*The Annals of Probability*
1989, Vol. 17, No. 3, 1083–1117

# STABLE HYDRODYNAMIC LIMIT FLUCTUATIONS OF A CRITICAL BRANCHING PARTICLE SYSTEM IN A RANDOM MEDIUM

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A particle system in Euclidean space is considered where the particles are subject to spatial motion according to a symmetric stable law and to a critical branching law in the domain of attraction of a stable law. The "branching intensity" may be position-dependent (varying medium) or be given by a realization of a random field (random medium). It is shown that under natural assumptions the hydrodynamic limit fluctuations around the macroscopic flow are the same as those given by the "averaged medium," the limit being a generalized stable Ornstein–Uhlenbeck process. The convergence proof is based on an analysis of a nonlinear integral equation with random coefficients.

**1. Introduction.** The question of the existence and properties of equilibrium states for critical branching particle systems has been extensively studied; see, for example, the comprehensive presentation by Kerstan, Matthes and Mecke (1982), as well as Dawson, Fleischmann, Foley and Peletier (1986), and the references therein. On the other hand, there is considerable current interest in understanding the behavior of distributed systems in *random media* (*environments*); see Kozlov (1985) and Papanicolaou (1983) for recent surveys.

In Dawson and Fleischmann (1983, 1985) a branching model in a random medium was introduced in a discrete space–time setting. Another model with branching was treated by Greven (1985). These branching models have the advantage of being more tractable than most interacting particle models such as those arising in statistical physics and consequently can serve as test cases.

In this article we will study the large scale fluctuations of a critical branching particle system in the spirit of Holley and Stroock (1978), Dawson (1981) and Dittrich (1987). Our model is different in that the branching law may have infinite variance and may also depend on a random medium. The treatment of the random medium involves ideas from the method of averaging [Kozlov (1985) gives a recent review of these ideas].

The model under consideration may be described roughly as follows. The states of the process are counting measures on Euclidean space $R^d$ of dimension

Received August 1987; revised June 1988.
[1]Research supported by NSERC A7750.
[2]Research supported in part by CONACyT grants ICEXNA-040319 and 140102 G203-006 (Mexico) and an NSERC (Canada) International Scientific Exchange Award.
*AMS* 1980 *subject classifications*. Primary 60J80; secondary 60F17, 60G20, 60G55.
*Key words and phrases*. Critical branching, random medium (environment), fluctuations, hydrodynamics, stable distribution, averaging, infinite particle system.



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$d$, representing a system of particles. These particles independently undergo spherically symmetric stable motions with exponent $\alpha$, $0 < \alpha \leq 2$. In addition each particle branches at rate $V > 0$ according to a critical law in the domain of attraction of a stable law with exponent $1 + \beta$, $0 < \beta \leq 1$. The branching law depends on a parameter $h$ which is location-dependent: If a particle branches at $x$, then it produces particles with *intensity* $h(x)$. This yields a branching particle system in a *spatially varying medium* $h = \{h(x): x \in R^d\}$. Finally, we assume that $h$ is given by a realization of an ergodic random field; we refer to $h$ as the *random medium*. A more precise description will be given in Section 2.

We denote by $N_t$ the state at time $t \geq 0$, and we consider the counting measure-valued process $N \equiv \{N_t, t \geq 0\}$. The distribution of the initial measure $N_0$ is chosen to depend on a parameter $\varepsilon > 0$ in such a way that as $\varepsilon \to 0$ the scaled random measure $\varepsilon^d N_0(\varepsilon^{-1} \cdot)$ converges in distribution to a nonrandom measure $\Lambda_0$. In this sense $N_0$ and $\Lambda_0$ describe the initial spatial distribution of particles at the microscopic and macroscopic levels, respectively.

The long-time behavior of the critical branching particle system $N$ involves a competition between the long-range "mixing" effect of the particle motions and the "clumping" effect of the branching. In order to exhibit this we consider the process viewed in the natural scaling associated with the particle motions, namely

$$X_t^\varepsilon(A) := \varepsilon^d N(\varepsilon^{-\alpha} t, \varepsilon^{-1} A).$$

One objective of this article is to show that in dimensions $d > \alpha/\beta$ we obtain the hydrodynamic limit

$$X_t^\varepsilon(\cdot) \to S_t \Lambda_0(\cdot) \quad \text{as } \varepsilon \to 0,$$

where $S_t$ is the semigroup of the particle motion. In other words in high dimensions the spatial diffusion dominates. In the case of low dimensions (which is not considered in this article) the branching dominates and the system degenerates to zero in this scaling [cf. Dawson and Fleischmann (1983, 1985) and Fleischmann and Gärtner (1988)]. For hydrodynamic limits of another branching model refer also to Dobrushin and Siegmund-Schultze (1982).

The main objective of this article is to investigate the fluctuations around this hydrodynamic limit. In the case $\alpha = 2$, $\beta = 1$ and the medium is deterministic the limit fluctuations are given by a well-known Gaussian generalized Ornlstein–Uhlenbeck process [see Holley and Stroock (1978), Gorostiza (1983), Bojdecki and Gorostiza (1986) and Dittrich (1987)]. In particular, we consider the rescaled fluctuation process

$$Y_t^\varepsilon := \varepsilon^{-k}\left[X_t^\varepsilon - S_t \Lambda_0\right],$$

where $k = (d\beta - \alpha)/(1 + \beta)$. The main result of this article is that $Y^\varepsilon$ converges as $\varepsilon \to 0$ to a generalized Ornstein–Uhlenbeck process. This limit process can also be described as the solution of a generalized Langevin equation which is driven by Gaussian noise when $\beta = 1$ and by stable noise when $\beta < 1$.

Our approach can be regarded as a generalization of Dittrich (1987). The key to the proof of these results is an analysis of the scaling properties of a nonlinear integral equation which enables us to establish the convergence of finite-dimensional distributions. [This key scaling argument is given in the proof of Proposition (5.3.1).] Concerning tightness, due to the lack of finite second moments, we cannot apply the usual methods based on the increasing process of martingales [as in Holley and Stroock (1978) and Gorostiza (1983)]. Instead we develop a method to verify Aldous' tightness criterion based on a moment inequality involving the characteristic functional.

The outline of the article is as follows. In Section 2 we describe in detail the branching system in a random medium. In Section 3 we introduce the state spaces for the process $N$ and the limiting fluctuation process $Y$. The precise statements of the main results described above are given in Section 4 and the proofs are given in Section 5.

**2. Description of the system.** The motion of the particles is assumed to be a spherically symmetric stable process of $R^d$ with exponent $\alpha$, $0 < \alpha \leq 2$.

Fix $\beta$, $0 < \beta \leq 1$. A *varying medium*, denoted by $h \equiv \{h(x): x \in R^d\}$, is an element of the space $H$ of real-valued measurable functions on $R^d$ satisfying $0 \leq h \leq (1+\beta)^{-1}$ (ther reason for imposing the upper bound is related to the branching law to be introduced). Let $\mathscr{H}$ be the $\sigma$-algebra of subsets of $H$ generated by the mappings $h \to \int_{R^d} f(x)h(x)\,dx$ with $f \in L^1(R^d)$.

The *random medium* is prescribed by a probability measure $m$ on $(H, \mathscr{H})$, not concentrated on $h \equiv 0$. We assume that $m$ is stationary and ergodic with respect to the translation group on $R^d$. The "classical" medium $h = \text{const.}$ is included as a special case. Another family of examples is given by $h(x) = (2/\pi(1+\beta))|\arctan g(x)|$, where $g$ is any continuous regular stationary Gaussian random field on $R^d$.

To each $h \in H$ is associated the family $g^h \equiv \{g^h(x): x \in R^d\}$ of offspring generating functions

$$\begin{aligned} g^h(x,s) &= s + h(x)(1-s)^{1+\beta} \\ &= s + h(x)\sum_{n=0}^{\infty}\binom{1+\beta}{n}(-s)^n, \qquad |s| < 1, \end{aligned} \tag{2.1}$$

where $s$ is a complex number. Here and in the following $z^{1+\beta} = \exp[(1+\beta)\log z]$ is always understood in the sense of the principal branch of the logarithm.

$g^h(x)$ describes the *local branching law*: If a particle located at $x$ splits, then $n$ particles are produced with probability

$$p_n^h(x) = \begin{cases} h(x)\binom{1+\beta}{n}(-1)^n & \text{if } n \neq 1, \\ 1 - h(x)(1+\beta) & \text{if } n = 1. \end{cases}$$

Note that the branching law is critical, that is, it has mean 1, and it belongs to the domain of normal attraction of a stable law with exponent $1+\beta$ [if $h(x) > 0$]; thus for $\beta < 1$ it has infinite variance. In the case $\beta = 1$, $p_n^h(x) = 0$ for $n > 2$ and the variance is $2h(x)$. The condition $p_1^h(x) \geq 0$ requires $h(x) \leq (1+\beta)^{-1}$. Since the probability that more than one particle is produced at $x$ is

$\beta h(x)$, we may interpret $h(x)$ as the *branching intensity* or *fertility* at $x$. Note also that $h(x) = p_0^h(x)$ is the probability of extinction at $x$.

Each particle has an exponentially distributed lifetime with parameter $V > 0$, and at the end of its life it branches according to $g^h$ given above. The offspring appear at the site where their parent branches.

The motions, lifetimes and branchings of all particles are independent of each other and of the random medium (except for the branching intensity). It is assumed that the random medium is sampled first (and fixed for all time) according to $m$ and then the system evolves in the resulting varying medium.

The description of the system is completed by specifying an initial distribution of particles which may possibly depend on the random medium.

Let $N_t(A)$ denote the number of particles in the system which lie in the Borel set $A \subset R^d$ at time $t \geq 0$. For fixed $h \in H$ and in the case of a finite number of initial particles the existence of process $N$ follows from Moyal (1962, 1964) and Ikeda, Nagasawa and Watanabe (1968a, b, 1969). In the next section we make precise the formulation of $N$ as a measure-valued process and its extension to an appropriate class of infinite measures.

## 3. State spaces for the basic processes.

3.1. *The counting process $N$.* The first objective of this section is to introduce a space $\mathscr{N}_p(R^d)$ of infinite counting measures with an appropriate topology which serves as the state space for $N$.

Given a topological space $M$ we define the following spaces of real-valued functions on $M$: $C(M)$ is the space of continuous bounded functions on $M$ with the topology induced by the sup-norm $\| \ \|_\infty$; $C_c(M)$ is the subset of functions having compact support, and if $M$ is locally compact $C_0(M)$ is the subset of functions vanishing at infinity. The collection of nonnegative elements of a real function space will be indicated by the index $+$.

When $M = R^d$ with its usual norm $|\cdot|$, we denote $\phi_p(x) = (1 + |x|^2)^{-p}$, $x \in R^d$, $p \geq 0$, and we define the spaces

$$C_p(R^d) = \left\{\phi \in C(R^d) \colon \|\phi/\phi_p\|_\infty < \infty\right\}, \qquad p > 0,$$

$$C_{p,0}(R^d) = \left\{\phi \in C(R^d) \colon \phi/\phi_p \in C_0(R^d)\right\}, \qquad p > 0.$$

Note that $C_{p,0}(R^d) \subset C_p(R^d) \subset C_0(R^d)$.

For $p \geq 0$, let $\mathscr{M}_p(R^d)$ denote the space of nonnegative Radon measures $\mu$ on $R^d$ such that $\int_{R^d} \phi_p \, d\mu < \infty$. We equip $\mathscr{M}_p(R^d)$ with the $p$-vague topology, that is, the smallest topology making the maps $\mu \to \int \phi \, d\mu$ continuous for all $\phi \in C_c(R^d)_+ \cup \{\phi_p\}$. We denote $\langle \mu, \phi \rangle = \int \phi \, d\mu$ for $\phi \in C_p(R^d)$ and $\mu \in \mathscr{M}_p(R^d)$. The space $\mathscr{M}_p(R^d)$ is Polish but not locally compact. The Lebesgue measure on $R^d$ belongs to $\mathscr{M}_p(R^d)$ for $p > d/2$. We denote by $\mathscr{N}_p(R^d)$ the space of counting measures in $\mathscr{M}_p(R^d)$, that is, the integer-valued measures finite on bounded Borel sets.

On $C_p(R^d)$, $p > 0$, we define the norm

$$\square\phi\square_p = \|\phi/\phi_p\|_\infty, \qquad \phi \in C_p(R^d),$$

making $C_p(R^d)$ a Banach space, and we consider the dual $C_p'(R^d)$ with norm

$$\square\xi\square_{-p} = \sup\left\{|\langle \xi, \phi\rangle| : \phi \in C_p(R^d), \square\phi\square_p \le 1\right\}, \qquad \xi \in C_p'(R^d).$$

Then $\mathscr{M}_p(R^d)$ can be regarded as a subspace of $C_p'(R^d)$. It is easy to verify that $\square\mu\square_{-p} = \langle \mu, \phi_p\rangle$ for all $\mu \in \mathscr{M}_p(R^d)$. The topology induced on $\mathscr{M}_p(R^d)$ by the norm $\square \cdot \square_{-p}$ is stronger than the $p$-vague topology.

Occasionally, we will use some of the function space symbols introduced above (and below) for the corresponding spaces of complex-valued functions.

Let $\{S_t^\alpha, t \ge 0\}$ denote the semigroup determined by the spherically symmetric stable process on $R^d$ with exponent $\alpha$, $0 < \alpha \le 2$ [see, e.g., Mijnheer (1975)]. By definition this is a time-homogeneous strong Markov process whose transition probability has a density $p_t^\alpha(x, y) \equiv p_t^\alpha(y - x)$ with characteristic function given by

$$\int_{R^d} e^{ix\cdot y} p_t^\alpha(x)\, dx = \exp\{-t|y|^\alpha\}, \qquad y \in R^d$$

($\cdot$ denotes the inner product in $R^d$). The case $\alpha = 2$ corresponds to the Wiener process with variance parameter 2. We will suppress the $\alpha$ in the notation of the semigroup and the transition density. We have

$$\begin{aligned} S_t\phi(t) &= \int_{R^d} \phi(y) p_t(y - x)\, dy \\ &= \int_{R^d} \phi(x + y) p_t(y)\, dy, \qquad t > 0, \\ S_0\phi &= \phi, \qquad \phi \in C(R^d) \qquad \text{(real- or complex-valued)}. \end{aligned}$$

For $\mu \in \mathscr{M}_p(R^d)$ with $p \ge 0$, we also define the Borel measure

$$\begin{aligned} S_t\mu(A) &= \int_{R^d}\left(\int_A p_t(y - x)\, dy\right)\mu(dx), \qquad t > 0, \\ S_0\mu &= \mu. \end{aligned}$$

Note that $\langle S_t\mu, \phi\rangle = \langle \mu, S_t\phi\rangle$ if $\phi \in C_p(R^d)$ and $\mu \in \mathscr{M}_p(R^d)$.

Throughout the remainder of this article, $p$ is a fixed number satisfying

(3.1a) $\quad p > d/2 \quad$ in all cases,

(3.1b) $\quad p < (d + \alpha)/2 \quad$ in the case $\alpha < 2$.

This condition is imposed in order to guarantee that Lebesgue measure belongs to $\mathscr{M}_p(R^d)$ and that $S_t$ maps $\mathscr{M}_p(R^d)$ into itself. The appropriate technical results given below are based on results of Iscoe (1986). [Note, however, that Iscoe defines $\phi_p(x) = (1 + |x|^p)^{-1}$ so that his results translate to our setting with $2p$ instead of $p$.]

(3.2) LEMMA. *If $p$ satisfies* (3.1), *then $S_t$ is a bounded linear operator from $(C_p(R^d), \Box \cdot \Box_p)$ into itself, and from $(\mathscr{M}_p(R^d), \Box \cdot \Box_{-p})$ into itself.*

PROOF. Since $\phi \in C_p(R^d)$ is uniformly continuous, then $S_t\phi$ is continuous and bounded. We have

$$|S_t\phi(x)| = |S_t(\phi_p\phi/\phi_p)(x)| \le \Box\phi\Box_p S_t\phi_p(x).$$

By Iscoe (1986), Corollary 2.4, $S_s\phi_p \le C_t\phi_p$ for $s \le t$ and some constant $C_t$. Hence $S_t\phi \in C_p(R^d)$ and $\Box S_t\phi\Box_p \le C_t\Box\phi\Box_p$.

For $\mu \in \mathscr{M}_p(R^d)$, by duality and the previous result,

$$\begin{aligned}\Box S_t\mu\Box_{-p} &= \sup\left\{|\langle\mu, S_t\phi\rangle|: \phi \in C_p(R^d), \Box\phi\Box_p \le 1\right\}\\ &\le \sup\left\{\Box\mu\Box_{-p}\Box S_t\phi\Box_p: \Box\phi\Box_p \le 1\right\} \le C_t\Box\mu\Box_{-p}.\end{aligned}$$

□

(3.3) LEMMA. *If $p$ satisfies* (3.1), *and $\phi \in C_p(R^d)$ is such that $\lim_{|x| \to \infty} \phi(x)/\phi_p(x)$ exists* [*e.g.*, $\phi \in C_c(R^d)_+ \cup \{\phi_p\}$], *then $t \to S_t\phi$ is a continuous curve in $(C_p(R^d), \Box \cdot \Box_p)$. Also, for $\mu \in \mathscr{M}_p(R^d)$, $t \to S_t\mu$ is $p$-vaguely continuous.*

PROOF. Let $0 \le s < t \le T$. By the proof of (3.2) and Iscoe (1986), Corollary 2.4,

$$\Box S_t\phi - S_s\phi\Box_p = \Box S_s(S_{t-s}\phi - \phi)\Box_p \le C_T\Box S_{t-s}\phi - \phi\Box_p$$

for some constant $C_T$, and $\Box S_{t-s}\phi - \phi\Box_p \to 0$ by Iscoe (1986), Lemma 2.4. By the first part of the lemma and dominated convergence,

$$\langle S_t\mu - S_s\mu, \phi\rangle = \langle\mu, S_t\phi - S_s\phi\rangle \to 0 \quad \text{as } t - s \to 0,$$

for $\phi \in C_c(R^d)_+ \cup \{\phi_p\}$. □

The infinitesimal generator of $\{S_t, t \ge 0\}$ is given by the fractional power of the Laplacian, $\Delta_\alpha \equiv -(-\Delta)^{\alpha/2}$ [see Pazy (1983)].

