The Annals of Probability
1989, Vol. 17, No. 3, 1063–1082

SOME "LIM INF" RESULTS FOR INCREMENTS OF A WIENER PROCESS

BY D. L. HANSON[1] AND RALPH P. RUSSO[2]

State University of New York at Binghamton and The University of Iowa

Let $W(t)$ for $0 \le t < \infty$ be a standard Wiener process, suppose $0 < a_T \le T$ for $T > 0$, and let $d(T, t) = \{2t[\log(T/t) + \log\log t]\}^{1/2}$. Quantities such as

$$\liminf_{T\to\infty} \sup_{a_T \le t \le T} \frac{W(T) - W(T-t)}{d(T,t)},$$

$$\liminf_{T\to\infty} \sup_{\substack{0 \le t \le T - a_T \\ 0 \le s \le a_T}} \frac{|W(t+s) - W(t)|}{d(t + a_T, a_T)}$$

and

$$\liminf_{T\to\infty} \sup_{\substack{0 \le u < v \le T \\ a_T \le v - u}} \frac{|W(v) - W(u)|}{d(v, v-u)}$$

are investigated.

1. Introduction. Let $W(t)$ for $0 \le t < \infty$ be a standard Wiener process. There is a considerable body of literature on the limiting behavior of properly weighted increments of such a process. Csörgő and Révész [6], for example, have given conditions on a_t under which

$$\limsup_{T\to\infty} \sup_{0 \le t \le T - a_T} \frac{|W(t + a_T) - W(t)|}{\{2a_T[\log(T/a_T) + \log\log T]\}^{1/2}} = 1 \quad \text{a.s.} \tag{1.1}$$

and

$$\limsup_{T\to\infty} \frac{|W(T + a_T) - W(T)|}{\{2a_T[\log(T/a_T) + \log\log T]\}^{1/2}} = 1 \quad \text{a.s.} \tag{1.2}$$

Variations on these results have been given, some involving weakened conditions on a_T, some involving a slight change in the denominator, some involving the "sup" over a different collection of increments, and so forth. (See, for example, [1], [2], [4], [6], [9], [10] and [15].)

Deo [8], Book and Shore [1], Csáki and Révész [5] and Chen, Hong and Hu [3] have investigated the same sorts of questions but using $\liminf_{T\to\infty}$ instead of $\limsup_{T\to\infty}$. In this article we continue that investigation. In Section 2 we state results. We make some remarks, give an example and compare our results with

Received June 1987; revised October 1988.
[1]Research supported by National Science Foundation Grant DMS-86-02565.
[2]Research supported by a grant from the Old Gold Foundation.
AMS 1980 *subject classifications*. Primary 60F15; secondary 60G15, 60G17.
Key words and phrases. Increments of a Wiener process, Wiener process.

the literature in Section 3. We give our proofs in Section 4.

2. Theorems. We assume that $0 < a_T \le T$ for all T and define

$$r_T = \left(\log(T/a_T)\right)/\log\log T \tag{2.0a}$$

so that

$$a_T = T/(\log T)^{r_T}. \tag{2.0b}$$

We also define

$$r = \liminf_{T\to\infty} r_T. \tag{2.0c}$$

Throughout this article we use the notation $\log\log x = \log\log(\max\{x, e\})$ and we let C denote various positive constants whose exact numerical values do not matter so that, for example, $1 + C = C$ might appear in this notation. Most of the time when square brackets are used they will be used only for variety and clarity; sometimes [...] will mean "the greatest integer in..."; we hope it is clear which is which.

There are results for three different, but related, ways of dealing with increments. We have put them into the following three theorems. As should be obvious, corresponding parts of the three theorems are analogous.

For notational convenience we define

$$d(T, t) = \left\{2t(\log(T/t) + \log\log t)\right\}^{1/2}. \tag{2.0d}$$

This is the denominator *in all three* theorems and (unless a "sup" is taken over increments of the Wiener process associated with a fixed denominator) the denominator $d(T, t)$ is associated with the increment $W(T) - W(T - t)$; the first variable T is the time (or index) of the leading term in the increment and the second variable t is the "length" (in time or index) of the increment.

THEOREM 1. *Suppose* $0 < a_T \le T$ *for* $T > 0$. *Then*

$$\liminf_{T\to\infty} \sup_{a_T \le t \le T} \frac{W(T) - W(T-t)}{d(T, t)} \begin{cases} \text{is in } [-1, F(a)] & a.s. \text{ if } \liminf_{T\to\infty} \frac{a_T}{T} = a, \\ = F(a) & a.s. \text{ if } a_T/T \to a, \end{cases} \tag{2.1}$$

where

$$F(a) = \begin{cases} 0 & \text{if } a = 0, \\ -1/\{1 - (\log a)/4\}^{1/2} & \text{if } 0 < a \le 1. \end{cases}$$

THEOREM 2. *Suppose* $0 < a_T \le T$ *for* $T > 0$ *and* $a_T \to \infty$. *Let*

$$\{r/(r+1)\}^{1/2} = 1 \quad \text{if } r = +\infty.$$

Then

(2.2a) $$\liminf_{T\to\infty} \sup_{0\le t\le T-a_T} \frac{|W(t+a_T)-W(t)|}{d(t+a_T,a_T)} = \{r/(r+1)\}^{1/2} \quad a.s.,$$

(2.2b) $$\liminf_{T\to\infty} \sup_{\substack{0\le t\le T-a_T\\ 0\le s\le a_T}} \frac{|W(t+s)-W(t)|}{d(t+a_T,a_T)} = \{r/(r+1)\}^{1/2} \quad a.s.,$$

(2.2c) $$\liminf_{T\to\infty} \sup_{0\le t\le T-a_T} \frac{W(t+a_T)-W(t)}{d(t+a_T,a_T)} \begin{cases} = \{r/(r+1)\}^{1/2} & a.s.\ \text{if } r\in(0,\infty], \\ \text{is in } [-1,] & a.s.\ \text{if } r=0, \\ = G(a) & a.s.\ \text{if } r=0 \text{ and } \frac{a_T}{T}\to a, \end{cases}$$

where

$$G(a) = \begin{cases} 0 & \text{if } a=0, \\ -\left\{\frac{(2n+1)a-1}{n(n+1)a}\right\}^{1/2} & \text{if } \frac{1}{n+1}\le a\le\frac{1}{n};\ n=1,2,\dots, \end{cases}$$

and

(2.2d) $$\liminf_{T\to\infty} \sup_{\substack{0\le t\le T-a_T\\ 0\le s\le a_T}} \frac{W(t+s)-W(t)}{d(t+a_T,a_T)} = \{r/(r+1)\}^{1/2} \quad a.s.$$

THEOREM 3. *Suppose* $0 < a_T \le T$ *for* $T>0$ *and* $a_T\to\infty$. *Let*

$$\{r/(r+1)\}^{1/2} = 1 \quad \text{if } r=+\infty.$$

Then

(2.3a) $$\liminf_{T\to\infty} \sup_{\substack{0\le u<v\le T\\ a_T\le v-u}} \frac{|W(v)-W(u)|}{d(v,v-u)} = \{r/(r+1)\}^{1/2} \quad a.s.,$$

(2.3b) $$\liminf_{T\to\infty} \sup_{\substack{0\le u\le s\le t\le v\le T\\ a_T\le v-u}} \frac{|W(t)-W(s)|}{d(v,v-u)} = \{r/(r+1)\}^{1/2} \quad a.s.,$$

