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KOLMOGOROV AND THE THEORY OF MARKOV PROCESSES

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1. Beginning. In 1906–1907 Markov [59, 60] discovered that limit theorems for independent random variables can be extended to variables "connected in a chain." About the same time Einstein [31] started to study mathematically a physical phenomenon—the Brownian motion. The synthesis of both directions in Kolmogorov's celebrated paper [K28][1] was the beginning of the theory of Markov processes. (Kolmogorov called them stochastically determined processes. The name Markov process was suggested in 1934 by Khintchine.)

Today we distinguish Markov transition functions from measures on path space which can be constructed starting from such functions and which we call Markov processes. Measure theory on functional spaces did not exist in 1931, and Kolmogorov [K28] deals with Markov transition functions; paths are used only as a heuristic background to motivate definitions and assumptions.

2. Kolmogorov's program. In modern terms, Kolmogorov introduced a Markov transition function as a family of stochastic kernels $P(s, x; t, E)$ such that

$$\int P(s, x; t, dy) P(t, y; u, E) = P(s, x; u, E) \quad \text{for every } s < t < u. \tag{1}$$

Formula (1) is usually called the Chapman–Kolmogorov equation. Kolmogorov himself used the name of Smoluchowski who had written (1) in a special situation. After discussing various particular cases, Kolmogorov showed that the ergodic principle established by Markov for chains holds for broad classes of general transition functions.

The central idea of the paper is the introduction of local characteristics at each time t and the construction of transition functions by solving certain differential equations involving these characteristics. If the state space is at most countable, then the local characteristic at time t is given by a matrix $A_{jk}(t)$, and the corresponding differential equations have the form

$$\frac{\partial}{\partial t} P_{ii}(s, t) = \sum_j P_{ij}(s, t) A_{jk}(t) \tag{2}$$

and

$$\frac{\partial}{\partial s} P_{ik}(s, t) = -\sum_j A_{ij}(s) P_{jk}(s, t). \tag{3}$$

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[1] Reference citations preceded by K refer to the list of Kolmogorov's publications on pages 945–964.

Following Feller, we call (2) and (3) the Kolmogorov forward and backward differential equations.

The last part of [K28] is devoted to a class of transition functions in $\mathbb{R}$ for which there exist limits

$$A(t,x) = \lim_{\delta \to 0} \delta^{-1} \int_{\mathbb{R}} (y-x) P(t,x;t+\delta,dy)$$

and

$$B(t,x) = \lim_{\delta \to 0} (2\delta)^{-1} \int_{\mathbb{R}} (y-x)^2 P(t,x;t+\delta,dy).$$

Feller suggested for these limits the names drift and diffusion coefficients. A property of the third moments needs to be postulated to exclude the possibility of jumps. Assuming, in addition, that the density function $f(s,x;t,y)$ of the measure $P(s,x;t,dy)$ is sufficiently smooth, Kolmogorov proves that it satisfies the forward differential equation

$$\begin{aligned} \frac{\partial}{\partial t} f(s,x;t,y) = &-\frac{\partial}{\partial y}[A(t,y) f(s,x;t,y)] \\ &+ \frac{\partial^2}{\partial y^2}[B(t,y) f(s,x;t,y)] \end{aligned} \tag{4}$$

and the backward differential equation

$$\frac{\partial}{\partial s} f(s,x;t,y) = -A(s,x)\frac{\partial}{\partial x} f(s,x;t,y) - B(s,x)\frac{\partial^2}{\partial x^2} f(s,x;t,y). \tag{5}$$

The equation (4) arises if one is interested in the time evolution of the probability distribution, and, in fact, a special form of (4) appeared earlier in papers of Fokker [37] and Planck [65]. Kolmogorov was not familiar with these papers in 1931, but since 1934 he called (4) the Fokker–Planck equation. The backward equation (5) had never appeared before [K28] even in the physics literature.

