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THE MINIMAL EIGENFUNCTIONS CHARACTERIZE THE ORNSTEIN–UHLENBECK PROCESS[1]

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A process (X_t) is equivalent to an Ornstein–Uhlenbeck process if and only if $e^{-\lambda t}f(X_t)$ is a martingale for every $f \geq 0$ on $\mathbb{R}^d$ such that $\Delta f(x) - \langle x, \nabla f(x)\rangle = \lambda f(x)$.

Introduction. In [6] it was shown that the minimal solutions of the parabolic equation $\Delta u(x,t) - \langle x, \nabla u(x,t)\rangle = u_t(x,t)$ on $\mathbb{R}^d \times \mathbb{R}$ determine martingales that characterize the Ornstein–Uhlenbeck process on $\mathbb{R}^d$. This is a property that the Ornstein–Uhlenbeck process on $\mathbb{R}^d$ shares with Brownian motion on a noncompact symmetric space [5] and several other examples [6].

The Ornstein–Uhlenbeck operator is the basic example of an operator L for which the minimal solutions of the corresponding "heat" equation do not factor into the product of a nonnegative eigenfunction of L times an exponential in t, which is the case for uniformly elliptic operators on $\mathbb{R}^d$ and Brownian motion on a homogeneous space [4].

However, if $u \geq 0$ is a solution of $Lu(x) = \Delta u(x) - \langle x, \nabla u(x)\rangle = \lambda u(x)$, then $e^{-\lambda t}u(x) = v(x,t)$ is a solution of $Lu + u_t = 0$. In addition, if $(X_t)_{t\geq 0}$ is an Ornstein–Uhlenbeck process with initial position x_0, then $(v(X_t, t))_{t\geq 0}$ is a martingale with expectation $u(x_0)$. In this note the minimal eigenfunctions (minimal nonnegative solutions u of $Lu = \lambda u$) are determined and it is shown that the corresponding martingales characterize the Ornstein–Uhlenbeck process.

For $d = 1$, it is shown that $(e^{nt}H_n(X_t))_{t\geq 0}$ is a martingale for each Hermite polynomial H_n if the minimal eigenfunctions determine martingales. These "Hermite" martingales are used to characterize the process.

For any dimension, the minimal eigenfunctions are given by the formula

$$\int_0^\infty r^{\lambda-1}\exp\{-r^2 + \sqrt{2r}\,\langle x, b\rangle\}\,dr = K(\lambda, b; x) \quad \text{for } \lambda > 0 \text{ and } b \in S^{d-1}.$$

Hence, the projection onto a line through the origin of a process $(X_t)_{t\geq 0}$ for which $K(\lambda, b; X_t)$ is always a martingale is necessarily equivalent to a real-valued Ornstein–Uhlenbeck process. From this it follows easily that $(X_t)_{t\geq 0}$ is equivalent to an Ornstein–Uhlenbeck process.

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It follows that the Ornstein–Uhlenbeck process on Weiner space $\mathbf{W} = \mathscr{C}([0,1], \mathbb{R})$ may be characterized by the martingales of the form

$$\int_0^{\infty} r^{\lambda-1} \exp\{-r^2 + \sqrt{2r}\langle\gamma, X_t\rangle\}\, dr = K(\lambda, \gamma; X_t),$$

for γ a continuous linear functional on Weiner space of norm 1. This leads to the open question of characterizing the functions $\int_0^{\infty} r^{\lambda-1}\exp\{-r^2 + \sqrt{2r}\langle\gamma, \cdot\rangle\}\, dr$ in terms of the generator of the process. Are they the minimal eigenfunctions?

1. Computation of the minimal eigenfunctions. Let $Lu(x) = \Delta u(x) - \langle x, \nabla u(x)\rangle$ denote the generator of the Ornstein–Uhlenbeck process on $\mathbb{R}^d$. Using a scaling in x by $\sqrt{2}$ and the computations in [3] the minimal solutions $K(y; x, t)$ of the equation $Lu = u_t$ on $\mathbb{R}^d \times \mathbb{R}$ are easily computed. One may also simply calculate by Martin's method as in [3] starting from the formula for the fundamental solution $G(x, t; y, s) = P_{t-s}(x, y)$ if $s < t$ and $= 0$ otherwise, where

$$P_t(x, y) = \left[1/2\pi(1 - e^{-2t})\right]^{d/2} \exp\{(-1/2(1 - e^{-2t}))\|e^{-t}x - y\|^2\}.$$

These solutions are the functions

$$K(y; x, t) = \exp\{-(e^{-2t} - 1)\|y\|^2 + \sqrt{2}\, e^{-t}\langle y, x\rangle\},$$

where $y \in \mathbb{R}^d$. One immediate consequence of the strict positivity of these functions is the following lemma.