Given a topological space $M$, we denote by $D_M \equiv D(R_+, M)$ the space of functions from $R_+$ into $M$ which are right-continuous and have left limits. $D_M$ is equipped with a Skorohod-type topology when $M$ is a metric space [see Ethier and Kurtz (1986)].

We now return to the counting process $N$ described in the previous section. Following the method of Iscoe (1986), this process can be extended to include initial measures $\mu \in \mathscr{N}_p(R^d)$ and with sample paths in $D_{\mathscr{M}_p(R^d)}$ [see the proof of Lemma (5.2.3)]. Then $N$ is a time-homogeneous Markov process with characteristic functional

$$F(t, \phi, \mu) = E[\exp\{-i\langle N_t, \phi\rangle\}|N_0 = \mu], \qquad t \ge 0, \phi \in C_p(R^d), \mu \in \mathscr{N}_p(R^d),$$

which has the following properties:

(3.4) (multiplicative property)

$$F(t, \phi, \mu_1 + \mu_2) = F(t, \phi, \mu_1)F(t, \phi, \mu_2), \qquad \mu_1, \mu_2 \in N_p(R^d),$$

(3.5) (branching property) $v(t) \equiv \{v(t,x) := F(t,\phi,\delta_x), x \in R^d\}$ satisfies the integral equation

$$v(t) = e^{-Vt}S_t e^{-i\phi} + \int_0^t \left[S_r g^h(v(t-r))\right] V e^{-Vr}\, dr, \qquad t \geq 0.$$

Property (3.4) is a consequence of the independence of the systems generated by different initial particles. Property (3.5) is a nonlinear renewal-type equation which arises as follows: If we start one particle at location $x$, this particle moves according to the stable law and does not branch before time $t$ (first term on the right-hand side), or it splits at time $r$ at rate $V$ according to $g^h$ and all its offspring evolve independently after birth in the same way (second term).

3.2. *The limit fluctuation process Y.* In the remainder of this section we discuss some technical points concerning the Schwartz space of tempered distributions, $\mathscr{S}'(R^d)$, which serves as the state space for the limiting fluctuation process $Y$. Further information on this material can be found in Gelfand and Vilenkin (1964), Itô (1984) and Mitoma (1983).

Let $\mathscr{S}(R^d)$ denote the space of infinitely differentiable functions which are rapidly decreasing at infinity together with all their derivatives. $\mathscr{S}(R^d)$ is topologized by either of the two following sequences of increasing norms:

$$\|\phi\|_n = \left( \sum_{|k|=0}^{n} \int_{R^d} \left(1+|x|^2\right)^n |D^k\phi(x)|^2\, dx \right)^{1/2},$$

$$|||\phi|||_n = \max_{0 \leq |k| \leq n} \sup_{x \in R^d} \left(1+|x|^2\right)^n |D^k\phi(x)|, \qquad \phi \in \mathscr{S}(R^d),$$

$n = 0, 1, 2, \dots,$ where $k = (k_1, \dots, k_d)$, $|k| = k_1 + \cdots + k_d$, $D^k = \partial^{|k|}/\partial x_1^{k_1} \dots \partial x_d^{k_d}$. The $\|\cdot\|_n$ are Hilbert norms, and for each $n \geq 1$ there exist constants $c_n$ and $l_n > 0$ such that $|||\phi|||_n \leq c_n \|\phi\|_{n+l_n}$ and $\|\phi\|_n \leq c_n |||\phi|||_{n+l_n}$, $\phi \in \mathscr{S}(R^d)$. We will denote by $S_n(R^d)$ the completion of $\mathscr{S}(R^d)$ with respect to $\|\cdot\|_n$. Then $\mathscr{S}(R^d) = \bigcap_{n=0}^{\infty} \mathscr{S}_n(R^d)$ is a countably Hilbert nuclear space; this fact makes possible the application of the results of Hida (1980) and Mitoma (1983). Note that $\mathscr{S}(R^d) \subset C_{p,0}(R^d)$ for all $p > 0$.

The topological dual of $\mathscr{S}(R^d)$ will be designated by $\mathscr{S}'(R^d)$; this is the space of tempered distributions. We have $\mathscr{S}'(R^d) = \bigcup_{n=0}^{\infty} \mathscr{S}_n'(R^d)$, where $\mathscr{S}_n'(R^d)$ is the dual of $\mathscr{S}_n(R^d)$, with the dual-norm written $\|\cdot\|_{-n}$. We will denote by $\langle \cdot, \cdot \rangle$ the duality on $(\mathscr{S}'(R^d), \mathscr{S}(R^d))$ and on $(\mathscr{S}_n'(R^d), \mathscr{S}_n(R^d))$, as well as other dualities.

If $\mathscr{S}'(R^d)_+$ designates the space of nonnegative members of $\mathscr{S}'(R^d)$, then $\mathscr{S}'(R^d)_+ = \bigcup_{p \geq 0} \mathscr{M}_p(R^d)$. Moreover, for each $p \geq 0$ there exists $n \geq 1$ such that $\mathscr{M}_p(R^d) \subset \mathscr{S}_n'(R^d)$; the topology induced on $\mathscr{M}_p(R^d)$ by $\mathscr{S}_n'(R^d)$ is weaker than the $p$-vague topology, and still weaker is the one induced by $\mathscr{S}'(R^d)$ [since $\mathscr{S}'(R^d)$ is not the strict inductive limit of the $\mathscr{S}_n'(R^d)$].

The space $D_{\mathscr{S}'(R^d)}$ is equipped with a Skorohod-type topology [see Mitoma (1983)].

(3.6) REMARKS. 1. As is now common with models of the present type, an appropriate setting for the fluctuation limit theorem is the nuclear triple

$$\mathscr{S}(R^d) \subset L^2(R^d) \subset \mathscr{S}'(R^d).$$

Since the Brownian motion semigroup and the Laplacian map $\mathscr{S}(R^d)$ into itself, the latter is therefore a natural space of test functions for our model when the particles undergo Brownian motion [e.g., Holley and Stroock (1978)]. However, a technical difficulty arises from the fact that, contrary to the Brownian motion case, the semigroup and the infinitesimal generator of the symmetric stable process with exponent $\alpha < 2$ do not map $\mathscr{S}(R^d)$ into itself. To verify this, note that (3.2) implies that $S_t(\mathscr{S}(R^d)) \subset C_{p,0}(R^d)$ for all $p$ which satisfy (3.1). Moreover, $S_t\phi$ is infinitely differentiable for all $\phi \in \mathscr{S}(R^d)$, since the stable density $p_t(x)$ is infinitely differentiable in $x$. Nevertheless $S_t(\mathscr{S}(R^d))$ is not contained in $\mathscr{S}(R^d)$ for $\alpha < 2$ because in this case fast decay at infinity fails due to the long tails of the stable law. Moreover, the domain of $\Delta_\alpha$ contains $\mathscr{S}(R^d)$ and $\Delta_\alpha(\mathscr{S}(R^d)) \subset C_{p,0}(R^d)$ but $\Delta_\alpha(\mathscr{S}(R^d)) \not\subset \mathscr{S}(R^d)$ again due to the long tails of the stable law [cf. Dawson and Gorostiza (1988)]. Since $S_t$ and $\Delta_\alpha$ do not map $\mathscr{S}(R^d)$ into itself for $\alpha < 2$, extra work is needed in order to use $\mathscr{S}(R^d)$ as a space of test functions. We will come back to this point in Remark (4.7).

2. An alternative approach would be to define an ad hoc nuclear space $F$ of test functions which is invariant under $S_t$ and $\Delta_\alpha$. Such a space can be constructed but turns out to be smaller than $\mathscr{S}(R^d)$. Consequently, $F'$ is larger than $\mathscr{S}'(R^d)$ thus yielding a weaker result than the one obtained by working with $\mathscr{S}'(R^d)$.

**4. Statement of the main results.** In order to set up the scaling limit we introduce a parameter $\varepsilon$, $0 < \varepsilon < 1$, and a corresponding family of processes $N^\varepsilon \equiv \{N_t, t \geq 0\}$. For fixed $h \in H$ we denote by $P_\varepsilon^h$ the distribution of $N^\varepsilon$ on $D_{\mathscr{S}'(R^d)}$. Then the distribution of $N^\varepsilon$ in the random medium is given by

$$P_\varepsilon^m(\cdot) = \int_H P_\varepsilon^h(\cdot)m(dh); \tag{4.1}$$

and the expectation with respect to $P_\varepsilon^m$ is given by

$$E(\cdot) = \int_H E_h(\cdot)m(dh), \tag{4.2}$$

where $E_h$ denotes the expectation with respect to $P_\varepsilon^h$ [see Dellacherie and Meyer (1978)].

We will assume that as $\varepsilon \to 0$ the scaled initial random measure $\varepsilon^d N_0^\varepsilon(\varepsilon^{-1}\cdot)$ converges weakly under $P_\varepsilon^m$ to a nonrandom measure $\Lambda_0 \in \mathscr{M}_p(R^d)$. An example of this is $N_0^\varepsilon$ = Poisson particle system with intensity measure $\varepsilon^{-d}\Lambda_0(\varepsilon\,\cdot)$, with $\Lambda_0 \in \mathscr{M}_p(R^d)$. The index $\varepsilon$ is introduced only in connection with this assumption on $N_0^\varepsilon$; the $\varepsilon$ has no bearing on the transition probability of $N^\varepsilon$; hence (3.4) and (3.5) hold under each $P_\varepsilon^h$.

Given $N^\varepsilon$ we define the $C_p'(R^d)$-valued process $Y^\varepsilon \equiv \{Y_t, t \geq 0\}$ by

$$\langle Y_t^\varepsilon, \phi\rangle = \varepsilon^{-k}[\langle X_t^\varepsilon, \phi\rangle - \langle \Lambda_t, \phi\rangle], \qquad t \geq 0, \phi \in C_p(R^d), \tag{4.3}$$

where

$$\langle X_t^\varepsilon, \phi\rangle = \varepsilon^d \langle N_{\varepsilon^{-\alpha}t}^\varepsilon, \phi(\varepsilon\,\cdot)\rangle, \qquad \Lambda_t = S_t\Lambda_0 \quad \text{and} \quad k = \frac{(d\beta - \alpha)}{(1+\beta)}. \tag{4.4}$$

$Y^\varepsilon$ is the *large scale fluctuation process* around the macroscopic deterministic flow $\Lambda$. The distribution of $Y^\varepsilon$ in $D_{\mathscr{S}'(R^d)}$ is determined by $P_\varepsilon^h$ for the varying medium $h$, and by $P_\varepsilon^m$ for the random medium. Note that the space–time scaling involved in (4.4) preserves the distribution of the particle motion, and that $d > 2k$ because $\beta \leq 1$ and $\alpha > 0$.

We will denote weak convergence by $\Rightarrow$.

Let us recall our assumptions: $d > \alpha/\beta$, $p$ satisfies (3.1a) and (3.1b), $N_0^\varepsilon$ may depend on the random medium and takes values in $\mathscr{M}_p(R^d)$ and $\varepsilon^d N_0^\varepsilon(\varepsilon^{-1}\,\cdot\,) \Rightarrow \Lambda_0$ under $P_\varepsilon^m$, where $\Lambda_0 \in \mathscr{M}_p(R^d)$ is deterministic. In addition, we will assume that $E\langle N_0^\varepsilon, \phi_p\rangle < \infty$ and $E\langle X_0^\varepsilon, \phi^\varepsilon\rangle \to 0$ as $\varepsilon \to 0$ if $\phi^\varepsilon \to 0$ in $C_p(R^d)$, where $E$ is given by (4.2) [see, however, statement 3 in Remark (5.3.8)]. Note that $Y_0^\varepsilon$ is a $C_p'(R^d)$-valued random variable. We will now formulate our main result.

(4.5) THEOREM. (a) *If $Y_0^\varepsilon \Rightarrow Y_0$ under $P_\varepsilon^m$ as $\varepsilon \to 0$, where $Y_0$ is a $C_p'(R^d)$-valued random variable, and* $\sup_\varepsilon E\|Y_0^\varepsilon\|_{-p}^{1+\beta} < \infty$, *then $Y^\varepsilon \Rightarrow Y$ under $P_\varepsilon^m$ in $D_{\mathscr{S}'(R^d)}$ as $\varepsilon \to 0$, where $Y \equiv \{Y_t, t \geq 0\}$ is an $\mathscr{S}'(R^d)$-valued time-homogeneous Markov process whose transition characteristic functional is given by*

$$\begin{aligned} &E[\exp\{-i\langle Y_t, \phi\rangle\}|Y_s] \\ &\quad \equiv E[\exp\{-i\langle Y_t, \phi\rangle\}|\langle Y_s, \psi\rangle, \psi \in \mathscr{S}(R^d)] \\ &\quad = \exp\left\{-i\langle Y_s, S_{t-s}\phi\rangle + \left\langle \Lambda_s, V\bar{h}\int_s^t S_{t-r}(iS_{r-s}\phi)^{1+\beta}\,dr\right\rangle\right\}, \end{aligned} \tag{4.6}$$

$$0 \leq s \leq t, \phi \in \mathscr{S}(R^d),$$

*where $\bar{h} = \int_H h(0)m(dh)$ is the mean of the random medium.*

(b) *If the assumptions hold with respect to $P_\varepsilon^h$ for m-almost all $h$ (in particular if $N_0^\varepsilon$ is independent of the random medium), then the limit result in* (a) *is valid for each fixed varying medium $h$ in a set of m-measure* 1, *the limit process $Y$ being the same as in* (a).

(4.7) REMARK. The notation $\langle Y_s, S_{t-s}\phi\rangle$ in (4.6) needs justification because $S_{t-s}\phi$ lies in $C_{p,0}(R^d)$ but not necessarily in $\mathscr{S}(R^d)$. It can be shown that $\langle Y_s, \cdot\rangle$ has a continuous linear extension to $C_{p,0}(R^d)$, and therefore $\phi \to \langle Y_s, S_{t-s}\phi\rangle$ defines a linear random functional on $\mathscr{S}(R^d)$ [see Lemma (3.2) and Remark (3.6)]. Then, by the regularization theorem [e.g., Itô (1984)], this linear random functional has a regular version, that is, a version which is an $\mathscr{S}'(R^d)$-valued random variable. We denote the regular version also by $\langle Y_s, S_{t-s}\phi\rangle$, $\phi \in \mathscr{S}(R^d)$, and this is what is meant in (4.6). It should be noted that the previous argument does not allow us to assert that $Y$ is a $C'_{p,0}(R^d)$-valued process because the regularization theorem does not hold on $C_{p,0}(R^d)$. This and related questions involve other techniques, and they are discussed in detail in Dawson and Gorostiza (1988).

The fluctuation limit theorem, more precisely Proposition (5.3.1) below, also yields the hydrodynamic behavior of the system, that is, the law of large numbers, namely:

(4.8) COROLLARY. *$\langle X_t^\varepsilon, \phi\rangle \Rightarrow \langle \Lambda_t, \phi\rangle$ under $P_\varepsilon^m$ as $\varepsilon \to 0$ for each $\phi \in C_p(R^d)$, and a corresponding a.s. result for condition* (b) *above.*

In the next theorem we will give some properties of the limit fluctuation process $Y$.

