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CUT POINTS ON BROWNIAN PATHS[1]

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Let X be a standard two-dimensional Brownian motion. There exists a.s. $t \in (0,1)$ such that $X([0,t)) \cap X((t,1]) = \varnothing$. It follows that $X([0,1])$ is not homeomorphic to the Sierpiński carpet a.s.

1. Introduction. Let X be a standard (i.e., continuous) n-dimensional Brownian motion, $n \geq 1$. A (random) point $z \in \mathbb{R}^n$ will be called a cut point if there exists $t \in (0,1)$ such that $X(t) = x$ and $X([0,t)) \cap X((t,1]) = \varnothing$.

QUESTION. *Do cut points exist?*

The answer depends on the dimension n. If $n = 1$, then cut points correspond to "points of increase" or "points of decrease" of the Brownian path. Dvoretzky, Erdös and Kakutani (1961) have shown that such points do not exist a.s. [see Adelman (1985) for a simple proof]. If $n \geq 4$, then Brownian paths have no double points [Dvoretzky, Erdös and Kakutani (1950)] and every $x = X(t)$, $t \in (0,1)$, is a cut point.

The main result of the article is the following

ANSWER. *Cut points exist if and only if* $n \geq 2$.

One consequence of this result is that for $n = 2$, the random set $X([0,1])$ is not homeomorphic to the Sierpiński carpet, as has been conjectured [Mandelbrot (1982)].

Recently, quite a few results have been proved about the geometric properties of the two-dimensional Brownian paths; see, for example, Burdzy (1985, 1987a, b), Cranston, Hsu and March (1989), El Bachir (1983), Evans (1985), Le Gall (1986, 1987), Mountford (1987) and Shimura (1984, 1985, 1988).

A rigorous statement of the results and an outline of the main proof appear in Section 2. The proofs are given in Section 4. Section 3 introduces some notation and presents three ideas (Lemmas 3.1–3.3) on which the proofs are based. The readers are referred to Doob (1984) for the review of the theory of h-processes; this may be a little unfair but even the shortest review of the basic concepts of Brownian motion, potential theory and their relationship would take enormous space.

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2. Main results. Let Ω be the set of all paths $\omega: [0, \infty) \to \mathbb{C} \cup \{\delta\}$ which are continuous on $[0, R)$ and equal to δ otherwise. The lifetime R may be infinite. The "coffin" state δ is outside $\mathbb{C}$. Let X be the canonical process on Ω, that is, $X_t(\omega) = \omega(t)$ for all t and ω. Let $\mathscr{F} = \sigma\{X_s,\ s \ge 0\}$ and let P^x denote a measure on $(\Omega, \mathscr{F})$ which makes X a standard Brownian motion starting from x.

The set of all complex numbers, the imaginary unit and the real and imaginary parts of x will be denoted $\mathbb{C}$, i, $\Re x$ and $\Im x$, respectively.

THEOREM 2.1. *For every $\varepsilon > 0$, the following event has a strictly positive P^0-probability*:

$$\begin{aligned}\{\exists\, t \in (0,1) \text{ such that } & X([0,t)) \cap X((t,1]) = \varnothing,\\ & X(s) \ne X(t) \text{ for all } s \in [0,1],\ s \ne t,\\ & \arg(X(s) - X(t)) \in [0, \pi] \text{ for all } s \in [0,t) \text{ and}\\ & \arg(X(s) - X(t)) \in [\pi - \varepsilon, 2\pi] \text{ for all } s \in (t,1]\}.\end{aligned}$$

THEOREM 2.2. *P^0-a.s., for every $\varepsilon > 0$ there exists $t \in (0, \varepsilon)$ such that $X(s) \ne X(t)$ for all $s \in [0,1]$, $s \ne t$, and*

$$X([0,t)) \cap X((t,1]) = \varnothing.$$

THEOREM 2.3. *P^0-a.s., for every $\varepsilon > 0$ there exist s and t such that $s \in (1/2 - \varepsilon, 1/2)$, $t \in (1/2, 1/2 + \varepsilon)$, $X(s) \ne X(u) \ne X(t)$ for all $u \in [0,1]$, $u \ne s, t$, and*

$$X((s,t)) \cap X([0,s) \cup (t,1]) = \varnothing.$$

COROLLARY 2.1. *Theorems 2.2 and 2.3 hold for the three-dimensional Brownian motion.*

The Sierpiński carpet, a two-dimensional analog of the Cantor set, is defined as [Mandelbrot (1982)]

$$\begin{aligned}&\{z \in \mathbb{C} : \Im z \in [0,1],\ \Re z \in [0,1]\}\\ &\quad \setminus \bigcup_{k=1}^{\infty} \bigcup_{n=0}^{3^{k-1}-1} \bigcup_{m=0}^{3^{k-1}-1} \{z \in \mathbb{C} : \Im z \in ((3n+1)3^{-k}, (3n+2)3^{-k}),\\ &\qquad\qquad\qquad\qquad \Re z \in ((3m+1)3^{-k}, (3m+2)3^{-k})\}.\end{aligned}$$

COROLLARY 2.2. *The Brownian trace $X([0,1])$ is not homeomorphic to the Sierpiński carpet P^0-a.s.*

Suppose that X has the distribution P^0 and $Z(t) = X(t) - tX(1)$ for $t \in [0,1]$. Then the process Z is the Brownian motion conditioned to return to its starting point at time 1. Mandelbrot (1982) calls it a "Brownian loop."

COROLLARY 2.3. *The trace $Z([0,1])$ of the Brownian loop is not homeomorphic to the Sierpiński carpet a.s.*

Here is an outline of the main proofs. Some details are changed for the sake of brevity and clarity.

(i) Consider an h-process X in a half-plane D, converging to a point $x \in \partial D$ (i.e., X is a Brownian motion conditioned to hit ∂D at x). Let $B_k = B_k(y_k, r_k)$ be a ball with $y_k \in \partial D$, $|y_k - x| = 2^{-k}$, $r_k = c2^{-k}$, where c is a small constant. Lemmas 4.1–4.4 show that the complement of $X([0, R))$ contains not only D^c but a sufficiently large (random) family of balls $\{B_{k_j}\}$.

(ii) Let X be the two-dimensional Brownian motion starting from 0. One would like to know the chance of an "approximate" cut point. An "approximate" cut point is a point $X(T)$ such that $X([0, T)) \cap X((T + \varepsilon, S]) = \varnothing$, where ε is small and T and S are random times.

Let L denote a horizontal line below 0 and let T be the hitting time of L by X. The trace $X([0, T))$ has the property described in (i), that is, its complement D_1 contains not only the half-plane D_2 below L but a sufficiently large family of balls $\{A_k\}$ analogous to B_k's and centered at points of L close to $X(T)$ as well. If $X(T + \varepsilon)$ happens to be in D_2, then $X(T + \varepsilon + \cdot\,)$ has some chance of traveling far below L before hitting ∂D_1. Lemmas 4.5–4.7 give estimates of the expected maximum of the vertical displacement of $X(T + \varepsilon + \cdot\,)$ before it hits ∂D_1. These estimates indicate that the distribution of this maximum displacement has a heavy tail. In other words, "approximate" cut points with relatively large $\Im(X(S) - X(T))$ are quite likely. At this point, it is crucial that D_1 contains not only D_2 but the balls $\{A_k\}$ as well.

(iii) An idea of Davis (1983) forms the basis of the main part of the proof of Theorem 2.1.

Let L_k be the horizontal line below 0, $\mathrm{dist}(0, L_k) = \sqrt{\varepsilon}\, 2^{-k}$. Let T_1 be the hitting time of L_1 and let S_1 be the hitting time of $X([0, T_1))$ by $X(T_1 + \varepsilon + \cdot\,)$. Define inductively T_k to be the hitting time of the first line L_j which lies below $X([0, S_{k-1}))$ and let S_k be the hitting time of $X([0, T_k))$ by $X(T_k + \varepsilon + \cdot\,)$. The vertical components of $X(T_k)$ form a process which resembles a renewal process. The estimates mentioned in (ii) are used to show that $\Im(X(T_{k+1}) - X(T_k))$ are likely to take large values even if $\varepsilon \to 0$. This means that even for small ε, it is likely that for some k the parts $X([0, T_k))$ and $X((T_k + \varepsilon, S_k))$ of the path are large and, by definition, disjoint. It is easy to see that the "approximate" cut points $X(T_k)$ converge to the "true" cut points as $\varepsilon \to 0$.

(iv) Statements similar to Theorems 2.2 and 2.3 are usually proved using the zero–one law which in the present case may suggest that: "If a cut point may exist then it must exist in every neighborhood of the starting point." Unfortunately, the events under consideration do not belong to the germ σ-field $\mathscr{F}_{0+}$ and, consequently, the zero–one law cannot be applied. Instead, an elementary argument based on scaling and the strong Markov property is supplied. It is shown that the events that a cut point occurs in the annulus

$\{x: 2^{-k} < |X(0) - x| < 2^{-k-1}\}$ for $k \geq 1$ are sufficiently independent and have probabilities bounded away from 0 so at least one of them must happen.

3. Preliminaries. The Doob theory of h-processes (i.e., conditioned Brownian motion) will be the main tool used in the proofs. The monograph of Doob (1984) contains a detailed review of this theory and will be quoted repeatedly below. The readers are advised to consult this book for the definitions of harmonic functions, the Martin boundary, h-processes, time-reversal, Harnack inequality, and so forth.

The set of all natural numbers (except 0) will be denoted $\mathbb{N}$. For a set $A \subset \mathbb{C}$, the interior of A and the translation of A by x will be denoted int A and $A + x$.