(2.3c) $$\liminf_{T\to\infty} \sup_{\substack{0\le u<v\le T\\ a_T\le v-u}} \frac{W(v)-W(u)}{d(v,v-u)} \begin{cases} = \{r/(r+1)\}^{1/2} & a.s.\ \text{if } r>0, \\ \text{is in } [-1,0] & a.s.\ \text{if } r=0, \\ = G(a) & a.s.\ \text{if } r=0 \text{ and } \frac{a_T}{T}\to a, \end{cases}$$

where $G(a)$ comes from Theorem 2, *and*

(2.3d) $$\liminf_{T\to\infty} \sup_{\substack{0\le u\le s\le t\le v\le T\\ a_T\le v-u}} \frac{W(t)-W(s)}{d(v,v-u)} = \{r/(r+1)\}^{1/2} \quad a.s.$$

Note that from (2.0a) and (2.0c), if $\liminf_{T\to\infty} a_T/T > 0$ then $r = 0$; also, from (2.0b) and (2.0c), if $r > 0$ then $a_T/T \to 0$. We do not need, or mention, r at all in Theorem 1 since $a_T/T \to 0$ completely determines what happens in (2.1); that is, the behavior when $a_T/T \to 0$ is the same for $r = 0$ and for all $r \in (0, \infty]$.

3. Remarks and comparison of our results with other results. For notational convenience we define

(3.1) $$d(T,t) = \{2t[\log(T/t) + \log\log t]\}^{1/2}$$

and

(3.2) $$d^*(T,t) = \{2t[\log(T/t) + \log\log T]\}^{1/2}.$$

We used the denominator $d(T, t)$ in our work in [9] while $d^*(T, t)$ was used in [10], by Csörgő and Révész [6] and by Book and Shore [1]. Now $d(T, t)/d^*(T, t) \to 1$ as $T \to \infty$ uniformly for $0 < t \le T$. [For fixed γ in (0, 1) consider separately the cases $t \le T^\gamma$ and $T^\gamma \le t$.] Thus, as long as $T \to \infty$, the same results are obtained with either denominator. In particular, we obtain exactly the same results in all three theorems if we use the denominator $d^*(\cdot,\cdot)$ instead of the denominator $d(\cdot,\cdot)$.

Our result in (2.2a) is very closely related to the "lim inf" part of Theorem 1 of Book and Shore [1]. We define $r = \liminf_{T\to\infty}(\log(T/a_T))/\log\log T$ and they define $r = \lim_{T\to\infty}(\log(T/a_T))/\log\log T$. If their $r < \infty$, then $a_T \to \infty$ as $T \to \infty$ automatically. Our assumptions on a_T are strictly weaker than theirs except when $r = \infty$. Because our denominator is smaller than theirs (strictly so except when $a_T = T$ and $t = T - a_T$), if we ignore the differences in assumptions our result implies that $\{r/(r+1)\}^{1/2}$ is a lower bound in their case, and their result implies that $\{r/(r+1)\}^{1/2}$ is an upper bound in ours.

Our result (2.2b) is essentially Theorem 2 of Chen, Hong and Hu [3], but proved under weaker assumptions. Their Theorem 1 is also related to our work.

The first work on "$\liminf_{T\to\infty}$" seems to have been done by Deo [8] and is cited by Csáki and Révész [5], page 38. As cited by Csáki and Révész, Deo's work says that if $\limsup_{T\to\infty} r_T < \infty$, then both expressions (1.1) and (1.2) change if $\limsup_{T\to\infty}$ is replaced by $\liminf_{T\to\infty}$.

As mentioned above, the result in (2.2a) is closely related to work in Theorem 1 of Book and Shore [1] and (2.2b) to work of Chen, Hong and Hu [3].

(2.2c) is essentially Theorem 2 of Csáki and Révész [5]. We use $d(t + a_T, a_T)$ for a denominator while they use $\{2T \log\log T\}^{1/2}$. Now

$$\lim_{T\to\infty} \frac{a_T\{\log((t+a_T)/a_T) + \log\log a_T\}}{T\log\log T} = a$$

uniformly for $0 \le t \le T - a_T$ if $a_T/T \to a > 0$. This accounts for the difference between the denominator in their C_α (or C_a) and the denominator in our $G(a)$.

It is not difficult to show that

$$\liminf_{T\to\infty} \sup_{0<t\le T} \frac{|W(T) - W(T-t)|}{d(T,t)} = 0 \quad \text{a.s.,} \tag{3.3a}$$

$$\liminf_{T\to\infty} \sup_{\substack{0<t\le T\\ 0\le s\le t}} \frac{|W(T) - W(T-s)|}{d(T,t)} = 0 \quad \text{a.s.} \tag{3.3b}$$

and

$$\liminf_{T\to\infty} \sup_{\substack{0<t\le T\\ 0\le s\le t}} \frac{W(T) - W(T-s)}{d(T,t)} = 0 \quad \text{a.s.} \tag{3.3c}$$

What denominator should be used, in each case, to obtain a positive, but finite, "lim inf"?

The following miscellaneous comments apply to the behavior of increments of the Wiener process.

It follows from Theorem 2.2 of [10] that our "lim sup" results (Theorems 3.1, 3.2B and 3.3B of [9]) remain true when absolute values are omitted. In addition, from Theorem 2.2 of [10] we get equality (not just inequality) in Theorem 3.2B of [9], with or without absolute values, without requiring the assumption that "a_T is onto."

If a_T is well behaved (e.g., if a_T is continuous), then the sets of limit points of our various functionals are intervals of the form [$\liminf_{T\to\infty}$ "functional," $\limsup_{T\to\infty}$ "functional"] a.s. In most cases we know what this interval is. For example, if the functional is $\sup_{a_T\le t\le T}[W(T) - W(T-t)]/d(T,t)$ and $a_T/T \to \frac{1}{2}$, then, from (2.1) of Theorem 1 of this article and Theorem 3.1 of [9], with absolute values removed as mentioned above, the set of limit points is $[-1/\{1 + 4^{-1}\log 2\}^{1/2}, 1]$.

The assumption that $a_T \to \infty$ seems to be required for Theorems 2 and 3. In particular, these theorems are both false if $a_T \equiv a$ for all $T \ge T_0$ where a and T_0 are constants. (See the argument in the middle of page 618 of [9].)

It seems clear that the results of this article, together with the invariance principle results of Komlós, Major and Tusnády ([11], [12], [13] and [14]), will give "lim inf" results for partial sums of certain i.i.d. sequences of random variables in much the same way that "lim sup" results were obtained in Section 5 of [9].

EXAMPLE. Suppose $0 < \alpha < c < \beta \le 1$. We show how to construct a continuous function a_T such that

$$\alpha \le a_T/T \le \beta \quad \text{for all } T > 0, \tag{3.4}$$

$$\liminf_{T\to\infty} a_T/T = \alpha \quad \text{and} \quad \limsup_{T\to\infty} a_T/T = \beta, \tag{3.5}$$

and such that the a.s. "lim inf's" in (2.1), (2.2c) and (2.3c) are, respectively, $F(c)$, $G(c)$ and $G(c)$.