(4.9) THEOREM. (i) *$Y$ satisfies the generalized Langevin equation*

$$dY_t = \Delta_\alpha Y_t\,dt + dZ_t, \qquad t \geq 0,$$

*where $Z \equiv \{Z_t, t \geq 0\}$ is an $\mathscr{S}'(R^d)$-valued process with independent increments such that*

$$E\exp\{-i(\langle Z_t, \phi\rangle - \langle Z_s, \phi\rangle)\}$$
$$= \exp\left\{V\bar{h}\int_s^t \langle \Lambda_r, (i\phi)^{1+\beta}\rangle\,dr\right\}, \qquad 0 \leq s \leq t,\ \phi \in \mathscr{S}(R^d).$$

(ii) *For $t \geq 0$ and $\phi \in \mathscr{S}(R^d)_+$, $\langle Y_t, \phi\rangle - \langle Y_o, S_t\phi\rangle$ has a distribution with characteristic functional*

$$E\exp\{iu(\langle Y_t, \phi\rangle - \langle Y_0, S_t\phi\rangle)\}$$
$$= \exp\left\{-c(\phi)|u|^{1+\beta}\left[1 - i\,\mathrm{sgn}\,u\tan\frac{\pi(1+\beta)}{2}\right]\right\}, \qquad u \in R,$$

*that is, a Gaussian distribution if $\beta = 1$, and if $\beta < 1$ it is a stable distribution with index $1 + \beta$, and anisotropy parameter $-1$ [see, e.g., Breiman* (1968), *page* 204]. *The normalization constant $c(\phi)$ is given by*

$$c(\phi) = -\cos\left(\frac{\pi(1+\beta)}{2}\right)\left\langle \Lambda_0, V\bar{h}\int_0^t S_{t-r}(S_r\phi)^{1+\beta}\,dr\right\rangle.$$

(iii) *$Y_t - S_tY_0$ is spatially homogeneous for each $t \geq 0$ if and only if $\Lambda_0(dx) = \lambda\,dx$, $\lambda > 0$. In this case, if $Y_0 = 0$ and $\beta = 1$ (the Gaussian case), the spectral measure of $Y_t$ is given by*

$$V\bar{h}\lambda|x|^{-\alpha}(1 - \exp\{-2t|x|^\alpha\})\,dx.$$

(iv) *There exists $n \geq 1$ such that $Y_t - S_tY_0$ takes values in $S_n'(R^d)$ for all $t \geq 0$.*

(4.10) REMARKS. 1. Under condition (b) in (4.5) the tightness proof is basically the same as in case (a), but simpler. We will do it only for (a). The condition $\sup_\varepsilon E\|Y_0^\varepsilon\|_{-p}^{1+\beta} < \infty$ can be replaced by the slightly weaker one: $\sup_\varepsilon E\|Y_o^\varepsilon\|_{-p}^{1+\theta} < \infty$ for some $0 < \theta < \beta$ but this complicates the calculations.

2. Under each of the two cases in (4.5), if the random medium is only invariant (not necessarily ergodic) with respect to the translation group, then the results hold with $\bar{h}$ replaced by $E[h|\mathscr{I}]$, where $\mathscr{I}$ is the $\sigma$-field of invariant sets.

3. In view of (4.9)(i) and (4.9)(ii) we refer to $Y$ as a *generalized stable Ornstein–Uhlenbeck process*. More precisely, $Y - SY_0$ is such a process. In the case $\beta = 1$, $Y - SY_0$ is a special case of a generalized Gaussian Ornstein–Uhlenbeck process with continuous paths [see Holley and Stroock (1978) and Dawson (1981)]; Bojdecki and Gorostiza (1986) discuss a more general class of such processes; in this case the continuity can be shown by an extension of the Dudley–Fernique theorem [cf. Mitoma (1981)]. On the other hand, for $\beta < 1$, $Y$ has discontinuous paths. Hence the properties of $Y$ being Gaussian or strictly stable, and correspondingly continuous or discontinuous, depend on whether the branching law has finite second moment or not ($\beta = 1$ or $\beta < 1$, respectively), and these properties are independent of whether the particle motion is Brownian motion or a strictly stable process ($\alpha = 2$ or $\alpha < 2$). It should also be noticed that (4.6) shows that the conditional distribution of the stable process $Y - SY_0$ is also stable; this is generally not true for symmetric stable non-Gaussian processes [Adler, Cambanis and Samorodnitsky (1987)].

4. The asymmetric behavior of $Y - SY_0$ in the case $\beta < 1$ occurs because the branching law is the distribution of a nonnegative random variable and belongs to the domain of attraction of an asymmetric stable law.

5. If $N_0^\varepsilon$ is a Poisson system with intensity measure $\varepsilon^{-d}\Lambda_0(\varepsilon\,\cdot\,)$, then $\varepsilon^{-d/2}(\varepsilon^d N_0^\varepsilon(\varepsilon^{-1}\,\cdot\,) - \Lambda_0)$ converges weakly to a Gaussian white noise on $R^d$ determined by $\Lambda_0$, and therefore $Y_0 = 0$ because $d > 2k$. In this case the normalization which yields a nondegenerate limit at times $t > 0$ is too strong to yield limit fluctuations at time $t = 0$.

6. In the Gaussian case ($\beta = 1$) and Brownian motion ($\alpha = 2$), the Langevin equation in (4.9)(i) can be obtained by applying Theorem 3.6 in Bojdecki and Gorostiza (1986). For $\alpha < 2$ and $\beta = 1$ the equation is formally the same, just replacing $\Delta$ by $\Delta_\alpha$. However, for $\alpha < 2$ (and $\beta \le 1$) the Langevin equation cannot be interpreted in the usual way, that is,

$$\langle Y_t, \phi\rangle = \langle Y_0, \phi\rangle + \int_0^t \langle Y_s, \Delta_\alpha \phi\rangle\, ds + \langle Z_t, \phi\rangle, \qquad t \ge 0,\ \phi \in \mathscr{S}(R^d),$$

because $\Delta_\alpha$ does not map $\mathscr{S}(R^d)$ into itself. A generalized interpretation of the equation is necessary. This turned out to be a fairly technical problem, and we preferred to discuss it separately [Dawson and Gorostiza (1988)].

## 5. Proofs.

5.1. *A nonlinear integral equation with random coefficients.* The proof of convergence is based on properties of solutions of an integral equation. In this section we will derive these properties.

Fix $T > 0$, $\phi \in C_p(R^d)$, $L > 0$ and $0 < \lambda \leq L$, and assume that for each $h$ in a subset $H_0$ of $H$ and $0 < \varepsilon < 1$ we have a bounded complex-valued function $u_\varepsilon(t) \equiv u_\varepsilon(t, x) \equiv u_\varepsilon(t, \phi\lambda, x) \equiv u_\varepsilon(t, \phi\lambda)$, $0 \leq t \leq T$, $x \in R^d$ (we omit the dependence on $h$ in the notation) which satisfies the integral equation

$$\begin{aligned} u_\varepsilon(t) &= S_t\big[\varepsilon^{k-d}\big(1 - \exp\{-i\varepsilon^{d-k}\phi\lambda\}\big)\big] \\ &\quad -\varepsilon^k V\int_0^t S_{t-s}\big[h(\varepsilon^{-1}\cdot)u_\varepsilon^{1+\beta}(s)\big]\,ds, \qquad 0 \leq t \leq T. \end{aligned} \tag{5.1.1}$$

(5.1.2) LEMMA. *There exist* $0 < \varepsilon_{T,L} < 1$ *and* $0 < K < \infty$ *such that*

$$\sup\big\{|u_\varepsilon(t, x)| : t \leq T, x \in R^d, \varepsilon < \varepsilon_{T,L}, h \in H_0\big\} \leq \text{const.}\|\phi\|_\infty\lambda \equiv K\lambda.$$

PROOF. Let $a_\varepsilon(t) = \sup\{|u_\varepsilon(t, x)| : x \in R^d\}$. Then, since $S_t$ is a contraction and $0 \leq h \leq 1$, (5.1.1) implies

$$0 \leq a_\varepsilon(t) \leq \|\phi\|_\infty\lambda + \varepsilon^k V\int_0^t a_\varepsilon^{1+\beta}(s)\,ds, \qquad 0 \leq t \leq T.$$

The equation

$$b_\varepsilon(t) = \|\phi\|_\infty\lambda + \varepsilon^k V\int_0^t b_\varepsilon^{1+\beta}(s)\,ds, \qquad 0 \leq t \leq T,$$

has unique solution $b_\varepsilon(t) = \|\phi\|_\infty\lambda(1 - \varepsilon^k V\beta\|\phi\|_\infty^\beta\lambda^\beta t)^{-1/\beta}$ for all $0 < \lambda \leq L$ provided that

$$\varepsilon < 1/\big(V\beta T\|\phi\|_\infty^\beta L^\beta\big)^{1/k}. \tag{5.1.3}$$

Clearly, there exists $\varepsilon_{T,L} > 0$ such that $b_\varepsilon(t) \leq \text{const.}\|\phi\|_\infty\lambda$ for $0 \leq t \leq T$ and $\varepsilon < \varepsilon_{T,L}$.

By a comparison theorem [Miller (1971), page 121], $a_\varepsilon(t) \leq b_\varepsilon(t)$ for $0 \leq t \leq T$; hence the assertion follows. □

(5.1.4) LEMMA.

$$\sup\big\{|u_\varepsilon(t)| : \varepsilon < \varepsilon_{T,L}, h \in H_0\big\} \leq \text{const.}\, S_t|\phi|\lambda, \qquad 0 \leq t \leq T.$$

PROOF. By (5.1.5) and (5.1.2),

$$\sup_h |u_\varepsilon(t)| \leq S_t|\phi|\lambda + \varepsilon^k V K^\beta\lambda^\beta\int_0^t S_{t-s}\sup_h|u_\varepsilon(s)|\,ds, \qquad 0 \leq t \leq T.$$

The comparison theorem used in the previous proof does not apply in this case; henceforth we assume $\varepsilon_{T,L}$ is small enough so that $c \equiv \varepsilon_{T,L}^k V K^\beta L^\beta T < 1$ where $K = \text{const.}\|\phi\|_\infty$ [this is consistent with (5.1.3)]; then by iteration of the inequality it can be shown that

$$\sup_h |u_\varepsilon(t)| \leq (1 - c)^{-1}S_t|\phi|\lambda, \qquad 0 \leq t \leq T. \qquad \square$$

(5.1.5) LEMMA. *There exist positive constants* $K_1$ *and* $K_2$ *such that for all* $t \leq T$ *and* $\varepsilon < \varepsilon_{T,L}$,

$$\sup_h |u_\varepsilon(t) - iS_t\phi\lambda| \leq K_1\varepsilon^k S_t|\phi|\lambda^{1+\beta} \leq K_2\varepsilon^k\lambda.$$

PROOF. Since

$$|\exp\{-i\varepsilon^{d-k}\phi\lambda\} - 1 + i\varepsilon^{d-k}\phi\lambda| \le \varepsilon^{2(d-k)}|\phi|^2\lambda^2 \le \text{const. } \varepsilon^{2(d-k)}|\phi|\lambda^{1+\beta},$$

then

$$\text{(a)} \qquad |S_t\{\varepsilon^{k-d}(1 - e^{-i\varepsilon^{d-k}\phi\lambda})\} - iS_t\phi\lambda| \le \text{const. } \varepsilon^{d-k}S_t|\phi|\lambda^{1+\beta}.$$

Then, by (5.1.1), (5.1.2), (5.1.4) and (a),

$$\begin{aligned}|u_\varepsilon(t) - iS_t\phi\lambda| &\le |S_t\{\varepsilon^{k-d}(1 - e^{-i\varepsilon^{d-k}\phi\lambda})\} - iS_t\phi\lambda| + \varepsilon^k V\int_0^t S_{t-s}|u_\varepsilon(s)|^{1+\beta}\,ds \\ &\le \text{const. } \varepsilon^{d-k}S_t|\phi|\lambda^{1+\beta} + \text{const. } \varepsilon^k V K^\beta \lambda^\beta \int_0^t S_{t-s}S_s|\phi|\lambda\,ds \\ &\le \text{const. } \varepsilon^{d-k}S_t|\phi|\lambda^{1+\beta} + \text{const. } \varepsilon^k S_t|\phi|\lambda^{1+\beta} \le \text{const. } \varepsilon^k S_t|\phi|\lambda^{1+\beta},\end{aligned}$$

the last inequality because $d - k > k$. This proves the first asserted inequality; the second one is obvious. □

(5.1.6) PROPOSITION. *There exists $\gamma > 1$ such that for all $t \le T$ and $\varepsilon < \varepsilon_{T,L}$,*

$$\begin{aligned}\sup_h \left| u_\varepsilon(t) - iS_t\phi\lambda + \varepsilon^k V\int_0^t S_{t-s}\left[h(\varepsilon^{-1}\cdot)(iS_s\phi\lambda)^{1+\beta}\right] ds\right| \\ \le \text{const. } \varepsilon^{\gamma k}S_t|\phi|\lambda^{1+\beta}.\end{aligned}$$

PROOF. Let $A = \{(s, y) \in [0, T] \times R^d : |S_s\phi(y)| \le 2K_2\varepsilon^k\}$, where $K_2$ is the constant in (5.1.5). Then, by (5.1.5), on $A$,

$$|u_\varepsilon(s, y)| \le |u_\varepsilon(s, y) - iS_s\phi(y)\lambda| + |S_s\phi(y)\lambda| \le K_2\varepsilon^k\lambda + 2K_2\varepsilon^k\lambda \le \text{const. } \varepsilon^k\lambda.$$

Hence, by (5.1.4), on $A$,

$$\text{(a)} \qquad \begin{aligned}|u_\varepsilon(s, y)|^{1+\beta} &= |u_\varepsilon(s, y)|\,|u_\varepsilon(s, y)|^\beta \\ &\le \text{const. } S_s|\phi|(y)\lambda\varepsilon^{k\beta}\lambda^\beta \le \text{const. } \varepsilon^{k\beta}S_s|\phi|(y)\lambda^{1+\beta}.\end{aligned}$$

Also, on $A$,

$$\text{(b)} \qquad \begin{aligned}|S_s\phi(y)\lambda|^{1+\beta} &\le |S_s\phi(y)|\,|S_s\phi(y)|^\beta\lambda^{1+\beta} \\ &\le \text{const. } \varepsilon^{k\beta}S_s|\phi|(y)\lambda^{1+\beta}.\end{aligned}$$

Using (a) and (b), we have

$$\begin{aligned}\varepsilon^k V\int_0^t ds\int_{R^d} dy\, 1_A(s, y)p_{t-s}(\cdot, y)|u_\varepsilon^{1+\beta}(s, y) - (iS_s\phi(y)\lambda)^{1+\beta}| \\ \le \text{const. } \varepsilon^{k(1+\beta)}\lambda^{1+\beta}\int_0^t ds\int_{R^d} dy\, p_{t-s}(\cdot, y)S_s|\phi|(y),\end{aligned}$$

so

$$\text{(c)}\quad \begin{aligned}&\varepsilon^k V\int_0^t ds\int_{R^d} dy\, 1_A(s,y)p_{t-s}(\cdot,y)|u_\varepsilon^{1+\beta}(s,y)-(iS_s\phi(y)\lambda)^{1+\beta}|\\ &\qquad\le \text{const.}\ \varepsilon^{k(1+\beta)}S_t|\phi|\lambda^{1+\beta}.\end{aligned}$$

On the other hand, on $A^c$, $|iS_s\phi(y)\lambda| > 2K_2\varepsilon^k\lambda$, but by (5.1.5) $|u_\varepsilon(s,y) - iS_s\phi(y)\lambda| \le K_2\varepsilon^k\lambda$. Hence on $A^c$ both $u_\varepsilon(s,y)$ and $iS_s\phi(y)\lambda$ lie on the same half-plane $\operatorname{Im}(z) > 0$ or $\operatorname{Im}(z) < 0$. On these half-planes the function $z^{1+\beta}$ is Lipschitz-continuous on compact sets. Now, $A^c$ is bounded since $\sup_{s\le T} S_s|\phi|(y) \to 0$ as $|y| \to \infty$, due to (3.2) and Iscoe (1986), Corollary 2.4. Hence $A^c$ has compact closure. Therefore by (5.1.5), on $A^c$,

$$\begin{aligned}|u_\varepsilon^{1+\beta}(s,y)-(iS_s\phi(y)\lambda)^{1+\beta}| &\le \text{const.}|u_\varepsilon(s,y)-iS_s\phi(y)\lambda|\\ &\le \text{const.}\ \varepsilon^k S_s|\phi|(y)\lambda^{1+\beta},\end{aligned}$$

consequently

$$\begin{aligned}&\varepsilon^k V\int_0^t ds\int_{R^d} dy\, 1_{A^c}(s,y)p_{t-s}(\cdot,y)|u_\varepsilon^{1+\beta}(s,y)-(iS_s\phi(y)\lambda)^{1+\beta}|\\ &\qquad\le \text{const.}\ \varepsilon^{2k}\lambda^{1+\beta}\int_0^t ds\int_{R^d} dy\, p_{t-s}(\cdot,y)S_s|\phi|(y),\end{aligned}$$

so

$$\text{(d)}\quad \begin{aligned}&\varepsilon^k V\int_0^t ds\int_{R^d} dy\, 1_{A^c}(s,y)p_{t-s}(\cdot,y)|u_\varepsilon^{1+\beta}(s,y)-(iS_s\phi(y)\lambda)^{1+\beta}|\\ &\qquad\le \text{const.}\ \varepsilon^{2k}S_t|\phi|\lambda^{1+\beta}.\end{aligned}$$

Finally, using (5.1.1), (5.1.5)(a), (c) and (d), we have

$$\begin{aligned}&\sup_h\left|u_\varepsilon(t) - iS_t\phi\lambda + \varepsilon^k V\int_0^t S_{t-s}\left[h(\varepsilon^{-1}\cdot)(iS_s\phi\lambda)^{1+\beta}\right]ds\right|\\ &\le |S_t\{\varepsilon^{k-d}(1-e^{i\varepsilon^{d-k}\phi\lambda})\} - iS_t\phi\lambda|\\ &\quad + \varepsilon^k V\int_0^t ds\int_{R^d} dy\, 1_A(s,y)p_{t-s}(\cdot,y)|(iS_s\phi(y)\lambda)^{1+\beta} - u_\varepsilon^{1+\beta}(s,y)|\\ &\quad + \varepsilon^k V\int_0^t ds\int_{R^d} dy\, 1_{A^c}(s,y)p_{t-s}(\cdot,y)|(iS_s\phi(y)\lambda)^{1+\beta} - u_\varepsilon^{1+\beta}(s,y)|\\ &\le \text{const.}(\varepsilon^{d-k}+\varepsilon^{k(1+\beta)}+\varepsilon^{2k})S_t|\phi|\lambda^{1+\beta}\\ &\le \text{const.}\ \varepsilon^{\gamma k}S_t|\phi|\lambda^{1+\beta},\end{aligned}$$

where $\gamma = \min\{(d-k)/k, 1+\beta, 2\}$; $\gamma > 1$ because $d-k > k$ and $\beta > 0$. □

REMARK. The randomness in (5.1.1) is due to $h$. The only fact which we used about $h$ is that it is bounded by 1, and therefore the previous results hold for all realizations of $h$ in $H_0$.