The space Ω and the canonical process X, introduced in Section 2, will be used most of the time as the underlying structure. Denote $A^c = \Omega \setminus A$, $X(R-) = \lim_{t \to R-} X(t)$ and $T(A) = \inf\{t > 0: X(t) \in A\}$. Let P_h^z and P_h^μ denote measures on $(\Omega, \mathscr{F})$ which make X an h-process starting from z or having μ as the initial distribution. Here h is a positive superharmonic function in a Greenian subdomain of $\mathbb{C}$. The corresponding expectations will be denoted E_h^z and E_h^μ. The distribution and expectation of Brownian motion in a Greenian set D, that is, Brownian motion killed at the hitting time of $\mathbb{C} \setminus D$, will be denoted P_D^z and E_D^z.

For $A \subset \mathbb{C}$ let

$$cA = \{z \in \mathbb{C}: \exists\, x \in A \text{ such that } z = cx\}.$$

LEMMA 3.1 (Scaling property). *Suppose that $c \in (0, \infty)$, $D \subset \mathbb{C}$ is a Greenian domain, h is a positive superharmonic function in D, μ is a measure supported in D and $A \in \mathscr{F}$. Define*

$$h_c(z) = h(c^{-1}z) \quad \textit{for } z \in cD,$$

$$\mu_c(B) = \mu(c^{-1}B) \quad \textit{for } B \subset cD,$$

$$A_c = \{\omega \in \Omega: \exists\, \omega_1 \in A \textit{ such that } \omega(t) = c\omega_1(t/c^2) \textit{ for all } t\}.$$

Then $P_{h_c}^{\mu_c}(A_c) = P_h^\mu(A)$.

PROOF. The result follows immediately from the scaling properties of Brownian motion and the definition of an h-process [Doob (1984), 2 VII 2 and 2 X 1). □

A domain $D \subset \mathbb{C}$ will be called Lipschitz if every point $x \in \partial D$ has a neighborhood U such that $\partial D \cap U$ is a graph of a Lipschitz function (in some coordinate system, depending on x).

LEMMA 3.2 [Boundary Harnack principle; Dahlberg (1977)]. *Suppose that* D_1, D_2 *and* D_3 *are bounded, connected and open subsets of* $\mathbb{C}$, $D_1 \subset D_2$, D_2 *is Lipschitz and the closure of* D_1 *is a subset of* D_3. *Then there exists a constant* $c > 0$ *such that*

$$h_1(x)/h_2(x) \geq c h_1(y)/h_2(y)$$

for all $x, y \in D_1$ *and all positive harmonic functions* h_1 *and* h_2 *in* D_2 *which vanish on* $\partial D_2 \cap D_3$.

The Martin topology and the minimal Martin boundary may be identified with the Euclidean topology and boundary in bounded Lipschitz domains [Hunt and Wheeden (1970)].

LEMMA 3.3. *Suppose that* h *is a positive harmonic function in a Greenian domain* $D \subset \mathbb{C}$, B *is a closed subset of* D *and* $x \in D \setminus B$. *Consider the process* $Y = \{X(t),\ t \in [0, \min(R, T(B)))\}$ *under* P_h^x.

(i) *The process* Y *is an* h*-process in* $D \setminus B$.

(ii) *Conditioned on* $\{T(B) < R\}$, *the process* Y *is an* h_1*-process in* $D \setminus B$, *where* h_1 *is a harmonic function in* $D \setminus B$ *which vanishes on* $\partial(D \setminus B) \setminus B$ *and is equal to* h *on* $\partial(D \setminus B) \cap B$.

(iii) *Conditioned on* $\{T(B) \geq R\}$ [*i.e.*, $\{T(B) = \infty\}$], *the process* Y *is an* h_2*-process in* $D \setminus B$, *where* $h_2 = h - h_1$.

PROOF. (i) See Doob (1984), page 675.

(ii) and (iii) These parts of the lemma follow immediately from (i) and the interpretation of the h-process Y as a mixture of g_z-processes where $\{g_z\}$ is the family of all minimal harmonic functions in $D \setminus B$ [see Doob (1984), page 691]. □

4. Proofs.

LEMMA 4.1. *Denote* $D = \{z \in \mathbb{C}: \Im z > 0\}$ *and for* $a \in (0, 1/8)$ *and* $k \in \mathbb{N}$ *let*

$$B_k = B_k(a) = \left\{z \in \mathbb{C}: |-5 \cdot 2^{-k-2} - z| \leq a 2^{-k}\right\}.$$

For $k \in \mathbb{N}$ *and* $K \subset \{1, 2, \ldots, k-1\}$ *let*

$$D_1 = D_1(k, K, a) = \{z \in D: |z| < 1\} \setminus \bigcup_{m \in K} B_m.$$

There exists a constant $c_1 < \infty$ (*which does not depend on* k, K *or* a) *such that for every positive harmonic function* h *in* D_1 *which vanishes on* $\{z \in \partial D_1: |z| < 1\}$

and all $x \in B_k \cap D_1$ one has

$$h(x) \le c_1 h(7i/8)\Im x.$$

PROOF. Let $D_2 = \{z \in D: 3/4 < |z| < 1\}$ and $S = \{z \in D: |z| = 7/8\}$. The functions h and $z \to \Im z$ are positive and harmonic in D_2 so the boundary Harnack principle implies that there exists $c_2 < \infty$ such that

$$h(y)/\Im y \le c_2 h(7i/8)/\Im(7i/8)$$

for all $y \in S$. Thus

$$h(y) \le (8c_2/7)h(7i/8)\Im y$$

for $y \in S$. This inequality holds also for $y \in \partial D_1$, $|y| < 1$, since h vanishes for such y. Use the averaging property of harmonic functions to see that

$$\begin{aligned} h(x) &= \int_{\partial D_1 \cup S} h(y) P^x(X(T(\partial D_1 \cup S)) \in dy) \\ &\le \int_{\partial D_1 \cup S} (8c_2/7)h(7i/8)\Im y P^x(X(T(\partial D_1 \cup S)) \in dy) \\ &= (8c_2/7)h(7i/8)\Im x \end{aligned}$$

for $x \in D_1$, $|x| < 7/8$, in particular for $x \in B_k \cap D_1$. □

LEMMA 4.2. *Let $D_3 \subset D$ be such that $\{z \in D_3: |z| \le 1\} = \{z \in D_1: |z| \le 1\}$. Suppose that h is positive harmonic in D_3 and vanishes on $\{z \in \partial D_3: |z| \le 1\}$. Then*

$$P_h^x(T(B_k) < \infty) \le c_3 a 2^{-k}$$

for all $x \in D_3$, $|x| > 1$, and some constant $c_3 < \infty$ (which does not depend on D_3, x, h, k, K or a).

PROOF. Apply the boundary Harnack principle to positive harmonic functions h and $x \to P_{D_3}^x(T(B_k) < \infty)$ in D_2 to obtain

$$\frac{P_{D_3}^x(T(B_k) < \infty)}{h(x)} \le c_2 \frac{P_{D_3}^{7i/8}(T(B_k) < \infty)}{h(7i/8)} \le \frac{c_2}{h(7i/8)} \tag{4.1}$$

for $x \in S$. The function $y \to P_{D_3}^y(T(B_k) < \infty)$ vanishes on $\{x \in \partial D_3: |z| \ge 7/8\}$, so (4.1) holds on the whole boundary of $\{z \in D_3: |z| > 7/8\}$ and, consequently, inside this region. In particular, (4.1) holds for $x \in D_3$, $|x| > 1$.

Note that $\max_{y \in B_k} \Im y = a2^{-k}$. This fact, (4.1), Lemma 4.1 and (2.1) from Section 2 X 2 of Doob (1984) imply that

$$\begin{aligned} P_h^x(T(B_k) < \infty) &= \int_{\partial B_k} [h(y)/h(x)]\, P_{D_3}^x(X(T(B_k)) \in dy) \\ &\le \int_{\partial B_k} [c_1 h(7i/8) \Im y/h(x)]\, P_{D_3}^x(X(T(B_k)) \in dy) \\ &\le \int_{\partial B_k} [c_1 h(7i/8) a2^{-k}/h(x)]\, P_{D_3}^x(X(T(B_k)) \in dy) \\ &= c_1 a 2^{-k} P_{D_3}^x(T(B_k) < \infty) h(7i/8)/h(x) \\ &\le c_1 a 2^{-k} c_2 \end{aligned}$$

for $x \in D_3$, $|x| > 1$. □

LEMMA 4.3. *Let $D_4 \subset D$ be such that*

$$\{z \in D_4 : |z| \le 2^{-k}\} = \{z \in D : |z| \le 2^{-k}\} \setminus \bigcup_{m \in K} B_m,$$

where $K \subset \{k+1, k+2, \dots, j-1\}$. Suppose that h is positive harmonic in D_4 and vanishes on $\{z \in \partial D_4 : |z| < 2^{-k}\}$. Then

$$P_h^{\mu}(T(B_j) < \infty) \le c_3 a 2^{-j+k}$$

for every measure μ supported in $\{z \in D_4 : |z| > 2^{-k}\}$.

PROOF. Lemma 4.2 and the scaling property imply that

$$P_h^x(T(B_j) < \infty) \le c_3 a 2^{-j+k}$$

for $x \in D_4$, $|x| > 2^{-k}$. The result follows by integration with respect to μ. □

LEMMA 4.4. *Denote $A_k = \{T(B_k) = \infty\}$ and let $\mathscr{F}_k$ be the σ-field generated by $\{A_1, A_2, \dots, A_k\}$. Let h be the minimal positive harmonic function in D corresponding to $0 \in \partial D$. For sufficiently small $a > 0$ there exists $p = p(a) > 0$ such that $P_h^x(A_{k+1} \mid \mathscr{F}_k) > p$ for all $x \in D$, $|x| \ge 1$.*

PROOF. Fix $k \in \mathbb{N}$. Every set in $\mathscr{F}_k$ is a disjoint union of events of the form $\bigcap_{m \in K} A_m \cap \bigcap_{m \in J} A_m^c$, where $J \cup K = \{1, 2, \dots, k\}$ and $J \cap K = \varnothing$. The event $\bigcap_{m \in K} A_m \cap \bigcap_{m \in J} A_m^c$ is in turn a disjoint union of events $F_{\mathbf{j}} \cap \bigcap_{m \in K} A_m$, where

$$F_{\mathbf{j}} = \{T(B_{j_1}) < T(B_{j_2}) < \cdots < T(B_{j_n}) < \infty\}$$

and $\mathbf{j} = (j_1, j_2, \dots, j_n)$ is a sequence of all elements of J. To prove the lemma, it is enough to prove that

$$P_h^x\Big(A_{k+1} \mid F_{\mathbf{j}} \cap \bigcap_{m \in K} A_m\Big) > p$$

for every choice of K and $\mathbf{j}$. Thus, fix some K and $\mathbf{j}$.

