit as $m \to -\infty$ and $n \to \infty$ using (3.3) of Chapter 7 of Liggett (1985). □

In order to obtain an upper bound for $\operatorname{gap}(\Omega)$, we will evaluate the expression inside the infimum in (6.2) for functions f of a particular form. Given the nearest particle nature of the interaction, it seems natural to try functions which depend on the distances from a fixed site to the nearest ones to the right and left of that site. So, given a function $a(n)$ on the positive integers which is constant from some point on, define the function $A(\eta)$ on Y by $A(\eta) = 0$ if $\eta(0) = 1$, $A(\eta) = a(l + r)$ if $\eta(-l) = \eta(r) = 1$ and $\eta(x) = 0$ for all $-l < x < r$, and $A(\eta) = \lim_n a(n)$ otherwise.

PROPOSITION 6.3. *The function A defined above is in D and satisfies*

$$\int A \, d\nu = M^{-1} \sum_{n=2}^{\infty} a(n)\beta(n)(n-1), \tag{6.4}$$

$$\int A^2 \, d\nu = M^{-1} \sum_{n=2}^{\infty} a^2(n)\beta(n)(n-1) \tag{6.5}$$

and

$$\begin{aligned} \sum_u \int [A(\eta^u) - A(\eta)]^2 \, d\nu &= M^{-1} \sum_{n=2}^{\infty} a^2(n) \sum_{l+r=n} \beta(l)\beta(r) \\ &\quad + 2M^{-1} \sum_{m,\, n \ge 1} [a(m+n) - a(n)]^2 \beta(m)\beta(n)(n-1). \end{aligned} \tag{6.6}$$

PROOF. A is in D because $a(n)$ is eventually constant. The proofs of (6.4) and (6.5) are identical, so we prove only the first of these. The sets $\{\eta: \eta(0) = 1\}$

and $\{\eta\colon \eta(-l) = \eta(r) = 1,\ \eta(x) = 0$ for all $-l < x < r\}$ for $l,\ r \geq 1$ form a partition of Y', and $v(Y') = 1$, so that

$$\int A\,dv = \sum_{l,r\geq 1} a(l+r)v\{\eta\colon \eta(-l) = \eta(r) = 1, \eta(x) = 0 \text{ for all } -l < x < r\}$$
$$= \sum_{l,r\geq 1} a(l+r)M^{-1}\beta(l+r).$$

Changing variables in the summation yields (6.4). To prove (6.6), write

$$\int [A(\eta^0) - A(\eta)]^2\,dv$$
$$= \sum_{l,r\geq 1} a^2(l+r)v\{\eta\colon \eta(-l) = \eta(0)$$
$$= \eta(r) = 1, \eta(x) = 0 \text{ for all } x \neq 0, -l < x < r\}$$
$$= M^{-1}\sum_{n=2}^{\infty} a^2(n) \sum_{l+r=n} \beta(l)\beta(r)$$

and

$$\sum_{r=1}^{\infty} \int [A(\eta^r) - A(\eta)]^2\,dv$$
$$= \sum_{l,r,m\geq 1} [a(l+r+m) - a(l+r)]^2 v\{\eta\colon \eta(-l) = \eta(r)$$
$$= \eta(r+m) = 1, \eta(x) = 0 \text{ for all } x \neq r, -l < x < r+m\}$$
$$= M^{-1}\sum_{m,n\geq 1} [a(m+n) - a(n)]^2 \beta(m)\beta(n)(n-1).$$

The terms on the left-hand side of (6.6) which correspond to negative u give the same total contribution to the sum as those which correspond to positive u. Thus we obtain (6.6). □

Next, we need to make an effective choice of $a(n)$. Define ρ as in the statement of Theorem 5.7 and choose $\gamma > \rho^{-1}$. For $N \geq 1$, let A_N be the function which corresponds to $a_N(n) = \gamma^n$ if $n \leq N$, and $a_N(n) = \gamma^N$ if $n \geq N$. We need to examine the asymptotic behavior of the expressions corresponding to this function in Proposition 6.3 as $N \to \infty$. The following lemmas consider each of these expressions separately. Let

$$c(N) = \left[\gamma^{2N}\sum_{n=N}^{\infty} n\beta(n)\right]^{-1} M.$$

Note that

(6.7) $$\lim_{N\to\infty} c(N) = 0$$

since $\gamma\rho > 1$.