LEMMA 1.1. *A positive solution u of $Lu + u_t = 0$ on $\mathbb{R}^d \times (a, b)$, $-\infty \le a < b \le +\infty$, which vanishes continuously on $\mathbb{R}^d \times \{b\}$ is identically* 0. *Consequently, if u is a positive solution of $Lu + u_t = 0$ on $\mathbb{R}^d \times \mathbb{R}$, then $\int P_t(x, dy)u(y, s) = u(x, s - t)$, where $(P_t)_{t \ge 0}$ is the transition semigroup for the Ornstein–Uhlenbeck process on $\mathbb{R}^d$.*

PROOF. To prove the first statement, it suffices to prove the analogous statement for the operator $Lu - u_t$. For this the proof of Theorem 3 in [4] applies without change. The second statement is an immediate consequence as $w(x, t) = u(x, s - t) - \int P_t(x, dy)u(y, s)$ is a positive solution on $\mathbb{R}^d \times (-\infty, s)$ that vanishes on $\mathbb{R}^d \times \{s\}$. □

Let u be a positive solution of the equation $Lu = \lambda u$ on $\mathbb{R}^d$. Assume $u(0) = 1$. As $v(x, t) = e^{\lambda t}u(x)$ is a solution of $Lv = v_t$ there is a unique probability μ on $\mathbb{R}^d$ such that $v(x, t) = \int K(y; x, t)\mu(dy)$. Now

$$e^{\lambda a}v(x, t) = v(x, t + a) = \int K(y; x, t + a)\mu(dy).$$

The minimal functions for the heat equation are normalized so that $K(y; 0, 0) = 1$. The shift in time determines an isomorphism of the cone of positive solutions so $K(y; x, t+a)/K(y; 0, a) = K(e^{-a}y; x, t)$. As a result the measure μ satisfies

$$(*) \qquad e^{\lambda a}\mu(dy) = K(e^{a}y; 0, a)\mu_a(dy),$$

where

$$\int f(y)\mu_a(dy) = \int f(e^{-a}y)\mu(dy), \qquad a \in \mathbb{R}.$$

Let $\nu + m\varepsilon_0 = \mu$, where $m = \mu(\{0\})$. Then $\nu_a + m\varepsilon_0 = \mu_a$ and so $\lambda = 0$ if $m \neq 0$. Assume that $\lambda \neq 0$.

For any Borel set $A \subset \mathbb{R}^d$ and $b \in S^{d-1}$ let $A(b) = \{s > 0 | sb \in A\}$. As $\mu(\{0\}) = 0$, there is a regular conditional probability π such that $\mu(A) = \int 1_{S^{d-1}}(b)\pi(b, A(b))\eta(db)$, where η is the projection onto S^{d-1} of μ. Condition $(*)$ implies that for every nonnegative measurable function f on $(0, \infty)$ and $a \in \mathbb{R}$,

$$e^{\lambda a}\int_{(0,\infty)} \pi(b, ds) f(s) = \int_{(0,\infty)} \pi(b, ds) f(e^{-a}s)\exp\{(1 - e^{-2a})s^2\},$$

from which it follows that for each $b \in S^{d-1}$, $\pi(b, ds)$ is absolutely continuous w.r.t. the Lebesgue measure on $(0, \infty)$ and, its density $\varphi(b, s)$ satisfies the equation

$$\varphi(b, s)\exp\{\lambda a - (e^{2a} - 1)s^2\} = e^{a}\varphi(b, se^{a}).$$

Setting $s = 1$ and $r = e^a$ gives $\varphi(b, r) = C(b)r^{\lambda-1}e^{-r^2}$. Therefore,

