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SELF-NORMALIZED LAWS OF THE ITERATED LOGARITHM

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Using suitable self-normalizations for partial sums of i.i.d. random variables, a law of the iterated logarithm, which generalizes the classical LIL, is proved for all distributions in the Feller class. A special case of these results applies to any distribution in the domain of attraction of some stable law.

1. Introduction. Let $X, X_1, X_2, \dots$ be nondegenerate, i.i.d., real-valued random variables and set $S_n = X_1 + \cdots + X_n$. If $\mathrm{Var}(X) = \sigma^2 < \infty$, it follows from the central limit theorem (CLT) that

$$\frac{S_n - nE(X)}{(n\sigma^2)^{1/2}} \to N(0,1). \tag{1.1}$$

In this case the a.s. deviations of S_n from $nE(X)$ are measured by the law of the iterated logarithm (LIL),

$$\limsup_{n\to\infty} \frac{S_n - nE(X)}{(2\sigma^2 n L_2 n)^{1/2}} = 1 \quad \text{a.s.} \tag{1.2}$$

Of course, there are many random variables with infinite variance for which the normalized partial sums are asymptotically normal. A well-known necessary and sufficient condition for this is given by

$$\lim_{x\to\infty} \frac{x^2 P(|X| > x)}{E\left(X^2 I(|X| \le x)\right)} = 0. \tag{1.3}$$

Assuming for simplicity that $E(X) = 0$ whenever $E(X^2) < \infty$, if we define

$$d(t) = \inf\left\{x \colon x^{-2} E\left(X^2 I(|X| \le x)\right) = t^{-1}\right\},$$

then under (1.3)

$$\frac{S_n - nE(X)}{d(n)} \to N(0,1). \tag{1.4}$$

In trying to extend (1.2) to this setting, it seems natural to ask whether

$$\limsup_{n\to\infty} \frac{S_n - nE(X)}{\left(2L_2 n d^2(n)\right)^{1/2}} = 1 \quad \text{a.s.} \tag{1.5}$$

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This turns out to be false as the following example of Feller shows [2]: Let X be symmetric with density $|x|^{-3}$ for $|x| > 1$. Then one can easily show (1.3) holds and $d(n) \sim (nLn)^{1/2}$. However, (1.5) fails since

$$\limsup_{n\to\infty} \frac{S_n - nE(X)}{(2nL_2nLn)^{1/2}} = \infty \quad \text{a.s.}$$

Moreover, Feller was able to make precise why (1.5) fails. To describe this, let ${}^{(1)}X_n, \dots, {}^{(n)}X_n$ be an arrangement of $X_1, \dots, X_n$ in decreasing order of magnitude, that is, $|{}^{(1)}X_n| \geq \cdots \geq |{}^{(n)}X_n|$. For $r \geq 1$ an integer set

$${}^{(r)}S_n = S_n - \sum_{i=1}^{r} {}^{(i)}X_n,$$

the sample sum with the r largest summands removed. Then Feller showed that in his example

$$\limsup_{n\to\infty} \frac{{}^{(1)}S_n - nE(X)}{(2nL_2nLn)^{1/2}} = 1 \quad \text{a.s.}$$

Thus it is the single largest summand which prevents (1.5) from holding in this case. The aim of this article is to show that while (1.5) fails, there is a natural generalization of (1.2) which does hold in this setting (and beyond). This will be done by self-normalizing or studentizing the partial sums, a procedure which is common in statistical practice.

To describe our results, let

$$V_n = \sum_{i=1}^{n} X_i^2.$$

If $E(X^2) < \infty$ [and recalling that we assume $E(X) = 0$ in this situation], it follows from the strong law of large numbers, (1.1) and (1.2) that

$$\frac{S_n - nE(X)}{V_n^{1/2}} \to N(0,1) \tag{1.6}$$

and

$$\limsup_{n\to\infty} \frac{S_n - nE(X)}{(2L_2nV_n)^{1/2}} = 1 \quad \text{a.s.} \tag{1.7}$$

It is not hard to see that if we replace the condition $E(X^2) < \infty$ by (1.3), then

$$\frac{V_n}{d^2(n)} \to_P 1 \tag{1.8}$$

and so (1.6) still holds. It is far from clear however that (1.7) should hold in this case. Indeed that it does, bearing in mind Feller's example, must depend on the fact that almost sure convergence fails in (1.8).

To see that there is some hope for (1.7) holding, observe that it cannot fail for the same reason as in Feller's example. This is simply because $|^{(1)}X_n| \le V_n$ and so

$$\limsup_{n\to\infty} \frac{S_n - nE(x)}{(2L_2nV_n)^{1/2}} = \limsup_{n\to\infty} \frac{^{(1)}S_n - nE(X)}{(2L_2nV_n)^{1/2}}.$$

This idea, which is really the basis of our approach, can be carried further. By the Cauchy–Schwarz inequality

$$\sum_{i=1}^{r_n} |^{(i)}X_n| \le r_n^{1/2}\left(\sum_{i=1}^{r_n} {}^{(i)}X_n^2\right)^{1/2} \le (r_nV_n)^{1/2}$$

and so if $r_n = o(L_2n)$, then

$$\limsup_{n\to\infty} \frac{S_n - nE(X)}{(2L_2nV_n)^{1/2}} = \limsup_{n\to\infty} \frac{^{(r_n)}S_n - nE(X)}{(2L_2nV_n)^{1/2}}. \tag{1.9}$$

Now Kuelbs and Ledoux [5] have shown that if (1.3) holds, there is a sequence $r_n = o(L_2n)$ such that

$$\limsup_{n\to\infty} \frac{^{(r_n)}S_n - nE(X)}{2^{1/2}L_2nd(n/L_2n)} = 1 \quad \text{a.s.} \tag{1.10}$$

Furthermore, since $V_n \ge T_n = \sum_{i=1}^n X_i^2 I(|X_i| \le d(n/L_2n))$ and $E(T_n) = L_2nd^2(n/L_2n)$, it seems reasonable to hope (and in fact it is true) that

$$\limsup_{n\to\infty} \frac{V_n}{L_2nd^2(n/L_2n)} \ge 1 \quad \text{a.s.} \tag{1.11}$$

This then implies the upper bound in (1.7) for any X satisfying (1.3). Interestingly, as the above shows, the large summands play no role in making $(S_n - nE(X))(2L_2nV_n)^{-1/2}$ big; indeed they tend to have the opposite effect of making it small, which is in direct contrast to the situation in (1.5). This suggests that to prove the lower bound in (1.7), one should look at times when the large summands are of moderate size. This idea will be made precise in the proof.