5.2. *Properties of the process N.* We will derive here some properties we need of the process $N^\varepsilon \equiv N \equiv \{N_t, t \ge 0\}$ under $P_\varepsilon^h$ ($h$ fixed). We denote by $E_h$ the expectation with respect to $P_\varepsilon^h$.

(5.2.1) LEMMA. *For $0 \le s < t$, $\phi \in C_p(R^d)$ and $\lambda \in R$,*

(i) $$E_h[\exp\{-i\lambda\langle N_t, \phi\rangle\}|N_s] = \exp\{\langle N_s, \log[1 - w(t-s)]\rangle\},$$

*where $w(t) \equiv w(t, \phi\lambda, x\}$ satisfies the equation*

(ii) $$w(t) = S_t[1 - e^{-i\phi\lambda}] - V\int_0^t S_{t-r}[hw^{1+\beta}(r)]\,dr, \qquad t \ge 0,$$

*and $w(t, 0, x) = 0$. Moreover,*

(iii) $$E_h \exp\{-i\lambda\langle N_t, \phi\rangle\} = E_h \exp\{\langle N_0, \log[1 - w(t)]\rangle\}.$$

PROOF. By (3.3),

(a) $$v(t) \equiv v(t, \phi\lambda, x) := E_h[\exp\{-i\langle N_t, \phi\lambda\rangle|N_0 = \delta_x\}], \qquad t \ge 0,\ x \in R^d,$$

satisfies

$$v(t) = e^{-Vt}S_t e^{-i\phi\lambda} + V\int_0^t e^{-V(t-s)}S_{t-s}g^h(v(s))\,ds, \qquad t \ge 0.$$

The generator of the semigroup $\{e^{-Vt}S_t, t \ge 0\}$ is $\Delta_\alpha - VI$, where $I$ is the identity operator. Hence $v$ also satisfies the following variation-of-constants equation with respect to the semigroup $S_t$ [see, e.g., Pazy (1983)]:

$$v(t) = S_t e^{-i\phi\lambda} + V\int_0^t S_{t-s}[g^h(v(s)) - v(s)]\,ds, \qquad t \ge 0.$$

By (2.1), $g^h(v(s)) - v(s) = h(1 - v(s))^{1+\beta}$; then setting

(b) $$w(t) \equiv w(t, \phi\lambda, x) := 1 - v(t), \qquad t \ge 0,\ x \in R^d,$$

we see that $w$ satisfies (ii) and $w(t, 0, x) = 0$.

Using the multiplicative property (3.4), the time homogeneity of $N$, (a) and (b), we have

$$\begin{aligned} &E_h[\exp\{-i\lambda\langle N_t, \phi\rangle\}|N_s] \\ &\quad = \exp\{\langle N_s, \log E_h[\exp\{-i\lambda\langle N_t, \phi\rangle\}|N_s = \delta_\cdot]\rangle\} \\ &\quad = \exp\{\langle N_s, \log[1 - w(t-s)]\rangle\}. \end{aligned}$$

Then (iii) follows by taking expectation. □

We have stated the previous lemma as if we already knew that $N_t$ takes values in $\mathscr{M}_p(R^d)$. In fact this will follow from the next results. For the moment we may assume the validity of the lemma for $\phi \in C_c(R^d)$. Moreover, the assumption $E\langle N_0, \phi_p\rangle < \infty$ implies $E_h\langle N_0, \phi_p\rangle < \infty$ for $m$-almost all $h$ [see (4.2)]; we may assume for simplicity that this holds for all $h$.

(5.2.2) LEMMA. *For* $0 \le s < t$ *and* $\phi \in C_p(R^d)$,

(i) $$E_h[\langle N_t, \phi\rangle | N_s] = \langle N_s, S_{t-s}\phi\rangle$$

*and*

(ii) $$E_h\langle N_t, \phi\rangle = E_h\langle N_0, S_t\phi\rangle < \infty.$$

PROOF. By (5.2.1)(i),

$$\begin{aligned} E_h[\langle N_t, \phi\rangle | N_s] &= i\partial/\partial\lambda\big(\exp\{\langle N_s, \log[1 - w(t-s)]\rangle\}\big)|_{\lambda=0} \\ &= i\langle N_s, -\partial/\partial\lambda(w(t-s))\rangle|_{\lambda=0}, \end{aligned}$$

because $w(t,0,x) = 0$, and from (5.2.1)(ii) we find $\partial/\partial\lambda(w(t-s))|_{\lambda=0} = iS_{t-s}\phi$, because $\beta > 0$. This yields (i), and (ii) follows from (i) by taking expectation. The results hold for $\phi \in C_p(R^d)$ due to (3.2) and the assumption $E_h\langle N_0, \phi_p\rangle < \infty$. □

Since $E_h\langle N_t, \phi_p\rangle < \infty$, then $\langle N_t, \phi_p\rangle < \infty$ a.s. Hence for each $t \ge 0$, $E_hN_t$ and $N_t$ take values in $\mathscr{M}_p(R^d)$ a.s. Since $\mathscr{M}_p(R^d) \subset \mathscr{S}_n'(R^d)$ for some $n \ge 1$, then for each $t \ge 0$, $N_t$ and $N_t - E_hN_t$ take values in $\mathscr{S}_n'(R^d)$ a.s. Moreover, we have the following result.

(5.2.3) LEMMA. *The processes* $N$ *and* $N - E_hN$ *can be realized in* $D_{\mathscr{S}_n'(R^d)}$ *for some* $n \ge 1$, *and they have the strong Markov property* (*under* $P_\varepsilon^h$).

PROOF. We assume the filtration generated by $N$ is completed and made right-continuous in the usual way. The result for $N$ follows by an application of Theorem 9.4 in Chapter 1 of Blumenthal and Getoor (1968) to the $\mathscr{M}_p(R^d)$-valued Markov process $N$. However, since $\mathscr{M}_p(R^d)$ is not locally compact (in the $p$-vague topology), in order to apply this theorem we embed $\mathscr{M}_p(R^d)$ in the locally compact space $\mathscr{M}_p(\dot{R}^d)$ introduced by Iscoe (1986).

Let $P_t(\mu, d\nu)$ denote the transition probability of the time-homogeneous Markov process $N$ in the medium $h$. For $\Phi \in C(\mathscr{M}_p(\dot{R}^d))$ we define

$$T_t\Phi(\mu) = \int_{\mathscr{N}_p(R^d)} \Phi(\nu)P_t(\mu, d\nu), \qquad t \ge 0,\ \mu \in \mathscr{N}_p(R^d).$$

As in Iscoe (1986) this definition can be extended to $\mu \in \mathscr{N}_p(\dot{R}^d)$, and it can be shown that $T_t$: $C_0(\mathscr{N}_p(\dot{R}^d)) \to C_0(\mathscr{N}_p(\dot{R}^d))$, and $T_t\Phi \to \Phi$ as $t \to 0$ uniformly on $\mathscr{N}_p(\dot{R}^d)$ for each $\Phi \in C_0(\mathscr{N}_p(\dot{R}^d))$. Then the quoted theorem applies, so that $N$ can be realized in $D_{\mathscr{N}_p(\dot{R}^d)}$ and has the strong Markov property. As in Iscoe (1986) it can be shown that $N$ is $\mathscr{N}_p(R^d)$-valued. Then from the results in Section 2, $N$ can also be realized in $D_{\mathscr{S}_n'(R^d)}$ for some $n \ge 1$ and the strong Markov property is maintained in this space.

The result for $N - E_hN$ now follows from the fact that by (5.2.2)(ii) and (3.3) $t \to E_hN_t = S_tE_hN_0$ is $p$-vaguely continuous. □

(5.2.4) LEMMA. *Let* $\phi \in C_p(R^d)$ *and assume* $\phi$ *is in the domain of* $\Delta_\alpha$. *Then the process*

$$\langle N_t, \phi\rangle - \int_0^t \langle N_r, \Delta_\alpha\phi\rangle\, dr, \qquad t \ge 0,$$

*is a $P_\varepsilon^h$-martingale with respect to the filtration* $\mathscr{F}_s = \sigma\{\langle N_r, \psi\rangle : r \le s, \psi \in \mathscr{S}(R^d)\}$, $s \ge 0$.

PROOF. Integrability follows from the previous results. The integral in $t$ exists because of (5.2.3). Let $s < t$. Using the Markov property of $N$, (5.2.2)(i) and $d/dt(S_t) = S_t\Delta_\alpha$, we have

$$\begin{aligned}
&E_h\left[\langle N_t, \phi\rangle - \int_0^t \langle N_r, \Delta_\alpha\phi\rangle\, dr | \mathscr{F}_s\right]\\
&\quad= E_h[\langle N_t, \phi\rangle | N_s] - \int_0^s \langle N_r, \Delta_\alpha\phi\rangle\, dr - \int_s^t E_h[\langle N_r, \Delta_\alpha\phi\rangle | N_s]\, dr\\
&\quad= \langle N_s, S_{t-s}\phi\rangle - \int_0^s \langle N_r, \Delta_\alpha\phi\rangle\, dr - \left\langle N_s, \int_s^t S_{r-s}\Delta_\alpha\phi\, dr\right\rangle\\
&\quad= \langle N_s, \phi\rangle - \int_0^s \langle N_r, \Delta_\alpha\phi\rangle\, dr. \qquad \square
\end{aligned}$$

Replacing $N$ by $N - E_hN$ in (5.2.4) also yields a martingale. Note that the assumptions in (5.2.4) hold for $\phi \in \mathscr{S}(R^d)$.

5.3. *Convergence of finite-dimensional distributions.*

(5.3.1) PROPOSITION. *If* $\langle Y_0^\varepsilon, \phi\rangle \Rightarrow Y_0(\phi)$ *under* $P_\varepsilon^m$ *as* $\varepsilon \to 0$ *for each* $\phi \in C_p(R^d)$, *then*

$$\left(\langle Y_{t_1}^\varepsilon, \phi_1\rangle, \dots, \langle Y_{t_n}^\varepsilon, \phi_n\rangle\right) \Rightarrow \left(Y_{t_1}(\phi_1), \dots, Y_{t_n}(\phi_n)\right)$$

*under* $P_\varepsilon^m$ *as* $\varepsilon \to 0$ *for all* $n \ge 1$, $0 \le t_1 < \cdots < t_n$ *and* $\phi_1, \dots, \phi_n \in C_p(R^d)$, *where* $Y_{t_k}(\phi_k)$, $k = 0, 1, \dots, n$, *denote the limit real random variables* ($t_0 = 0$).

PROOF. It suffices to prove that

$$\text{(a)}\qquad E_\varepsilon \exp\left\{-i\sum_{j=0}^k \langle Y_{t_j}^\varepsilon, \phi_j\rangle\right\} \to E\exp\left\{-i\sum_{j=0}^k Y_{t_j}(\phi_j)\right\} \quad \text{as } \varepsilon \to 0$$

for all $k \ge 0$, $0 = t_0 < t_1 < \cdots < t_k$ and $\phi_0, \phi_1, \dots, \phi_k \in C_p(R^d)$, where $E_\varepsilon$ denotes expectation under $P_\varepsilon^m$ and $E$ expectation under the limit distribution.

For $k = 0$ (a) holds by hypothesis. Assuming (a) is true for $k = n - 1$, we will show it holds for $k = n$.

Using the Markov property of $N$ under $P_\varepsilon^h$, by (5.2.1) with $\lambda = 1$, we have [with the notation of (5.2.1), $N \equiv N^\varepsilon$]

$$\begin{aligned}
&E_h \exp\left\{-i\sum_{j=0}^n \langle N_{t_j}, \phi_j\rangle\right\}\\
&\quad= E_h\left(\exp\left\{-i\sum_{j=0}^{n-1}\langle N_{t_j}, \phi_j\rangle\right\} E_h\left[\exp\{-i\langle N_{t_n}, \phi_n\rangle\} | N_{t_{n-1}}\right]\right)\\
&\quad= E_h\left(\exp\left\{-i\sum_{j=0}^{n-1}\langle N_{t_j}, \phi_j\rangle + \langle N_{t_{n-1}}, \log[1 - w(t_n - t_{n-1}, \phi_n)]\rangle\right\}\right).
\end{aligned}$$

Hence, by (4.4),

$$E_h \exp\left\{-i\sum_{j=0}^{n}\langle \varepsilon^{-k}X^{\varepsilon}_{t_j}, \phi_j\rangle\right\}$$

$$= E_h \exp\left\{-i\sum_{j=0}^{n}\langle N_{\varepsilon^{-\alpha}t_j}, \varepsilon^{d-k}\phi_j(\varepsilon\,\cdot)\rangle\right\}$$

$$= E_h \exp\left\{-i\sum_{j=0}^{n-1}\langle N_{\varepsilon^{-\alpha}t_j}, \varepsilon^{d-k}\phi_j(\varepsilon\,\cdot)\rangle\right.$$

$$\left. + \left\langle N_{\varepsilon^{-\alpha}t_{n-1}}, \log\left[1 - w\left(\varepsilon^{-\alpha}(t_n - t_{n-1}), \varepsilon^{d-k}\phi_n(\varepsilon\,\cdot)\right)\right]\right\rangle\right\},$$

and therefore

$$E_h \exp\left\{-i\sum_{j=0}^{n}\langle \varepsilon^{-k}X^{\varepsilon}_{t_j}, \phi_j\rangle\right\}$$

$$= E_h \exp\left\{-i\sum_{j=0}^{n-1}\langle \varepsilon^{-k}X^{\varepsilon}_{t_j}, \phi_j\rangle\right.$$

$$\left. + \left\langle X^{\varepsilon}_{t_{n-1}}, \varepsilon^{-d}\log\left[1 - w\left(\varepsilon^{-\alpha}(t_n - t_{n-1}), \varepsilon^{d-k}\phi_n(\varepsilon\,\cdot), \varepsilon^{-1}\,\cdot\right)\right]\right\rangle\right\}.$$

Hence, denoting

$$\text{(b)} \qquad z(t) \equiv z(t,x) \equiv z(t,\phi_n,x) := w\left(\varepsilon^{-\alpha}t, \varepsilon^{d-k}\phi_n(\varepsilon\,\cdot), x\right),$$

we have

$$E_h \exp\left\{-i\sum_{j=0}^{n}\langle \varepsilon^{-k}X^{\varepsilon}_{t_j}, \phi_j\rangle\right\}$$

$$\text{(c)} \qquad = E_h \exp\left\{-i\sum_{j=0}^{n-1}\left\langle \varepsilon^{-k}X^{\varepsilon}_{t_j}, \phi_j\right\rangle\right.$$

$$\left. + \left\langle X^{\varepsilon}_{t_{n-1}}, \varepsilon^{-d}\log\left[1 - z\left(t_n - t_{n-1}, \varepsilon^{-1}\,\cdot\right)\right]\right\rangle\right\},$$

where $z$ satisfies, by (5.2.1)(ii),

$$z(t) = S_{\varepsilon^{-\alpha}t}\left[1 - \exp\{-i\varepsilon^{d-k}\phi_n(\varepsilon\,\cdot)\}\right] - V\int_0^{\varepsilon^{-\alpha}t} S_{\varepsilon^{-\alpha}t-s}\left[hz^{1+\beta}(\varepsilon^{\alpha}s)\right] ds$$

$$\text{(d)} \qquad = S_{\varepsilon^{-\alpha}t}\left[1 - \exp\{-i\varepsilon^{d-k}\phi_n(\varepsilon\,\cdot)\}\right]$$

$$-\varepsilon^{-\alpha}V\int_0^t S_{\varepsilon^{-\alpha}(t-r)}\left[hz^{1+\beta}(r)\right] dr, \qquad 0 \le t \le T.$$

Now, using the self-similarity of the stable density, that is,

$$p_t(x) = \varepsilon^{-d} p_{\varepsilon^{-\alpha}t}(\varepsilon^{-1}x), \qquad t > 0,\ x \in R^d,$$

which is a simple consequence of the stable characteristic function,

$$S_t f(x) = \left(S_{\varepsilon^{-\alpha}t} f(\varepsilon\,\cdot)\right)(\varepsilon^{-1}x);$$

hence by (d), $z$ satisfies

$$\text{(e)}\qquad \begin{aligned} z(t, \varepsilon^{-1}\,\cdot) &= S_t\left[1 - \exp\{-i\varepsilon^{d-k}\phi_n\}\right] \\ &\quad - \varepsilon^{-\alpha} V \int_0^t S_{t-s}\left[h(\varepsilon^{-1}\,\cdot) z^{1+\beta}(s, \varepsilon^{-1}\,\cdot)\right] ds, \qquad 0 \le t \le T. \end{aligned}$$

Letting

$$\text{(f)}\qquad u_\varepsilon(t) \equiv u_\varepsilon(t, \phi_n, x) := \varepsilon^{k-d} z(t, \phi_n, \varepsilon^{-1}x),$$

by (c) we have

$$\text{(g)}\qquad \begin{aligned} &E_h \exp\left\{-i \sum_{j=0}^{n} \langle \varepsilon^{-k} X_{t_j}^{\varepsilon}, \phi_j \rangle\right\} \\ &= E_h \exp\left\{-i \sum_{j=0}^{n-1} \langle \varepsilon^{-k} X_{t_j}^{\varepsilon}, \phi_j \rangle \right. \\ &\qquad \left. + \left\langle X_{t_{n-1}}^{\varepsilon}, \varepsilon^{-d} \log\left[1 - \varepsilon^{d-k} u_\varepsilon(t_n - t_{n-1})\right] \right\rangle \right\}, \end{aligned}$$

where, observing that $\varepsilon^{-\alpha+k-d+(d-k)(1+\beta)} = \varepsilon^k$, by (e) $u_\varepsilon$ satisfies

$$\text{(5.3.2)}\qquad \begin{aligned} u_\varepsilon(t) &= S_t\left[\varepsilon^{k-d}\left(1 - \exp\{-i\varepsilon^{d-k}\phi_n\}\right)\right] \\ &\quad - \varepsilon^k V \int_0^t S_{t-s}\left[h(\varepsilon^{-1}\,\cdot) u_\varepsilon^{1+\beta}(s)\right] ds, \qquad 0 \le t \le T. \end{aligned}$$

Note that this coincides with (5.1.1) if $\phi_n = \phi$ and $\lambda = 1$, and that (5.2.1)(a) and (5.2.1)(b), (b) and (f) imply the boundedness required of $u_\varepsilon$ in Section 5.1.