$$\begin{aligned} e^{\lambda t}u(x) &= \int_{S^{d-1}}\left[\int_0^\infty \varphi(b, s)\exp\{-(e^{-2t} - 1)s^2 + \sqrt{2}\,e^{-t}s\langle b, x\rangle\}\,ds\right]\eta(db) \\ &= \int_{S^{d-1}} C(b)\left[\int_0^\infty s^{\lambda-1}\exp\{-s^2e^{-2t} + \sqrt{2}\,e^{-t}s\langle b, x\rangle\}\,ds\right]\eta(db) \\ &= \int_{S^{d-1}} e^{\lambda t}C(b)\left[\int_0^\infty r^{\lambda-1}\exp\{-r^2 + \sqrt{2}\,r\langle b, x\rangle\}\,dr\right]\eta(db). \end{aligned}$$

From this it follows that $\lambda > 0$. Thus there is a measure η on S^{d-1} such that

$$u(x) = \int_{S^{d-1}} K(\lambda, b; x)\eta(db),$$

where

$$K(\lambda, b; x) = \int_0^\infty r^{\lambda-1}\exp\{-r^2 + \sqrt{2}\,r\langle b, x\rangle\}\,dr.$$

On the other hand, if η is a measure on S^{d-1} and

$$u(x) = \int K(\lambda, b; x)\eta(ds) \quad \text{and if} \quad v(x, t) = \int K(y; x, t)\mu(dy),$$

where

$$\nu(A) = \int 1_S^{d-1}(b)\pi(b, A(b))\eta(db)$$

it follows from the above that $v(x, t) = e^{\lambda t}u(x)$.

Therefore, since η is the projection of ν onto S^{d-1}, it follows that there is bijection (order-preserving) between the cone of positive solutions u of $Lu = \lambda u$ and the cone of positive measures on S^{d-1}. As a result, the minimal solutions of the equation $Lu = \lambda u$, $\lambda > 0$, are the functions $K(\lambda, b; \cdot)$. This completes the proof of the following theorem.

THEOREM 1.2. *Let u be a positive solution of the equation $Lu = \lambda u$. If $\lambda \neq 0$, then $\lambda > 0$ and*

$$u(x) = \int_{S^{d-1}} C(b)\left[\int_0^{\infty} r^{\lambda-1}\exp\{-r^2 + \sqrt{2}\, r\langle b, x\rangle\}\, dr\right]\eta(db).$$

Up to a multiplicative constant, the minimal solutions of the equation $Lu = \lambda u$, $\lambda > 0$, on $\mathbb{R}^d$ are the functions $K(\lambda, b; x)$, $b \in S^{d-1}$, where

$$K(\lambda, b; x) = \int_0^{\infty} r^{\lambda-1}\exp\{-r^2 + \sqrt{2}\, r\langle b, x\rangle\}\, dr.$$

Further, if $\lambda = 0$ the only nonnegative solutions are the constants.

REMARK 1.3. When $d = 1$, there are two minimal functions $K_\lambda^{\pm}$, where

$$K_\lambda^{\pm}(x) = K(\lambda, \pm; x) = \int_0^{\infty} r^{\lambda-1}\exp\{-r^2 \pm \sqrt{2}\, rx\}\, dr.$$

This formula appears on page 60 of Titchmarsh [7] with a difference due to scaling.

2. Two corresponding entire families of martingales. Assume that $d = 1$. Then, for each $\lambda > 0$, $K(\lambda, +; x)$ and $K(\lambda, -; x)$ form a fundamental set of solutions of the equation $y'' - xy' - \lambda y = 0$. They are analytic in λ if $\operatorname{Re}\lambda > 0$. Another set of fundamental solutions $y_0(\lambda, x)$ and $y_1(\lambda, x)$ are given by the series solutions of the equation $y'' - xy' - \lambda y = 0$. They are

$$(0)\qquad \begin{aligned} y_0 = 1 &+ \lambda x^2/2! + (\lambda+2)\lambda x^4/4! + \cdots \\ &+ (\lambda + 2n - 2)(\lambda + 2n - 4)\cdots(\lambda+2)\lambda x^{2n}/(2n)! + \cdots \end{aligned}$$

and

$$(1)\qquad \begin{aligned} y_1 = x &+ (\lambda+1)x^3/3! + (\lambda+3)(\lambda+1)x^5/5! + \cdots \\ &+ (\lambda + 2n - 1)(\lambda + 2n - 3)\cdots(\lambda+1)x^{2n+1}/(2n+1)! + \cdots \end{aligned}$$

(cf. [1], page 157, with $\sqrt{2}x$ replaced by x and $-\alpha/2$ by λ).