At the end of the paper, Kolmogorov demonstrates on examples that, for processes with jumps, integral terms must be added to the right sides of (4) and (5). He also writes the following multidimensional analogs for equations (4) and (5):

$$\frac{\partial f}{\partial t} = -\sum_i \frac{\partial (A_i f)}{\partial y_i} + \sum_{i,j} \frac{\partial^2 (B_{ij} f)}{\partial y_i \, \partial y_j} \tag{4a}$$

and

$$\frac{\partial f}{\partial s} = -\sum_i A_i \frac{\partial f}{\partial x_i} - \sum_{i,j} B_{ij} \frac{\partial^2 f}{\partial x_i \, \partial x_j}. \tag{5a}$$

(With the coefficients depending only on time, these equations appeared first in 1900 in a paper of Bachelier [2] who already had an intuitive idea of the Brownian motion.)

3. Progress in study of Markov transition functions. The program outlined in [K28] influenced deeply the development of probability theory and related topics in analysis during the next decades. Construction of transition functions corresponding to given drift and diffusion coefficients motivated the work in PDE on fundamental solutions of parabolic differential equations. The first step was taken by Kolmogorov himself in [K45] where he investigated (4a) and (5a) on compact Riemannian manifolds. Substantial progress in the study of these equations (and their generalizations with integral terms) was accomplished by Feller [33].

Separate local characteristics were introduced in [K28] for various concrete classes of transition functions. In 1934 Kolmogorov [K58] suggested the following unification. A Markov transition function $P(s, x; t, E)$ can be interpreted as a family of operators T_t^s which transform probability distributions at time s to probability distributions at time $t > s$. The equation (1) can be rewritten in the form $T_t^s T_u^t = T_u^s$ for $s < t < u$. In the stationary case, the operators $T_t^s = T_{t-s}$ satisfy the equation $T_s T_t = T_{s+t}$, and one can expect that $T_t = e^{tA}$ for some linear (in general, unbounded) operator A. The infinitesimal operator (or the generator) A can be defined by the formula

$$A = \lim_{t \downarrow 0} (T_t - I)/t,$$

where I is the identity operator. (In the nonstationary case an analogous limit depends on s and its existence is a certain regularity assumption.) Kolmogorov has not given any precise definition of the generator A. This was a done later. Generators became an important tool in the theory of Markov processes after Feller's work [34] on one-dimensional diffusions. It turned out to be more fruitful to interpret the T_t^s as operators

$$T_t^s f(x) = \int P(s, x; t, dy) f(y)$$

acting on functions rather than on measures. Feller restricted himself to semigroups T_t which preserve the space of continuous functions. In general, T_t can be considered as acting on the space of all bounded measurable functions.

A stationary Markov transition function $P_t(x, dy)$ is called symmetric (with respect to a measure m) if

$$\int_A m(dx) P_t(x, B) = \int_B m(dx) P_t(x, A)$$

for all t, A and B. For such functions, the semigroup T_t preserves the space $L^2(m)$ and a natural substitute for the infinitesimal operator A is the Dirichlet form $\mathscr{E}(f, g)$ which is an extension of the bilinear form $-\int f A g \, dm$. The theory of symmetric Markov processes based on using Dirichlet forms, started by Fukushima, has been developed by a number of authors (see the bibliography in [39]).

The symmetry of a transition function $P_t(x, dy)$ is closely related to the time reversibility of the corresponding Markov process. Kolmogorov [K85] investigated these properties for diffusions on compact differentiable manifolds. Under mild restrictions on the coefficients A_i and B_{ij}, he proved the existence and uniqueness of a stationary probability distribution and he evaluated the backward transition function $\hat{P}_t(x, dy)$. Then he established necessary and sufficient conditions for time reversibility which he defined by the equation $\hat{P} = P$. To find these conditions, he rewrites (5a) in an invariant form by introducing the Riemannian metric associated with B_{ij} and by using the covariant derivatives in this metric. Time reversal of Markov chains had been introduced earlier in [K70]. (Still earlier Schrödinger [69] investigated time reversal for some one-dimensional diffusions.)

An important class of degenerate diffusions [with $\det(B_{ij}) = 0$] appeared in [K57]. Let a physical system be described by n coordinates $q_1, \ldots, q_n$ and n velocities $\dot{q}_1, \ldots, \dot{q}_n$. Since the velocities determine the change of coordinates during any time interval, the process is a diffusion in a $2n$-dimensional space for which the matrix B_{ij} has rank n. (In the simplest case of a free Brownian particle, such a model was first introduced by Uhlenbeck and Ornstein in [75]). Time reversal for the processes studied in [K57] was the subject of the dissertation [82] of Yaglom, who was at that time Kolmogorov's student. An analytic study of degenerate parabolic PDE was started in the thesis of Piskunov [64], another student of Kolmogorov. A survey of recent progress in this field can be found in [38].