Choose $\varepsilon > 0$ so that $-\varepsilon > \max\{F(c), G(c)\}$. Fix $\theta > 1$, let a_T be any function satisfying (3.4), $a_T = \alpha T$ would do, and define

$$A(T) = \left\{ \sup_{T \le T^* \le \theta T} \sup_{\substack{0 \le u \le v \le T^* \\ a_{T^*} \le v - u}} \frac{|W(v) - W(u)|}{d(v, v-u)} \ge \varepsilon \right\}. \tag{3.6}$$

Then

$$P(A(T)) \le P\left\{ \sup_{\substack{0 \le u \le v \le \theta T \\ \alpha T \le v - u}} |W(v) - W(u)| \ge \varepsilon d(\alpha T, \alpha T) \right\}$$

$$= P\left\{ \sup_{\substack{0 \le s \le t \le 1/2 \\ \alpha/2\theta \le t - s}} |W(t) - W(s)| \ge \varepsilon d(\alpha T, \alpha T)/\sqrt{2\theta T} \right\} \tag{3.7}$$

$$\le P\left\{ \sup_{\substack{0 \le s \le 1 \\ 0 < t - s = \Delta \le 1/2}} |W(s + \Delta) - W(s)| \ge \varepsilon\sqrt{\alpha \log\log(\alpha T)}\,/\sqrt{\theta} \right\}.$$

This last probability goes to 0 as $T \to \infty$. (See, e.g., Lemma 1.1.1 on page 24 of [7].) Choose a sequence $T_i \to \infty$ so that $\Sigma P(A(T_i)) < \infty$ and $T_{i+1} > \theta T_i$ for all i. Then

$$P(\limsup A(T_i)) = 0. \tag{3.8}$$

Let

$$S = [0, \infty) - \bigcup_{i=1}^{\infty} [T_i, \theta T_i] \quad \text{and} \quad S^c = \bigcup_{i=1}^{\infty} [T_i, \theta T_i]. \tag{3.9}$$

Define $a_T^c \equiv cT$ and let a_T be any continuous function from $(0, \infty)$ to $(0, \infty)$ such that

$$a_T = cT = a_T^c \quad \text{for all } T \text{ in } S \tag{3.10}$$

and such that (3.4) and (3.5) are satisfied. (I.e., $a_T = cT$ except on S^c, and on S^c it "wobbles" enough that $\liminf_{T \to \infty} a_T/T = \alpha$ and $\limsup_{T \to \infty} a_T/T = \beta$.)

Let $B = (\limsup A(T_i))^c$. Because of (3.8) we have $P(B) = 1$. It follows from (3.6) that if $\omega \in B$, then there is a $T(\omega)$ such that if $T \in S^c$ and $T \ge T(\omega)$, then all the weighted increments appearing in (2.1), (2.2c) and (2.3c), for both a_T and a_T^c, lie in the interval $[-\varepsilon, \varepsilon]$. Thus, because we chose $\varepsilon < \min\{-F(c), -G(c)\}$, if we apply Theorems 1, 2 and 3 to a_T^c we get the same results in (2.1), (2.2c) and (2.3c) whether we use "$\liminf_{T \to \infty,\, T \in S}$" or "$\liminf_{T \to \infty}$". Since $a_T = a_T^c$ on S, if we take "$\liminf_{T \to \infty,\, T \in S}$" and use a_T, we get the almost sure results $F(c)$, $G(c)$ and $G(c)$ in (2.1), (2.2c) and (2.3c), respectively. As in the argument above, when we use a_T, what happens for $T \in S^c$ does not change the lim inf, except possibly on a set of probability 0.

4. Proofs. The proofs are of two entirely different forms. We will prove the "c" parts of the three theorems first. For these, the "heavy machinery" is a slight variation of a result due to Strassen [16] which can also be found in Csörgő and Révész [6], Theorem 1.3.2, page 37. The form stated here as a lemma, without proof, comes from [2], Corollary to Theorem 1, page 74, with $a_T = T$.

Let $C(0,1)$ be the space (uniform convergence topology) of continuous real-valued functions on $[0,1]$ and let K be the subset of functions f from $C(0,1)$ such that f is absolutely continuous with respect to Lebesgue measure, $f(0) = 0$, and

$$\int_0^1 (f'(x))^2\,dx \le 1, \tag{4.1}$$

where "dx" indicates integration with respect to Lebesgue measure. K is compact.

LEMMA 1. *For x in $[0,1]$ and $T > e$ let $\eta_T(x) = W(Tx)/\{2T \log\log T\}^{1/2}$. There is a set Ω_0 of probability 1 such that if ω is in Ω_0, then*

(4.2) *the net $\eta_T(\cdot) = \eta_T(\cdot, \omega)$ is relatively compact in $C(0,1)$ and*

(4.3) *K is the set of its limit points (as $T \to \infty$).*

LEMMA 2. *Suppose $a < b$ and $f(x) = \alpha x + \beta$ for $x = a$ and $x = b$. Suppose also that f is absolutely continuous on $[a, b]$ with Radon-Nikodym derivative f'. Then*

$$\int_a^b (f'(x))^2\,dx \ge \int_a^b \alpha^2\,dx$$

and equality holds if and only if $f(x) = \alpha x + \beta$ for all x in $[a, b]$.

PROOF. Let μ be Lebesgue measure and let $P = \mu/(b-a)$. Then $X = f'$ is a random variable on the probability space $([a, b], \Sigma, P)$, where Σ is the collection of Lebesgue measurable subsets of $[a, b]$.

$$\frac{1}{b-a}\int_a^b (f'(x))^2\,dx = EX^2 = E(X - EX)^2 + (EX)^2$$

$$= E(X-\alpha)^2 + \alpha^2 \ge \alpha^2 = \frac{1}{b-a}\int_a^b \alpha^2\,dx$$

and the inequality is strict unless $f' = X = \alpha$ a.s. □

PROOF THAT

$$\liminf_{T\to\infty} \sup_{a_T \le t \le T} \frac{W(T) - W(T-t)}{d(T,t)} = F(a) \quad \text{if } a_T/T \to a > 0.$$

Assume $a_T/T \to a > 0$; then for T large enough $\log(T/t)$ is uniformly bounded in t for $a_T \le t \le T$. Thus we can replace $d(T, t)$ by $\{2t \log\log t\}^{1/2}$. Fix ω_0 in the Ω_0 from Lemma 1.

Suppose $\{T_n\}$ is chosen so that $T_n \to \infty$ and so that

$$\lim_{n\to\infty} \sup_{a_{T_N} \le t \le T_N} \frac{W(T_n) - W(T_n - t)}{\{2t \log\log t\}^{1/2}} = \liminf_{T\to\infty} \sup_{a_T \le t \le T} \frac{W(T) - W(T-t)}{d(T,t)}.$$

From (4.2) of Lemma 1 there exists a subsequence of T_n (which we will still call T_n) and an f_0 in K such that

$$\frac{W(T_n x)}{\sqrt{2T_n \log\log T_n}} - f_0(x) \to 0 \quad \text{uniformly for } x \text{ in } [0,1].$$

Thus

$$\frac{W(T_n \cdot 1) - W(T_n(1 - t/T_n))}{\sqrt{2T_n \log\log T_n}} - \left(f_0(1) - f_0\left(1 - \frac{t}{T_n}\right)\right) \to 0$$

uniformly for $0 \le t \le T_n$ as $n \to \infty$. It follows that (for ω_0)

$$\begin{aligned} &\liminf_{T\to\infty} \sup_{a_T \le t \le T} \frac{W(T) - W(T-t)}{d(T,t)} \\ &\quad = \lim_{n\to\infty} \sup_{a_{T_n} \le t \le T_n} \frac{W(T_n) - W(T_n - T)}{\sqrt{2T_n \log\log T_n}} \sqrt{\frac{T_n}{t}} \\ &\quad = \sup_{a \le s \le 1} \frac{f_0(1) - f_0(1-s)}{s^{1/2}} \\ &\quad \ge \inf_{f\in K} \sup_{a \le s \le 1} \frac{f(1) - f(1-s)}{s^{1/2}}. \end{aligned} \tag{4.4}$$