Studentized or self-normalized sums have been studied previously in connection with weak convergence; see [1] and [6]. In [6] the asymptotic distribution of $S/V_n^{1/2}$ is found for X in the domain of attraction of a stable law. The limit law turns out to have tails which are very much like those of the standard normal distribution. Motivated by these findings, we consider extensions of (1.7) to this case and more generally to the setting in which X belongs to the Feller class $\mathscr{F}$, that is, X is in $\mathscr{F}$ if there exist sequences a_n and b_n such that $(S_n - b_n)a_n^{-1}$ is tight with all subsequential limits nondegenerate, or, equivalently, if the analytic condition $\limsup_{x\to\infty} x^2P(|X| > x)/E(X^2I(|X| \le x)) < \infty$ holds.

The class $\mathscr{F}$ is natural to consider in the sense that there are symmetric X outside $\mathscr{F}$ such that

$$\lim_n S_n/^{(1)}X_n = 1 \quad \text{a.s.;} \tag{1.12}$$

see, for example, [7] and [9] for details and other references. Thus when (1.12) holds we have the degenerate result

$$\lim_n \frac{|S_n|}{(2L_2nV_n)^{1/2}} = 0 \quad \text{a.s.}$$

The precise statements of our LIL results will be given in Section 2.

NOTATION. The following symbols and relationships are used throughout the article.

$X, X_1, X_2, \ldots$ is an i.i.d. sequence of real-valued, nondegenerate random variables such that $E(X) = 0$ whenever $E(X^2) < \infty$.

$S_n = X_1 + \cdots + X_n$, $n \geq 1$.

$G(x) = P(|X| > x)$ for $x \geq 0$.

$K(x) = x^{-2}E(X^2I(|X| \leq x))$ for $x > 0$.

$Q(x) = G(x) + K(x)$ for $x > 0$.

$d(t) = \inf\{s \geq b + 1: K(s) \leq 1/t\}$, where $b = \inf\{x \geq 1: K(x) > 0\}$.

$d_n(\lambda) = d(\lambda n/L_2n)$.

$\beta_n^2 = nd_n^2(\lambda)K(d_n(\lambda)) = L_2nd_n^2(\lambda)/\lambda$ since $K(d(x)) = 1/x$ for $x \geq b + 2$.

$T_n(\lambda) = \sum_{i=1}^n X_i^2 I(|X_i| \leq d_n(\lambda))$.

$\gamma_n(\lambda) = (2L_2n\beta_n^2(\lambda))^{1/2} = (2/\lambda)^{1/2}L_2nd_n(\lambda)$ since $K(d(x)) = 1/x$ for $x \geq b + 2$.

$V_n = \sum_{i=1}^n X_i^2$.

$\mathscr{F} = \{X: \limsup_{x \to \infty} G(x)/K(x) < \infty\}$.

$\mathscr{F}(\theta) = \{X: \limsup_{x \to \infty} G(x)/K(x) < \theta\}$ for $\theta > 0$.

2. Statement of results. The most complete result is our first theorem. If $x \in \mathbb{R}^1$, $A \subseteq \mathbb{R}^1$, then the distance from x to A is defined as

$$d(x, A) = \inf_{y \in A} |x - y|.$$

If $\{x_n\}$ is a real sequence, then $C(\{x_n\})$ denotes its cluster set, that is, $C(\{x_n\}) = \{y: \liminf_n |x_n - y| = 0\}$. We write $\{x_n\} \twoheadrightarrow A$ if both $\lim_n d(x_n, A) = 0$ and $C(\{x_n\}) = A$.

THEOREM 1. *If X is in the domain of attraction of a Gaussian law, then $E(X)$ exists and*

$$\left\{ \frac{S_n - nE(X)}{(2L_2nV_n)^{1/2}} \right\} \twoheadrightarrow [-1, 1] \quad a.s. \tag{2.1}$$

REMARK. The cluster set $[-1, 1]$ in (2.1) is correct even if $E(X^2) < \infty$ since we are assuming $E(X) = 0$ in this situation. Of course, if Y is nondegenerate and $E(Y^2) < \infty$, then Theorem 1 applies to $X = Y - E(Y)$, and hence no loss of generality occurs in this assumption.

An extension of Theorem 1 to random variables in the domain of attraction of a stable law is given by the following theorem.

THEOREM 2. *If X is in the domain of attraction of a stable law of index $\alpha \in (0,2]$, denoted $X \in D(\alpha)$, then*

$$\left\{ \frac{S_n - nE(XI(|X| \le d_n(\lambda)))}{(2L_2 n V_n)^{1/2}} \right\} \twoheadrightarrow [-k_1(X,\lambda), k_2(X,\lambda)] \quad a.s., \tag{2.2}$$

where $k_i(X,\lambda)$ are finite for every λ and strictly positive if λ is sufficiently large. Furthermore, it is possible to choose $\lambda = \lambda(\alpha)$ such that for $i = 1, 2$,

$$\lim_{\alpha \uparrow 2} \sup_{X \in D(\alpha)} k_i(X,\lambda) = 1 \quad a.s. \tag{2.3}$$

and

$$\lim_{\alpha \uparrow 2} \inf_{X \in D(\alpha)} k_i(X,\lambda) = 1 \quad a.s. \tag{2.4}$$

REMARK. One might suspect that the constants $k_i(X,\lambda)$ do not depend on X but just on α. However the manner in which the centerings depend upon the distribution of X makes this seem unlikely.