By (4.3), (g) and (4.2) we have

$$\text{(5.3.3)}\qquad \begin{aligned} C_\varepsilon &:= E_\varepsilon \exp\left\{-i \sum_{j=0}^{n} \langle Y_{t_j}^{\varepsilon}, \phi_j \rangle\right\} \\ &= E_\varepsilon \exp\left\{-i \sum_{j=0}^{n-1} \langle Y_{t_j}^{\varepsilon}, \phi_j \rangle + \langle X_{t_{n-1}}^{\varepsilon}, \varepsilon^{-d} \log\left[1 - \varepsilon^{d-k} u_\varepsilon(\tau)\right] \rangle \right. \\ &\qquad \left. + i\varepsilon^{-k} \langle \Lambda_{t_{n-1}}, S_\tau \phi_n \rangle \right\}, \end{aligned}$$

where we have denoted $\tau = t_n - t_{n-1}$ and $u_\varepsilon$ solves (5.3.2). This can be written as

$$\text{(h)}\qquad C_\varepsilon = E_\varepsilon \exp\{-iA_{1,\varepsilon} - iA_{2,\varepsilon} + G_\varepsilon - B_{1,\varepsilon} + B_{2,\varepsilon}\}$$

with

$$A_{1,\varepsilon} = \sum_{j=0}^{n-1} \langle Y^\varepsilon_{t_j}, \phi_j \rangle, \qquad A_{2,\varepsilon} = \langle Y^\varepsilon_{t_{n-1}}, S_\tau \phi_n \rangle,$$

$$G_\varepsilon = \left\langle X^\varepsilon_{t_{n-1}}, V \int_0^\tau S_{\tau-s}\left[h(\varepsilon^{-1}\cdot)(iS_s\phi_n)^{1+\beta}\right] ds \right\rangle,$$

$$B_{1,\varepsilon} = \varepsilon^{-k}\left\langle X^\varepsilon_{t_{n-1}}, u_\varepsilon(\tau) - iS_\tau\phi_n + \varepsilon^k V \int_0^\tau S_{\tau-s}\left[h(\varepsilon^{-1}\cdot)(iS_s\phi_n)^{1+\beta}\right] ds \right\rangle$$

and

$$B_{2,\varepsilon} = \langle X^\varepsilon_{t_{n-1}}, \varepsilon^{-d}\{\log[1 - \varepsilon^{d-k}u_\varepsilon(\tau)] + \varepsilon^{d-k}u_\varepsilon(\tau)\}\rangle.$$

We will show that

$$C_\varepsilon \to C := E\exp\{-iA_1 - iA_2 + G\} \quad \text{as } \varepsilon \to 0,$$

with

$$A_1 = \sum_{j=1}^{n-1} Y_{t_j}(\phi_j), \qquad A_2 = Y_{t_{n-1}}(S_\tau \phi_n)$$

and

$$G = \left\langle \Lambda_{t_{n-1}}, V\bar{h}\int_0^\tau S_{\tau-s}(iS_s\phi_n)^{1+\beta}\, ds \right\rangle,$$

where $E$ is the expectation with respect to the limit distribution. Note that $G$ is nonrandom.

The convergence statements below refer to $P^m_\varepsilon$. Suppose for the moment that we have already proved the following lemma.

(5.3.4) LEMMA. *$B_{1,\varepsilon}$ and $B_{2,\varepsilon}$ converge to 0 in probability as $\varepsilon \to 0$ and $G_\varepsilon$ converges to $G$ in probability as $\varepsilon \to 0$.*

Now, since $A_{1,\varepsilon}$ and $A_{2,\varepsilon}$ are real, $|\exp\{G_\varepsilon - B_{1,\varepsilon} + B_{2,\varepsilon}\}| = 1$, and hence $|\exp\{G_\varepsilon - G - B_{1,\varepsilon} + B_{2,\varepsilon}\}| \le e^{|G|}$, which is bounded. Therefore for any $\delta > 0$, by (h)

$$\begin{aligned} &|C_\varepsilon - E_\varepsilon \exp\{-iA_{1,\varepsilon} - iA_{2,\varepsilon} + G\}| \\ &\quad \le e^{|G|}E_\varepsilon|\exp\{G_\varepsilon - G - B_{1,\varepsilon} + B_{2,\varepsilon}\} - 1| \\ &\quad \le \text{const.}\left(P^m_\varepsilon[|G_\varepsilon - G - B_{1,\varepsilon} + B_{2,\varepsilon}| > \delta] + e^\delta - 1\right). \end{aligned}$$

Taking limits as $\varepsilon \to 0$, using (5.3.4) and then letting $\delta$ tend to 0, we have

$$C_\varepsilon - E_\varepsilon \exp\{-iA_{1,\varepsilon} - iA_{2,\varepsilon} + G\} \to 0 \quad \text{as } \varepsilon \to 0.$$

Then, to finish the proof of the proposition, it suffices to show that

$$E_\varepsilon \exp\{-iA_{1,\varepsilon} - iA_{2,\varepsilon} + G\} \to C \quad \text{as } \varepsilon \to 0,$$

but this follows from the induction hypothesis and Lemma 3.2.

PROOF OF LEMMA (5.3.4). By the induction hypothesis, $\langle Y^\varepsilon_{t_{n-1}}, \phi\rangle \Rightarrow Y_{t_{n-1}}(\phi)$ as $\varepsilon \to 0$, $\phi \in C_p(R^d)$. Hence

(i) $$\langle X^\varepsilon_{t_{n-1}}, \phi\rangle \Rightarrow \langle \Lambda_{t_{n-1}}, \phi\rangle \quad \text{as } \varepsilon \to 0,\ \phi \in C_p(R^d).$$

By Proposition (5.1.6), for $\varepsilon < \varepsilon_{T,1}$, we have

$$|B_{1,\varepsilon}| \le \text{const. } \varepsilon^{(\gamma-1)k}\langle X^\varepsilon_{t_{n-1}}, S_\tau|\phi_n|\rangle,$$

where $\gamma > 1$. Then, by (2.1) and (i), $B_{1,\varepsilon} \Rightarrow 0$ as $\varepsilon \to 0$.

Now we turn to $B_{2,\varepsilon}$. According to (5.1.2), $\varepsilon^{d-k}u_\varepsilon(\tau) \to 0$ as $\varepsilon \to 0$ uniformly in $x$ and $h$, since $d - k > 0$. Hence, for sufficiently small $\varepsilon < \varepsilon_{T,1}$,

$$|B_{2,\varepsilon}| \le \text{const.}\langle X^\varepsilon_{t_{n-1}}, \varepsilon^{-d}\big(\varepsilon^{d-k}|u_\varepsilon(\tau)|\big)^2\rangle,$$

and then, by (5.1.4),

$$|B_{2,\varepsilon}| \le \text{const. } \varepsilon^{d-2k}\langle X^\varepsilon_{t_{n-1}}, (S_\tau|\phi_n|)^2\rangle \le \text{const. } \varepsilon^{d-2k}\langle X^\varepsilon_{t_{n-1}}, S_\tau|\phi_n|\rangle.$$

Hence, since $d > 2k$, by (3.2) and (i), $B_{2,\varepsilon} \Rightarrow 0$ as $\varepsilon \to 0$.

In order to show that $G_\varepsilon \Rightarrow G$, suppose that we have already proved the following lemma.

(5.3.5) LEMMA. *For $\phi \in C_p(R^d)$,*

$$\Box\int_0^\tau S_{\tau-s}\big[h(\varepsilon^{-1}\cdot)(iS_s\phi)^{1+\beta}\big]\,ds - \bar{h}\int_0^\tau S_{\tau-s}(iS_s\phi)^{1+\beta}\,ds\Box_p \to 0 \quad \text{as } \varepsilon \to 0$$

*for $m$ a.e. $h$.*

[We will see in Lemma (5.3.6) that the integrals above belong to $C_{p,0}(R^d)$.]

Let

(j) $$g_\varepsilon = \int_0^\tau S_{\tau-s}\big[h(\varepsilon^{-1}\cdot)(iS_s\phi)^{1+\beta}\big]\,ds, \qquad g = \bar{h}\int_0^\tau S_{\tau-s}(iS_s\phi)^{1+\beta}\,ds.$$

By (i) we have $\langle X^\varepsilon_{t_{n-1}}, g\rangle \Rightarrow \langle \Lambda_{t_{n-1}}, g\rangle$ as $\varepsilon \to 0$. Therefore to show that $G_\varepsilon \Rightarrow G$, it suffices to prove that $\langle X^\varepsilon_{t_{n-1}}, g_\varepsilon - g\rangle \Rightarrow 0$ as $\varepsilon \to 0$. Now (see Section 3.1)

$$|\langle X^\varepsilon_{t_{n-1}}, g_\varepsilon - g\rangle| \le \Box X^\varepsilon_{t_{n-1}}\Box_{-p}\Box g_\varepsilon - g\Box_p,$$

and by (i) $\Box X^\varepsilon_{t_{n-1}}\Box_{-p} \Rightarrow \Box\Lambda_{t_{n-1}}\Box_{-p}$; hence the result follows from (5.3.5). □

Before proving Lemma (5.3.5) we need the following lemma.

(5.3.6) LEMMA. *For each $0 < \varepsilon < 1$, $g_\varepsilon$ and $g$ defined by* (j) *belong to* $C_{p,0}(R^d)$.

PROOF. $g$ is a special case of $g_\varepsilon$ with $h \equiv \bar{h}$. We will prove first the continuity of $g_\varepsilon$. Since $|iS_s\phi|^\beta \le \|S_s\phi\|^\beta_\infty \le \|\phi\|^\beta_\infty$, then

(k) $$|S_{\tau-s}h(\varepsilon^{-1}\cdot)(iS_s\phi)^{1+\beta}| \le \text{const.}\|\phi\|^{1+\beta}_\infty.$$

Let $f_s \equiv (iS_s\phi)^{1+\beta}$. For each $0 < \delta < \tau$ and $x, z \in R^d$, since $0 \le h \le 1$, by (k),

$$
\begin{aligned}
&|g_\varepsilon(x+z) - g_\varepsilon(x)| \\
(1) \qquad &\le \int_0^{\tau-\delta}\int_{R^d} |f_s(y)|\,|p_{\tau-s}(x+z, y) - p_{\tau-s}(x,y)|\,dy\,ds \\
&\quad + 2\delta\,\text{const.}\|\phi\|_\infty^{1+\beta}.
\end{aligned}
$$

The stable density $p_s(x)$ is uniformly continuous in $\{s\colon s \ge \delta\} \times R^d$. Hence for any such $\delta$ above there is an $\eta > 0$ such that if $|z| < \eta$ the first term on the right in (1) can be estimated by

$$
\begin{aligned}
\delta \int_0^{\tau-\delta}\int_{R^d} |f_s(y)|\,dy\,ds &\le \delta\,\text{const.}\int_0^{\tau}\int_{R^d} S_s|\phi|(y)\,dy\,ds \\
&= \delta\,\text{const.}\,\tau\int_{R^d}|\phi|(y)\,dy = \text{const.}\,\delta.
\end{aligned}
$$

Hence the continuity of $g_\varepsilon$ is obtained.

For all $0 \le s \le \tau$, by (k) we have

$$
\text{(m)} \quad |S_{\tau-s}h(\varepsilon^{-1}\cdot)(iS_s\phi)^{1+\beta}|/\phi_p \le \text{const.}(S_\tau|\phi|)/\phi_p \le \text{const.}\,\Box S_\tau|\phi|\Box_p,
$$

which is finite by (3.2).

By (3.2),

$$
|S_s\phi|^{1+\beta}/\phi_p \le |S_s\phi|^{\beta}|S_s\phi|/\phi_p \le \Box S_s\phi\Box_p|S_s\phi|^{\beta},
$$

and since $S_s\phi \in C_0(R^d)$, then $|S_s\phi|^{1+\beta} \in C_{p,0}(R^d)$; consequently

$$
\text{(n)} \qquad \lim_{|x|\to\infty} |x|^{2p}|S_s\phi(x)|^{1+\beta} = 0.
$$

Now, $|S_{\tau-s}h(\varepsilon^{-1}\cdot)(iS_s\phi)^{1+\beta}| \le S_{\tau-s}|S_s\phi|^{1+\beta}$; therefore by (n) and Iscoe (1986), Proposition 2.3,

$$
\lim_{|x|\to\infty} |x|^{2p}|S_{\tau-s}h(\varepsilon^{-1}\cdot)(iS_s\phi)^{1+\beta}(x)| = 0,
$$

so

$$
\text{(o)} \qquad \lim_{|x|\to\infty} |S_{\tau-s}h(\varepsilon^{-1}\cdot)(iS_s\phi)^{1+\beta}(x)|/\phi_p(x) = 0.
$$

The dominated convergence theorem can be applied due to (m), hence (o) implies $g_\varepsilon \in C_{p,0}(R^d)$. □

**Proof of Lemma** (5.3.5). It follows from the proof of (5.3.6) that $(1 + |x|^2)^p|g_\varepsilon(x) - g(x)|$ is uniformly continuous on $R^d$ uniformly in $\varepsilon$. Hence, to prove the lemma, it suffices to show that $(1 + |x|^2)^p|g_\varepsilon(x) - g(x)| \to 0$, $m$ a.e. for fixed $x \in R^d$, and therefore it is enough to prove that $g_\varepsilon(x) - g(x) \to 0$, $m$ a.e. for fixed $x$ (the exceptional set may depend on $x$, but only countably many $x$ are needed).

From (j) we have, with $f_s \equiv (iS_s\phi)^{1+\beta}$,

$$g_\varepsilon(x) - g(x) = \int_0^\tau \int_{R^d} (h(\varepsilon^{-1}y) - \bar{h}) f_s(y) p_{\tau-s}(y - x)\, dy\, ds$$
$$= \varepsilon^d \int_0^\tau \int_{R^d} (h(z) - \bar{h}) f_s(\varepsilon z) p_{\tau-s}(\varepsilon z - x)\, dz\, ds.$$

Hence

$$|g_\varepsilon(x) - g(x)| = \varepsilon^d \left| \int_{R^d} (h(z) - \bar{h}) \theta(\varepsilon z)\, dz \right|,$$

where

$$\theta(z) = \int_0^\tau p_s(z - x) f_s(z)\, ds.$$

Note that $\theta(z) \geq 0$ and $\int_{R^d} \theta(z)\, dz = \tau$.

Now,

$$\varepsilon^d \left| \int_{R^d} (h(z) - \bar{h}) \theta(\varepsilon z)\, dz \right| \leq A_\varepsilon + B_\varepsilon,$$

where

$$A_\varepsilon = \varepsilon^d \int_{R^d} |h(z) - \bar{h}| \theta(\varepsilon z) 1_{\{|\varepsilon z| > a\}}\, dz,$$
$$B_\varepsilon = \varepsilon^d \left| \int_{R^d} (h(z) - \bar{h}) \theta(\varepsilon z) 1_{\{|\varepsilon z| \leq a\}}\, dz \right|,$$

and $a > 0$ is arbitrary.

Given arbitrary $\delta > 0$, taking $a$ sufficiently large, we have

$$A_\varepsilon \leq 2 \int_{R^d} \theta(y) 1_{\{|y| \geq a\}}\, dy < \delta.$$

Define $F$: $H \to R$ by $F(h) = (h(0) - \bar{h})$. Hence $F(T_z h) = (h(z) - \bar{h})$, where $T_z$ is the translation by $z$ on $H$. Thus

$$B_\varepsilon = \varepsilon^d \int_{R^d} F(T_z h) \theta(\varepsilon z) 1_{\{|\varepsilon z| \leq a\}}\, dz.$$

The function $\theta(y) 1_{\{|y| \leq a\}}$ satisfies the conditions of Tempel'man's individual ergodic theorem [Tempel'man (1972), Corollary 7.2]. By this theorem,

$$B_\varepsilon \to \int_{R^d} \theta(y) 1_{\{|y| \leq a\}}\, dy\, \hat{F}(h)$$

as $\varepsilon \to 0$ for $m$ a.e. $h$. Since $m$ is ergodic with respect to $T_z$, then $\hat{F}$ is constant on $H$, and we have $\hat{F} \equiv 0$.