See Lemma 3.3 for the results on conditioned h-processes which will be used below.

The distribution P_h^x conditioned by $\bigcap_{m \in K} A_m$ is equal to $P_{h_1}^x$ where h_1 is the minimal harmonic function in $D_5 \stackrel{\text{df}}{=} D \setminus \bigcup_{m \in K} B_m$ corresponding to $0 \in \partial D_5$. Let Q denote the distribution $P_{h_1}^x$ conditioned by $\{T(B_{j_n}) < \infty\}$. By the strong Markov property of the h_1-process, the process $\{X(t),\ t \in [T(B_{j_n}), R)\}$ under Q is an h_1-process in D_5 and the process $\{X(t),\ t \in [0, T(B_{j_n}))\}$ under Q is an h_2-process in $D_6 \stackrel{\text{df}}{=} D_5 \setminus B_{j_n}$. Moreover, the two processes are independent, given $X(T(B_{j_n}))$. It follows that if Q_1 is Q conditioned by $\{T(B_{j_1}) < T(B_{j_2}) < \cdots < T(B_{j_{n-1}}) < \infty\}$, then $\{X(t),\ t \in [T(B_{j_n}), R)\}$ under Q_1 is an h_1-process in D_5 and $\{X(t),\ t \in [0, T(B_{j_n}))\}$ under Q_1 is an h_2-process in D_6 conditioned by $\{T(B_{j_1}) < T(B_{j_2}) < \cdots < T(B_{j_{n-1}}) < \infty\}$.

Repeat the same argument for $\{X(t),\ t \in [0, T(B_{j_n}))\}$ under Q in place of $\{X(t),\ t \in [0, R)\}$ under $P_{h_1}^x$ and then proceed by induction to see that for all $m = 1, 2, \dots, n-1$ the process $\{X(t),\ t \in [T(B_{j_m}), T(B_{j_{m+1}}))\}$ under Q_1 is a g_m-process in $D_6 = D_6(m) \stackrel{\text{df}}{=} D_5 \setminus \bigcup_{r=m+1}^n B_{j_r}$. The initial distribution of this process is supported by B_{j_m} and g_m is a positive harmonic function in D_6 which vanished on $\partial D_6 \setminus B_{j_{m+1}}$. The above remains true for $m = 0$ if one defines $j_0 = 0$, $B_{j_0} = \{x\}$ and $T(B_{j_0}) = 0$.

Now Lemma 4.3 will be applied to $\{X(t),\ t \in [T(B_{j_m}), T(B_{j_{m+1}}))\}$ under Q_1. Substitute $\min(j_m, j_{m+1})$ and $k+1$ for k and j in the statement of Lemma 4.3 to obtain

$$\begin{aligned} Q\big(X(t) \in B_{j_{k+1}} \text{ for some } t \in \big[T(B_{j_m}), T(B_{j_{m+1}})\big)\big) &\\ \le c_3 a 2^{-k-1+\min(j_m, j_{m+1})} &\\ \le c_3 a 2^{-k-1+j_m}. & \end{aligned}$$

Then

$$\begin{aligned} &Q\big(T(B_{j_{k+1}}) < T(B_{j_n})\big) \\ &\quad\le \sum_{m=0}^{n-1} Q\big(X(t) \in B_{k+1} \text{ for some } t \in \big[T(B_{j_m}), T(B_{j_{m+1}})\big)\big) \\ &\quad\le \sum_{m=0}^{n-1} c_3 a 2^{-k-1+j_m} \\ &\quad\le c_3 a 2^{-k-1} \sum_{m=0}^{n-1} 2^{j_m} \\ &\quad\le c_3 a 2^{-k-1} \sum_{m=0}^{k} 2^m \\ &\quad\le c_3 a 2^{-k-1} 2^{k+1} = c_3 a. \end{aligned} \tag{4.2}$$

Denote $D_7 = \{z \in D\colon\ 3 \cdot 2^{-k-2} < |z| < 2^{-k}\}$, $D_8 = \{z \in D\colon\ |z| < 2^{-k}\}$, $S_1 = \{z \in D\colon |z| = 7 \cdot 2^{-k-3}\}$. Recall the harmonic function h_1 such that $\{X(t), t \in [T(B_{j_n}), R)\}$ under Q_1 is an h_1-process in D_5. Apply the boundary Harnack principle in D_7 to the functions h_1 and $x \to P^x_{D_5}(X(T(B_{k+1})) \in dy)$, where $dy \subset B_{k+1}$, to see that

$$\frac{P^x_{D_5}(X(T(B_{k+1})) \in dy)}{h_1(x)} \le c_2 \frac{P^v_{D_5}(X(T(B_{k+1})) \in dy)}{h_1(v)} \tag{4.3}$$

for all $x \in S_1$. Here $v = 7 \cdot 2^{-k-3}i$ and $c_2 < \infty$ is the same constant as in Lemma 4.1; it does not depend on k, by scaling.

Now apply the boundary Harnack principle in D_8 to the functions h_1 and $y \to \Im y$ to obtain

$$h_1(y)/h_1(v) \le c_4 \Im y / \Im v = \left(c_4 2^{k+3}/7\right) \Im y \tag{4.4}$$

for $y \in S_1$. The constant $c_4 < \infty$ does not depend on k.

By (4.3), (4.4) and formula 2 X 2 (2.1) of Doob (1984),

$$\begin{aligned}
P^x_{h_1}(T(B_{k+1}) < \infty) &= \int_{B_{k+1}} (h_1(y)/h_1(x)) P^x_{D_5}(X(T(B_{k+1})) \in dy) \\
&\le \int_{B_{k+1}} (h_1(y)/h_1(v)) c_2 P^v_{D_5}(X(T(B_{k+1})) \in dy) \\
&\le \int_{B_{k+1}} \left(c_2 c_4 2^{k+3}/7\right) \Im y P^v_{D_5}(X(T(B_{k+1})) \in dy) \\
&\le \int_{B_{k+1}} \left(c_2 c_4 2^{k+3}/7\right) a 2^{-k-1} P^v_{D_5}(X(T(B_{k+1})) \in dy) \\
&\le (c_2 c_4 4a/7) P^v_{D_5}(T(B_{k+1}) < \infty) \\
&\le (c_2 c_4 4/7) a.
\end{aligned}$$

Use the strong Markov property at $T_1 \overset{\mathrm{df}}{=} \inf\{t > T(B_{j_n})\colon X(t) \in S_1\}$ to obtain

$$\begin{aligned}
Q_1\big(X(t) \in B_{k+1} \text{ for some } t > T(B_{j_n})\big) &= \int_{S_1} P^x_{D_5}(T(B_{k+1}) < \infty) Q_1(X(T_1) \in dx) \\
&\le \int_{S_1} (c_2 c_4 4/7) a Q_1(X(T_1) \in dx) \\
&\le (c_2 c_4 4/7) a.
\end{aligned}$$

This and (4.2) imply that

$$Q_1(T(B_{k+1}) < \infty) \le c_3 a + (c_2 c_4 4/7) a.$$

Now choose $a > 0$ so that the last expression is less than 1/2. Recall the definition of Q_1 to see that the last inequality may be rewritten as

$$P_h^x\Big(A_{k+1} \mid F_j \cap \bigcap_{m \in K} A_m\Big) > 1/2$$

and this completes the proof. □

LEMMA 4.5. *Fix some $k \in \mathbb{N}$ and $K \subset \mathbb{N}$ and let*

$$D_9 = \{z \in \mathbb{C}: \Im z < 0\} \cup \operatorname{int} B_k \cup \bigcup_{m \in K} \operatorname{int} B_m.$$

Denote $M(n) = \{z \in \mathbb{C}: \Im z = -2^{-n}\}$. Then

$$P_{D_9}^x(T(M(k-2)) < \infty) \ge b P_{D_9}^x(T(M(k-1)) < \infty)/2$$

for all $x \in D_9$, $|x| \le 2^{-k-1}$, and a constant $b = b(a) > 1$ which does not depend on k or K.