If $E(X)$ exists, it seems plausible that the centerings used in (2.2) should be replaceable by $nE(X)$. This is indeed the case if $\alpha > 1$ and follows from Theorem 3 where a more general setting is considered.

THEOREM 3. *If $X \in \mathscr{F}$ with*

$$\limsup_{x \to \infty} G(x)/K(x) < \theta < 1, \tag{2.5}$$

then $E(|X|) < \infty$ and

$$\left\{ \frac{S_n - nE(X)}{(2L_2 n V_n)^{1/2}} \right\} \twoheadrightarrow [-k_1(X), k_2(X)] \quad w.p.1, \tag{2.6}$$

where $k_1(X)$ and $k_2(X)$ are strictly positive finite numbers. Furthermore, with $\mathscr{F}(\theta) = \{X: \limsup_{x\to\infty} G(x)/K(x) < \theta\}$ we have for $i = 1, 2$,

$$\lim_{\theta \downarrow 0} \sup_{X \in \mathscr{F}(\theta)} k_i(X) = \lim_{\theta \downarrow 0} \inf_{X \in \mathscr{F}(\theta)} k_i(X) = 1 \quad a.s. \tag{2.7}$$

REMARK. If $X \in D(\alpha)$, $0 < \alpha \le 2$, then

$$\lim_{x\to\infty} G(x)/K(x) = (2-\alpha)/\alpha$$

and hence when $1 < \alpha \le 2$ there is a $\theta < 1$ such that (2.5) holds. Thus Theorem 3 applies to this situation. Observe also that when $\alpha = 2$, θ may be chosen arbitrarily small and so (2.6) and (2.7) imply Theorem 1.

The major technical results required in the proof of the previous theorems are contained in the next theorem.

THEOREM 4. *Let* $X \in \mathscr{F}(\theta)$ *and assume* $E(X) = 0$ *when* $E(X^2) < \infty$. *Then*:

(I) *For all* $\lambda > 0$,

$$\limsup_n \frac{S_n - nE(XI(|X| \le d_n(\lambda)))}{(2L_2 nV_n)^{1/2}} = C_1(X, \lambda) \quad a.s., \tag{2.8}$$

where $C_1(X, \lambda)$ *is a finite constant.*

(II) *If* $X \in \mathscr{F}$ *is fixed, then for* λ *sufficiently large* $C_1(X, \lambda)$ *is strictly positive.*

(III) *In addition,*

$$\lim_{\substack{\theta \downarrow 0 \\ \lambda = \theta^{1/2}}} \sup_{X \in \mathscr{F}(\theta)} C_1(X, \lambda) = \lim_{\substack{\theta \downarrow 0 \\ \lambda = \theta^{1/2}}} \inf_{X \in \mathscr{F}(\theta)} C_1(X, \lambda) = 1. \tag{2.9}$$

REMARKS. 1. The assumption that $E(X) = 0$ when $E(X^2) < \infty$ is only important for (III). That is, if $0 < E(X^2) < \infty$ and $E(X) = \mu \neq 0$, then for $\lambda > 0$,

$$\begin{aligned} &\limsup_n \frac{S_n - nE(XI(|X| \le d_n(\lambda)))}{(2L_2 nV_n)^{1/2}} \\ &\quad = \left(E((X - \mu)^2)/E(X^2)\right)^{1/2} < 1. \end{aligned} \tag{2.10}$$

This is obvious from the classical LIL since $0 < E(X^2) < \infty$ implies $V_n \sim nE(X^2)$ and

$$\begin{aligned} &\limsup_n nE(|X|I(|X| > d_n(\lambda)))/(2L_2 nV_n)^{1/2} \\ &\quad \le \limsup_n \frac{n(E(X^2))^{1/2}(G(d_n(\lambda)))^{1/2}}{(2nE(X^2)L_2 n)^{1/2}} = 0 \end{aligned} \tag{2.11}$$

by (1.3).

2. The constant $C_1(X, \lambda)$ in (2.8) can be estimated via knowledge of those numbers θ which satisfy

$$\limsup_{x \to \infty} G(x)/K(x) < \theta < \infty.$$

The detailed inequalities for these estimates are given throughout the proof of Theorem 4.

3. We do not know whether a clustering result holds in this general setting. It does if the centering terms δ_n satisfy the regularity condition $(\delta_n - \delta_{n-1})(2L_2 nV_n)^{-1/2} \to 0$ a.s., as we now show.

PROPOSITION 2.1. *Assume δ_n is a centering sequence such that*

$$(\delta_n - \delta_{n-1})/(2L_2 nV_n)^{1/2} \to 0 \quad a.s.$$

If

$$\liminf_n \frac{S_n - \delta_n}{(2L_2 nV_n)^{1/2}} = C_1, \qquad \limsup_n \frac{S_n - \delta_n}{(2L_2 nV_n)^{1/2}} = C_2,$$

then

$$\left\{\frac{S_n - \delta_n}{(2L_2 nV_n)^{1/2}}\right\} \twoheadrightarrow [C_1, C_2]. \tag{2.12}$$

PROOF. Let $\Gamma_n = (2L_2 nV_n)^{1/2}$. Then for any n,

$$\begin{aligned}\frac{S_n - \delta_n}{\Gamma_n} &= \frac{S_{n-1} - \delta_{n-1}}{\Gamma_n} + \frac{X_n}{\Gamma_n} - \frac{\delta_n - \delta_{n-1}}{\Gamma_n} \\ &= \frac{S_{n-1} - \delta_{n-1}}{\Gamma_n} + o(1) \quad \text{a.s.}\end{aligned} \tag{2.13}$$