LEMMA 6.8.

$$\lim_{N\to\infty} c(N)\int A_N^2\,dv = \frac{\gamma^2\rho-\rho}{\gamma^2\rho-1},$$

PROOF. By (6.5) and the definition of A_N,

$$M\int A_N^2\,dv = \sum_{n=2}^{N-1}\gamma^{2n}\beta(n)(n-1) + \gamma^{2N}\sum_{n=N}^{\infty}\beta(n)(n-1),$$

so that

$$\lim_{N\to\infty} c(N)\int A_N^2\,dv = 1 + \lim_{N\to\infty}\frac{\sum_{k=1}^{N-2}\gamma^{-2k}\beta(N-k)(N-k-1)}{\sum_{n=N}^{\infty} n\beta(n)}.$$

Since $\beta(n+1)/\beta(n)$ increases to ρ,

$$\lim_{N\to\infty}\frac{\sum_{n=N}^{\infty} n\beta(n)}{N\beta(N)} = \lim_{N\to\infty}\sum_{n=0}^{\infty}\frac{(N+n)\beta(N+n)}{N\beta(N)} = \sum_{n=0}^{\infty}\rho^n = \frac{1}{1-\rho}$$

by dominated convergence if $\rho < 1$, and by Fatou's lemma if $\rho = 1$. On the other hand, since $\gamma\rho > 1$, dominated convergence gives

$$\lim_{N\to\infty}\sum_{k=1}^{N-2}\gamma^{-2k}\frac{(N-k-1)\beta(N-k)}{N\beta(N)} = \sum_{k=1}^{\infty}\gamma^{-2k}\rho^{-k} = \frac{1}{\gamma^2\rho-1}.$$

The statement of the lemma follows by combining these limiting statements. □

LEMMA 6.9.

$$\lim_{N\to\infty} c(N)\left[\int A_N\,dv\right]^2 = 0.$$

PROOF. Using (6.4), (6.5) and the Schwarz inequality, we see that for any a and corresponding A,

$$M\int A\,dv \le \sum_{n=2}^{L-1} a(n)\beta(n)(n-1) + \sqrt{M\int A^2\,dv\sum_{n=L}^{\infty}\beta(n)n}\,, \tag{6.10}$$

where L is any large integer. Apply (6.10) to the function A_N, and pass to the limit in N, using (6.7) and Lemma 6.8 to get

$$\limsup_{N\to\infty}\sqrt{c(N)}\,M\int A_N\,dv \le \sqrt{M\frac{\gamma^2\rho-\rho}{\gamma^2\rho-1}\sum_{n=L}^{\infty} n\beta(n)}\,.$$

Now let $L\to\infty$, recalling that β has a finite mean, to complete the proof. □

LEMMA 6.11. *Suppose that*

$$\sum_{n=1}^{\infty} \frac{\beta^2(n)}{\beta(2n)} < \infty. \tag{6.12}$$

Then

$$\lim_{N\to\infty} c(N)\sum_u \int [A_N(\eta^u) - A_N(\eta)]^2\, d\nu = 2(1-\rho)\sum_{k=1}^{\infty} \delta(k)\rho^{-k},$$

where

$$\delta(k) = \sum_{m=1}^{k} \beta(m)[\gamma^{m-k} - \gamma^{-k}]^2 + \sum_{m=k+1}^{\infty} \beta(m)[1-\gamma^{-k}]^2.$$

PROOF. We will use the expression in (6.6). First note that

$$\lim_{N\to\infty} c(N) \sum_{n=2}^{\infty} a_N^2(n) \sum_{l+r=n} \beta(l)\beta(r) = 0$$

by (6.7) and Lemma 6.8. To see this, compare the expression above with the right-hand side of (6.5) applied to a_N and use the fact that

$$\frac{\sum_{l+r=n}\beta(l)\beta(r)}{\beta(n)} \le 2\sum_{2l\le n} \frac{\beta^2(l)}{\beta(2l)},$$

which is bounded in n by (6.12). The inequality above is a consequence of the log-convexity of β, since $l \le r$ implies that

$$\frac{\beta(r)}{\beta(l+r)} \le \frac{\beta(l)}{\beta(2l)}.$$

This takes care of the first term on the right-hand side of (6.6). For the second term, write

$$\sum_{m,\,n\ge 1} [a_N(m+n) - a_N(n)]^2 \beta(m)\beta(n)(n-1)$$
$$= \sum_{k=1}^{N-1} (N-k-1)\beta(N-k)\gamma^{2N}\delta(k),$$

where we have made the substitution $k = N - n$ in the summation and used the fact that the summand on the left-hand side above is zero unless $n < N$. Therefore,

$$\lim_{N\to\infty} c(N)\sum_u \int [A_N(\eta^u) - A_N(\eta)]^2\, d\nu$$
$$= 2\lim_{N\to\infty} \frac{\sum_{k=1}^{N-1}(N-k-1)\beta(N-k)\,\delta(k)}{\sum_{n=N}^{\infty} n\beta(n)}.$$