PROOF. Denote

$$S_2 = S_2(k) = \{z \in D_9: |z| = 7 \cdot 2^{-k-3}\},$$

$$D_{10} = D_{10}(k) = \{z \in D_9: 3 \cdot 2^{-k-2} < |z| < 2^{-k}\},$$

$$A = A(k) = \{z \in \mathbb{C}: \Im z > 0, 3 \cdot 2^{-k-2} < |x| < 2^{-k}\},$$

$$D_{11} = D_{11}(k) = \{z \in \mathbb{C}: \Im z < 0, |z| < 2^{-k+2}\}.$$

By the boundary Harnack principle applied in D_{10}, one has

$$\frac{P^x(T(M(k-1)) < T(\partial D_{11}))}{P^x(T(M(k-1)) < T(A))} \le c_5 \frac{P^v(T(M(k-1)) < T(\partial D_{11}))}{P^v(T(M(k-1)) < T(A))} \stackrel{\text{df}}{=} c_5 c_6 \tag{4.5}$$

for $x \in S_2$ and $v = -7 \cdot 2^{-k-3} i$. Note that $c_5 c_6 > 0$ and these constants do not depend on k, by scaling. Observe that, for $x \in S_2$,

$$P_{D_9}^x(X(T(M(k-1))) \in D_{11}) \ge P^x(T(M(k-1)) < T(\partial D_{11}))$$

and

$$P_{D_9}^x(T(M(k-1)) < \infty) \le P^x(T(M(k-1)) < T(A)).$$

This and (4.5) imply that

$$\frac{P^x_{D_9}(X(T(M(k-1))) \in D_{11})}{P^x_{D_9}(T(M(k-1)) < \infty)} \geq c_5 c_6 \tag{4.6}$$

for $x \in S_2$. This inequality holds also for $x \in D_9$, $|x| < 2^{-k-1}$, by the strong Markov property applied at $T(S_2)$. Denote $D_{12} = \{z \in \mathbb{C}: \Im z < 0\}$ and $D_{13} = D_{12} \cup \operatorname{int} B_k$. Then

$$z \to P^z_{D_{13}}(T(M(k-2)) < \infty) - P^z_{D_{12}}(T(M(k-2)) < \infty)$$

is a strictly positive harmonic function for $x \in D_{12}$, $-2^{-k+2} < \Im z < 0$, and, therefore, it has a strictly positive minimum ε_1 on $M(k-1) \cap D_{11}$. The constant ε_1 does not depend on k, by scaling. It follows that for $x \in M(k-1) \cap D_{11}$

$$\begin{aligned} P^x_{D_9}(T(M(k-2)) < \infty) &\geq P^x_{D_{13}}(T(M(k-2)) < \infty) \\ &\geq \varepsilon_1 + P^x_{D_{12}}(T(M(k-2)) < \infty) \\ &= \varepsilon_1 + 1/2. \end{aligned}$$

This, the strong Markov property applied at $T(M(k-1))$ and (4.6) imply for $x \in D_9$, $|x| \leq 2^{-k-1}$,

$$\begin{aligned} &P^x_{D_9}(T(M(k-2)) < \infty) \\ &\quad = \int_{M(k-1)\cap D_{11}} P^y_{D_9}(T(M(k-2)) < \infty) P^x_{D_9}(X(T(M(k-1))) \in dy) \\ &\qquad + \int_{M(k-1)\setminus D_{11}} P^y_{D_9}(T(M(k-2)) < \infty) P^x_{D_9}(X(T(M(k-1))) \in dy) \\ &\quad \geq \int_{M(k-1)\cap D_{11}} (\varepsilon_1 + 1/2) P^x_{D_9}(X(T(M(k-1))) \in dy) \\ &\qquad + \int_{M(k-1)\setminus D_{11}} P^y_{D_{12}}(T(M(k-2)) < \infty) P^x_{D_9}(X(T(M(k-1))) \in dy) \\ &\quad = \int_{M(k-1)\cap D_{11}} (\varepsilon_1 + 1/2) P^x_{D_9}(X(T(M(k-1))) \in dy) \\ &\qquad + \int_{M(k-1)\setminus D_{11}} 1/2 P^x_{D_9}(X(T(M(k-1))) \in dy) \\ &\quad = (\varepsilon_1 + 1/2) P^x_{D_9}(T(M(k-1)) \in D_{11}) + 1/2 P^x_{D_9}(T(M(k-1)) \notin D_{11}) \\ &\quad = 1/2 P^x_{D_9}(T(M(k-1)) < \infty) + \varepsilon_1 P^x_{D_9}(T(M(k-1)) \in D_{11}) \\ &\quad \geq 1/2 P^x_{D_9}(T(M(k-1)) < \infty) + \varepsilon_1 c_5 c_6 P^x_{D_9}(T(M(k-1)) < \infty) \\ &\quad = (1/2 + \varepsilon_1 c_5 c_6) P^x_{D_9}(T(M(k-1)) \leq \infty). \end{aligned}$$

□

LEMMA 4.6. *Let*

$$D_{14} = \{z \in \mathbb{C}: -1 < \Im z < 0\} \cup \bigcup_{k=1}^{\infty} \operatorname{int} B_{j_k},$$

where $2 = j_0 < j_1 < j_2 < \cdots$. *Then*

$$\frac{E^x_{D_{14}}(|\min_{t \in (0,R)} \Im X(t)|)}{P^x_{D_{14}}(\Im(X(R-)) = -1)} \le c_7 \left(8 + 2\sum_{n=0}^{\infty} b^n (j_{n+1} - j_n)\right)$$

for all $x \in \{z \in \mathbb{C}: |-2^{-m}i - z| \le 2^{-m-1}\} \overset{\text{df}}{=} D_{15} = D_{15}(m)$, *and all* $m \ge 3$, $m \in \mathbb{N}$. *The constant* $b = b(a) > 1$ *is the same as in Lemma* 4.5. *The constant* $c_7 < \infty$ *does not depend on* a, b, m *or the* j*'s*.

PROOF. Fix an $m \ge 3$, $m \in \mathbb{N}$ and $x = -2^{-m}i$. Denote $p_k = P^x_{D_{14}}(T(M(k)) < \infty)$, $k \in \mathbb{N}$, $p_0 = P^x_{D_{14}}(\Im(X(R-)) = -1)$. Let $D_{16} = \{z \in \mathbb{C}: -1 < \Im z < 0\}$. Then

$$\begin{aligned} p_k &= \int_{M(k+1)} P^y_{D_{14}}(T(M(k)) < \infty) P^x_{D_{14}}(X(T(M(k+1))) \in dy) \\ &\ge \int_{M(k+1)} P^y_{D_{16}}(T(M(k)) < \infty) P^x_{D_{14}}(X(T(M(k+1))) \in dy) \\ &= \int_{M(k+1)} 1/2 P^x_{D_{14}}(X(T(M(k+1))) \in dy) \\ &= p_{k+1}/2 \end{aligned}$$

for $1 \le k \le m-1$; the inequality is valid for $k = 0$ for similar reasons.

By Lemma 4.5, $p_k \ge b p_{k+1}/2$ if $k + 2 = j_n$ for some $n \ge 1$, $k \le m-3$.

Let $s(k) = n$ if $j_n - 1 \le k < j_{n+1} - 1$. Then $p_k/p_0 \le 2^k b^{-s(k)}$ for $1 \le k \le m-3$.

The Harnack principle applied in $\{z \in \mathbb{C}: |x - z| < 2^{-m}\}$ shows that for some constant $c_8 < \infty$,

$$P^y_{D_{14}}(T(M(k)) < \infty) \le c_8 P^x_{D_{14}}(T(M(k)) < \infty) = c_8 p_k$$

and

$$P^y_{D_{14}}(\Im X(R-) = -1) \ge P^x_{D_{14}}(\Im X(R-) = -1)/c_8 = p_0/c_8$$

for $k \le m-2$ and $y \in D_{15}$. One has, for $y \in D_{15}$,

$$\begin{aligned} E^y_{D_{14}}\left(\left|\min_{t \in (0,R)} \Im X(t)\right|\right) &\le \sum_{k=0}^{m-2} 2 \cdot 2^{-k} P^y_{D_{14}}(T(M(k)) < \infty) \\ &\le \sum_{k=0}^{m-2} 2 \cdot 2^{-k} c_8 P^x_{D_{14}}(T(M(k)) < \infty) \\ &= \sum_{k=0}^{m-2} 2 \cdot 2^{-k} c_8 p_k \\ &\le \sum_{k=0}^{m-3} c_8 2^{-k+1} p_k + c_8 2^{-m+3}. \end{aligned}$$

Note that $p_0 \geq 2^{-m}$ and, therefore, $2^{-m+3}/p_0 \leq 8$. Thus, for $y \in D_{15}$,

$$\frac{E^y_{D_{14}}(|\min_{t\in(0,R)} \Im X(t)|)}{P^y_{D_{14}}(\Im X(R-) = -1)} \leq \frac{\sum_{k=0}^{m-3} c_8 2^{-k+1} p_k + c_8 2^{-m+3}}{p_0/c_8}$$

$$\leq \sum_{k=0}^{m-3} c_8^2 2^{-k+1} p_k/p_0 + 8c_8^2$$

$$\leq 8c_8^2 + \sum_{k=0}^{m-3} c_8^2 2^{-k+1} 2^k b^{-s(k)}$$

$$\leq 8c_8^2 + c_8^2 \sum_{k=0}^{m-3} 2b^{-s(k)}$$

$$\leq 8c_8^2 + c_8^2 \sum_{k=0}^{\infty} 2b^{-s(k)}$$

$$\leq c_8^2\left(8 + 2\sum_{n=0}^{\infty} b^n(j_{n+1} - j_n)\right). \qquad \square$$

Lemma 4.7. *Denote $W(d) = \{z \in \mathbb{C}: \Im z = d\}$ and $D_{17} = D_{17}(x, \rho) = \{z \in \mathbb{C}: |x - \rho i - z| < \rho/2\}$. If a domain D contains D_{17}, then let*

$$G(x, \rho, D) = \max_{y\in D_{17}}\left[\frac{E^y_D(|\min_{t\in(0,R)} \Im X(t)|)}{P^y_D(\Im X(R-) = -1)}\right].$$

Define

$$D_{18}(d) = \{z \in \mathbb{C}: d - 1 < \Im z < d\} \cup \bigcup_{k\in K} \operatorname{int}(B_k + X(T(W(d)))),$$

where

$$K = \{k \in \mathbb{N}: T(B_k + X(T(W(d)))) > T(W(d))\}.$$

For sufficiently small $a > 0$ and all $d \leq -1$, $m \geq 3$, $m \in \mathbb{N}$, $q > 0$, one has

$$E^0(G(X(T(W(d))), 2^{-m}, D_{18}(d))) \leq c_9$$

and, consequently,

$$P^0(G(X(T(W(d))), 2^{-m}, D_{18}(d)) \leq q) \geq 1 - c_9/q.$$

The constant $c_9 = c_9(a) < \infty$ does not depend on d or m.