Now let $C \in (C_1, C_2)$. If $C \geq 0$ we define

$$\begin{aligned} n_1 &= \inf\{n: (S_n - \delta_n)\Gamma_n^{-1} < C\}, \\ m_k &= \inf\{n > n_k: (S_n - \delta_n)\Gamma_n^{-1} \geq C\}, \qquad k \geq 1, \\ n_{k+1} &= \inf\{n > m_k: (S_n - \delta_n)\Gamma_n^{-1} < C\}, \qquad k \geq 1. \end{aligned}$$

Then $m_k, n_k \to \infty$ a.s. Since Γ_n is increasing a.s. and $C \geq 0$, we have by (2.15)

$$C \leq \frac{S_{m_k} - \delta_{m_k}}{\Gamma_{m_k}} = \frac{S_{m_k - 1} - \delta_{m_k - 1}}{\Gamma_{m_k - 1}} \frac{\Gamma_{m_k - 1}}{\Gamma_{m_k}} + o(1) \leq C + o(1).$$

Thus $(S_{m_k} - \delta_{m_k})\Gamma_{m_k}^{-1} \to C$ a.s.

The case $C < 0$ is handled analogously. □

3. Some lemmas. Here we prove some results used in the proof of Theorem 4. The notation of this section is that established at the end of Section 1, and we always assume $X \in \mathscr{F}(\theta)$.

LEMMA 3.1. *If $X \in \mathscr{F}(\theta)$, then for $y \geq x$ and all x sufficiently large*

$$y^2 K(y) - x^2 K(x) \leq x^2 K(x)\{(y/x)^{2-p}(1 + \theta) - 1\} \tag{3.1}$$

and

$$1 \le d(y)/d(x) \le ((y/x)(1+\theta))^{1/p}, \tag{3.2}$$

where

$$p = 2/(1+\theta). \tag{3.3}$$

PROOF. By Lemma 2.4 of [8], $x^p Q(x)$ decreases for all x sufficiently large where p is as in (3.3). Hence for $y \ge x$ and x sufficiently large

$$\begin{aligned} y^2K(y) - x^2K(x) &\le y^2Q(y) - x^2K(x) \\ &= y^{2-p}y^pQ(y) - x^2K(x) \\ &\le y^{2-p}x^pQ(x) - x^2K(x) \\ &\le y^{2-p}x^p(1+\theta)K(x) - x^2K(x) \\ &= x^2K(x)\{(y/x)^{2-p}(1+\theta) - 1\} \end{aligned}$$

and (3.1) holds. Similarly, since $K(d(x)) = 1/x$,

$$\begin{aligned} d^2(y) &= yd^2(y)K(d(y)) \\ &\le d^2(x)(d(y)/d(x))^{2-p}(y/x)(1+\theta) \end{aligned}$$

and this yields (3.2) since $d(x)$ is nondecreasing. □

LEMMA 3.2. *Let $X \in \mathscr{F}(\theta)$, $a > 1$, $n_k = [a^k]$ for $k \ge 1$ and p be as in* (3.3); *then for each $\lambda > 0$,*

$$\begin{aligned} 1 &\le \limsup_k d^2_{n_{k+1}}(\lambda)/d^2_{n_k}(\lambda) \\ &= \limsup_k \beta^2_{n_{k+1}}(\lambda)/\beta^2_{n_k(\lambda)} \le (a(1+\theta))^{2/p} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\limsup_k n_{k+1}E\big(|X|I\big(d_{n_k}(\lambda) < |X| \le d_{n_{k+1}}(\lambda)\big)\big)/\gamma_{n_k}(\lambda) \\ &\quad\le (\theta/2\lambda)^{1/2}a\{(a(1+\theta))^{(2-p)/p}(1+\theta) - 1\}^{1/2}. \end{aligned} \tag{3.5}$$

PROOF. Since $\beta_n(\lambda)$ and $d_n(\lambda)$ increase as n increases and

$$\lim_k L_2 n_{k+1}/L_2 n_k = 1,$$

it suffices to verify

$$\limsup_k d^2_{n_{k+1}}(\lambda)/d^2_{n_k}(\lambda) \le (a(1+\theta))^{2/p}.$$

This follows immediately from (3.2), so (3.4) is proved.

To verify (3.5), apply the Cauchy–Schwarz inequality to obtain

$$\limsup_k n_{k+1} E\big(|X| I\big(d_{n_k}(\lambda) < |X| \le d_{n_{k+1}}(\lambda)\big)\big)/\gamma_{n_k}(\lambda)$$

$$\le \limsup_k n_{k+1}\Big(E\Big(X^2 I\big(d_{n_k}(\lambda) < |X| \le d_{n_{k+1}}(\lambda)\big)\Big)\Big)^{1/2} G\big(d_{n_k}(\lambda)\big)^{1/2}/\gamma_{n_k}(\lambda)$$

$$\le \limsup_k n_{k+1}\theta^{1/2}\big(K\big(d_{n_k}(\lambda)\big)\big)^{1/2}$$

$$\times\left\{d^2_{n_{k+1}}(\lambda)K\big(d_{n_{k+1}}(\lambda)\big) - d^2_{n_k}(\lambda)K\big(d_{n_k}(\lambda)\big)\right\}^{1/2}/\gamma_{n_k}(\lambda)$$

[since $\limsup G(x)/K(x) < \theta$]

$$\le \limsup_k n_{k+1}\theta^{1/2}K\big(d_{n_k}(\lambda)\big)d_{n_k}(\lambda)$$

$$\times\left\{\big(d_{n_{k+1}}(\lambda)/d_{n_k}(\lambda)\big)^{2-p}(1+\theta) - 1\right\}^{1/2}/\gamma_{n_k}(\lambda)$$

[by (3.1)]

$$\le (\theta/2\lambda)^{1/2} a\left\{\big(a(1+\theta)\big)^{(2-p)/p}(1+\theta) - 1\right\}^{1/2}$$

[by (3.2)].