The result will follow if we show that we can take the limit inside the summation. Since $\gamma\rho > 1$, $\beta(m)\gamma^m$ is ultimately increasing in m. Therefore, $\delta(k)$ is bounded by a constant multiple of the tail of β. So, we need to show that

$$\lim_{L\to\infty} \limsup_{N\to\infty} \frac{\sum_{k=L}^{N-1}(N-k)\beta(N-k)\sum_{m=k}^{\infty}\beta(m)}{\sum_{n=N}^{\infty} n\beta(n)} = 0.$$

By the log-convexity of β,

$$\frac{\sum_{m=k}^{\infty}\beta(m)}{\beta(k)} \le \frac{\sum_{m=N}^{\infty}\beta(m)}{\beta(N)}$$

for $k \le N$. Therefore, it is enough to show that

$$\lim_{L\to\infty} \limsup_{N\to\infty} \sum_{k=L}^{N-1} \frac{(N-k)\beta(N-k)\beta(k)}{N\beta(N)} = 0.$$

To show this, use the log-convexity of β a final time to show that the above sum is bounded by

$$\sum_{2k\le N} \frac{\beta^2(k)}{\beta(2k)}\left[\frac{k}{N} + 1_{\{k\ge L\}}\right].$$

Now use (6.12) and dominated convergence. □

We can now use the above results to obtain the following explicit upper bound for gap(Ω).

Theorem 6.13. *Suppose that* (6.12) *holds. If* $1 \le \delta \le 2$, *then*

$$\text{gap}(\Omega) \le 4(1-\rho)^{\delta} \sum_{m=1}^{\infty} m^{\delta}\beta(m)\rho^{-m}.$$

Proof. By Theorem 6.1,

$$\text{gap}(\Omega) \le \inf_N \frac{\sum_u \int [A_N(\eta^u) - A_N(\eta)]^2 \, d\nu}{\int [A_N(\eta) - \int A_N(\eta)\, d\nu]^2 \, d\nu}.$$

Multiplying the numerator and denominator above by $c(N)$ and passing to the limit as $N \to \infty$, using Lemmas 6.8, 6.9 and 6.11, gives

$$\text{gap}(\Omega) \le 2\frac{1-\rho}{\rho}\frac{\gamma^2\rho - 1}{\gamma^2 - 1}\sum_{k=1}^{\infty}\delta(k)\rho^{-k} \tag{6.14}$$

for any γ such that $\gamma\rho > 1$. Using the definition of $\delta(k)$ in the statement of Lemma 6.11, one computes

$$\sum_{k=1}^{\infty}\delta(k)\rho^{-k} = \sum_{m=1}^{\infty}\beta(m)\left\{\sum_{k=1}^{m-1}\left[1-\gamma^{-k}\right]^2\rho^{-k} + \frac{\left[1-\gamma^{-m}\right]^2\rho^{-m}}{1-\gamma^{-2}\rho^{-1}}\right\}.$$

Now we can take $\gamma = \rho^{-1}$ in (6.14) to get

$$\text{(6.15)}\quad \operatorname{gap}(\Omega) \le 2 \sum_{m=1}^{\infty} \beta(m)\left\{(1-\rho)\sum_{k=1}^{m-1}(1-\rho^k)^2\rho^{-k} + (1-\rho^m)^2\rho^{-m}\right\}.$$

Note that

$$1 - \rho^k \le \min\{1, k(1-\rho)\}.$$

Therefore, the statement of the theorem can be obtained from (6.15) by replacing $(1-\rho^k)^2$ by $[k(1-\rho)]^{\delta-1}$, ρ^{-k} by ρ^{-m} and $(1-\rho^m)^2$ by $[m(1-\rho)]^{\delta}$. □

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