Using the same approximation argument, using (4.3) from Lemma 1, if f^* is in K, then there is a subsequence $T_n \to \infty$ such that

$$\lim_{n\to\infty} \sup_{a_{T_n} \le t \le T_n} \frac{W(T_n) - W(T_n - t)}{d(T_n, t)} = \sup_{a \le s \le 1} \frac{f^*(1) - f^*(1-s)}{s^{1/2}}.$$

Since K is compact there is an f^* in K such that

$$\sup_{a \le s \le 1} \frac{f^*(1) - f^*(1-s)}{s^{1/2}} = \inf_{f\in K} \sup_{a \le s \le 1} \frac{f(1) - f(1-s)}{s^{1/2}}. \tag{4.5}$$

This gives the reverse inequality in (4.4). Note that (4.5) depends on a, but not otherwise on the sequence a_T; it is our function $F(a)$ whose functional form we have yet to determine. From symmetry

$$F(a) = -\sup_{f\in K}\inf_{a\le s\le 1}\frac{f(1)-f(1-s)}{\sqrt{s}}$$

$$= -\sup_{f\in K}\inf_{a\le s\le 1}\frac{f(s)}{\sqrt{s}}.$$

Suppose h is a function in K which achieves the sup, that is,

$$-F(a) = \sup_{f\in K}\inf_{a\le s\le 1}\frac{f(s)}{\sqrt{s}} = \inf_{a\le s\le 1}\frac{h(s)}{\sqrt{s}}.$$

Then:

(4.6) $-F(a) > 0$. [Consider $f(s) = s$.]

(4.7) By definition $h(s)/\sqrt{s} \ge -F(a)$ for all s in $[a,1]$ and there is equality for at least one s in $[a,1]$.

(4.8) h is linear on $[0, a]$. [If not, we could define

$$h_1(s) = \begin{cases} sh(a)/a, & 0\le s\le a,\\ h(s), & a\le s\le 1.\end{cases}$$

Then

$$\inf_{a\le s\le 1}\frac{h_1(s)}{\sqrt{s}} = \inf_{a\le s\le 1}\frac{h(s)}{\sqrt{s}}$$

and, from Lemma 2, $\int_0^1(h_1'(s))^2\,ds < \int_0^1(h'(s))^2\,ds \le 1$. The function $h_2(s) = h_1(s)/\{\int_0^1(h_1'(t))^2\,dt\}^{1/2}$ would then be in K but $\inf_{a\le s\le 1}(h_2(s)/\sqrt{s}) > -F(a)$ would give a contradiction.]

(4.9) $h(s) = -F(a)\sqrt{s}$ for $a\le s\le 1$. [Suppose not, that $h(s^*) > -F(a)\sqrt{s^*}$. Let $h_1(s)$ be the tangent to $y = -F(a)\sqrt{s}$ at $s = s^*$ and let $h_2 = \min\{h, h_1\}$. The function h_2 would satisfy $h_2(s)/\sqrt{s} \ge -F(a)$ for all s in $[a,1]$ and, again because of Lemma 2, $\int_0^1(h_2'(s))^2\,ds < \int_0^1(h'(s))^2\,ds \le 1$. Then if we define $h_3 = h_2/\{\int_0^1(h_2'(s))^2\,ds\}^{1/2}$, h_3 would be in K and $\inf_{a\le s\le 1}(h_3(s)/\sqrt{s}) > -F(a)$ would give a contradiction.]

(4.10) $\int_0^1 (h'(s))^2\,ds = 1$ [otherwise we can multiply $h(s)$ by a constant and improve upon it].

$$(4.11)\quad h(s) = \begin{cases} \dfrac{-F(a)}{\sqrt{a}}s, & 0 \le s \le a, \\ -F(a)\sqrt{s}, & a \le s \le 1, \end{cases} \quad \text{[from (4.8) and (4.9)]}.$$

$$(4.12)\quad F(a) = -\{(1 - \log a)/4\}^{-1/2} \quad \text{[from (4.10) and (4.11)]}.$$

PROOF THAT

$$\liminf_{T\to\infty} \sup_{0 \le t \le T - a_T} \frac{W(t + a_T) - W(t)}{d(t + a_T, a_T)} = G(a) \quad \text{if } a_T/T \to a > 0.$$

As mentioned elsewhere in this article, this particular result was first discovered by Csáki and Révész and is Theorem 2 in their article [5]. Because of that, our proof—which proceeds along the lines of the proof in the preceding couple of pages, exhibits the extremal function in K, and proves directly that it is extremal—is omitted.

PROOF THAT

$$\liminf_{T\to\infty} \sup_{\substack{0 \le u < v \le T \\ a_T \le v - u}} \frac{W(v) - W(u)}{d(v, v - u)} = G(a) \quad \text{if } a_T/T \to a > 0.$$

Fix a. For some positive integer n we have $1/(n+1) \le a \le 1/n$. Let $L = \{0, a, \dots, na\} \cup \{1 - na, \dots, 1 - a, 1\}$. Define $b = 1 - na$ and $c = (n+1)a - 1$; note that $a = b + c$ and that $1 = (n+1)b + nc$. [$b = 0$ if $a = 1/n$ and $c = 0$ if $a = 1/(n+1)$.] Define h in K so that

$$(4.13)\quad h'(s) = \begin{cases} -\alpha, & ka < s < ka + b, \qquad k = 0, \dots, n, \\ -\beta, & ka + b < s < (k+1)a, \qquad k = 0, \dots, n-1, \end{cases}$$

$$(4.14)\quad \int_0^1 (h'(s))^2\,ds = (n+1)\alpha^2 b + n\beta^2 c = 1$$

and

$$(4.15)\quad b\alpha + c\beta \text{ is maximized subject to (4.14).}$$

Because of (4.13) $h(s)$ and $h(s - a)$ are always traveling along parallel line segments so that $h(s) - h(s - a)$ is constant for $a \le s \le 1$ and, in fact,

$$(4.16)\quad -b\alpha - c\beta = h(s) - h(s - a) \quad \text{for } a \le s \le 1.$$

The maximization in (4.15) subject to (4.14) gives

$$(4.17)\quad \begin{aligned} \alpha &= n^{1/2}/\{(n+1)[(2n+1)a - 1]\}^{1/2}, \\ \beta &= (n+1)^{1/2}/\{n[(2n+1)a - 1]\}^{1/2} \end{aligned}$$

and

$$(4.18)\qquad \begin{aligned} -b\alpha - c\beta &= -\{[(2n+1)a-1]/[n(n+1)]\}^{1/2} \\ &= h(s) - h(s-\alpha) \quad \text{for all } s \text{ in } [a,1]. \end{aligned}$$

This function is produced by Csáki and Révész in [5]. Clearly (using Strassen's result and the approximation argument given previously)

$$(4.19)\qquad \begin{aligned} &\liminf_{T\to\infty} \sup_{0\le t\le T-a_T} \frac{W[T(t/T+a_T/T)] - W[T(t/T)]}{\{2T\log\log T\}^{1/2}} \sqrt{\frac{T}{\alpha_T}} \\ &= \min_{f\in K} \max_{0\le s\le 1-a} \frac{f(s+a)-f(s)}{\sqrt{a}} \\ &\le \frac{1}{\sqrt{a}} \max_{a\le s\le 1} (h(s)-h(s-a)) = G(a) \end{aligned}$$

from (4.18). Csáki and Révész proved, using a probabilistic argument, that the inequality in (4.19) is actually an equality.