4. Stochastic processes and measures on functional spaces. The interaction between Markov processes and PDE is much richer than just the relationship between transition functions and their local characteristics. Already in 1933, Kolmogorov and Leontovich [K42] evaluated the probability of hitting a unit disk by the planar Brownian motion during time t and the probability distribution of the position at the hitting time. These probabilities, as functions of the initial state, satisfy the heat equation outside the disk, with certain boundary conditions. The authors used this result to express the mathematical expectation of the area for the Wiener sausage and to get an asymptotic formula as $t \to \infty$ for it (a problem proposed by S. I. Vavilov). Many interesting results on the size of the Wiener sausage have been obtained by the next generations of mathematicians (see [54] for references).

In the beginning, probabilistic intuition played about the same role in the theory of stochastic processes as physical intuition in analytical mechanics or mathematical physics. Only after a rigorous theory of measure on functional spaces was developed, did probabilistic arguments gain the power of mathematical proof. The first construction of a measure on the space of continuous functions is due to Wiener [79]. However, this construction is rather special and not applicable to more general situations. Only Kolmogorov's book [K40] and, especially, his theorem on measures on infinite products, have given a start to a general theory. Still Kolmogorov's theorem leads to a measure on the space of all functions without any regularity properties and a fundamental question was how to restrict it to a space of decent functions without changing the finite-dimensional distributions. One of the first results in this direction was Kolmogorov's criterion of continuity in terms of two-dimensional distributions. (Kolmogorov's proof was first published in [72].) The key role in developing measure theory on functional spaces was played by Doob (see [13, 16]). Doob's monograph [17] opened a new period in theory of stochastic processes.

Two classes of processes play a special role: processes with independent increments and martingales. The study of the first class has been inspired by their relationship to sums of independent random variables, and it has been facilitated by the invariance of the set of exponential functions under the corresponding semigroups. Pioneering results were obtained by de Finetti in 1929. In 1932, Kolmogorov [K34, K35] described all homogeneous processes with independent increments and finite variance. The general description is due to Lévy. Soon thereafter Khintchine gave a simple direct proof of Lévy's formula. (See [41] and [55] for references.)

Martingale theory was initiated and developed by Doob and substantially advanced by Meyer and others (some elements can be found in earlier work of Lévy and Ville; see bibliographies in [17], [47], [61] or [80]). The theory is based on the general conditioning introduced in [K40]. Kolmogorov also proved an important estimate for the maximum of a submartingale in terms of the expectation at the final moment (he worked with sums S_n of independent random variables, but used only the fact that $(S_n - ES_n)^2$ is a submartingale; the extension to what we call now submartingales is due to Bernstein [3]).

In the theory of Markov processes, measures corresponding to all possible initial times t and states x are considered and the dependence of probabilities on t and x plays a crucial role. Dealing simultaneously with many measures presents some additional problems which have been discussed in [23, 24] and, more recently, in [71].

5. Stochastic analysis. Progress in the theory of stochastic processes made it possible to apply direct probabilistic arguments for solving analytical problems. Stochastic analysis has evolved (based on the integration on functional spaces), thereby enriching classical analytical methods in mathematics and creating new possibilities for applications to science and engineering.

The central place in stochastic analysis belongs to Itô's stochastic differential equations which allow one to construct paths of a diffusion directly from paths of the Brownian motion without using PDE. The construction can be carried out in many dimensions as simply as in one dimension and it works for degenerate diffusions as well. First published in [48], Itô's theory was extended to processes with jumps, with the Brownian motion replaced by processes with independent increments in [49]. [A different version of stochastic differential equations was discovered independently by Gihman (see references in [40]).] Itô's stochastic calculus has gained new strength and flexibility from the introduction of integration with respect to a martingale (Kunita and Watanabe; Meyer) and from the martingale approach of Stroock and Varadhan [74] to stochastic differential equations. An excellent presentation of the subject and its history can be found in books of Ikeda and Watanabe [47] and of Williams and Rogers [80].