It is not hard to see that $y_0(x) = y_0(\lambda, x)$ and $y_1(x) = y_1(\lambda, x)$ are entire functions of λ for x fixed.

PROPOSITION 2.1. *Let $(\Omega, \mathscr{F}, P)$ be a probability space with an increasing filtration $(\mathscr{F}_t)_{t\geq 0}$ and let $(X_t)_{t\geq 0}$ be a process on $\mathbb{R}$ adapted to the filtration. If for all $\lambda > 0$, $e^{-\lambda t}K(\lambda, X_t) = M_t^\lambda$ is a martingale with respect to the filtration, then $\forall\, z \in \mathbb{C}$, $e^{-zt}y_0(z, X_t)$ and $e^{-zt}y_1(z, X_t)$ are also martingales.*

PROOF. Since all the terms in the series for $y_0(\lambda, x)$ are positive, the fact that $e^{-\lambda t}y_0(\lambda, X_t)$ is a martingale implies that the series

$$1 + zE\left[X_t^2\right]/2! + (z+2)zE\left[X_t^4\right]/4! + \cdots$$
$$+ (z+2n-2)(z+2n-4)\cdots(z+2)zE\left[X_t^{2n}\right]/(2n)! + \cdots$$

is an entire function of z. Consequently, for any set $B \in \mathscr{F}$, $z \to \int_B y_0(z, X_t)\,dP$ is an entire function and so $e^{-zt}y_0(z, X_t)$ is a martingale.

Let

$$Y_t(z) = y_1(z, X_t) = X_t + (z+1)X_t^3/3! + (z+3)(z+1)X_t^5/5! + \cdots$$
$$+ (z+2n-1)(z+2n-3)\cdots$$
$$\times (z+3)(z+1)X_t^{2n+1}/(2n+1)! + \cdots.$$

First consider the integral of Y_t over the sets $A(\pm) = \{\pm X_t > 1\}$. By hypothesis, $K(\lambda, \pm; X_t) \in L^1$ and since $2K(\lambda, +; X_t) = C_0 y_0(\lambda, X_t) + C_1 y_1(\lambda, X_t)$, this implies that $Y_t = y_1(\lambda, X_t) \in L^1$. Consequently,

$$\int_{A(\pm)} |Y_t(z)|\,dP \le \int_{A(\pm)} |y_1(|z|, X_t)|\,dP < \infty.$$

Consequently, for any set $B \in \mathscr{F}$, $B \subset \{|X_t| > 1\}$, $z \to \int_B Y_t(z)\,dP$ is an entire function.

To complete the proof, it suffices to consider the integral of $Y_t(z)$ over $\{|X_t| \le 1\}$. Since $|X_t| \le 1$, the series $\int_B Y_t(z)\,dP$, $B \in \mathscr{F}$, $B \subset \{|X_t| \le 1\}$ converges for all z by comparison with the series

$$1 + (|z|+1)/3! + (|z|+3)(|z|+1)/5! + \cdots$$
$$+ (|z|+2n-1)(|z|+2n-3)\cdots(|z|+3)(|z|+1)/(2n+1)! + \cdots. \quad \square$$

COROLLARY 2.2. *If for all $\lambda > 0$, $e^{-\lambda t}K(\lambda, X_t) = M_t^\lambda$ is a martingale with respect to the filtration, then $\forall\, n \ge 0$, $e^{nt}H_n(X_t)$ is a martingale, where $H_n(x)$ is the nth Hermite polynomial.*

PROOF. $Cy_0(-2n, x) = H_{2n}(x)$ and $Cy_1(-2n-1, x) = H_{2n+1}(x)$. $\square$

REMARK 2.3. If for all $\lambda > 0$, $e^{-\lambda t}K(\lambda, X_t) = M_t^\lambda$ is a martingale with respect to the filtration, then $e^{-t - X_t^2/2}$ is a martingale.