For purposes of our current argument we get

$$(4.20)\qquad \begin{aligned} &\liminf_{T\to\infty} \sup_{\substack{0\le u<v\le T\\ a_T\le v-u}} \frac{W[T(v/T)] - W[T(u/T)]}{\{2T\log\log T\}^{1/2}} \sqrt{\frac{T}{v-u}} \\ &= \min_{f\in K} \max_{\substack{0\le s<t\le 1\\ a\le t-s}} G\frac{f(t)-f(s)}{\sqrt{t-s}}. \end{aligned}$$

Clearly

$$(4.21)\qquad \min_{f\in K} \max_{\substack{0\le s<t\le 1\\ a\le t-s}} \frac{f(t)-f(s)}{\sqrt{t-s}} \ge \min_{f\in K} \max_{0\le s\le 1-a} \frac{f(s+a)-f(s)}{\sqrt{a}} = G(a).$$

We will argue that

$$\max_{\substack{0\le s<t\le 1\\ a\le t-s}} \frac{h(t)-h(s)}{\sqrt{t-s}} \le G(a)$$

from which it follows that (4.20) $= G(a)$ and we will be done.

If $a = 1/n$, then $b = 0$ and $\beta = 1$. If $a = 1/(n+1)$, then $c = 0$ and $\alpha = 1$. In either case, by (4.13) and (4.14) we have $h(s) = -s$ so that

$$\max_{\substack{0\le s<t\le 1\\ a\le t-s}} \frac{h(t)-h(s)}{\sqrt{t-s}} = \max_{\substack{0\le s<t\le 1\\ a\le t-s}} \frac{-(t-s)}{\sqrt{t-s}} \doteq -\sqrt{a} = G(a).$$

Now assume that $1/(n+1) < a < 1/n$. Let $t = s + a + \delta$ with $\delta \ge 0$.

$$(4.22)\qquad \sqrt{1+\delta/a} - 1 \le (1/2)(\delta/a) \quad \text{for all } \delta \ge 0.$$

From (4.17) we get $\beta = (n+1)\alpha/n$ so that $\alpha < \beta \le 2\alpha$ for every n. Then from

(4.13) we get $-2\alpha \le h'(s) \le -\alpha$ so that

$$\alpha \ge -h(a)/(2a). \tag{4.23}$$

Putting (4.22) and (4.23) together gives $\delta\alpha \ge -h(a)[\sqrt{1+\delta/a} - 1]$ or

$$\frac{h(a) - \delta\alpha}{\sqrt{a+\delta}} \le \frac{h(a)}{\sqrt{a}} \quad \text{for all } \delta > 0. \tag{4.24}$$

Now

$$\begin{aligned} \frac{h(t) - h(s)}{\sqrt{t-s}} &= \frac{[h(s+a+\delta) - h(s+a)] + [h(s+a) - h(s)]}{\sqrt{a+\delta}} \\ &\le \frac{\delta \max\{-\alpha, -\beta\} + h(a)}{\sqrt{a+\delta}} = \frac{-\delta\alpha + h(a)}{\sqrt{a+\delta}}. \end{aligned} \tag{4.25}$$

Thus

$$\max_{\substack{0 \le s < t \le 1 \\ a \le t-s}} \frac{h(t) - h(s)}{\sqrt{t-s}} \le \frac{h(a)}{\sqrt{a}} = G(a) \tag{4.26}$$

as was to be shown.

Remainder of the proof of Theorem 1. If $a_T/T \to 0$, then a.s.

$$\begin{aligned} &\liminf_{T\to\infty} \sup_{a_T \le t \le T} \frac{W(T) - W(T-t)}{d(T,t)} \\ &\ge \liminf_{T\to\infty} \sup_{aT \le t \le T} \frac{W(T) - W(T-t)}{d(T,t)} = F(a) \end{aligned} \tag{4.27}$$

for every $a > 0$ so (4.27) is bounded below by $0 = F(0)$. On the other hand

$$\sup\Big\{T \mid W(T) = \min_{0 \le t \le T} W(t)\Big\} = +\infty \quad \text{a.s.} \tag{4.28}$$

so (4.27) is always bounded above by 0 a.s.

We always have a.s.

$$\begin{aligned} -1 = F(1) &= \liminf_{T\to\infty} \sup_{1\cdot T \le t \le T} \frac{W(T) - W(T-t)}{d(T,t)} \\ &\le \liminf_{T\to\infty} \sup_{a_T \le t \le T} \frac{W(T) - W(T-t)}{d(T,t)}. \end{aligned} \tag{4.29}$$

If $\liminf_{T\to\infty} a_T/T = a = 0$, then, from (4.28), we see that (4.29) is bounded above by $0 = F(0)$. If $\liminf_{T\to\infty} a_T/T = a > 0$, then for every $0 < \varepsilon < a$ we have

$$(4.29) \le \liminf_{T\to\infty} \sup_{(a-\varepsilon)T \le t \le T} \frac{W(T) - W(T-t)}{d(T,t)} = F(a-\varepsilon)$$

and the last part of the proof of Theorem 1 is completed by letting $\varepsilon \to 0$. □

LEMMA 3. *If* $0 < a_T \le a_T^* \le T$ *for* $T > T_0$, *then for all* ω,

$$\liminf_{T\to\infty} \sup_{\substack{0\le u<v\le T\\ a_T^*\le v-u}} \frac{W(v)-W(u)}{d(v,v-u)} \le \liminf_{T\to\infty} \sup_{\substack{0\le u<v\le T\\ a_T\le v-u}} \frac{W(v)-W(u)}{d(v,v-u)}. \tag{4.30}$$

If, in addition, $a_T \to \infty$ *and* a_T^* *is continuous, then*

$$\liminf_{T\to\infty} \sup_{0\le t\le T-a_T^*} \frac{W(t+a_T^*)-W(t)}{d(t+a_T^*,a_T^*)} \le \liminf_{T\to\infty} \sup_{0\le t\le T-a_T} \frac{W(t+a_T)-W(t)}{d(t+a_T,a_T)}. \tag{4.31}$$

PROOF. (4.30) is obvious. From the continuity of a_T^* there exists, for each T sufficiently large, a T' such that $0 < T' \le T$ and $a_{T'}^* = a_T$. Since $a_T \to \infty$ we have $T' \to \infty$. For fixed ω and T sufficiently large

$$\alpha(T) = \sup_{0\le t\le T-a_T} \frac{W(t+a_T)-W(t)}{d(t+a_T,a_T)} \ge \sup_{0\le t\le T'-a_{T'}^*} \frac{W(t+a_{T'}^*)-W(t)}{d(t+a_{T'}^*,a_{T'}^*)} = \beta(T).$$

Thus

$$\liminf_{T\to\infty}\alpha(T) \ge \liminf_{T\to\infty}\beta(T) \ge \liminf_{T\to\infty} \sup_{0\le t\le T-a_T^*} \frac{W(t+a_T^*)-W(t)}{d(t+a_T^*,a_T^*)}. \qquad \square$$

LEMMA 4. *Suppose* $0 < a_T \le T$ *for* $T > 0$, $a_T \to \infty$ *as* $T \to \infty$ *and, for some* r *in* $(0,\infty]$,

$$\liminf_{T\to\infty} \frac{\log(T/a_T)}{\log\log T} = r. \tag{4.32}$$

Then

$$\liminf_{T\to\infty} \sup_{0\le t\le T-a_T} \frac{W(t+a_T)-W(t)}{d(t+a_T,a_T)} \ge \{r/(r+1)\}^{1/2} \quad a.s., \tag{4.33}$$

where we set $\infty/(\infty+1) = 1$.