Thus the lemma is proved. □

LEMMA 3.3. *Let* $X \in \mathscr{F}(\theta)$, $a > 1$, $n_k = [a^k]$ *for* $k \ge 1$, $\lambda > 0$, $I_k = (n_k, n_{k+1}]$ *and*

(3.6) $$J_n(\lambda) = \sum_{j=1}^{n} I\big(|X_j| > d_{n_{k+1}}(\lambda)\big)$$

for $n \in I(k)$. *Then*

(3.7) $$\limsup_k \sup_{n\in I(k)} J_n(\lambda)/L_2 n \le \alpha_1(\theta/\lambda),$$

where

(3.8) $$\alpha_1(\theta/\lambda) = \inf\{\rho > \theta/\lambda : \rho(\log(\rho\lambda/\theta) - 1) + \theta/\lambda > 1\}.$$

Furthermore,

(3.9) $$\lim_{\theta/\lambda\to 0} \alpha_1(\theta/\lambda) = 0.$$

REMARK. One can prove an unblocked version of Lemma 3.3, but the above is directly applicable for use in Lemma 4.3.

PROOF. For $\rho > 0$, $\varepsilon > 0$, $u > 0$,

$$\begin{aligned}
P\Big(\sup_{n\in I(k)} J_n(\lambda)/L_2 n > \rho\Big) &\le P\big(J_{n_{k+1}}(\lambda) > \rho L_2 n_k\big) \\
&\le E\big(e^{uJ_{n_{k+1}}(\lambda)}\big)e^{-u\rho L_2 n_k} \\
&= \Big(1 + (e^u - 1)G\big(d_{n_{k+1}}(\lambda)\big)\Big)^{n_{k+1}} e^{-u\rho L_2 n_k} \\
&\le \exp\Big\{n_{k+1}G\big(d_{n_{k+1}}(\lambda)\big)(e^u - 1) - u\rho L_2 n_k\Big\} \\
&\le \exp\Big\{n_{k+1}\theta K\big(d_{n_{k+1}}(\lambda)\big)(e^u - 1) - u\rho L_2 n_k\Big\} \\
&= \exp\{(\theta/\lambda)L_2 n_{k+1}(e^u - 1) - u\rho L_2 n_k\} \\
&\le \exp\{-L_2 n_{k+1}\{(\rho - \varepsilon)u - (\theta/\lambda)e^u + \theta/\lambda\}\}
\end{aligned}$$

if k is sufficiently large. Now assume $\rho > \alpha_1(\theta/\lambda)$ and set $u = \log(\rho\lambda/\theta)$. Then $u > 0$ and hence

$$P\Big(\sup_{n\in I(k)} J_n(\lambda)/L_2 n > \rho\Big) \le \exp\{-L_2 n_{k+1}\{(\rho - \varepsilon)\log(\rho\lambda/\theta) - \rho + \theta/\lambda\}\}.$$

By choosing $\varepsilon > 0$ sufficiently small that $(\rho - \varepsilon)\log(\rho\lambda/\theta) - \rho + \theta/\lambda > 1$, this then implies

$$\sum_k P\Big(\sup_{n\in I(k)} J_n(\lambda)/L_2 n > \rho\Big) < \infty.$$

Thus (3.7) holds, and (3.9) is obvious. □

LEMMA 3.4. *Let $Y, Y_1, Y_2, \dots$ be an arbitrary i.i.d. sequence of random variables. Then for any $b > 0$, $v > 0$, $s > 0$ and $n \ge 1$,*

$$P\left(\left|\sum_{j=1}^n Y_j I(|Y_j| \le b) - E\left(\sum_{j=1}^n Y_j I(|Y_j| \le b)\right)\right| > ve^v nbK_Y(b)/2 + sb/v\right) \le 2e^{-s} \tag{3.10}$$

where $K_Y(b) = E(Y^2 I(|Y| \le b))/b^2$.

PROOF. Let $W_n = \Sigma_{j=1}^n Y_j I(|Y_j| \le b)$ and apply the one-sided inequality of Lemma 3.1 in [8] to both W_n and its negative. Then (3.10) follows immediately since W_n is a sum of truncated variables rather than the censored variables of Pruitt's Lemma 3.1. □

LEMMA 3.5. *Let $\lambda > 0$, $X \in \mathscr{F}(\theta)$ and set*

$$\alpha_2(\lambda) = (1/2)\lambda^{1/2}e^{\lambda^{1/2}} + 2\lambda^{1/2}. \tag{3.11}$$

Let p be as in (3.3). *Then, for each* $\lambda > 0$,

$$\limsup_n T_n(\lambda)/\beta_n^2(\lambda) \le \alpha_3(\lambda, \theta), \tag{3.12}$$

where

$$\alpha_3(\lambda, \theta) = (1 + \theta)^{2/p}(\alpha_2(\lambda) + 1). \tag{3.13}$$

Furthermore, for each $X \in \mathscr{F}(\theta)$,

$$\liminf_n T_n(\lambda)/\beta_n^2(\lambda) \ge \alpha_4(\lambda, \theta), \tag{3.14}$$

where

$$\alpha_4(\lambda, \theta) = (1 - \alpha_2(\lambda))/(1 + \theta)^{2/p} > 0 \tag{3.15}$$

if $\lambda > 0$ *is sufficiently small so that* $\alpha_2(\lambda) < 1$. *Finally, if* $\lambda_0 > 0$ *is such that* $\alpha_2(\lambda_0) < 1$, *then for all* $\lambda \ge \lambda_0$,

$$\liminf_n V_n/\beta_n^2(\lambda) \ge \alpha_4(\lambda_0, \theta)(\lambda/\lambda_0)^{(p-2)/p}(1 + \theta)^{-2/p} > 0. \tag{3.16}$$