Since $\delta$ is arbitrary we have the desired result. □

The proof of Proposition (5.3.1) is finally complete. □

(5.3.7) **Corollary.** *The distributions of the limit random vectors* $(Y_{t_0}(\phi_0), Y_{t_1}(\phi_1), \dots, Y_{t_n}(\phi_n))$ *in Proposition* (5.3.1) *satisfy the relations*

$$E \exp\left\{ -i \sum_{j=0}^{n} Y_{t_j}(\phi_j) \right\}$$

$$\text{(i)} \qquad = E \exp\left\{ -i \left[ \sum_{j=0}^{n-1} Y_{t_j}(\phi_j) + Y_{t_{n-1}}(S_{t_n - t_{n-1}}\phi_n) \right] + \left\langle \Lambda_{t_{n-1}}, V\bar{h} \int_{t_{n-1}}^{t_n} S_{t_n - r}(iS_{r - t_{n-1}}\phi_n)^{1+\beta}\, dr \right\rangle \right\}$$

*and*

$$E\left[\exp\{-iY_{t_n}(\phi_n)\} \mid Y_{t_0}(\phi_0), \dots, Y_{t_{n-1}}(\phi_{n-1});\ \phi_0, \dots, \phi_{n-1} \in C_p(R^d)\right]$$

$$\text{(ii)} \qquad = \exp\left\{ -iY_{t_{n-1}}(S_{t_n - t_{n-1}}\phi_n) + \left\langle \Lambda_{t_{n-1}}, V\bar{h} \int_{t_{n-1}}^{t_n} S_{t_n - r}(iS_{r - t_{n-1}}\phi_n)^{1+\beta}\, dr \right\rangle \right\},$$

$$\phi_n \in C_p(R^d).$$

**Proof.** (i) follows directly from the argument before Lemma (5.3.4). (i) implies that $Y_{t_n}(\phi_n) - Y_{t_{n-1}}(S_{t_n - t_{n-1}}\phi_n)$ and $\{Y_{t_0}(\phi_0), \dots, Y_{t_{n-1}}(\phi_{n-1})\}$ are independent, which yields (ii). □

(5.3.8) **Remarks.** 1. Expression (i) in the proof is the law of large numbers for $N^\varepsilon$ (Corollary 4.8).

2. According to Proposition (5.3.1), the finite-dimensional distributions of $Y^\varepsilon$ converge if and only if the initial fluctuations converge.

3. For Proposition (5.3.1) the assumptions $E\langle N_0^\varepsilon, \phi_p \rangle < \infty$ and $E\langle X_0^\varepsilon, \phi^\varepsilon \rangle \to 0$ if $\phi^\varepsilon \to 0$ are not needed.

4. A weaker form of Lemma (5.3.5), namely mean square convergence, can be proved using a statistical ergodic theorem [Tempel'man (1972) and Fleischmann (1978)]. This suffices for the proof of assertion (a) of Theorem (4.5).

5. Once it is proved that $Y^\varepsilon \Rightarrow Y$ in $D_{\mathscr{S}'(R^d)}$ (Section 5.4), we can assert that the finite-dimensional distributions of the process $Y$ satisfy

$$\left(\langle Y_{t_0}, \phi_0 \rangle, \dots, \langle Y_{t_n}, \phi_n \rangle\right) = \left(Y_{t_0}(\phi_0), \dots, Y_{t_n}(\phi_n)\right)$$

in distribution for $0 = t_0 < t_1 < \cdots < t_n$, $\phi_0, \dots, \phi_n \in \mathscr{S}(R^d)$, and therefore we may write (5.3.7) (i) and (5.3.7)(ii) with $Y_{t_j}(\phi_j)$ replaced by $\langle Y_{t_j}, \phi_j \rangle$. Note that (5.3.7)(ii) implies the Markov property of $Y$, and therefore we have (4.6), where the meaning of $\langle Y_s, S_{t-s}\phi \rangle$ has been explained in Remark (4.7).

5.4. *Tightness and convergence in* $D_{\mathscr{S}'(R^d)}$ [*Proof of Theorem* (4.5)]. In this section we complete the proof that $Y^\varepsilon$ converges under $P_\varepsilon^m$ in $D_{\mathscr{S}'(R^d)}$ as $\varepsilon \to 0$ to an $\mathscr{S}'(R^d)$-valued process whose finite-dimensional distributions are given by (5.3.7). Since (5.3.1) yields the uniqueness of subsequential limits, it suffices to establish the tightness of $\{Y^{\varepsilon_n}\}_n$ in $D_{\mathscr{S}'(R^d)}$ for $\varepsilon_n \to 0$. By Mitoma (1983) it suffices to prove that the sequence $\{\langle Y^{\varepsilon_n}, \phi\rangle\}_n$ is tight in $D_R$ for each $\phi \in \mathscr{S}(R^d)$. According to the criterion of Aldous (1978), to prove the latter, it suffices to show that

(i) for each fixed $t$, $\{\langle Y_t^{\varepsilon_n}, \phi\rangle\}_n$ is tight in $R$,

(ii) given stopping times $\tau_n$ bounded by $T$ and $\delta_n \to 0$ as $n \to \infty$, then $\langle Y^{\varepsilon_n}_{\tau_n+\delta_n}, \phi\rangle - \langle Y^{\varepsilon_n}_{\tau_n}, \phi\rangle \to 0$ in probability as $n \to \infty$.

Condition (i) follows immediately from the convergence of finite-dimensional distributions [Proposition (5.3.1)]. Condition (ii) is established in Proposition (5.4.8) below.

We will recall first some general facts. Let $X$ be a random variable and $0 < \theta < \infty$. If $\int_K^\infty r^\theta P[|X| \geq r]\, dr < \infty$ for an arbitrary $0 < K < \infty$, then $E|X|^{1+\theta} < \infty$ and

$$E|X|^{1+\theta} \leq K^{1+\theta} + (1+\theta)\int_K^\infty r^\theta P[|X| \geq r]\, dr.$$

If $f_X$ denotes the characteristic function of $X$, then

$$P[|X| \geq r] \leq Cr\int_0^{1/r}[1 - \operatorname{Re} f_X(\lambda)]\, d\lambda, \tag{5.4.1}$$

where $C$ is a constant [see, e.g., Breiman (1968)]. Therefore in order to show that $E|X|^{1+\theta} < \infty$, it suffices to prove that

$$\int_K^\infty r^{1+\theta}\int_0^{1/r}[1 - \operatorname{Re} f_X(\lambda)]\, d\lambda\, dr < \infty \tag{5.4.2}$$

for some $K$, and combining (5.4.1) and (5.4.2), we have

$$E|X|^{1+\theta} \leq K^{1+\theta} + C(1+\theta)\int_K^\infty r^{1+\theta}\int_0^{1/r}[1 - \operatorname{Re} f_X(\lambda)]\, d\lambda\, dr \tag{5.4.3}$$

for arbitrary $K$. These results hold also for conditional expectation (with conditional characteristic function).

In order to apply Aldous' tightness criterion we need the following lemmas. The constants $T$, $L$ and $\varepsilon_{T,L}$ are those of Section 5.1.

(5.4.4) LEMMA. *For $\phi \in C_p(R^d)$, $\varepsilon < \varepsilon_{T,L}$, $0 < t \leq T$, $K \geq 1/L$, $0 < \theta < \beta$ and $h$ fixed,*

(i)
$$E_h\left[|\langle Y_t^\varepsilon, \phi\rangle|^{1+\theta}|X_0^\varepsilon\right] \leq K^{1+\theta} + \text{const.}\, K^{\theta-\beta}\left[\langle X_0^\varepsilon, S_t|\phi|\rangle + H_t^\varepsilon(\phi)\right],$$

*where*

(ii)
$$H_t^\varepsilon := \sup_{\lambda\geq 0}\lambda^{-(1+\beta)}(1 - \cos\langle Y_0^\varepsilon, S_t\phi\rangle\lambda) = K_\beta|\langle Y_0^\varepsilon, S_t\phi\rangle|^{1+\beta},$$

*$K_\beta$ is a constant depending on $\beta$, and*

(iii)
$$E_h\left[|\langle X_t^\varepsilon, \phi\rangle|^{1+\theta}|X_0^\varepsilon\right] \leq K^{1+\theta} + \text{const.}\, K^{\theta-\beta}\left[\varepsilon^k\langle X_0^\varepsilon, S_t|\phi|\rangle + J_t^\varepsilon(\varepsilon^k\phi)\right],$$

*where*

$$\text{(iv)}\qquad \begin{aligned} J_t^\varepsilon(\psi) &:= \sup_{\lambda\ge 0}\lambda^{-(1+\beta)}\big[1-\cos\big(\langle Y_0^\varepsilon,\psi\rangle+\langle\Lambda_t,\varepsilon^{-k}\psi\rangle\big)\lambda\big]\\ &= K_\beta|\langle Y_0^\varepsilon,\psi\rangle+\langle\Lambda_t,\varepsilon^{-k}\psi\rangle|^{1+\beta},\qquad \psi\in C_p(R^d). \end{aligned}$$

PROOF. Let $f(\lambda)=E_h[\exp\{i\lambda\langle Y_t^\varepsilon,\phi\rangle\}|X_0^\varepsilon]$, $\lambda\ge 0$. Then, using (5.2.1) and proceeding as in Section 5.3 [see (5.3.2) and (5.3.3)], we have

$$\text{(a)}\quad f(\lambda)=\exp\big\{\langle X_0^\varepsilon,\varepsilon^{-d}\log\big[1-\varepsilon^{d-k}u_\varepsilon(t,-\phi\lambda)\big]\rangle-i\lambda\langle\varepsilon^{-k}\Lambda_0,S_t\phi\rangle\big\},$$

where $u_\varepsilon(t)\equiv u_\varepsilon(t,\psi)$, $\psi\in C_p(R^d)$, satisfies

$$\text{(b)}\qquad \begin{aligned} u_\varepsilon(t)&=S_t\big[\varepsilon^{k-d}\big(1-\exp\{-i\varepsilon^{d-k}\psi\}\big)\big]\\ &\quad-\varepsilon^kV\int_0^t S_{t-s}\big[h(\varepsilon^{-1}\cdot)u_\varepsilon^{1+\beta}(s)\big]\,ds. \end{aligned}$$

We can write (a) as

$$\text{(c)}\qquad f(\lambda)=\exp\{\langle X_0^\varepsilon,A_\varepsilon(\lambda)+B_\varepsilon(\lambda)+C_\varepsilon(\lambda)\rangle\}\exp\{iD_\varepsilon\lambda\},$$

where

$$A_\varepsilon(\lambda)=\varepsilon^{-d}\big[\log\{1-\varepsilon^{d-k}u_\varepsilon(t,-\phi\lambda)\}+\varepsilon^{d-k}u_\varepsilon(t,-\phi\lambda)\big],$$

$$B_\varepsilon(\lambda)=-\varepsilon^{-k}\Big\{u_\varepsilon(t,-\phi\lambda)-iS_t(-\phi\lambda)+\varepsilon^kV\int_0^tS_{t-s}\big[h(\varepsilon^{-1}\cdot)(-iS_s\phi)^{1+\beta}\big]\,ds\Big\},$$

$$C_\varepsilon(\lambda)=V\int_0^tS_{t-s}\big[h(\varepsilon^{-1}\cdot)(-iS_s\phi\lambda)^{1+\beta}\big]\,ds,$$

$$D_\varepsilon=\langle Y_0^\varepsilon,S_t\phi\rangle.$$

Hence $f(\lambda)=(1+Z)(1+W)=1+Z(1+W)+W$, where

$$Z=\exp\{\langle X_0^\varepsilon,A_\varepsilon(\lambda)+B_\varepsilon(\lambda)+C_\varepsilon(\lambda)\rangle\}-1,\qquad W=\exp\{iD_\varepsilon\lambda\}-1,$$

and therefore

$$\text{(d)}\qquad \begin{aligned} 1-\operatorname{Re}f(\lambda)&=-\operatorname{Re}Z(1+W)-\operatorname{Re}W\\ &\le|Z|\,|1+W|-\operatorname{Re}W\le|Z|-\operatorname{Re}W. \end{aligned}$$

We will now estimate $|Z|$ and $-\operatorname{Re}W$.

By (a) and (c), $|\exp\{\langle X_0^\varepsilon,A_\varepsilon(\lambda)+B_\varepsilon(\lambda)+C_\varepsilon(\lambda)\rangle\}|\le 1$. If $\xi$ is a complex number such that $|e^\xi|\le 1$, then $|e^\xi-1|\le 2|\xi|$. Hence

$$\text{(e)}\qquad |Z|\le 2\langle X_0^\varepsilon,|A_\varepsilon(\lambda)|+|B_\varepsilon(\lambda)|+|C_\varepsilon(\lambda)|\rangle.$$

Now, by (5.1.2) and (5.1.4),

$$\begin{aligned} \langle X_0^\varepsilon,|A_\varepsilon(\lambda)|\rangle&\le\text{const.}\,\varepsilon^{-d}\varepsilon^{2(d-k)}\langle X_0^\varepsilon,|u_\varepsilon(t,-\phi\lambda)|^2\rangle\\ &\le\text{const.}\,\varepsilon^{d-2k}\langle X_0^\varepsilon,(S_t|\phi|\lambda)^2\rangle\le\text{const.}\,\varepsilon^{d-2k}\langle X_0^\varepsilon,S_t|\phi|\rangle\lambda^{1+\beta}, \end{aligned}$$

with $\lambda\le L$; recall that $d>2k$. By (5.1.6),

$$\langle X_0^\varepsilon,|B_\varepsilon(\lambda)|\rangle\le\text{const.}\,\varepsilon^{(\gamma-1)k}\langle X_0^\varepsilon,S_t|\phi|\rangle\lambda^{1+\beta},$$

where $\gamma > 1$. Finally,

$$\langle X_0^\varepsilon, |C_\varepsilon(\lambda)|\rangle \le \left\langle X_0^\varepsilon, V\int_0^t S_{t-s}|S_s\phi|^{1+\beta}\,ds\right\rangle\lambda^{1+\beta}$$
$$\le \text{const.}\|\phi\|_\infty^\beta\langle X_0^\varepsilon, S_t|\phi|\rangle\lambda^{1+\beta}.$$

Putting these results into (e), we obtain

$$|Z| \le \text{const.}\langle X_0^\varepsilon, S_t|\phi|\rangle\lambda^{1+\beta} \quad \text{with } \lambda \le L.$$

For $\operatorname{Re} W$ we have

$$-\operatorname{Re} W = -\operatorname{Re}\exp\{iD_\varepsilon\lambda\} + 1$$
$$= -\cos D_\varepsilon\lambda + 1 = \lambda^{1+\beta}(1-\cos D_\varepsilon\lambda)/\lambda^{1+\beta} \le H_t^\varepsilon(\phi)\lambda^{1+\beta},$$

where $H_t^\varepsilon(\phi)$ is given by (ii).

Taking these inequalities into (d), we obtain

$$1 - \operatorname{Re} f(\lambda) \le \text{const.}[\langle X_0^\varepsilon, S_t|\phi|\rangle + H_t^\varepsilon(\phi)]\lambda^{1+\beta}.$$

Therefore, by (5.4.3), for $0 \le \theta < \beta$ and $K \ge 1/L$ (so that $\lambda \le L$) we have

$$E_h\left[|\langle Y_t^\varepsilon, \phi\rangle|^{1+\theta}|X_0^\varepsilon\right]$$
$$\le K^{1+\theta} + \text{const.}[\langle X_0^\varepsilon, S_t|\phi|\rangle + H_t^\varepsilon(\phi)]\int_K^\infty r^{1+\theta}\int_0^{1/r}\lambda^{1+\beta}\,d\lambda\,dr$$
$$= K^{1+\theta} + \text{const.}\,K^{\theta-\beta}[\langle X_0^\varepsilon, S_t|\phi|\rangle + H_t^\varepsilon(\phi)],$$

which proves (i).

(iii) and (iv) are obtained similarly to (i) and (ii), using

$$E\left[\exp\{i\langle X_t^\varepsilon, \phi\rangle\}|X_0^\varepsilon\right] = E\left[\exp\{i(\langle Y_t^\varepsilon, \varepsilon^k\phi\rangle + \lambda\langle \Lambda_t, \phi\rangle)\}|X_0^\varepsilon\right]. \qquad \square$$

(5.4.5) Lemma. *Let $\phi^\varepsilon \in C_p(R^d)$ and assume $\phi^\varepsilon \to 0$ as $\varepsilon \to 0$. Then for $0 < \theta < \beta$ the following limits hold in $L^1(H, m)$:*

(i) $\sup_{0\le t\le T} E_h\left[|\langle Y_t^\varepsilon, \phi^\varepsilon\rangle|^{1+\theta}|X_0^\varepsilon\right] \to 0$ *in probability* $(P_\varepsilon^h)$ *as* $\varepsilon \to 0$

*and*

(ii) $\sup_{0\le t\le T} E_h\left[|\langle X_t^\varepsilon, \phi^\varepsilon\rangle|^{1+\theta}|X_0^\varepsilon\right] \to 0$ *in probability* $(P_\varepsilon^h)$ *as* $\varepsilon \to 0$.