PROOF. It is sufficient to show that the left-hand side of (4.33) is $\ge \{s/(s+1)\}^{1/2}$ for every s in $(0, r)$. Fix such an s, fix $0 < b < b_0 < \{s/(s+1)\}^{1/2}$ and fix θ in $(1, \frac{3}{2})$. For $T > e$ let $a(T) = T(\log T)^{-s}$. For $k > (\log\theta)^{-1}$ and sufficiently large, and for $j = 1, \dots, [(k \log \theta)^s] = \alpha_k$, we define

$$A_{jk} = \left\{ \frac{W(ja(\theta^k)) - W((j-1)a(\theta^k))}{d(ja(\theta^k), a(\theta^k))} < b \right\}$$

and note that (for k sufficiently large)

$$\begin{aligned} P(A_{jk}) &\le 1 - C\frac{\exp\{-b^2[\log j + \log\log a(\theta^k)]\}}{b\{2[\log j + \log\log a(\theta^k)]\}^{1/2}} \\ &\le 1 - \exp\{-b_0^2[\log(k^s) + \log\log e^k]\} \\ &= 1 - k^{-b_0^2(s+1)} \le \exp\{-k^{-b_0^2(s+1)}\}. \end{aligned}$$

Thus, since for each fixed k the A_{jk}'s are independent, we have

$$\begin{aligned} \sum_{k=1}^{\infty} P\left(\bigcap_{j=1}^{\alpha_k} A_{jk} \right) &\le C + \sum_{k=1}^{\infty} \left\{ \exp(-k^{-b_0^2(s+1)}) \right\}^{\alpha_k} \\ &\le C + \sum_k \exp\{-Ck^{s-b_0^2(s+1)}\} \end{aligned}$$

which is finite since $b_0 < \{s/(s+1)\}^{1/2}$. It follows that

$$P\left\{ \omega \in \bigcap_{j=1}^{\alpha_k} A_{jk} \text{ infinitely often} \right\} = 0. \tag{4.34}$$

Suppose $\theta^k < T \le \theta^{k+1}$ and k is sufficiently large. Then

$$\begin{aligned} &\sup_{0 \le t \le T - a(T)} \frac{W(t + a(T)) - W(t)}{d(t + a(T), a(T))} \\ &\ge \sup_{0 \le t \le \theta^k - a(\theta^k)} \left\{ \frac{W(t + a(\theta^k)) - W(t)}{d(t + a(\theta^k), a(\theta^k))} \times \frac{d(t + a(\theta^k), a(\theta^k))}{d(t + a(T), a(T))} \right. \\ &\qquad + \frac{W(t + a(T)) - W(t + a(\theta^k))}{d(t + a(T), a(\theta^{k+1}) - a(\theta^k))} \\ &\qquad \left. \times \frac{d(t + a(T), a(\theta^{k+1}) - a(\theta^k))}{d(t + a(T), a(T))} \right\} \\ &= \sup_{0 \le t \le \theta^k - a(\theta^k)} \{B_{tk}C_{Ttk} + D_{Ttk}E_{Ttk}\}. \end{aligned}$$

From (4.34), with probability 1, for all k sufficiently large there is a $t = t(k)$ of the form $(j-1)a(\theta^k)$ in $[0, \theta^k - a(\theta^k)]$ for which $B_{tk} \ge b$. Using Lemma 3.1 from [9] and some analysis, we see that

$$\begin{aligned} \liminf_{k \to \infty} \inf_{\substack{\theta^k < T \le \theta^{k+1} \\ 0 \le t \le \theta^k - a(\theta^k)}} C_{Ttk} &\ge \liminf_{k \to \infty} \inf_{0 \le t \le \theta^k - a(\theta^k)} \frac{d(t + a(\theta^k), a(\theta^k))}{d(t + a(\theta^{k+1}), a(\theta^{k+1}))} \\ &= \theta^{-1/2}. \end{aligned}$$

For k large enough $a(\theta^{k+1}) - a(\theta^k) < a(\theta^k) < t + a(T)$. We have $a(\theta^{k+1}) - a(\theta^k) \to \infty$ as $k \to \infty$. Thinking of $a(\theta^{k+1}) - a(\theta^k)$ as a value of "a," we can use (3.13b) from Theorem 3.3a of [9] to get

$$\limsup_{k\to\infty} \sup_{\substack{\theta^k < T \le \theta^{k+1} \\ 0 \le t \le \theta^k - a(\theta^k)}} |D_{Ttk}| \le 1 \quad \text{a.s.}$$

Finally, Lemma 3.1 of [9] and some analysis give

$$\begin{aligned}\limsup_{k\to\infty} & \sup_{\substack{\theta^k < T \le \theta^{k+1} \\ 0 \le t \le \theta^k - a(\theta^k)}} E_{Ttk} \\ &\le \limsup_{k\to\infty} \sup_{0 \le t \le \theta^k - a(\theta^k)} \frac{d\left(t + a(\theta^{k+1}), a(\theta^{k+1}) - a(\theta^k)\right)}{d\left(t + a(\theta^k), a(\theta^k)\right)} \\ &= (\theta - 1)^{1/2}.\end{aligned}$$

Thus

$$\liminf_{T\to\infty} \sup_{0 \le t \le T - a(T)} \frac{W(t + a(T)) - W(t)}{d(t + a(T), a(T))} \ge b\theta^{-1/2} - (\theta - 1)^{1/2}$$

for every b such that $0 < b < \{s/(s+1)\}^{1/2}$ and every θ in $(1, \frac{3}{2})$. Letting $\theta \downarrow 1$ and $b \uparrow \{s/(s+1)\}^{1/2}$ gives the desired result, but with a_T replaced by $a(T)$. For all T large enough, say $T > T_0$, $\log(T/a_T)/\log\log T > s$ so $a_T < T(\log T)^{-s} = a(T)$. An application of Lemma 3 completes the proof of this lemma. □

LEMMA 5. *Suppose that* $0 < a_T \le T$ *for* $T > 0$, *that* $a_T \to \infty$, *and that*

$$\liminf_{T\to\infty} \frac{\log(T/a_T)}{\log\log T} < r < \infty. \tag{4.35}$$

Then

$$\liminf_{T\to\infty} \sup_{\substack{0 \le u \le s \le t \le v \le T \\ a_T \le v - u}} \frac{|W(t) - W(s)|}{d(v, v-u)} \le \{r/(r+1)\}^{1/2} \quad a.s. \tag{4.36}$$

PROOF. Fix $r/(r+1) < b < 1$, fix δ in $(0, b)$, and set $\theta = 1 + \delta$. For $T > e^e$ let $b_T = T(\log T)^{-r}$, $k_T = [r(\log\log T)/\log\theta] + 1$ and $n_T = [(\theta\delta)^{-1}(\log T)^r] + 1$. Define

$$E_{Tkn} = \left\{ \frac{|W(n\delta\theta^k b_T + \theta^k b_T) - W(n\delta\theta^k b_T)|}{d(n\delta\theta^k b_T + \theta^k b_T, \theta^k b_T)} > b^{1/2} \right\}.$$

Then for T large enough we get for all $k = 0, \ldots, k_T$ and $n = 0, \ldots, n_T$,

$$\begin{aligned} P(E_{Tkn}) &\le \exp\left\{-b\left[\log(n\delta + 1) + \log\log(\theta^k b_T)\right]\right\} \\ &\le C\left\{(n+1)\log(\theta^k b_T)\right\}^{-b} \le C\{(n+1)\log b_T\}^{-b}. \end{aligned} \tag{4.37}$$

Define $E_T = \bigcup_{k=0}^{k_T} \bigcup_{n=0}^{n_T} E_{Tkn}$. From (4.37), for $T > e^e$ we get

$$\begin{aligned} P(E_T) &\le C \sum_{k=0}^{k_T} \sum_{n=0}^{n_T} \{(n+1)\log b_T\}^{-b} \\ &\le C(k_T+1)(\log b_T)^{-b} \int_0^{n_T+1} x^{-b}\,dx \\ &\le C(\log\log T)(\log b_T)^{-b}(n_T)^{1-b} \\ &\le C(\log\log T)(\log T)^{-b+(1-b)r} \end{aligned}$$

and $P(E_T) \to 0$ since $-b + (1-b)r < 0$ is implied by $b > r/(r+1)$.