PROOF. If $u(x) = e^{x^2/2}$, then $u''(x) - xu'(x) = u(x)$. Hence, $u(x) = a_0 y_0(1, x) + a_1 y_1(1, x)$. $\square$

3. A characterization of the Ornstein–Uhlenbeck process on $\mathbb{R}$. Corollary 2.2 and Remark 2.3 make it possible to use "Hermite" martingales to analyze the process in $L^2(\mathbb{R}, e^{-x^2/2})$.

If $\varphi \in \mathscr{C}_C^\infty(\mathbb{R})$ and $\varphi(x) = \phi(x)e^{-x^2/2}$, then by Cramér [2]

$$\phi(x) = \sum_{n=0}^{\infty} \{a_n/n!\} H_n(x) e^{-x^2/2},$$

where the series converges uniformly and absolutely to ϕ. The coefficients $a_n = \langle \varphi, H_n \rangle$, where $\langle f, g \rangle = (1/2\pi)^{1/2} \int f(x) g(x) e^{-x^2/2}\,dx$. In addition, if $(P_t)_{t \geq 0}$ is the transition semigroup of the Ornstein–Uhlenbeck process, then

$$(*) \qquad P_t\varphi(x) = \sum_{n=0}^{\infty} \{a_n/n!\} P_t H_n(x) = \sum_{n=0}^{\infty} \{a_n/n!\} e^{-nt} H_n(x).$$

Theorem 3.1. *Let $(\Omega, \mathscr{F}, P)$ be a probability space with an increasing filtration $(\mathscr{F}_t)_{t \geq 0}$ and let $(X_t)_{t \geq 0}$ be a process on $\mathbb{R}$ adapted to the filtration.*

For $\lambda > 0$ let $M_t^{\pm\lambda} = e^{-\lambda t} K(\lambda, \pm X_t)$. The following conditions are equivalent:

1. *The process is equivalent to the Ornstein–Uhlenbeck process with initial position x_0.*
2. *$\forall\, \lambda > 0$, $(M_t^{+\lambda})_{t \geq 0}$ is a martingale with expectation*

$$\int_0^\infty r^{\lambda - 1} \exp\{-r^2 + \sqrt{2}\, r x_0\}\, dr$$

and $(M_t^{-\lambda})_{t \geq 0}$ is a martingale with expectation

$$\int_0^\infty r^{\lambda - 1} \exp\{-r^2 - \sqrt{2}\, r x_0\}\, dr.$$

Proof. Lemma 1.1 shows that (1) ⇒ (2). To show the converse, note that for any $\varphi \in \mathscr{C}_C^\infty(\mathbb{R})$,

$$\left| E[\varphi(X_t) | \mathscr{F}_s] - \sum_{k=0}^{n} \{a_k/k!\} E[H_k(X_t) | \mathscr{F}_s] \right| < \varepsilon E\left[e^{X_t^2/2} | \mathscr{F}_s\right] = \varepsilon e^{(t-s) + X_s^2/2}$$

if

$$\left| \varphi(x) - \sum_{k=0}^{n} \{a_k/k!\} H_k(x) \right| < \varepsilon e^{x^2/2}.$$

Hence, by $(*)$

$$(1) \qquad \begin{aligned} E[\varphi(X_t)] &= \sum_{n=0}^{\infty} \{a_n/n!\} E[H_n(X_t)] \\ &= \sum_{n=0}^{\infty} \{a_n/n!\} e^{-nt} H_n(x_0) = P_t\varphi(x_0) \end{aligned}$$

and

$$(2) \qquad E[\varphi(X_t) | \mathscr{F}_s] = P_{t-s}\varphi(X_s).$$

From (2) it follows that the process is Markov and by (1) it has the correct distributions. □

REMARK 3.2. The theorem remains true if the time interval is restricted to say $[0, T]$.