Proof. Fix some $d \leq -1$ and let $D_{19} = \{z \in \mathbb{C}: \Im z > d\}$. Let h_x denote the minimal harmonic function in D_{19} corresponding to $x \in \partial D_{19}$.

The process $\{X(t),\ t \in [0, T(W(d)))\}$ under P^0 is a mixture of h_x-processes in D_{19} [see Doob (1984), 2 X 8]. Thus, it will suffice to prove the lemma for each h_x-process separately.

Fix an $x \in \partial D_{19}$ and let $3 \leq j_1 < j_2 \dots$ be the sequence of all integers greater than 2 such that $\{T(B_{j_k} + x) = \infty\}$.

Choose an $a > 0$ so that Lemma 4.4 holds for some $p > 0$. Lemma 4.4 says that no matter which balls $B_3 + x, B_4 + x, \dots, B_{k-1} + x$ were hit by X, the conditional $P^0_{h_x}$-probability of $\{T(B_k + x) = \infty\}$ is at least p. Thus, for each $n \in \mathbb{N}$, the distribution of $j_{n+1} - j_n$ is stochastically smaller than the geometric distribution with the parameter p and, consequently, the expectations of $j_{n+1} - j_n$, $n \in \mathbb{N}$, are uniformly bounded, say,

$$E^0_{h_x}(j_{n+1} - j_n) \le c_{10} < \infty$$

for $n \in \mathbb{N}$ and also $n = 0$ (here $j_0 = 2$). Let

$$D_{20} = \{z \in \mathbb{C}: d - 1 < \Im z < d\} \cup \bigcup_{k=1}^{\infty} \operatorname{int}(B_{j_k} + x).$$

Lemma 4.6 implies that

$$\begin{aligned} E^0_{h_x}(G(x, 2^{-m}, D_{20})) &\le E^0_{h_x}\left(c_7\left(8 + 2\sum_{n=0}^{\infty} b^n(j_{n+1} - j_n)\right)\right) \\ &= 8c_7 + 2\sum_{n=0}^{\infty} b^n E^0_{h_x}(j_{n+1} - j_n) \\ &\le 8c_7 + 2\sum_{n=0}^{\infty} b^n c_{10} \\ &= 8c_7 + 2c_{10}/(1 - b) \overset{\text{df}}{=} c_9 < \infty. \end{aligned}$$ □

PROOF OF THEOREM 2.1. Fix $\varepsilon > 0$ and choose $a > 0$ so that Lemma 4.4 holds with some $p > 0$ and $B_k \subset \{z \in \mathbb{C}: \arg z \in (\pi - \varepsilon, 2\pi)\}$ for $k \in \mathbb{N}$.

Fix an $m \in \mathbb{N}$. Recall that $W(d) = \{z \in \mathbb{C}: \Im z = d\}$. Let

$$D_{21}(k) = \{z \in \mathbb{C}: |X(T(W(-1 - k2^{-m}))) - 2^{-m-1}i - z| \le 2^{-m-2}\},$$

$$D_{22}(k) = \{z \in \mathbb{C}: -2 - k2^{-m} < \Im z < -1 - k2^{-m}\}$$
$$\cup \bigcup (B_n + X(T(W(1 - k2^{-m})))),$$

where the union is taken over n such that

$$(B_n + X(T(W(-1 - k2^{-m})))) \cap X([0, T(W(-1 - k2^{-m}))]) = \varnothing,$$

$$G(k) = \max_{y \in D_{21}(k)} \left[\frac{E^y_{D_{22}(k)}(|\min_{t \in (0,R)} \Im X(T) + 1 + k2^{-m}|)}{P^y_{D_{22}(k)}(\Im X(R-) = -2 - k2^{-m})}\right],$$

$$A_k = \{G(k) \le q, T(W(-1 - (k+1)2^{-m})) \ge T(W(-1 - k2^{-m})) + 2^{-2m},$$
$$\text{and } X(T(W(-1 - k2^{-m})) + 2^{-2m}) \in D_{21}(k)\},$$

$$j_1 = \inf\{k \in \mathbb{N}: A_k \text{ holds}\},$$

$$S_1 = T(W(-1 - j_1 2^{-m})) + 2^{-2m},$$

$$T_1 = \inf\{t > S_1: X(t) \in \partial D_{22}(j_1)\},$$

$$N_1 = \min_{t \le T_1} \Im X(t).$$

Define by induction

$$\tilde{j}_k = \inf\{n > j_{k-1}\colon -1 - n2^{-m} < N_{k-1}\},$$
$$j_k = \inf\{n > j_{k-1}\colon -1 - n2^{-m} < N_{k-1} \text{ and } A_n \text{ holds}\},$$
$$S_k = T(W(-1 - j_k 2^{-m})) + 2^{-2m},$$
$$T_k = \inf\{t > S_k\colon X(t) \in \partial D_{22}(j_k)\},$$
$$N_k = \min_{t \le T_k} \Im X(t).$$

Let $\gamma = P^0(T(W(-2^{-m})) \ge 2^{-2m},\ |X(2^{-2m}) + 2^{-m-1}i| < 2^{-m-2})$. The constant $\gamma > 0$ does not depend on m, by scaling. Apply the strong Markov property at $T(W(-1 - k2^{-m}))$ and use Lemma 4.7 to see that

$$P^0(A_k) \ge (1 - c_9/q)\gamma$$

and

$$P^0(A_k^c) \le 1 - (1 - c_9/q)\gamma.$$

It follows that

$$E^0 \sum_{k=1}^{2^m} \mathbf{1}_{A_k^c} \le 2^m(1 - (1 - c_9/q)\gamma)$$
$$= 2^m(1 - \gamma + c_9\gamma/q)$$

and

$$P^0\left(\sum_{k=1}^{2^m} \mathbf{1}_{A_k^c} \ge 2^m(1 - \gamma + c_9\gamma/q)\alpha \right) \le 1/\alpha.$$

Choose some $\alpha > 1$ and $q < \infty$ so that $\beta \overset{\mathrm{df}}{=} (1 - \gamma + c_9\gamma/q)\alpha < 1$. Then

$$P^0\left(\sum_{k=1}^{2^m} \mathbf{1}_{A_k^c} \le 2^m\beta \right) \ge 1 - 1/\alpha. \tag{4.7}$$

Denote $V_k = -1 - j_k 2^{-m} - N_k + 2^{-m}$ and note that $V_k \le 2 \cdot 2^{-m} + \Im X(S_k) - N_k$. Let

$$D_{23}(j_k) = \{z \in \mathbb{C}\colon \Im z > -2 - j_k 2^{-m}\}$$
$$\setminus \{z \in \mathbb{C}\colon \Im z \ge -1 - j_k 2^{-m},\ \Re z = \Re X(T(W(-1 - j_k 2^{-m})))\}.$$

Then $D_{22}(j_k) \subset D_{23}(j_k)$ and

$$E^0 V_k \le 2 \cdot 2^{-m} + E^0(\Im X(S_k) - N_k)$$
$$\le 2 \cdot 2^{-m} + E_{D_{23}}^{X(S_k)}\Big(\Im X(S_k) - \min_{t \in (0, R)} \Im X(t) \Big)$$
$$\le 2 \cdot 2^{-m} + \max_{z \in D_{21}(j_k)} E_{D_{23}}^{z}\Big(\Im z - \min_{t \in (0, R)} \Im X(t) \Big)$$
$$= c_{10}.$$

The constant c_{10} does not depend on k and it is easy to see that $c_{10} \to 0$ as $m \to \infty$. Observe that the constants c_9 and γ and, consequently, α and β, may be chosen independently of m. Thus, for sufficiently large m there exists $n_1 = n_1(m)$ such that

$$(4.8) \qquad \left(\frac{1-\beta}{2}\right)\left(\frac{\alpha-1}{4\alpha}\right) \le \sum_{k=1}^{n_1} E^0 V_k \le \left(\frac{1-\beta}{2}\right)\left(\frac{\alpha-1}{2\alpha}\right)$$

and, therefore,

$$(4.9) \qquad P^0\left(\sum_{k=1}^{n_1} V_k \le (1-\beta)/2\right) \ge 1-(\alpha-1)/(2\alpha) = (\alpha+1)/(2\alpha).$$

If $\sum_{k=1}^{2^m} \mathbf{1}_{A_k^c} \le 2^m\beta$, then $\sum_{j_k \le 2^m}(j_k - \tilde{j}_k) \le 2^m\beta$. It follows from (4.7) that

$$(4.10) \qquad P^0\left(\sum_{j_k \le 2^m} \left(j_k - \tilde{j}_k\right) \le 2^m\beta\right) \ge 1 - 1/\alpha.$$

One has $2^{-m}(\tilde{j}_k - j_{k-1}) \le V_{k-1}$ so $\sum_{k=2}^{n_1}(\tilde{j}_k - j_{k-1}) \le \sum_{k=2}^{n_1} 2^m V_{k-1}$ and, by (4.9),

$$P^0\left(\sum_{k=2}^{n_1} \left(\tilde{j}_k - j_{k-1}\right) \le 2^m(1-\beta)/2\right) \ge (\alpha+1)/(2\alpha).$$

This and (4.10) imply that

$$P^0\left(\sum_{j_k \le 2^m} \left(j_k - \tilde{j}_k\right) \le 2^m\beta \text{ and } \sum_{k=1}^{n_1} \left(\tilde{j}_k - j_{k-1}\right) \le 2^m(1-\beta)/2\right) \ge (\alpha-1)/(2\alpha).$$

The event appearing in the above expression implies that $j_{n_1} < 2^m$ and, consequently, $\mathfrak{J}X(S_k) \ge -2$ for $k \le n_1$. Thus $P^0(\mathfrak{J}X(S_k) \ge -2) \ge (\alpha-1)/(2\alpha)$ for $k \le n_1$.