Proof. Let $\delta > 0$. By (5.4.4)(i) and (5.4.4)(ii), (3.2) and (4.1),

$$\int_H P_\varepsilon^h\left[\sup_{t\le T} E_h\left[|\langle Y_t^\varepsilon, \phi^\varepsilon\rangle|^{1+\theta}|X_0^\varepsilon\right] > \delta\right]m(dh)$$
$$\le \int_H P_\varepsilon^h\left\{K^{1+\theta} + \text{const.}\,K^{\theta-\beta}\left[\langle X_0^\varepsilon, \sup_{t\le T} S_t|\phi^\varepsilon|\rangle + \sup_{t\le T} H_t^\varepsilon(\phi^\varepsilon)\right] > \delta\right\}m(dh)$$
$$\le 1_{\{K^{1+\theta} > \delta_1\}} + P_\varepsilon^m\left[K^{\theta-\beta}\langle X_0^\varepsilon, |\phi^\varepsilon|\rangle > \delta_2\right]$$
$$+ P_\varepsilon^m\left[K^{\theta-\beta}\sup_{t\le T} H_t^\varepsilon(\phi^\varepsilon) > \delta_3\right],$$

where $\delta_1$, $\delta_2$ and $\delta_3$ are some positive constants. Now,

$$\langle X_0^\varepsilon, |\phi^\varepsilon| \rangle \le \square X_0^\varepsilon \square_{-p} \square \phi^\varepsilon \square_p, \qquad \square X_0^\varepsilon \square_{-p} \Rightarrow \square \Lambda_0 \square_{-p} \quad \text{and} \quad \square \phi^\varepsilon \square_p \to 0$$

by assumption; hence $P_\varepsilon^m[K^{\theta-\beta}\langle X_0^\varepsilon, |\phi^\varepsilon|\rangle > \delta_2] \to 0$. Since

$$\sup_{t \le T} H_t^\varepsilon(\phi^\varepsilon) \le \text{const.} \square Y_0^\varepsilon \square_{-p}^{1+\beta} \square \phi^\varepsilon \square_p^{1+\beta} \quad \text{and} \quad \square Y_0^\varepsilon \square_{-p} \Rightarrow \square Y_0 \square_{-p},$$

then

$$P_\varepsilon^m\left[K^{\theta-\beta} \sup_{t \le T} H_t^\varepsilon(\phi^\varepsilon) > \delta_3\right] \to 0.$$

Hence assertion (i) will be proved if we can let $K \to 0$.

Recall that the results we have used were proved for fixed $\phi \in C_p(R^d)$,

$$\text{(a)} \quad \varepsilon < 1/\big(V\beta T \|\phi\|_\infty^\beta L^\beta\big)^{1/k} \quad \text{and} \quad \varepsilon_{T,L} < \text{const.}/\big(VT\|\phi\|_\infty^\beta L^\beta\big)^{1/k}$$

[see (5.1.3) and (5.1.4)], and $K \ge 1/L$. If we now let $\phi = \phi^\varepsilon$ depend on $\varepsilon$, we may also let $L = L_\varepsilon$ depend on $\varepsilon$ in such a way that conditions (a) are not altered. For example, we may define $L_\varepsilon = \|\phi^\varepsilon\|_\infty^{-1}$. Now, since $\phi^\varepsilon \to 0$ in $C_p(R^d)$ implies $\|\phi^\varepsilon\|_\infty \to 0$, then $L_\varepsilon \to \infty$ and therefore we may let $K \to 0$ as desired.

Assertion (ii) is proved similarly, using (5.4.4)(iii) and (5.4.4)(iv). □

We will need the following result, which can be proved from (4.2): If $\mathscr{G}$ is a sub-$\sigma$-algebra of the basic $\sigma$-algebra, then for any $A \in \mathscr{G}$,

$$\text{(5.4.6)} \qquad \int_A E[\,\cdot\,|\mathscr{G}]\, dP = \int_H \int_A E_h[\,\cdot\,|\mathscr{G}]\, dP_\varepsilon^h m(dh).$$

(5.4.7) **Lemma.** *Let $\phi^\varepsilon \in C_p(R^d)$ and assume that $\phi^\varepsilon$ is in the domain of $\Delta_\alpha$, $\Delta_\alpha\phi^\varepsilon \in C_p(R^d)$, $\phi^\varepsilon \to 0$ and $\Delta_\alpha\phi^\varepsilon \to 0$ as $\varepsilon \to 0$. Then for $T > 0$ and $0 < \theta < \beta$,*

$$\text{(i)} \quad E\left[\sup_{0 \le t \le T} |\langle Y_t^\varepsilon, \phi^\varepsilon\rangle|^{1+\theta} \Big| X_0^\varepsilon\right] \to 0 \quad \textit{in probability } (P_\varepsilon^m) \textit{ as } \varepsilon \to 0$$

*and*

$$\text{(ii)} \quad E\left[\sup_{0 \le t \le T} |\langle X_t^\varepsilon, \phi^\varepsilon\rangle|^{1+\theta} \Big| X_0^\varepsilon\right] \to 0 \quad \textit{in probability } (P_\varepsilon^m) \textit{ as } \varepsilon \to 0.$$

**Proof.** First we will estimate $E[\sup_{0 \le t \le T}|\langle Y_t^\varepsilon, \phi\rangle|^{1+\theta}|X_0^\varepsilon]$, $\phi \in \mathscr{S}(R^d)$. We start by showing that

$$\text{(a)} \qquad \langle Y_t^\varepsilon, \phi\rangle - \int_0^t \langle Y_s^\varepsilon, \Delta_\alpha \phi\rangle\, ds, \qquad t \ge 0,$$

is a $P_\varepsilon^m$-martingale. By (4.3) and (4.4),

$$\begin{aligned}
\langle Y_t^\varepsilon, \phi\rangle &= \langle \varepsilon^{d-k} N_{\varepsilon^{-\alpha}t}, \phi(\varepsilon\,\cdot)\rangle - \langle \varepsilon^{-k}\Lambda_0, S_t\phi\rangle \\
&= \varepsilon^{d-k}\left[\langle N_{\varepsilon^{-\alpha}t}, \phi(\varepsilon\,\cdot)\rangle - \int_0^{\varepsilon^{-\alpha}t} \langle N_s, (\Delta_\alpha)\phi(\varepsilon\,\cdot)\rangle\, ds\right] \\
&\quad + \varepsilon^{d-k}\int_0^{\varepsilon^{-\alpha}t} \langle N_s, (\Delta_\alpha)\phi(\varepsilon\,\cdot)\rangle\, ds - \left\langle \varepsilon^{-k}\Lambda_0, \int_0^t S_s \Delta_\alpha \phi\, ds\right\rangle - \langle \varepsilon^{-k}\Lambda_0, \phi\rangle,
\end{aligned}$$

where we have used $d/ds(S_s) = S_s \Delta_\alpha$. Since the term in brackets is a $P_\varepsilon^h$-martingale for fixed $h$ because of (5.2.4), and the last term is independent of $t$, we have that the difference of the remaining terms constitutes a $P_\varepsilon^h$-martingale. Since $\varepsilon^{-\alpha}(\Delta_\alpha)\phi(\varepsilon\,\cdot\,) = (\Delta_\alpha\phi)(\varepsilon\,\cdot\,)$ [this follows from the self-similarity of $p_t(x)$], then

$$\begin{aligned}
&\varepsilon^{d-k}\int_0^{\varepsilon^{-\alpha}t}\langle N_s, (\Delta_\alpha)\phi(\varepsilon\,\cdot)\rangle ds - \left\langle \varepsilon^{-k}\Lambda_0, \int_0^t S_s\Delta_\alpha\phi\, ds\right\rangle \\
&\quad = \varepsilon^{d-k}\int_0^t \langle N_{\varepsilon^{-\alpha}s}, (\Delta_\alpha\phi)(\varepsilon\,\cdot)\rangle\, ds - \left\langle \varepsilon^{-k}\Lambda_0, \int_0^t S_s\Delta_\alpha\phi\, ds\right\rangle \\
&\quad = \int_0^t \langle Y_s^\varepsilon, \Delta_\alpha\phi\rangle\, ds,
\end{aligned}$$

and therefore (a) is a $P_\varepsilon^h$-martingale for each $h$. Using (5.4.6), one can then show that (a) is a $P_\varepsilon^m$-martingale.

Applying the conditional form of Doob's inequality to the martingale (a), we have

$$\begin{aligned}
&E\left[\sup_{0\le t\le T}|\langle Y_t^\varepsilon, \phi\rangle|^{1+\theta}|X_0^\varepsilon\right] \\
&\quad \le \text{const.}\left\{E\left[\sup_{0\le t\le T}\left|\langle Y_t^\varepsilon, \phi\rangle - \int_0^t\langle Y_s^\varepsilon, \Delta_\alpha\phi\rangle\, ds\right|^{1+\theta}|X_0^\varepsilon\right]\right. \\
&\qquad\qquad \left. + E\left[\sup_{0\le t\le T}\left|\int_0^t\langle Y_s^\varepsilon, \Delta_\alpha\phi\rangle\, ds\right|^{1+\theta}|X_0^\varepsilon\right]\right\} \\
&\quad \le \text{const.}\left\{E\left[\left|\langle Y_T^\varepsilon, \phi\rangle - \int_0^T\langle Y_s^\varepsilon, \Delta_\alpha\phi\rangle\, ds\right|^{1+\theta}|X_0^\varepsilon\right]\right. \\
&\qquad\qquad \left. + E\left[\left(\int_0^T|\langle Y_s^\varepsilon, \Delta_\alpha\phi\rangle|\, ds\right)^{1+\theta}|X_0^\varepsilon\right]\right\} \\
&\quad \le \text{const.}\left\{E\left[|\langle Y_T^\varepsilon, \phi\rangle|^{1+\theta}|X_0^\varepsilon\right]\right. \\
&\qquad\qquad \left. + E\left[\left(\int_0^T|\langle Y_s^\varepsilon, \Delta_\alpha\phi\rangle|\, ds\right)^{1+\theta}|X_0^\varepsilon\right]\right\},
\end{aligned}$$

and, using Hölder's inequality we obtain

$$\begin{aligned}
&E\left[\sup_{0\le t\le T}|\langle Y_t^\varepsilon, \phi\rangle|^{1+\theta}|X_0^\varepsilon\right] \\
&\quad \le \text{const.}\left\{E\left[|\langle Y_T^\varepsilon, \phi\rangle|^{1+\theta}|X_0^\varepsilon\right] + \int_0^T E\left[|\langle Y_s^\varepsilon, \Delta_\alpha\phi\rangle|^{1+\theta}|X_0^\varepsilon\right] ds\right\} \\
&\quad \le \text{const.}\left\{E\left[|\langle Y_T^\varepsilon, \phi\rangle|^{1+\theta}|X_0^\varepsilon\right] + \sup_{0\le t\le T} E\left[|\langle Y_t^\varepsilon, \Delta_\alpha\phi\rangle|^{1+\theta}|X_0^\varepsilon\right]\right\}.
\end{aligned}$$

Therefore assertion (i) will follow if we show that

$$\sup_{0\le t\le T} E\big[|\langle Y_t^\varepsilon, \psi^\varepsilon\rangle|^{1+\theta}|X_0^\varepsilon\big] \to 0$$

in probability as $\varepsilon \to 0$, where $\psi^\varepsilon \to 0$ in $C_p(R^d)$.

Let $\delta > 0$. We have

$$P\Big\{\sup_{t\le T} E\big[|\langle Y_t^\varepsilon, \psi^\varepsilon\rangle|^{1+\theta}|X_0^\varepsilon\big] > \delta\Big\} = P\Big[\sup_{t\le T} d\mu_t^\varepsilon/dP > \delta\Big],$$

where $\mu_t^\varepsilon$ is the measure

$$\mu_t^\varepsilon(\cdot) = \int_H\int_\cdot E_h\big[|\langle Y_t^\varepsilon, \psi^\varepsilon\rangle|^{1+\theta}|X_0^\varepsilon\big]\, dP_\varepsilon^h m(dh)$$

[see (5.4.6)] and $d\mu_t^\varepsilon/dP$ in the Radon–Nikodym derivative. Now,

$$\mu_t^\varepsilon(\cdot) \le \nu^\varepsilon(\cdot) \equiv \int_H\int_\cdot \sup_{t\le T} E_h\big[|\langle Y_t^\varepsilon, \psi^\varepsilon\rangle|^{1+\theta}|X_0^\varepsilon\big]\, dP_\varepsilon^h m(dh),$$

and, using (5.4.4)(i) and (3.2), we obtain

$$\nu^\varepsilon(\cdot) \le \int_\cdot \text{const.}\Big(K^{1+\theta} + K^{\theta-\beta}\Big[\langle X_0^\varepsilon, |\psi^\varepsilon|\rangle + \sup_{t\le T} H_t^\varepsilon(\psi^\varepsilon)\Big]\Big)\, dP.$$

Hence $d\mu_t^\varepsilon/dP = (d\mu_t^\varepsilon/d\nu^\varepsilon)(d\nu^\varepsilon/dP) \le d\nu^\varepsilon/dP$. Therefore

$$P\Big\{\sup_{t\le T} E\big[|\langle Y_t^\varepsilon, \psi^\varepsilon\rangle|^{1+\theta}|X_0^\varepsilon\big] > \delta\Big\} \le P(A),$$

where $A \equiv [d\nu^\varepsilon/dP > \delta]$; so

$$\nu^\varepsilon(A) = \int_A (d\nu^\varepsilon/dP)\, dP \ge \delta P(A),$$

and, using the previous results, we have

$$\begin{aligned} &P\Big\{\sup_{t\le T} E\big[|\langle Y_t^\varepsilon, \psi^\varepsilon\rangle|^{1+\theta}|X_0^\varepsilon\big] > \delta\Big\} \\ &\quad\le \delta^{-1}\nu^\varepsilon(A) \\ &\quad\le \delta^{-1}\,\text{const.}\Big\{K^{1+\theta} + K^{\theta-\beta}\Big[E\langle X_0^\varepsilon, |\psi^\varepsilon|\rangle + E\sup_{t\le T} H_t^\varepsilon(\psi^\varepsilon)\Big]\Big\}. \end{aligned}$$

Since $\psi^\varepsilon \to 0$, by assumption we have $E\langle X_0^\varepsilon, |\psi^\varepsilon|\rangle \to 0$. Also, $E\,\sup_{t\le T} H_t^\varepsilon(\psi^\varepsilon) \to 0$ since

$$\sup_{t\le T} H_t^\varepsilon(\psi^\varepsilon) \le \text{const.}\square Y_0^\varepsilon \square_{-p}^{1+\beta} \square \psi^\varepsilon \square_p^{1+\beta} \quad\text{and}\quad \sup_\varepsilon E\square Y_0^\varepsilon\square_{-p}^{1+\beta} < \infty.$$

Finally, we can let $K \to 0$. Assertion (i) is proved.

Assertion (ii) is proved similarly, using (5.4.4)(iii), (5.4.4)(iv) and (5.4.5)(ii). □

(5.4.8) **Lemma.** *For $0 < \varepsilon < 1$, let $\tau_\varepsilon$ be stopping times bounded by $T$ with respect to the filtration $(\sigma\{X_s^\varepsilon, s \le t; h\})_{t\ge 0}$, and $\delta_\varepsilon > 0$ such that $\delta_\varepsilon \to 0$ as*

$\varepsilon \to 0$. *Then for each* $\phi \in \mathscr{S}(R^d)$,

$$\langle Y^{\varepsilon}_{\tau_\varepsilon+\delta_\varepsilon}, \phi\rangle - \langle Y^{\varepsilon}_{\tau_\varepsilon}, \phi\rangle \to 0 \quad \textit{in probability } (P^m_\varepsilon) \textit{ as } \varepsilon \to 0.$$

PROOF. Recall we denote $P \equiv P^m_\varepsilon$ and $E$ the corresponding expectation. Let $\tau \equiv \tau_\varepsilon$, $\delta \equiv \delta_\varepsilon$, $\eta > 0$. Since

$$\begin{aligned} P\big[|\langle Y^{\varepsilon}_{\tau+\delta}, \phi\rangle - \langle Y^{\varepsilon}_{\tau}, \phi\rangle| > \eta\big] &= E\big(P\big[|\langle Y^{\varepsilon}_{\tau+\delta}, \phi\rangle - \langle Y^{\varepsilon}_{\tau}, \phi\rangle| > \eta|X^\varepsilon_0, h\big]\big) \\ &= E\big(P^h_\varepsilon\big[|\langle Y^{\varepsilon}_{\tau+\delta}, \phi\rangle - \langle Y^{\varepsilon}_{\tau}, \phi\rangle| > \eta|X^\varepsilon_0\big]\big), \end{aligned}$$

by dominated convergence it suffices to prove that

(a) $P^h_\varepsilon\big[|\langle Y^{\varepsilon}_{\tau+\delta}, \phi\rangle - \langle Y^{\varepsilon}_{\tau}, \phi\rangle| > \eta|X^\varepsilon_0\big] \to 0$ in probability $(P)$ as $\varepsilon \to 0$.