PROOF. Applying Lemma 3.4 to the random variables $\{Y_j = X_j^2 \colon j \ge 1\}$ with $b = d_n^2(\lambda)$, $s = 2L_2 n$, and $v > 0$, we obtain

$$\begin{aligned} P\big(|T_n(\lambda) - \beta_n^2(\lambda)| > ve^v nd_n^2(\lambda)K_Y\big(d_n^2(\lambda)\big)/2 \\ + 2L_2 nd_n^2(\lambda)/v\big) \le 2e^{-2L_2 n}. \end{aligned} \tag{3.17}$$

Setting $v = \lambda^{1/2}$ and noting that

$$\begin{aligned} K_Y(x^2) &= E\big(Y^2 I(|Y| \le x^2)\big)/x^4 \\ &\le E\big(X^2 I(|X| \le x)\big)/x^2 \\ &= K(x), \end{aligned}$$

we obtain

$$P\big(|T_n(\lambda) - \beta_n^2(\lambda)| > \beta_n^2(\lambda)\alpha_2(\lambda)\big) \le 2e^{-2L_2 n}. \tag{3.18}$$

To verify (3.12), let $a > 1$, $n_k = [a^k]$ and $I(k) = (n_k, n_{k+1}]$. If $\rho(a(1 + \theta))^{-2/p} - 1 > \alpha_2(\lambda)$, then for each $\varepsilon > 0$ and all k sufficiently large

$$\begin{aligned} &P\big(T_n(\lambda)/\beta_n^2(\lambda) > \rho + \varepsilon \text{ for some } n \in I(k)\big) \\ &\quad \le P\big(T_{n_{k+1}}(\lambda) > (\rho + \varepsilon)\beta_{n_k}^2(\lambda)\big) \\ &\quad = P\left(T_{n_{k+1}}(\lambda) - \beta_{n_{k+1}}^2(\lambda) > \beta_{n_{k+1}}^2(\lambda)\left((\rho + \varepsilon)\frac{\beta_{n_k}^2(\lambda)}{\beta_{n_{k+1}}^2(\lambda)} - 1\right)\right) \\ &\quad \le P\Big(\big|T_{n_{k+1}}(\lambda) - \beta_{n_{k+1}}^2(\lambda)\big| > \beta_{n_{k+1}}^2(\lambda)\big(\rho(a(1 + \theta))^{-2/p} - 1\big)\Big) \\ &\quad \le 2\exp\{-2L_2 n_{k+1}\}. \end{aligned} \tag{3.19}$$

By the Borel–Cantelli lemma (3.19) implies (3.12) [and (3.13)] since $a > 1$ and $\varepsilon > 0$ are arbitrary in (3.19).

To verify (3.14) and (3.15), let $\lambda > 0$ be sufficiently small that $\alpha_2(\lambda) < 1$. Fix $a > 1$ and assume $\rho > 0$ satisfies

$$1 - \rho(a(1+\theta))^{2/p} > \alpha_2(\lambda). \tag{3.20}$$

Then for any $\varepsilon \in (0, \rho)$ and all k sufficiently large

$$\begin{aligned} &P\big(T_n(\lambda)/\beta_n^2(\lambda) < \rho - \varepsilon \text{ for some } n \in I(k)\big) \\ &\quad \le P\big(T_{n_k}(\lambda) < (\rho-\varepsilon)\beta^2_{n_{k+1}}(\lambda)\big) \\ &\quad = P\Big(T_{n_k}(\lambda) - \beta^2_{n_k}(\lambda) < \beta^2_{n_k}(\lambda)\big((\rho-\varepsilon)\beta^2_{n_{k+1}}(\lambda)/\beta^2_{n_k}(\lambda) - 1\big)\Big) \\ &\quad \le P\Big(\big|T_{n_k}(\lambda) - \beta^2_{n_k}(\lambda)\big| > \beta^2_{n_k}(\lambda)\big(1 - \rho(a(1+\theta))^{2/p}\big)\Big) \\ &\quad \le 2\exp\{-2L_2 n_k\} \end{aligned} \tag{3.21}$$

by (3.18) and (3.20).

The Borel–Cantelli lemma, (3.20) and (3.21) now imply (3.14) and (3.15) since $a > 1$ and $\varepsilon \in (0, \rho)$ are arbitrary in (3.20) and (3.21). Finally, fix $\lambda_0 > 0$ so that $\alpha_2(\lambda_0) < 1$ and let $\lambda \ge \lambda_0$. Then by (3.14) and (3.2)

$$\begin{aligned} \liminf_n V_n/\beta_n^2(\lambda) &\ge \liminf_n \frac{T_n(\lambda_0)}{\beta_n^2(\lambda_0)} \frac{\beta_n^2(\lambda_0)}{\beta_n^2(\lambda)} \\ &\ge \alpha_4(\lambda_0, \theta) \liminf_n \frac{d_n^2(\lambda_0)}{d_n^2(\lambda)} \frac{\lambda}{\lambda_0} \\ &\ge \alpha_4(\lambda_0, \theta) \left(\frac{\lambda}{\lambda_0}\right)^{(p-2)/p} (1+\theta)^{-2/p} > 0. \end{aligned}$$