Recall that, by definition, A_{j_k} holds, $X(S_k) \in D_{21}(j_k)$ and $G(j_k) \le q$. Note that $\mathfrak{J}X(S_k) \le -1 - j_k 2^{-m} - 2^{-m-2}$ so $N_k \le -1 - j_k 2^{-m} - 2^{-m-2}$ and $V_k \le 8(-1 - j_k 2^{-m} - N_k)$. These facts, together with the strong Markov property applied at S_k, imply that

$$\begin{aligned}
&P^0(\mathfrak{J}X(S_k) \ge -2 \text{ and } -1 - j_k 2^{-m} - N_k = 1)\\
&\quad \ge (\alpha-1)/(2\alpha) P^0(-1 - j_k 2^{-m} - N_k = 1)\\
&\quad = E^0\left((\alpha-1)/(2\alpha) P_{D_{22}(k)}^{X(S_k)}(\mathfrak{J}X(R-) = -2 - j_k 2^{-m})\right)\\
&\quad \ge E^0\left((\alpha-1)/(2\alpha) q^{-1} E_{D_{22}(k)}^{X(S_k)}\left(\left|\min_{t\in(0,R)} \mathfrak{J}X(t) + 1 + j_k 2^{-m}\right|\right)\right)\\
&\quad \ge (\alpha-1)/(2\alpha) q^{-1}(1/8) E^0 V_k.
\end{aligned}$$

Denote

$$F_k = \left\{\mathfrak{J}X(S_k) \ge -2 \text{ and } -1 - j_k 2^{-m} - N_k = 1\right\}.$$

The events F_k are disjoint. Thus, by the left-hand side of (4.8),

$$P^0\left(\bigcup_{k=1}^{n_1} F_k\right) = \sum_{k=1}^{n_1} P^0(F_k)$$
$$\geq \sum_{k=1}^{n_1} E^0 V_k(\alpha - 1)/(16\alpha q)$$
$$\geq \left[\left(\frac{1-\beta}{2}\right)\left(\frac{\alpha-1}{4\alpha}\right)\right]\frac{\alpha - 1}{16\alpha q} \overset{\text{df}}{=} c_{11} > 0.$$

Note that c_{11} does not depend on m, at least for m large enough so that $n_1(m)$ is well defined. The event $\bigcup_{k=1}^{n_1} F_k$ implies the event

$$H(m) = \{\exists s \leq 2^{-2m} \exists d \in [-2, -1] \text{ such that}$$
$$X([0, T(W(d))]) \cap X([T(W(d)) + s, T(W(d-1))]) = \varnothing \text{ and}$$
$$X([T(W(d)) + s, T(W(d-1))]) \subset D_{24}(X(T(W(d))))\},$$

where

$$D_{24}(x) = \{z \in \mathbb{C}: \arg(z - x) \in [\pi - \varepsilon, 2\pi]\} \cup \{x\}.$$

The events $H(m)$ are decreasing as $m \to \infty$ and they all have P^0-probabilities greater or equal to c_{11}, so the same may be said about their intersection.

Let s_m and d_m be some random numbers (if they exist) which satisfy the definition of $H(m)$. By compactness, a subsequence of $\{d_m\}$ converges to a point $d_\infty \in [-2, -1]$. This and the continuity of Brownian paths imply that

$$\bigcap_{m=1}^{\infty} H(m) \subset \{\exists d_\infty \in [-2, -1] \exists t > 0 \text{ such that } \mathfrak{I}X(t) = d_\infty,$$
$$\mathfrak{I}X(s) \geq d_\infty \text{ for all } s < t,$$
$$X([0, t)) \cap X((t, T(W(d_\infty - 1))]) = \varnothing \text{ and}$$
$$X((t, T(W(d_\infty - 1))]) \subset D_{24}(X(t))\}$$
$$\overset{\text{df}}{=} H(\infty)$$

and $P^0(H(\infty)) \geq c_{11} > 0$.

Suppose that $H(\infty)$ holds and $X(s) = X(t)$ for some $s \neq t$, $s \in [0, T(W(d_\infty - 1))]$. Then, for some $\varepsilon_1 > 0$, each set $X([s - \varepsilon_1, s])$, $X([s, s + \varepsilon_1])$, $X([t - \varepsilon_1, t])$ and $X([t, t + \varepsilon_1])$ would lie in a cone with the vertex $X(t)$ and opening not greater than $\pi + \varepsilon < 2\pi$. It follows easily from Theorem 1 of Evans (1985) that this event has probability 0. Thus $P^0(H(\infty)$ and $X(t) \neq X(s)$ for all $s \in [0, T(W(d_\infty - 1))]$, $s \neq t) \geq c_{11} > 0$ and this essentially completes the proof.

In order to translate this result into the statement given in Theorem 1, one may use standard techniques, such as scaling and the strong Markov property. □

Proof of Theorem 2.2. A statement somewhat stronger than Theorem 2.2 will be proved, in preparation for the proof of Theorem 2.3.

Let $D_1 = \{z \in \mathbb{C}: |z| < 1\}$. The distribution of the process

$$\left\{Y(t) \stackrel{\text{df}}{=} X(T(\partial D_1 - t)),\ t \in (0, T(\partial D_1))\right\}$$

under P^0 will be called Q. By the time reversal, the process Y is an h-process in D_1 with $h(x) = -\log|x|$ and with the initial distribution uniform on ∂D_1 [see Doob (1984), 3 III 2].

Fix some $a \in (0,1)$ and denote

$$\begin{aligned}
D_2(x) &= \{z \in \mathbb{C}: |x|/4 < |z| < 2|x|, |\arg x - \arg z| < \pi/4\},\\
T_1 &= \inf\{t > 0: |Y(t)| = a/2\},\\
U_1 &= \inf\{t > 0: |Y(t)| = a/8\},\\
A_1 &= \big\{Y([T_1, U_1)) \subset D_2(Y(T_1)),\\
&\qquad \exists\, t \in (T_1, U_1) \text{ such that } |Y(t)| \in (a/4, a/2),\\
&\qquad Y([T_1, t)) \cap Y((t, U_1]) = \varnothing, Y^{-1}(Y(t)) \cap [T_1, U_1] = \{t\},\\
&\qquad |Y(s)| > a/4 \text{ for } s \in [T_1, t], |Y(s)| < a/2 \text{ for } s \in [t, U_1]\big\}.
\end{aligned}$$

If A_1 holds, then let

$$V_1 = \inf\{t > U_1: |Y(t)| = a/4\}$$

and if $V_1 < \infty$, then let

$$M_1 = \inf_{t \in (U_1, V_1)} |Y(t)|/2.$$

If A_1 does not hold, then let $M_1 = a/16$.

Now define some more objects inductively, for $k \geq 1$. If $V_k = \infty$, then do not define any new objects with the subscript $k+1$. Otherwise, let

$$\begin{aligned}
T_{k+1} &= \inf\{t > 0: |Y(t)| = M_k/2\},\\
U_{k+1} &= \inf\{t > 0: |Y(t)| = M_k/8\},\\
A_{k+1} &= \big\{Y([T_{k+1}, U_{k+1})) \subset D_2(Y(T_{k+1})),\\
&\qquad \exists\, t \in (T_{k+1}, U_{k+1}) \text{ such that } |Y(t)| \in (M_k/4, M_k/2),\\
&\qquad Y([T_{k+1}, t)) \cap Y((t, U_{k+1}]) = \varnothing, Y^{-1}(Y(t)) \cap [T_{k+1}, U_{k+1}] = \{t\},\\
&\qquad |Y(s)| > M_k/4 \text{ for } s \in [T_{k+1}, t], |Y(s)| < M_k/2 \text{ for } s \in [t, U_{k+1}]\big\}.
\end{aligned}$$

If A_{k+1} holds, then let

$$V_{k+1} = \inf\{t > U_{k+1}: |Y(t)| = M_k/4\}$$

and if $V_{k+1} < \infty$, then let

$$M_{k+1} = \inf_{t \in (U_{k+1}, V_{k+1})} |Y(t)|/2.$$

If A_{k+1} does not hold, then let $M_{k+1} = M_k/16$.

The scaling property implies that $Q(A_k^c$ or $V_k < \infty | V_{k-1} < \infty)$ does not depend on k. It follows easily from Theorem 2.1 that this conditional probability is strictly less than 1, equal to, say, $p < 1$. The random times T_k are stopping times with respect to the filtration generated by Y. Apply the strong Markov property at these stopping times to see that

$$Q\left(\bigcap_{k=1}^{n} (A_k^c \text{ or } V_k < \infty)\right) = p^n.$$

Let $n \to \infty$ to obtain

$$Q(\exists\, k \in \mathbb{N}\colon A_k \text{ and } V_k = \infty) = 1.$$

The event $(A_k$ and $V_k = \infty)$ implies that there exist t and $x \in D_1$ such that

$$\begin{gathered} |x| < a, \\ |Y(t)| \in (|x|/2, |x|), \\ Y([0,t)) \cap Y((t,\rho)) = \varnothing, \\ Y^{-1}(Y(t)) \cap [0,\rho] = \{t\}, \qquad (4.11) \\ |Y(s)| > |x|/2 \quad \text{for } s \in [0,t], \\ |Y(s)| < |x| \quad \text{for } s \in [t,\rho), \\ Y([0,\rho)) \cap \{z \in \mathbb{C}\colon |x|/2 < |z| < |x|\} \subset \{z \in \mathbb{C}\colon |\arg z - \arg x| < \pi/4\}, \end{gathered}$$

where $\rho = T(\partial D_1)$. With Q-probability 1, simultaneously for all rational $a \in (0,1)$, such pairs (t,x) exist. The continuity of Brownian paths shows that for small $a > 0$, t is arbitrarily close to ρ. In terms of the original process X, this says that P^0-a.s., for every $\varepsilon > 0$ there exists $t \in (0,\varepsilon)$ such that $X^{-1}(X(t)) \cap [0, T(\partial D_1)] = \{t\}$ and

$$X([0,t)) \cap X((t, T(\partial D_1)]) = \varnothing.$$

By scaling, a similar result holds for each hitting time $T_k = \inf\{t > 0\colon |X(t)| = k\}$ in place of $T(\partial D_1)$. Since $1 < T_k$ for some $k \in \mathbb{N}$ P^0-a.s., the theorem follows. □

Proof of Theorem 2.3. Let $D_1 = \{z \in \mathbb{C}\colon |z| < 1\}$ and suppose that $x_n \in D_1$ for $n \in \mathbb{N}$, $|x_{n+1}| < |x_n|/2$. Denote

$$S_n = \{z \in \mathbb{C}\colon |z| = |x_n|\},$$

$$\tilde{S}_n = \{z \in S_n\colon |\arg z - \arg(-x_n)| < \pi/4\}.$$

First it will be proved that for every $y \in D_1$, $y \neq 0$,

$$P_h^y\left(\exists\, n\colon T(S_n) = T(\tilde{S}_n),\ |x_n| < |y|\right) = 1, \qquad (4.12)$$

where $h(z) = -\log|z|$ for $z \in D_1$.