To work with the state of a Markov process at any stopping time, a strong version of the Markov property is needed. Doob [15] proved such a property for processes in denumerable spaces. Broad conditions under which the strong Markov property holds have been established in [30] and, independently, in [4] (particular cases have been treated in [43] and [66]). For strong Markov processes, a local characteristic can be defined by passing to the limit in space rather than in time. The characteristic operator $\mathfrak{A}$, introduced this way in [24], has some advantages in comparison with the infinitesimal operator: Obviously, $\mathfrak{A}$ is local if paths are continuous; the diffusion, corresponding to a differential operator L can be defined as a continuous stochastic process for which $\mathfrak{A} f = Lf$ for all $f \in C^2$ (which implies that L must be the second order elliptic differential operator, possibly, degenerate); using characteristic operators it is easier to investigate mathematical expectations related to a Markov process.

A fundamental role in stochastic analysis is played by additive functionals and transformations related to these functionals, such as random time change, killing or mass creation (the Feynman–Kac formula), the Girsanov transformation.... A number of results on this subject and related topics have been obtained at a seminar on Markov processes at Moscow University in the 1950's and 1960's. Work done in the 1950's was reported in [22]. A substantial part of the monograph [24] was based on these results.

By using random time change and killing, it is possible to construct, starting from the Brownian motion, all one-dimensional continuous strong Markov processes (the corresponding class of generators was introduced first analytically by Feller in [35]). The construction is described in the monographs [24] and [50]. The latter contains also the first rigorous proof of Lévy's results [56] on the fine structure of the Brownian motion in $\mathbb{R}^1$.

Ideas of Kakutani [52, 53] and Doob [18, 19] prepared the soil for Hunt's beautiful probabilistic potential theory [44]. A systematic presentation of this theory with many important contributions of the authors is contained in [5]. Spectacular progress in probabilistic potential theory was made possible by the "théorie générale" of stochastic processes developed by Meyer and Dellacherie (see [61, 10] and, for a more recent presentation, [11]). The most complete exposition of probabilistic potential theory in its relation to classical analytic theory is contained in [21].

6. The countable case. The study of processes in a countable state space had a great impact on the earlier development of Markov theory. The countable case first appeared in [K28], and its systematic study was launched in 1936 in [K70] and [K68] (see [K81] for complete proofs). The main objective was to describe the asymptotic behavior of the transition probabilities $P_{ij}(n)$ in n steps as $n \to \infty$. Kolmogorov introduces the partition of the state space into an inessential part and classes of essential states. Each class K consists of d periodic subclasses and, for every i, $j \in K$, and in the same periodic subclass

$$\lim_{k \to \infty} P_{ij}(kd) = M_{jj}^{-1} \quad \text{as } k \to \infty, \tag{6}$$

where M_{jj} is the mean recurrence time to j [if n is not divisible by d, then $P_{ij}(n) = 0$]. Within one class, either all $M_{jj} < \infty$ (positive class) or all $M_{jj} = \infty$

(null class). For a positive class the limits (6) define the only stationary probability distribution. This was a very far-reaching extension of Markov's original results for finite chains consisting of one class. Doeblin worked independently in the same direction. He also solved some problems posed by Kolmogorov (see [12]). A nice presentation of Kolmogorov's theory is given in [36]. More precise asymptotic results have been obtained by Chung, Derman, Orey and others (see [6] for references). Generalizations to the uncountable case due to Doeblin, Chung and others are described in [67].

Processes in a countable space, with a continuous time parameter were investigated in [K186]. The author was interested in differential properties (in particular, in local characteristics) of a stationary transition function $P_{ij}(t)$ (assuming that it tends to the unit matrix as $t \downarrow 0$). Earlier Doob [14, 15] had proved the existence of the (finite or infinite) derivatives $q_{ij} = P'_{ij}(0)$. It is clear that

$$q_{ij} \geq 0 \quad \text{for } i \neq j \tag{7}$$

and, for every i,

$$\sum_{j \neq i} q_{ij} \leq q_i \leq \infty, \tag{8}$$

where $q_i = -q_{ii}$. If the number of states is finite, then a stronger statement than (8) holds, namely