Thus (3.17) holds and the lemma is proved. □

4. Proof of the upper-bound portion of Theorem 4. Throughout this section we assume the notation established at the end of Section 1. In addition, for $a > 1$ set $n_k = [a^k]$ and $I(k) = (n_k, n_{k+1}]$ for $k \ge 1$. For $n \in I(k)$ set

$$U_n(\lambda) = \sum_{j=1}^{n} u_j, \tag{4.1}$$

where

$$u_j = u_j(k, \lambda) = X_j I\big(|X_j| \le d_{n_{k+1}}(\lambda)\big), \qquad 1 \le j \le n_{k+1}. \tag{4.2}$$

LEMMA 4.1. *For each* $\lambda > 0$, $a > 1$ *and* $X \in \mathscr{F}(\theta)$,

$$\limsup_n |U_n(\lambda) - E(U_n(\lambda))|/\gamma_n(\lambda) = C_2(\lambda, X, a) \quad w.p.1, \tag{4.3}$$

where $C_2(\lambda, X, a)$ *is a finite constant such that*

$$C_2(\lambda, X, a) \le (a(1+\theta))^{1/p}\{\exp\{(2\lambda)^{1/2}\} + 1\}/2. \tag{4.4}$$

PROOF. Since $\gamma_n(\lambda)\uparrow\infty$ for each $\lambda > 0$ as $n\uparrow\infty$, $C_2(\lambda, X, a)$ is measurable with respect to the tail σ-field of $\{X_j: j \ge 1\}$, and hence is a constant. To show it is finite, we proceed as follows.

For each $\varepsilon > 0$, Chebyshev's inequality and (3.4) imply

$$\begin{aligned}
&\limsup_k \sup_{n\in I(k)} P\Big(\big|U_{n_{k+1}}(\lambda) - E\big(U_{n_{k+1}}(\lambda)\big) - \big(U_n(\lambda) - E(U_n(\lambda))\big)\big| > \varepsilon\gamma_{n_k}(\lambda)\Big)\\
&\quad\le \limsup_k \frac{n_{k+1}E\big(X^2 I(|X| \le d_{n_{k+1}}(\lambda))\big)(a(1+\theta))^{2/p}}{2\big(\varepsilon L_2 n_{k+1} d_{n_{k+1}}(\lambda)\big)^2/\lambda}\\
&\quad= \limsup_k \lambda n_{k+1} K\big(d_{n_{k+1}}(\lambda)\big)(a(1+\theta))^{2/p}/\big(2(\varepsilon L_2 n_{k+1})^2\big)\\
&\quad= 0.
\end{aligned}$$

Hence by Ottaviani's inequality, for each $\delta > \varepsilon > 0$ and all k sufficiently large

$$\begin{aligned}
&P\Big(\sup_{n\in I(k)} |U_n(\lambda) - E(U_n(\lambda))| > \delta\gamma_{n_k}(\lambda)\Big)\\
&\quad\le 2P\Big(\big|U_{n_{k+1}}(\lambda) - E\big(U_{n_{k+1}}(\lambda)\big)\big| > (\delta - \varepsilon/2)\gamma_{n_k}(\lambda)\Big)\\
&\quad\le 2P\Big(\big|U_{n_{k+1}}(\lambda) - E\big(U_{n_{k+1}}(\lambda)\big)\big| > (\delta - \varepsilon)\gamma_{n_{k+1}}(\lambda)(a(1+\theta))^{-1/p}\Big)\\
&\qquad\qquad [\text{by } (3.4)].
\end{aligned}$$

Thus $C_2(\lambda, X, a)$ will be finite if for some $\rho > 0$,

$$\sum_k P\Big(|W_k(\lambda)| > \rho\gamma_{n_{k+1}}(\lambda)(a(1+\theta))^{-1/p}\Big) < \infty, \tag{4.5}$$

where

$$W_k(\lambda) = U_{n_{k+1}}(\lambda) - E\big(U_{n_{k+1}}(\lambda)\big). \tag{4.6}$$

In fact, since $\varepsilon > 0$ is arbitrary

$$C_2(\lambda, X, a) \le \inf\{\rho > 0; (4.5)\text{ converges}\}. \tag{4.7}$$

By Lemma 3.4 for any $\xi > 0$ and $v > 0$, with $s = (1 + \xi)L_2 n_{k+1}$,

$$\begin{aligned}
&P\Big(|W_k(\lambda)| > ve^v n_{k+1} d_{n_{k+1}}(\lambda) K\big(d_{n_{k+1}}(\lambda)\big)/2\\
&\qquad\qquad + (1+\xi)L_2 n_{k+1} d_{n_{k+1}}(\lambda)/v\Big)\\
&\quad\le 2\exp(-(1+\xi)L_2 n_{k+1}).
\end{aligned} \tag{4.8}$$

Now

$$
\begin{aligned}
&ve^v n_{k+1} d_{n_{k+1}}(\lambda) K\big(d_{n_{k+1}}(\lambda)\big)/2 + (1+\xi) L_2 n_{k+1} d_{n_{k+1}}(\lambda)/v \\
&\quad = \gamma_{n_{k+1}}(\lambda)\big\{(\lambda/2)^{1/2} ve^v/(2\lambda) + (1+\xi)(\lambda/2)^{1/2}/v\big\} \\
&\quad = \gamma_{n_{k+1}}(\lambda)\big\{\exp\big\{(2\lambda)^{1/2}\big\} + (1+\xi)\big\}/2
\end{aligned}
\tag{4.9}
$$

by setting $v = (2\lambda)^{1/2}$. Since $\xi > 0$ is arbitrary, (4.5) will follow from (4.6), (4.8) and (4.9) if

$$
\rho(a(1+\theta))^{-1/p} > \big\{\exp\big\{(2\lambda)^{1/2}\big\} + 1\big\}/2.
$$