Fix some $y \in D_1$, $y \neq 0$, and find m such that $|x_m| < |y|/2$. Let μ be the uniform probability distribution on $S = \{z \in \mathbb{C}\colon |z| = |y|\}$. Then $P_h^\mu(T(S_m) = T(\tilde{S}_m)) = 1/4$, by symmetry. It follows that $P_h^x(T(S_m) = T(\tilde{S}_m))| \geq 1/4$ for some

$x \in S$. The Harnack inequality applied in $\{z \in \mathbb{C}: |y|/2 < |z| < 2|y|\}$ implies that

$$P^y(T(S_m) = T(\tilde{S}_m)) \geq c_1 P^x(T(S_m) = T(\tilde{S}_m))$$

and

$$P^y(T(S_m) < T(\partial D_1)) \leq (1/c_1) P^x(T(S_m) < T(\partial D_1)),$$

where $c_1 > 0$ may be chosen independently of y, x and m. Then

$$\begin{aligned} P_h^y(T(S_m) = T(\tilde{S}_m)) &= P^y(T(S_m) = T(\tilde{S}_m))/P^y(T(S_m) < T(\partial D_1)) \\ &\geq c_1^2 P^x(T(S_m) = T(\tilde{S}_m))/P^x(T(S_m) < T(\partial D_1)) \\ &= c_1^2 P_h^x(T(S_m) = T(\tilde{S}_m)) \\ &\geq c_1^2/4 > 0. \end{aligned}$$

By analogy, $P_h^x(T(S_n) = T(\tilde{S}_n)) \geq c_1^2/4$ for $x \in S_{n-1}$, $n - 1 > m$. Apply the strong Markov property at the hitting times of S_n's to see that

$$P_h^y\left(\bigcap_{k=m}^{n} \{T(S_k) \neq T(\tilde{S}_k)\} \right) \leq (1 - c_1^2/4)^{n-m}.$$

Let $n \to \infty$ to obtain (4.12).

Suppose that X has the distribution P_h^μ and Z is an independent, standard two-dimensional Brownian motion, starting from 0. Denote $T_Z = \inf\{t > 0: |Z(t)| = 1\}$. It has been shown in the proof of Theorem 2.2 [see (4.11)] that a.s. there exist sequences $\{x_n\}$ and $\{t_n\}$ such that

$$\begin{gathered} |x_{n+1}| < |x_n|/2, \\ |Z(t_n)| \in (|x_n|/2, |x_n|), \\ Z([0, t_n)) \cap Z((t_n, T_Z]) = \varnothing, \\ Z^{-1}(Z(t_n)) \cap [0, T_Z] = \{t_n\}, \\ |Z(s)| < |x_n| \quad \text{for } s \in [0, t_n], \\ |Z(s)| > |x_n|/2 \quad \text{for } s \in [t_n, T_Z], \\ Z([0, T_Z]) \cap \{z \in \mathbb{C}: |x_n|/2 < |z| < |x_n|\} \subset \{z \in \mathbb{C}: |\arg z - \arg x_n| < \pi/4\}. \end{gathered}$$

Fix a "typical" path of Z, such that there exist sequences $\{x_n\}$ and $\{t_n\}$ satisfying the above conditions. Choose some $a \in (0, 1)$ and recall the definitions of S_n and $\tilde{S}_n$ from the beginning of the proof.

The next part of the proof is very similar to the proof of Theorem 2.2. Let

$$\begin{aligned} D_2(x) &= \{z \in \mathbb{C}: |x|/4 < |z| < 2|x|, |\arg x - \arg z| < \pi/4\}, \\ T_1 &= \inf\{t > 0: t = T(S_n) = T(\tilde{S}_n), |x_n| < a/2\}. \end{aligned}$$

Let n_1 be defined simultaneously with T_1 by $|X(T_1)| = |x_{n_1}|$.

$$U_1 = \inf\{t > T_1: |X(t)| = |x_{n_1}|/4\},$$

$$\begin{aligned} A_1 = \Big\{ & X([T_1, U_1)) \subset D_2(X(T_1)), \\ & \exists\, t \in (T_1, U_1) \text{ such that } |X(t)| \in \big(|x_{n_1}|/2, |x_{n_1}|\big), \\ & X([T_1, t)) \cap X((t, U_1]) = \varnothing, \\ & X^{-1}(X(t)) \cap [T_1, U_1] = \{t\}, \\ & |X(s)| > |x_{n_1}|/2 \text{ for } s \in [T_1, t], \\ & |X(s)| < |x_{n_1}| \text{ for } s \in [t, U_1]\Big\}. \end{aligned}$$

If A_1 holds, then let

$$V_1 = \inf\{t > U_1: |X(t)| = |x_{n_1}|/2\}$$

and if $V_1 < \infty$, then let

$$M_1 = \inf_{t \in (U_1, V_1)} |X(t)|/2.$$

If A_1 does not hold, then let $M_1 = |x_{n_1}|/8$.

Make the following inductive definitions for $k \geq 1$, unless A_k and $\{V_k = \infty\}$ hold:

$$T_{k+1} = \inf\{t > 0: t = T(S_n) = T(\tilde{S}_n), |x_n| < M_k\},$$

n_{k+1} is defined by $|x_{n_{k+1}}| = |X(T_{k+1})|$,

$$U_{k+1} = \inf\{t > T_{k+1}: |X(t)| = |x_{n_{k+1}}|/4\},$$

$$\begin{aligned} A_{k+1} = \Big\{ & X([T_{k+1}, U_{k+1})) \subset D_2(X(T_{k+1})), \\ & \exists\, t \in (T_{k+1}, U_{k+1}) \text{ such that } |X(t)| \in \big(|x_{n_{k+1}}|/2, |x_{n_{k+1}}|\big), \\ & X([T_{k+1}, t)) \cap X((t, U_{k+1}]) = \varnothing, \\ & X^{-1}(X(t)) \cap [T_{k+1}, U_{k+1}] = \{t\}, \\ & |X(s)| > |x_{n_{k+1}}|/2 \text{ for } s \in [T_{k+1}, t], \\ & |X(s)| < |x_{n_{k+1}}| \text{ for } s \in [t, U_{k+1}]\Big\}. \end{aligned}$$

If A_{k+1} holds, then let

$$V_{k+1} = \inf\{t > U_{k+1}: |X(t)| = |x_{n_{k+1}}|/2\}$$

and if $V_{k+1} < \infty$, then let

$$M_{k+1} = \inf_{t \in (U_{k+1}, V_{k+1})} |X(t)|/2.$$

If A_{k+1} does not hold, then let $M_{k+1} = |x_{n_{k+1}}|/8$.

Theorem 2.1 and (4.12) imply that

$$P_h^\mu(A_k^c \text{ or } V_k < \infty | V_{k-1} < \infty) = p < 1,$$

where p does not depend on k, by scaling. This implies, as in the proof of Theorem 2.2, that

$$P_h^\mu(\exists\, k \in \mathbb{N}: A_k \text{ and } V_k = \infty) = 1.$$

It is elementary to check that the event $(A_k$ and $V_k = \infty)$ implies that there exist s and t such that $|Z(s)| < a$, $|X(t)| < a$, $Z^{-1}(Z(s)) \cap [0, T_Z] = \{s\}$, $X^{-1}(X(t)) \cap [0, R) = \{t\}$ and

$$(Z([0,s)) \cup X((t,R))) \cap (Z((s,T_Z]) \cup X((0,t))) = \varnothing.$$

By the time reversal applied to X, as in the proof of Theorem 2.2, one obtains the following result.

Let Y_1 and Y_2 be independent standard Brownian motions with $Y_1(0) = Y_2(0) = 0$. Let

$$T_1^k = \inf\{t > 0: |Y_1(t)| = k\},$$
$$T_2^k = \inf\{t > 0: |Y_2(t)| = k\}.$$

Then with probability 1, for every rational $a > 0$ there exist $s > 0$ and $t > 0$ such that $|Y_1(s)| < a$, $|Y_2(t)| < a$, $Y_1^{-1}(Y_1(s)) \cap [0, T_1^1] = \{s\}$, $Y_2^{-1}(Y_2(t)) \cap [0, T_2^1] = \{t\}$ and

$$(4.13) \qquad (Y_1([0,s)) \cup Y_2([0,t))) \cap \left(Y_1((s,T_1^1]) \cup Y_2((t,T_2^1])\right) = \varnothing.$$

The same holds if T_1^1 and T_2^1 are replaced by T_1^k and T_2^k, by scaling.