$$\sum_{j \neq i} q_{ij} = q_i < \infty. \tag{9}$$

Doob established that (9) is equivalent to the following property: For every separable process corresponding to the matrix $Q = (q_{ij})$, the discontinuity times form a well-ordered set. Moreover Q determines the process uniquely until the first accumulation time T of discontinuities. Doob also has given examples of Q to which many processes correspond (they differ by their behavior at time T). In [K186] Kolmogorov proved that $q_{ij} < \infty$ for $i \neq j$ and he constructed examples where, for some i, $q_i = \infty$ or $q_i < \infty$, but the sum in (8) is smaller than q_i. He also conjectured that finite derivatives $P'_{ij}(t)$ exist for all $t > 0$. [K186] is written without any reference to paths, but behind Kolmogorov's examples there was an intuitive idea of an entrance boundary of a process. The present author was a student in Kolmogorov's seminars on Markov processes in 1945 and 1946 where this idea was discussed on a heuristic level. It took several years before Kolmogorov published rigorous proofs. Meanwhile Lévy [57] presented a deep informal analysis of possible path behavior. He suggested calling a state i stable if $q_i < \infty$ and instantaneous if $q_i = \infty$. He thought that no process with only instantaneous states exists. Later counterexamples to this statement were published by several authors. Kolmogorov's conjecture on the differentiability of $P_{ij}(t)$ was proved, under some restrictions, by Yushkevich, Austin, Chung and Reuter. The first complete proof is due to Ornstein [63]. More detail on work inspired by [K186] can be found in [6] and in Yushkevich's comments on [K186] in [K471].

7. Boundary theory. Let Q satisfy (9). To get all right-continuous processes corresponding to Q (all of them have Q as their characteristic operator), one needs to know the possible behavior of paths at the first accumulation time of discontinuities. Similarly, one can be interested in the possible behavior of a diffusion in a domain D after the first exit time from D. Both problems can be stated in terms of boundary conditions determining generators. Feller [34] succeeded in describing all such boundary conditions in the simplest case of a one-dimensional diffusion. Wentzell [78] considered the multidimensional case. The work on Wentzell's boundary conditions (see references in [68]) lead to Motoo's theory of the boundary processes [62]. In the countable case, the first step is to construct a suitable boundary, in fact, two boundaries—exit and entrance. Such a construction, motivated by Martin's theory of positive harmonic functions, was suggested by Doob in [20] and substantially improved by Hunt in [45] (see also [27]; for an extension to general state spaces see [28, 29] and references there). Behaviour of Markov processes on the boundary was studied in [7] and [25] in the countable case and in [26] for a class of processes related to the classical boundary value problem with oblique derivative. Martin boundary theory is an important tool for investigating the limit behavior of paths as $t \to \infty$ and for determining all stationary distributions.

8. Branching processes. Initially, Kolmogorov became interested in branching processes because of their applications to genetics. Two papers [K101, K286] are devoted to such applications.

Branching processes were the subject of Kolmogorov's seminar at Moscow University in 1946–1947. In fact, the general concept of a branching process with many types of particles and even the name "branching processes" appeared at this seminar for the first time. The results obtained there (see [K139], [K140] and [81]) stimulated intensive work in the years to come. The progress in the 1950's and 1960's was presented in monographs of Harris [42], Sevast'yanov [70] and Athreya and Ney [1] which also contain their own significant contributions to the subject. Processes with several types of particles introduced in Kolmogorov's seminar are related to particle systems which combine branching with a Markov transition mechanism in a finite state space. Analogous theory for an arbitrary state space was developed in [51] and [46]. A related class of measure-valued processes has been studied by Watanabe [77], Dawson [8] and others (see the bibliography in [9]).

9. Concluding remarks. Kolmogorov's ideas have influenced, directly or indirectly, almost all work on Markov processes. Only a small part of it could be covered in one article and, inevitably, the choice was conditioned by my own interest and limited knowledge. I apologize for omissions and possible unintentional misrepresentations. Some important directions in the modern theory of Markov processes were left out in our survey. They are covered in the following recent monographs: diffusion on manifolds, Malliavin's calculus [47]; infinite particle systems [58]; limit theorems for sequences of Markov processes [32];

large deviations [73, 76]; applications of Markov processes to various problems in PDE (small parameter, quasilinear equations, etc.) [38].

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