Since

$$\left| \frac{\log(T/a_T)}{\log\log T} - \frac{\log(T/(\delta a_T))}{\log\log T} \right| \to 0,$$

it follows from assumption (4.35) that there is an $r_0 < r$ and a sequence $T_i \uparrow \infty$ such that $\delta a_{T_i} > e^e$ and such that $\delta a_{T_i} > T_i(\log T_i)^{-r_0}$ for all i. Since $P(E_{T_i}) \to 0$ we will assume that $\sum P(E_{T_i}) < \infty$—by choosing a subsequence of T_i, if necessary—so that $P(\limsup E_{T_i}) = 0$. Let

$$F = \{\omega\colon \omega \in E_{T_i} \text{ only finitely often}\} \cap \left\{ \limsup_{T\to\infty} \sup_{\substack{1\le t\le T \\ 0\le s\le t}} \frac{|W(T) - W(T-s)|}{d(T,t)} \le 1 \right\}.$$

By what we have just argued and (3.9b) from Theorem 3.1 of [9] we get $P(F) = 1$.

Fix ω in F and fix j large enough that (for our fixed ω)

$$\sup_{T\ge\delta b_{T_j}} \sup_{1\le t\le T} \sup_{0\le s\le t} \frac{|W(T) - W(T-s)|}{d(T,t)} \le \theta, \tag{4.38}$$

$$b_{T_j} > e^e/\delta, \quad T_j(\log T_j)^{-r_0} > \theta b_{T_j} \quad \text{and} \quad \omega \in (E_{T_j})^c. \tag{4.39}$$

Suppose u, v, s and t are fixed with $0 \le u \le s \le t \le v \le T_j$ and $t - s \ge \delta a_{T_j}$ which is $> T_j(\log T_j)^{-r_0}$ because of the choice of $\{T_i\}$ which was made earlier. From (4.39) and the definitions of b_T, k_T and n_T it follows that there exist integers k_0 in $\{1, \dots, k_{T_j} - 1\}$ and n_0 in $\{1, n_{T_j}\}$ such that

$$\theta^{k_0} b_{T_j} < t - s \le \theta^{k_0+1} b_{T_j}$$

and (note that we now define x)

$$(n_0 - 1)\delta\theta^{k_0} b_{T_j} \le s \le n_0 \delta\theta^{k_0} b_{T_j} = x.$$

Define $y = x + \theta^{k_0} b_{T_j}$. It follows from the above that $|x - s| \le \delta\theta^{k_0} b_{T_j}$ and that $|y - t| \le \delta\theta^{k_0} b_{Tj}$. We have that

$$\begin{aligned}
\frac{|W(t) - W(s)|}{d(v, v-u)} &\le \frac{|W(t) - W(s)|}{d(t, t-s)} \\
&\le \frac{|W(t) - W(y)|}{d\left(t, \delta\theta^{k_0} b_{T_j}\right)} \frac{d\left(t, \delta\theta^{k_0} b_{T_j}\right)}{d(t, t-s)} \\
&\quad + \frac{|W(y) - W(x)|}{d(y, y-x)} \frac{d(y, y-x)}{d(t, t-s)} \\
&\quad + \frac{|W(x) - W(s)|}{d\left(x, \delta\theta^{k_0} b_{T_j}\right)} \frac{d\left(x, \delta\theta^{k_0} b_{T_j}\right)}{d(t, t-s)} \\
&= AD + BE + CG.
\end{aligned}$$

From (4.39) we have $\delta\theta^{k_0} b_{T_j} > \theta > 1$; also, $t > x \ge \delta\theta^{k_0} b_{T_j} > \delta b_{T_j}$; hence (4.38) can be used to give $A \le \theta$ and $C \le \theta$.

Since $\omega \in (E_{T_j})^c$ we have $B \le b^{1/2}$. Now from Lemma 3.1 of [9]

$$D \le \frac{d\left(t, \delta\theta^{k_0} b_{T_j}\right)}{d\left(t, \theta^{k_0} b_{T_j}\right)}$$

so that

$$\limsup_{j \to \infty} \sup_{\substack{0 \le s \le t \le T_j \\ \delta a_{T_j} \le t-s}} D \le \delta^{1/2}.$$

The same argument works for G and a similar argument gives

$$\limsup_{j \to \infty} \sup_{\substack{0 \le s \le t \le T_j \\ \delta a_{T_j} \le t-s}} E = 1.$$

Thus for $\omega \in F$,

$$\limsup_{j \to \infty} \sup_{\substack{0 \le u \le s \le t \le v \le T_j \\ \delta a_{T_j} \le t-s}} |W(t) - W(s)|/d(v, v-u)$$

$$\le \limsup_{j \to \infty} \sup_{\substack{0 \le u \le s \le t \le v \le T_j \\ \delta a_{T_j} \le t-s}} (AD + BE + CG) \tag{4.40}$$

$$\le \theta\delta^{1/2} + b^{1/2} + \theta\delta^{1/2}.$$

Now

$$\liminf_{T\to\infty} \sup_{\substack{0\le u\le s\le t\le v\le T\\ a_T\le v-u}} \frac{\leqq (t) - W(s)|}{d(v, v-u)}$$

(4.41)
$$\le \limsup_{j\to\infty} \sup_{\substack{0\le u\le s\le t\le v\le T_j\\ aT_j\le v-u}} \frac{|W(t) - W(s)|}{d(v, v-u)}$$

$$= \max\{\mathrm{I}, \mathrm{II}\},$$

where

$$\mathrm{I} = \limsup_{j\to\infty} \sup_{\substack{0\le u\le s\le t\le v\le T_j\\ \delta a_{T_j}\le t-s\\ a_{T_j}\le v-u}} \frac{|W(t) - W(s)|}{d(v, v-u)}$$

and

$$\mathrm{II} = \limsup_{j\to\infty} \sup_{\substack{0\le u\le s\le t\le v\le T_j\\ a_{T_j}\le v-u\\ t-s\le \delta a_{T_j}}} \frac{|W(t) - W(s)|}{d(v, v-u)}.$$

From (4.40), $\mathrm{I} \le b^{1/2} + 2\theta\delta^{1/2}$ a.s. Now from (3.14b) of Theorem 3.3B of [9]

(4.42)
$$\limsup_{j\to\infty} \sup_{0\le v'-\delta a_{T_j}\le s\le t\le v'\le T_j} \frac{|W(t) - W(s)|}{d\left(v', \delta a_{T_j}\right)} \le 1 \quad \text{a.s.}$$

In addition,

(4.43)
$$\limsup_{j\to\infty} \sup_{\substack{0\le u\le v\le T_j\\ a_{T_j}\le v-u}} d\left(v, \delta a_{T_j}\right)/d(v, v-u) = \delta^{1/2}.$$