Now let X have the distribution P^0 and let

$$\tilde{Y}_1(t) = X(1/2 + t) - X(1/2),$$
$$\tilde{Y}_2(t) = X(1/2 - t) - X(1/2).$$

The processes $(\tilde{Y}_1(t),\ t \in [0,1/2])$ and $(\tilde{Y}_2(t),\ t \in [0,1/2])$ are independent standard Brownian motions, $\tilde{Y}_1(0) = \tilde{Y}_2(0) = 0$. With probability 1, $T_1^k \geq 1/2$ and $T_2^k \geq 1/2$ for some $k \in \mathbb{N}$. Thus (4.13) applies also to $\tilde{Y}_1$ and $\tilde{Y}_2$. In other words, with P^0-probability 1, for every rational $a > 0$ there exist $s > 0$ and $t > 0$ such that $|\tilde{Y}_1(s)| < a$, $|\tilde{Y}_2(s)| < a$, $\tilde{Y}_1^{-1}(\tilde{Y}_1(s)) \cap [0,1/2] = \{s\}$, $\tilde{Y}_2^{-1}(\tilde{Y}_2(t)) \cap [0,1/2] = \{t\}$ and

$$\left(\tilde{Y}_1([0,s)) \cup \tilde{Y}_2([0,t))\right) \cap \left(\tilde{Y}_1((s,1/2]) \cup \tilde{Y}_2((t,1/2])\right) = \varnothing.$$

By continuity of Brownian paths and their point nonrecurrence, the times s and t are arbitrarily close to 0, for small $a > 0$. This completes the proof. □

PROOF OF COROLLARY 2.1. The corollary follows immediately from Theorems 2.2 and 2.3 and the fact that the orthogonal projection of the three-dimensional Brownian motion on a plane is a two-dimensional Brownian motion. □

PROOF OF COROLLARY 2.2. It is elementary to check that the Sierpiński carpet has not cut points [see Mandelbrot (1982), Section 14]. The result follows from Theorem 2.2. □

PROOF OF COROLLARY 2.3. Let X have the distribution P^0 and $Z(t) = X(t) - tX(1)$. It is easy to see that the distributions of $\{X(t),\ t \in [1/4, 3/4]\}$ and $\{Z(t),\ t \in [1/4, 3/4]\}$ are mutually absolutely continuous. By Theorem 2.3, for each $\varepsilon > 0$ there exist $s \in (1/2 - \varepsilon, 1/2)$ and $t \in (1/2, 1/2 + \varepsilon)$ such that $Z^{-1}(Z(s)) \cap [0,1] = \{s\}$, $Z^{-1}(Z(t)) \cap [0,1] = \{t\}$ and

$$Z([1/4, s) \cup (t, 3/4]) \cap Z((s,t)) = \varnothing.$$

With probability 1, the distance between $Z(1/2)$ and $Z([0,1/4] \cup [3/4,1])$ is greater than 0. It follows easily that there exist s and t, $0 < s \le 1/2 < t \le 1$, such that $Z^{-1}(Z(s)) \cap [0,1] = \{s\}$, $Z^{-1}(Z(t)) \cap [0,1] = \{t\}$ and

$$Z([0, s) \cup (t, 1]) \cap Z((s,t)) = \varnothing.$$

This means that with probability 1, the set $Z([0,1])$ becomes disconnected after removing certain two points. The Sierpiński carpet does not have this property. □

REMARKS. (i) The above results raise many questions.

(a) Is the set of all cut points uncountable? What is the Hausdorff dimension of this set? The Associate Editor suggested that the methods of Orey and Taylor (1974) are likely to give an affirmative answer to the first question.

(b) Are there any cut points which are *not* two-sided cone points at the same time? Common sense suggests that such points exist, since by relaxing, in a sense, the condition one makes it more likely for a point to exist. Supplying a rigorous proof, however, does not seem trivial.

(c) One cannot extend Theorem 2.1 to $\varepsilon = 0$ because this would mean that the one-dimensional Brownian motion $\Im X(t)$ had a point of decrease, which is impossible [Dvoretzky, Erdös and Kakutani (1961)]. On may ask, however [Taylor (1986), Problem 8], whether there exist a random straight line L and $s \in (0,1)$ such that $X([0,s))$ and $X((s,1])$ lie on the opposite sides of L. Shimura (1988) has some results related to this problem.

(d) Theorem 2.3 implies that for every $t \in (0,1)$, the ramification order of the point $X(t)$ of the set $X([0,1])$ is equal to 2 a.s. [see Blumenthal and Menger (1970), Section 13.2, or Mandelbrot (1982), Section 14, for the definition of the ramification order]. This implies that the Hausdorff dimension of the set of all points of order 2 is 2 a.s. It seems that $X([0,1])$ contains a.s. points with ramification order greater than 2, even uncountably infinite. For $k > 2$, what is the Hausdorff dimension of the set of all points of order k?

(e) A related, possibly much more difficult task is to find a nice topological description of the Brownian trace $X([0,1])$. Is it possible to find a nonrandom set A with a simple geometric definition such that $X([0,1])$ is homeomorphic to A

a.s.? It is not obvious whether any nonrandom set A (not necessarily "simple") has this property.

(f) Does the Brownian path contain any double cut points, that is, are there s and t, $0 < s < t < 1$, such that $X(s) = X(t)$ and $X([0, s) \cup (t, 1]) \cap X((s, t)) = \varnothing$? This is related to the question whether the "self-avoiding Brownian motion" is self-avoiding, that is, whether it is homeomorphic to a circle. Mandelbrot (1982) defines a "self-avoiding Brownian motion" as the boundary of the unbounded connected component of the complement of the Brownian loop.

(ii) The existence of cut points is closely related to the problem of nonintersection of two independent Brownian motions X and Y starting at a close distance, say $|X(0) - Y(0)| = \varepsilon$. One may be interested in the rate with which the probability of $\{X([0,1]) \cap Y([0,1]) = \varnothing\}$ goes to 0 as $\varepsilon \to 0$. Lawler (1985, 1986) has many results in this area.

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REFERENCES

ADELMAN, O. (1985). Brownian motion never increases: A new proof to a result of Dvoretzky, Erdös and Kakutani. *Israel J. Math.* **50** 189–192.

BLUMENTHAL, L. M. and MENGER, K. (1970). *Studies in Geometry*. Freeman, San Francisco.

BURDZY, K. (1985). Brownian paths and cones. *Ann. Probab.* **13** 1006–1010.

BURDZY, K. (1987a). Geometric properties of the complement of 2-dimensional Brownian path. *Probab. Theory Related Fields*. To appear.

BURDZY, K. (1987b). *Multidimensional Brownian Excursions and Potential Theory*. Longman, London.

CRANSTON, M., HSU, P. and MARCH, P. (1989). Smoothness of the convex hull of planar Brownian motion. *Ann. Probab.* **17** 144–150.

DAHLBERG, B. E. J. (1977). Estimates of harmonic measure. *Arch. Rational Mech. Anal.* **65** 275–288.

DAVIS, B. (1983). On Brownian slow points. *Z. Wahrsch. verw. Gebiete* **64** 359–367.

DOOB, J. L. (1984). *Classical Potential Theory and Its Probabilistic Counterpart*. Springer, New York.

DVORETZKY, A., ERDÖS, P. and KAKUTANI, S. (1950). Double points of paths of Brownian motion in n-space. *Acta Sci. Math. (Szeged)* **12** 75–81.

DVORETZKY, A., ERDÖS, P. and KAKUTANI, S. (1961). Nonincrease everywhere of the Brownian motion process. *Proc. Fourth Berkeley Symp. Math. Statist. Probab.* **2** 103–116. Univ. California Press.

EL BACHIR, M. (1983). L'enveloppe convexe du mouvement brownien. Thèse 3-ème cycle. Université Toulouse III, France.

EVANS, S. N. (1985). On the Hausdorff dimension of Brownian cone points. *Math. Proc. Cambridge Philos. Soc.* **98** 343–353.

HUNT, R. A. and WHEEDEN, R. L. (1970). Positive harmonic functions on Lipschitz domains. *Trans. Amer. Math. Soc.* **147** 507–527.

LAWLER, G. F. (1985). Intersections of simple random walks. In *Particle Systems, Random Media and Large Deviations* (R. Durrett, ed.). *Contemp. Math.* **41** 281–289. Amer. Math. Soc., Providence, R.I.

LAWLER, G. F. (1986). Gaussian behavior of loop-erased self-avoiding random walk in four dimensions. *Duke Math. J.* **53** 249–269.

LE GALL, J.-F. (1986). Mouvement brownien plan, cônes et processus stables. *C. R. Acad. Sci. Paris. Sér. I Math.* **302** 641–643.

LE GALL, J.-F. (1987). Mouvement brownien, cônes et processus stables. *Probab. Theory Related Fields* **76** 587–627.

MANDELBROT, B. B. (1982). *The Fractal Geometry of Nature*. Freeman, New York.

MOUNTFORD, T. S. (1987). On the asymptotic number of small components created by planar Brownian motion. Preprint.

OREY, S. and TAYLOR, S. J. (1974). How often on a Brownian path does the law of iterated logarithm fail? *Proc. London Math. Soc.* (3) **28** 174–192.

SHIMURA, M. (1984). A limit theorem for two-dimensional conditional random walk. *Nagoya Math. J.* **95** 105–116.

SHIMURA, M. (1985). Excursions in a cone for two-dimensional Brownian motion. *J. Math. Kyoto Univ.* **25** 433–443.

SHIMURA, M. (1988). Meandering points of two-dimensional Brownian motion. *Kodai Math. J.* **11** 169–176.

TAYLOR, S. J. (1986). The measure theory of random fractals. *Math. Proc. Cambridge Philos. Soc.* **100** 383–406.

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