4. A characterization of the Ornstein–Uhlenbeck process on $\mathbb{R}^d$.

THEOREM 4.1. *Let $(\Omega, \mathscr{F}, P)$ be a probability space with an increasing filtration $(\mathscr{F}_t)_{t \geq 0}$ and let $(X_t)_{t \geq 0}$ be a process on $\mathbb{R}^d$ adapted to the filtration.*

For $\lambda > 0$ and $b \in S^{d-1}$, let $M_t^{\lambda, b} = e^{-\lambda t} K(\lambda, b; X_t)$. The following conditions are equivalent:

1. *The process is equivalent to the Ornstein–Uhlenbeck process with initial position x_0.*
2. *$\forall\ \lambda > 0$ and $b \in S^{d-1}$, $(M_t^{\lambda, b})_{t \geq 0}$ is a martingale with expectation*

$$\int_0^{\infty} r^{\lambda - 1} \exp\left\{ -r^2 + \sqrt{2}\, r\langle b, x_0 \rangle \right\} dr.$$

PROOF. (1) $\Rightarrow$ (2) by Lemma 1.1.

(2) $\Rightarrow$ (1). For every $b \in S^{d-1}$, $(\langle b, X_t \rangle)_{t \geq 0}$ is equivalent to a one-dimensional Ornstein–Uhlenbeck process. Consequently, the paths of $(X_t)_{t \geq 0}$ are almost surely continuous. From this (1) follows immediately because: (i) $(X_t)_{t \geq 0}$ is an Ornstein–Uhlenbeck process on $\mathbb{R}^d$ if and only if $(B_s)_{s \geq 0}$ is a standard Brownian motion, where $e^t X_t + X_0 = B_s$, $s = e^{2t} - 1$; and (ii) a standard Brownian motion is characterized by having its projection on any unit vector a one-dimensional Brownian motion. $\square$

REMARK 4.2. Also, by an obvious change of coordinates, the above result may be used to characterize the Ornstein–Uhlenbeck process on $\mathbb{R}^n$ with generator

$$Lu(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \frac{\partial^2 u}{\partial x_i\, \partial x_j}(x) - \langle x, \nabla u(x) \rangle,$$

where (c_{ij}) is positive definite.

5. The Ornstein–Uhlenbeck process on Weiner space. The Ornstein–Uhlenbeck process on Weiner space is a process $(X_t)_{t \geq 0}$ on the Banach space $\mathbf{W} = \mathscr{C}([0, T], \mathbb{R}^d)$ with the following property. For any finite number of continuous linear functionals γ_i, $1 \leq i \leq n$, on $\mathbf{W}$ the process $(X_t)_{t \geq 0}$, where

$$X_t(\omega) = \left(\langle \gamma_1, \mathbf{X}_t(\omega) \rangle, \langle \gamma_2, \mathbf{X}_t(\omega) \rangle, \dots, \langle \gamma_n, \mathbf{X}_t(\omega) \rangle \right)$$

is an Ornstein–Uhlenbeck process on $\mathbb{R}^n$ with generator

$$Lu(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \frac{\partial^2 u}{\partial x_i\, \partial x_j}(x) - \langle x, \nabla u(x) \rangle,$$

and $c_{ij} = \langle \gamma_i, \gamma_j \rangle_H$ (see [8]). In particular, for any $\gamma \in \mathbf{W}'$, $(\langle \gamma, X_t \rangle)_{t \geq 0}$ is an Ornstein–Uhlenbeck process on $\mathbb{R}$ with generator $Lu(x) = \|\gamma\|_H^2 u''(x) - x u'(x)$.

Assume the process on $\mathbf{W}$ starts from $w_0 = \mathbf{X}_0$. Then $(X_t)_{t \geq 0}$ is equivalent to the Ornstein–Uhlenbeck process on $\mathbf{W}$ started from $\mathbf{w}_0$ if and only if for all $\gamma \in \mathbf{W}'$ and $\lambda > 0$, $e^{-\lambda t} K(\lambda, (1/\|\gamma\|_H)\langle \gamma, \mathbf{X}_t \rangle)$ is a martingale with expectation $K(\lambda, (1/\|\gamma\|_H)\langle \gamma, \mathbf{w}_0 \rangle)$.

This raises the question as to whether the functions $K(\lambda, \langle \gamma, \cdot \rangle)$ on $\mathbf{W}$, with $\|\gamma\|_H = 1$ and fixed $\lambda > 0$, are the minimal solutions of the equation $Lu = \lambda u$ in some sense, where L is the generator of the Ornstein–Uhlenbeck process on $\mathbf{W}$.

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