Whenever s, t, u and v satisfy $0 \le u \le s \le t \le v \le T_j$, $a_{T_j} \le t - s$ and $t - s \le \delta a_{T_j}$, we can (in addition) find v' so that (in addition) $0 \le u \le v' - \delta a_{T_j} \le s \le t \le v' \le v \le T_j$. Since $d(v, \delta a_{T_j})$ is increasing in v for $v \ge \delta a_{T_j}$ (see Lemma 3.1 of [10]), (4.42) and (4.43) can be combined to give $\mathrm{II} \le 1 \cdot \delta^{1/2}$ a.s. Thus $(4.41) \le \max\{b^{1/2} + 2\theta\delta^{1/2}, \delta^{1/2}\}$ a.s. Recall that $0 < \delta < b$, that $\theta = \delta + 1$, and now let $\delta \downarrow 0$ and $b \downarrow r/(r+1)$ to complete the proof of the lemma. □

REMAINDER OF PROOFS OF THEOREMS 2 AND 3. Consider the left-hand sides of expressions (2.2a)–(2.2d) and (2.3a)–(2.3d). For notational convenience let $L(2.ni)$ refer to the left-hand side of expression $(2.ni)$. Just by comparing the terms involved in the various sup's we see that

$$L(2.2\mathrm{c}) = \min\{L(2.2\mathrm{a}), L(2.2\mathrm{b}), L(2.2\mathrm{c}), L(2.2\mathrm{d}), L(2.3\mathrm{a}), L(2.3\mathrm{b}), L(2.3\mathrm{c}), L(2.3\mathrm{d})\}$$

and that

$$L(2.3\text{b}) = \max\{L(2.2\text{a}), L(2.2\text{b}), L(2.2\text{c}), L(2.2\text{d}), L(2.3\text{a}), L(2.3\text{b}), L(2.3\text{c}), L(2.3\text{d})\}.$$

It follows from Lemma 4 that when $0 < r \le \infty$ we have the lower bound $\{r/(r+1)\}^{1/2}$ for all eight left-hand sides. Lemma 5 shows that $\{r/(r+1)\}^{1/2}$ is an upper bound when $0 < r < \infty$; (3.14b) from Theorem 3.3B of [9] shows that when $r = \infty$ we get the upper bound $1 = \{r/(r+1)\}^{1/2}$ for all eight left-hand sides.

Suppose $r = 0$. It is easy to see that $L(2.3\text{b}) \le \{\varepsilon/(\varepsilon+1)\}^{1/2}$ for every $\varepsilon > 0$ so that $L(2.3\text{b}) = 0$, and hence that $L(2.ni) = 0$ for $n = 1, 2$ and $i = \text{a}, \text{b}, \text{d}$.

We are done except for the cases where $r = 0$ and $a_T/T \nrightarrow a > 0$ in (2.2c) and (2.3c). Suppose $r = 0$. Fix $\varepsilon > 0$ and let $a_T^* = \min\{a_T, T/(\log T)^\varepsilon\} = T/(\log T)^{\max\{r_T, \varepsilon\}}$. Then, using the law of the iterated logarithm and (2.3b) of Theorem 3 with $r = \varepsilon$ (in that order) gives

$$\begin{aligned} -1 &\le \liminf_{T\to\infty} \sup_{t=0} \frac{W(t+a_T) - W(t)}{d(t+a_T, a_T)} \\ &\le L(2.2\text{c}) \le L(2.3\text{c}) \le L(2.3\text{b}) \\ &\le \liminf_{T\to\infty} \sup_{\substack{0\le u\le s\le t\le v\le T \\ a_T^* \le v-u}} \frac{|W(t) - W(s)|}{d(v, v-u)} = \{\varepsilon/(1+\varepsilon)\}^{1/2} \quad \text{a.s.} \end{aligned}$$

Letting $\varepsilon \downarrow 0$ shows that both $L(2.2\text{c})$ and $L(2.3\text{c})$ are in $[-1, 0]$ a.s. if $r = 0$.

Now let $a_T^* = \varepsilon T$. Then using that part of (2.2c) already proved and (4.31) from Lemma 3 gives

$$\begin{aligned} G(\varepsilon) &= \liminf_{T\to\infty} \sup_{0\le t\le T-a_T^*} \frac{W(t+a_T^*) - W(t)}{d(t+a_T^*, a_T^*)} \\ &\le L(2.2\text{c}) \le L(2.3\text{c}) \le 0 \quad \text{a.s.} \end{aligned}$$

Letting $\varepsilon \downarrow 0$ so that $G(\varepsilon) \uparrow 0$ completes our proof that $L(2.2\text{c}) = L(2.3\text{c}) = G(0)$ a.s. when $r = 0$ and $a_T/T \to 0$. It also completes our proofs of Theorems 2 and 3. □

REFERENCES

[1] Book, S. A. and Shore, T. R. (1978). On large intervals in the Csörgő–Révész theorem on increments of a Wiener process. *Z. Wahrsch. verw. Gebiete* **46** 1–11.

[2] Chan, A. H. C., Csörgő, M. and Révész, P. (1978). Strassen type limit points for moving averages of a Wiener process. *Canad. J. Statist.* **6** 57–75.

[3] Chen, G., Hong, S. and Hu, S. (1987). Some convergence problems about the increments of a Wiener process. *Acta Math. Sinica* **3** 38–44.

[4] Chen, G., Kong, F. and Lin, Z. (1986). Answers to some questions about increments of a Wiener process. *Ann. Probab.* **14** 1252–1261.

[5] Csáki, E. and Révész, P. (1979). How big must be the increments of a Wiener process? *Acta Math. Acad. Sci. Hungar.* **33** 37–49.

[6] Csörgő, M. and Révész, P. (1979). How big are the increments of a Wiener process? *Ann. Probab.* **7** 731–737.

[7] CSÖRGŐ, M. and RÉVÉSZ, P. (1981). *Strong Approximations in Probability and Statistics.* Academic, New York.

[8] DEO, C. M. (1977). A note on increments of a Wiener process. Technical Report, Univ. Ottawa.

[9] HANSON, D. L. and RUSSO, R. P. (1983). Some results on increments of the Wiener process with applications to lag sums of i.i.d. random variables. *Ann. Probab.* **11** 609–623.

[10] HANSON, D. L. and RUSSO, R. P. (1983). Some more results on increments of the Wiener process. *Ann. Probab.* **11** 1009–1015.

[11] KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1975). An approximation of partial sums of independent r.v.'s and the sample df. I. *Z. Wahrsch. verw. Gebiete* **32** 111–131.

[12] KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1976). An approximation of partial sums of independent r.v.'s and the sample df. II. *Z. Wahrsch. verw. Gebiete* **34** 33–58.

[13] MAJOR, P. (1976). The approximation of partial sums of independent r.v's. *Z. Wahrsch. verw. Gebiete* **35** 213–220.

[14] MAJOR, P. (1976). Approximation of partial sums of i.i.d. r.v.'s when the summands have only two moments. *Z. Wahrsch. verw. Gebiete* **35** 221–229.

[15] ORTEGA, J. and WSCHEBOR, M. (1984). On the increments of the Wiener process. *Z. Wahrsch. verw. Gebiete* **65** 329–339.

[16] STRASSEN, V. (1964). An invariance principle for the law of the iterated logarithm. *Z. Wahrsch. verw. Gebiete* **3** 211–226.

DEPARTMENT OF MATHEMATICAL SCIENCES
STATE UNIVERSITY OF NEW YORK AT BINGHAMTON
BINGHAMTON, NEW YORK 13901

DEPARTMENT OF STATISTICS
THE UNIVERSITY OF IOWA
IOWA CITY, IOWA 52242