As a consequence of (5.2.3) the process $(Y^\varepsilon, h) \equiv \{(Y^\varepsilon_t, h), t \geq 0\}$ is strong Markov with respect to the given filtration; hence we have

$$\text{(b)} \qquad \begin{aligned} &P^h_\varepsilon\big[|\langle Y^{\varepsilon}_{\tau+\delta}, \phi\rangle - \langle Y^{\varepsilon}_{\tau}, \phi\rangle| > \eta|X^\varepsilon_0\big] \\ &\quad = E_h\big[P^h_\varepsilon\big[|\langle Y^{\varepsilon}_{\tau+\delta}, \phi\rangle - \langle Y^{\varepsilon}_{\tau}, \phi\rangle| > \eta|X^\varepsilon_\tau\big]|X^\varepsilon_0\big]. \end{aligned}$$

Now, by (5.4.1),

$$\text{(c)} \quad P^h_\varepsilon\big[|\langle Y^{\varepsilon}_{\tau+\delta}, \phi\rangle - \langle Y^{\varepsilon}_{\tau}, \phi\rangle| > \eta|X^\varepsilon_\tau\big] \leq C\eta\int_0^{1/\eta}[1 - \operatorname{Re} f(\lambda)]\, d\lambda,$$

where

$$f(\lambda) := E_h\big[\exp\{i\lambda\langle Y^{\varepsilon}_{\tau+\delta}, \phi\rangle - \langle Y^{\varepsilon}_{\tau}, \phi\rangle\}|X^\varepsilon_\tau\big].$$

Proceeding as in Section 5.3 [see (5.3.2) and (5.3.3)], with the strong Markov property, $f(\lambda)$ is given by

$$\begin{aligned} f(\lambda) = \exp\big\{-i\lambda\langle Y^\varepsilon_\tau, \phi\rangle &+ \langle X^\varepsilon_\tau, \varepsilon^{-d}\log\big[1 - \varepsilon^{d-k}u_\varepsilon(\delta, -\phi\lambda)\big]\rangle \\ &-i\lambda\langle \varepsilon^{-k}\Lambda_\tau, S_\delta\phi\rangle\big\}, \end{aligned}$$

where $u_\varepsilon(t) \equiv u_\varepsilon(t, \psi)$, $\psi \in \mathscr{S}(R^d)$, satisfies

$$u_\varepsilon(t) = S_t\big[\varepsilon^{k-d}\big(1 - \exp\{-i\varepsilon^{d-k}\psi\}\big)\big] - \varepsilon^k V\int_0^t S_{t-s}\big[h(\varepsilon^{-1}\cdot)u_\varepsilon^{1+\beta}(s)\big]\, ds;$$

in particular, $u_\varepsilon(0, \psi) = \varepsilon^{k-d}(1 - \exp\{-i\varepsilon^{d-k}\psi\})$, and therefore

$$\log\big[1 - \varepsilon^{d-k}u_\varepsilon(0, -\phi\lambda)\big] = i\varepsilon^{d-k}\phi\lambda.$$

$f(\lambda)$ can then be written as

$$f(\lambda) = \exp\{i\langle Y^\varepsilon_\tau, S_\delta\phi - \phi\rangle\lambda + \langle X^\varepsilon_\tau, a^\varepsilon_1(\lambda) - a^\varepsilon_2(\lambda) - a^\varepsilon_3(\lambda)\rangle\},$$

where

$$\begin{aligned} a^\varepsilon_1(\lambda) &= \varepsilon^{-d}\big\{\log\big[1 - \varepsilon^{d-k}u_\varepsilon(\delta, -\phi\lambda)\big] + \varepsilon^{d-k}u_\varepsilon(\delta, -\phi\lambda)\big\}, \\ a^\varepsilon_2(\lambda) &= \varepsilon^{-d}\big\{\log\big[1 - \varepsilon^{d-k}u_\varepsilon(0, -S_\delta\phi\lambda)\big] + \varepsilon^{d-k}u_\varepsilon(0, -S_\delta\phi\lambda)\big\}, \\ a^\varepsilon_3(\lambda) &= \varepsilon^{-k}\big[u_\varepsilon(\delta, -\phi\lambda) - u_\varepsilon(0, -S_\delta\phi\lambda)\big]. \end{aligned}$$

By (a), (b) and (c) we have to show that

$$\int_0^{1/\eta}(1 - E_h[\operatorname{Re} f(\lambda)|X_0^\varepsilon])\,d\lambda \to 0 \quad \text{in probability } (P) \text{ as } \varepsilon \to 0,$$

and since $f(\lambda)$ is bounded, it suffices to prove that

(d) $$E\Big[\sup_{0\le t\le T} |\langle Y_t^\varepsilon, S_\delta\phi - \phi\rangle|^{1+\theta}|X_0^\varepsilon\Big] \to 0 \quad \text{in probability as } \varepsilon \to 0,$$

(e) $$E\Big[\sup_{0\le\lambda\le 1/\eta}\ \sup_{0\le t\le T} |\langle X_t^\varepsilon, a_j^\varepsilon(\lambda)\rangle|^{1+\theta}|X_0^\varepsilon\Big] \to 0 \quad \text{in probability as } \varepsilon \to 0,$$

for $j = 1, 2, 3$. (d) holds by (5.4.7)(i) since $S_\delta\phi - \phi \to 0$ and $\Delta_\alpha(S_\delta\phi - \phi) \to 0$ in $C_p(R^d)$. Indeed, $\phi \in \mathscr{S}(R^d)$ implies $\Delta_\alpha\phi \in C_{p,0}(R^d)$ [Dawson and Gorostiza (1988)], and $\Delta_\alpha(S_\delta\phi - \phi) = S_\delta\Delta_\alpha\phi - \Delta_\alpha\phi \to 0$ as $\delta \to 0$ by (3.3). For (e) with $j = 1$, proceeding as in the proof of (5.3.4), we have

$$\begin{aligned}\varepsilon^{-d}\{\log[1 - \varepsilon^{d-k}u_\varepsilon(t, -\phi\lambda)] + \varepsilon^{d-k}u_\varepsilon(t, -\phi\lambda)\} &\le \text{const. } \varepsilon^{d-2k}S_t|\phi|\lambda \\ &\le \text{const. } \varepsilon^{d-2k}S_t\psi,\end{aligned}$$

where $\psi \in \mathscr{S}(R^d)_+$ is such that $|\phi| \le \psi$ and $\lambda \le 1/\eta$. We can now apply (5.4.7)(ii) with $\phi^\varepsilon = \varepsilon^{d-2k}S_t\psi$, hence (e) holds for $j = 1$, since $d > 2k$. The proof for $j = 2$ is similar. For $j = 3$ by (5.1.6) we have

$$\begin{aligned}|a_3^\varepsilon(\lambda)| &\le \varepsilon^{-k}\Big|u_\varepsilon(\delta, -\phi\lambda) + iS_\delta\phi\lambda + \varepsilon^k V\int_0^\delta S_{\delta-s}\big[h(\varepsilon^{-1}\cdot)(-iS_s\phi\lambda)^{1+\beta}\big]\,ds\Big| \\ &\quad + V\int_0^\delta S_{\delta-s}|S_s\phi\lambda|^{1+\beta}\,ds + \varepsilon^{-k}| - iS_\delta\phi\lambda - u(0, S_\delta\phi\lambda)| \\ &\le \text{const.}\big\{\varepsilon^{(\gamma-1)k}S_t|\phi|\lambda^{1+\beta} + \delta S_\delta|\phi|\lambda^{1+\beta} + \varepsilon^{d-2k}(S_\delta|\phi|)^2\lambda^2\big\} \\ &\le \text{const.}\big\{\varepsilon^{(\gamma-1)k}S_t\psi + \delta S_\delta\psi + \varepsilon^{d-2k}S_\delta\psi\big\},\end{aligned}$$

where $\psi$ is as above and $\lambda \le 1/\eta$. Again we can apply (5.4.7)(ii) since $\gamma > 1$, $d > 2k$ and $\delta \to 0$. □

5.5. *Properties of Y [Proof of Theorem* (4.9)]. We have already mentioned that (4.9)(i) is proved in Dawson and Gorostiza (1988) [Remark (4.10.6)].

The results (4.9)(ii) and (4.9)(iii) are straightforward. We will give only an outline of the proofs.

The form (4.9)(ii) of the stable characteristic function is standard [see Mijnheer (1975)], and it follows easily from (4.6).

If $\Lambda_0(dx) = \lambda\,dx$, $\lambda > 0$, it is easy to verify that $\langle\Lambda_0, S_{t-r}(iS_r\phi)^{1+\beta}\rangle = \langle\Lambda_0, (iS_r\phi)^{1+\beta}\rangle$. The translation invariance of the stable distribution implies that $\langle\Lambda_0, (iS_r\phi)^{1+\beta}\rangle$ is invariant under translations of $\phi$. By (4.6) this yields the spatial homogeneity of $Y_t - S_tY_0$.

Now suppose that $Y_t - S_tY_0$ is spatially homogeneous. Then, denoting $\phi_h = \phi(h + \cdot)$, $h \in R^d$, by (4.6) we have

$$\int_0^t\langle\Lambda_0, S_{t-r}(iS_r\phi)^{1+\beta}\rangle\,dr = \int_0^t\langle\Lambda_0, S_{t-r}(iS_r\phi_h)^{1+\beta}\rangle\,dr,$$

and by the translation invariance of the stable distribution,

$$\int_0^t \langle \Lambda_0, S_{t-r}(iS_r\phi)^{1+\beta} \rangle \, dr = \int_0^t \langle \Lambda_{0,h}, S_{t-r}(iS_r\phi)^{1+\beta} \rangle \, dr,$$

where $\Lambda_{0,h}(dx) := \Lambda_0(dx - h)$. Differentiating the latter expression at $t = 0$ gives $\langle \Lambda_0, (i\phi)^{1+\beta} \rangle = \langle \Lambda_{0,h}, (i\phi)^{1+\beta} \rangle$. This implies that $\Lambda_0$ is invariant under translations, and hence is of the form $\Lambda_0(dx) = \lambda \, dx$, $\lambda > 0$.

For $\Lambda_0(dx) = \lambda \, dx$, $Y_0 = 0$ and $\beta = 1$, from (4.6) we have

$$E\langle Y_t, \phi \rangle^2 = 2\lambda V\bar{h} \int_0^t \int_{R^d} (S_r\phi(x))^2 \, dx \, dr = \int_{R^d} \int_{R^d} \phi(x)\phi(y) k_t(x, y) \, dx \, dy,$$

where

$$k_t(x, y) = 2\lambda V\bar{h} \int_0^t p_{2r}(y - x) \, dr,$$

and $p_t(y)$ denotes the stable density. The spectral density is given by

$$\sigma_t(x) = \int_{R^d} e^{-ix\cdot y} k_t(0, y) \, dy = \lambda V\bar{h} |x|^{-\alpha}(1 - \exp\{-2t|x|^\alpha\}).$$

To prove (4.9)(iv), since $\mathscr{S}(R^d)$ is a countably Hilbert nuclear space, by Hida (1980), Chapter 3, Theorem 3.1, it suffices to show that for all $t \geq 0$, the characteristic functional of $Y_t - S_tY_0$ is continuous in some norm $\|\cdot\|_n$. By (4.6) it suffices to prove that $\int_0^t \langle \Lambda_0, S_{t-r}(S_r\phi)^{1+\beta} \rangle \, dr$ is continuous in $\phi$ in some norm $\|\cdot\|_n$. By (3.2) and $\langle \Lambda_0, \phi_p \rangle = \Box \Lambda_0 \Box_{-p}$ (Section 3.1),

$$\begin{aligned} \langle \Lambda_0, |S_{t-r}(S_r\phi)|^{1+\beta} \rangle &\leq \|\phi\|_\infty^\beta \langle \Lambda_0, S_t|\phi| \rangle \\ &\leq \text{const.} \|\phi\|_\infty^\beta \Box \phi \Box_p \langle \Lambda_0, \phi_p \rangle \\ &= \text{const.} \Box \lambda_0 \Box_{-p} \|\phi\|_\infty^\beta \Box \phi \Box_p \\ &\leq \text{const.} |||\phi|||_m^{1+\beta}, \end{aligned}$$

for some $m \geq 1$, because $\|\phi\|_\infty = |||\phi|||_0$ and $\Box \phi \Box_p \leq |||\phi|||_m$ for $m \geq p$. This yields the assertion for some $n > m$ (see Section 3.2). □

**Acknowledgments.** We thank a referee and an Associate Editor for comments and suggestions which improved the exposition.

## REFERENCES

ADLER, R. J., CAMBANIS, S. and SAMORODNITSKY, G. (1987). On stable Markov processes. Technical Report 203, Center for Stochastic Processes, Univ. North Carolina.

ALDOUS, D. (1978). Stopping times and tightness. *Ann. Probab.* **6** 335–340.

BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory.* Academic, New York.

BOJDECKI, T. and GOROSTIZA, L. G. (1986). Langevin equations for $\mathscr{S}'$-valued Gaussian processes and fluctuation limits of infinite particle system. *Probab. Theory Related Fields* **73** 227–244.

BREIMAN, L. (1968). *Probability.* Addison-Wesley, Reading, Mass.

DAWSON, D. A. (1981). Limit theorems for interaction free geostochastic systems. In *Point Processes and Queueing Problems* (*Colloq. Keszthely 1978*). *Colloq. Math. Soc. János Bolyai* **24** 27–47. North-Holland, Amsterdam.

DAWSON, D. A. and FLEISCHMANN, K. (1983). On spatially homogeneous branching processes in a random environment. *Math. Nachr.* **113** 249–257.

DAWSON, D. A. and FLEISCHMANN, K. (1985). Critical dimension for a model of branching in a random medium. *Z. Wahrsch. verw. Gebiete* **70** 315–334.

DAWSON, D. A. and GOROSTIZA, L. G. (1988). Generalized solutions of a class of nuclear space valued stochastic evolution equations. Technical Report 225, Center for Stochastic Processes, Univ. North Carolina.

DAWSON, D. A., FLEISCHMANN, K., FOLEY, R. D. and PELETIER, L. A. (1986). A critical measure-valued branching process with infinite mean. *Stochastic Anal. Appl.* **4** 117–129.

DELLACHERIE, C. and MEYER, P.-A. (1978). *Probabilities and Potential.* North-Holland, Amsterdam.

DITTRICH, P. (1987). Limit theorems for branching diffusions in hydrodynamical scaling. *Math. Nachr.* **131** 59–72.

DOBRUSHIN, R. L. and SIEGMUND-SCHULTZE, R. (1982). The hydrodynamic limit for systems of particles with independent evolution. *Math. Nachr.* **105** 199–224.

ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence.* Wiley, New York.

FLEISCHMANN, K. (1978). Mixing properties of cluster-invariant distributions. *Litovsk. Mat. Sb.* **18** 191–199.

FLEISCHMANN, K. and GÄRTNER, J. (1986). Occupation time processes at a critical point. *Math. Nachr.* **125** 275–290.

GEL'FAND, I. M. and VILENKIN, N. YA. (1964). *Generalized Functions* **4**. Academic, New York.

GOROSTIZA, L. G. (1983). High density limit theorems for infinite systems of unscaled branching Brownian motions. *Ann. Probab.* **11** 374–392.

GREVEN, A. (1985). The coupled branching process in random environment. *Ann. Probab.* **13** 1133–1147.

HIDA, T. (1980). *Brownian Motion.* Springer, New York.

HOLLEY, R. A. and STROOCK, D. W. (1978). Generalized Ornstein–Uhlenbeck processes and infinite particle branching Brownian motions. *Publ. Res. Inst. Math. Sci.* **14** 741–788.

IKEDA, N., NAGASAWA, M. and WATANABE, S. (1968a). Branching Markov processes. I. *J. Math. Kyoto Univ.* **8** 233–278.

IKEDA, N., NAGASAWA, M. and WATANABE, S. (1968b). Branching Markov processes. II. *J. Math. Kyoto Univ.* **8** 365–410.

IKEDA, N., NAGASAWA, M. and WATANABE, S. (1969). Branching Markov processes. III. *J. Math. Kyoto Univ.* **9** 95–160.

ISCOE, I. (1986). A weighted occupation time for a class of measure-valued branching processes. *Probab. Theory Related Fields* **71** 85–116.

ITÔ, K. (1984). *Foundations of Stochastic Differential Equations in Infinite Dimensional Spaces.* SIAM, Philadelphia.

KERSTAN, J., MATTHES, K. and MECKE, J. (1982). *Infinitely Divisible Point Processes.* Nauka, Moscow. (In Russian.)

KOZLOV, S. M. (1985). The method of averaging and walks in inhomogeneous environments. *Russian Math. Surveys* **40**(2) 73–145.

MIJNHEER, J. L. (1975). Sample path properties of stable processes. *Math. Centre Tracts* **59**. Math. Centrum, Amsterdam.

MILLER, R. K. (1971). *Nonlinear Volterra Integral Equations.* Benjamin, New York.

MITOMA, I. (1981). On the norm continuity of $\mathscr{S}'$-valued Gaussian processes. *Nagoya Math. J.* **82** 209–220.

MITOMA, I. (1983). Tightness of probabilities on $C([0,1]; \mathscr{S}')$ and $D([0,1]; \mathscr{S}')$. *Ann. Probab.* **11** 989–999.

MOYAL, J. E. (1962). The general theory of stochastic population processes. *Acta Math.* **108** 1–31.

MOYAL, J. E. (1964). Multiplicative population processes. *J. Appl. Probab.* **1** 267–283.

PAPANICOLAOU, G. C. (1983). Diffusions and random walks in random media. *The Mathematics and Physics of Disordered Media. Lecture Notes in Math.* **1035** 391–399. Springer, New York.

PAZY, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, New York.

TEMPEL'MAN, A. A. (1972). Ergodic theorems for general dynamical systems. *Trans. Moscow Math. Soc.* **26** 94–132.

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