Thus by (4.7)

$$
C_2(\lambda, X, a) \le (a(1+\theta))^{1/p}\big(\exp\big\{(2\lambda)^{1/2}\big\} + 1\big)/2
$$

and Lemma 4.1 is proved. □

LEMMA 4.2. *If $X \in \mathscr{F}(\theta)$, then for all $\lambda > 0$,*

$$
\limsup_n \left| \frac{S_n - nE(XI(|X| \le d_n(\lambda)))}{(2L_2 nV_n)^{1/2}} \right| = C_3(\lambda, X), \tag{4.10}
$$

where $C_3(\lambda, X)$ is a finite constant. Moreover,

$$
\limsup_{\substack{\theta \downarrow 0 \\ \lambda = \theta^{1/2}}} \sup_{X \in \mathscr{F}(\theta)} C_3(\lambda, X) \le 1. \tag{4.11}
$$

PROOF. Recalling the definition of $\alpha_2(\lambda)$ in (3.11), we see that for any $\lambda > 0$ we can find $\lambda_0 \in (0, \lambda]$ such that $\alpha_2(\lambda_0) < 1$. Fix such a λ_0. Then for $a > 1$ and $n \in I_k$,

$$
\begin{aligned}
S_n - nE(XI(|X| \le d_n(\lambda))) &= (U_n(\lambda) - E(U_n(\lambda))) + (S_n - U_n(\lambda)) \\
&\quad + (E(U_n(\lambda)) - nE(XI(|X| \le d_n(\lambda)))) \\
&= A_n(\lambda) + B_n(\lambda) + C_n(\lambda).
\end{aligned}
\tag{4.12}
$$

Setting $\alpha_5(\theta, \lambda, \lambda_0) = \alpha_4(\theta, \lambda_0)(\lambda\lambda_0^{-1})^{(p-2)/p}(1+\theta)^{-2/p}$, we have by (3.16) and (4.3) that

$$
\begin{aligned}
&\limsup_n |A_n(\lambda)|/(2L_2 nV_n)^{1/2} \\
&\quad \le \limsup_n |A_n(\lambda)|/\big(2L_2 n\beta_n^2(\lambda)\alpha_5(\theta, \lambda, \lambda_0)\big)^{1/2} \\
&\quad = C_2(\lambda, X, a)\big(\alpha_5(\theta, \lambda, \lambda_0)\big)^{-1/2} < \infty,
\end{aligned}
\tag{4.13}
$$

where $C_2(\lambda, X, a)$ satisfies (4.4). Now (3.5) implies in a similar fashion that

$$
\begin{aligned}
\limsup_n |C_n(\lambda)|/(2L_2 nV_n)^{1/2} &\le \big(\alpha_5(\theta, \lambda, \lambda_0)\big)^{-1/2} \limsup_n |C_n(\lambda)|/\gamma_n(\lambda) \\
&\le \big(\alpha_5(\theta, \lambda, \lambda_0)\big)^{-1/2}(\theta/2\lambda)^{1/2} \\
&\quad \times a\big\{(a(1+\theta))^{(2-p)/p}(1+\theta) - 1\big\}^{1/2} \\
&< \infty.
\end{aligned}
\tag{4.14}
$$

Next, by the Cauchy–Schwarz inequality

$$|B_n(\lambda)| \le (J_n(\lambda))^{1/2} V_n^{1/2},$$

where $J_n(\lambda)$ is as in Lemma 3.3. Thus by Lemma 3.3

$$\limsup_n |B_n(\lambda)|/(2L_2 n V_n)^{1/2} \le 2^{-1/2}\alpha_1(\theta/\lambda) < \infty. \tag{4.15}$$

Combining (4.13), (4.14) and (4.15), it follows that

$$\limsup_n |S_n - nE(XI(|X| \le d_n(\lambda)))|/(2L_2 n V_n)^{1/2} < \infty \quad \text{a.s.} \tag{4.16}$$

Hence by the Kolmogorov zero–one law (4.10) holds with $C_3(\lambda, X)$ a finite constant, and furthermore,

$$\begin{aligned} C_3(\lambda, X) &\le (\alpha_5(\theta, \lambda, \lambda_0))^{-1/2} \\ &\quad \times \left\{ C_2(\lambda, X, a) + (\theta/2\lambda)^{1/2} a \left\{ (a(1+\theta))^{(2-p)/p}(1+\theta) - 1 \right\}^{1/2} \right\} \\ &\quad + 2^{-1/2}\alpha_1(\theta/\lambda), \end{aligned} \tag{4.17}$$

where $\lambda_0 \in (0, \lambda]$ is such that $\alpha_2(\lambda_0) < 1$. Thus, if λ is sufficiently small that $\alpha_2(\lambda) < 1$, then we may choose $\lambda_0 = \lambda$ in the above. Hence by (3.9), (3.11), (3.15), the definition of $\alpha_5(\theta, \lambda, \lambda_0)$ and (4.4) (since $a > 1$ is arbitrary),

$$\limsup_{\substack{\theta \downarrow 0 \\ \lambda = \theta^{1/2}}} \sup_{X \in \mathscr{F}(\theta)} C_3(\lambda, X) \le 1. \tag{4.18}$$ □

PROOF OF (I) AND PART OF (III) OF THEOREM 4. Lemma 4.2 and the Kolmogorov zero–one law combine to give (I) with

$$|C_1(X, \lambda)| \le C_3(\lambda, X) < \infty. \tag{4.19}$$

Hence (4.18) implies

$$\limsup_{\substack{\theta \downarrow 0 \\ \lambda = \theta^{1/2}}} \sup_{X \in \mathscr{F}(\theta)} |C_1(X, \lambda)| \le 1. \tag{4.20}$$

The remaining aspects of Theorem 4 are proved in Section 5. □

5. The lower-bound portion of Theorem 4. Our first objective is to develop a suitable probability estimate useful in connection with the establishment of nontrivial lower bounds in Theorem 4. We first develop some lemmas, and the estimate itself appears in Proposition 5.1.

For $m \ge 1$, $\lambda > 0$, let $Z(\lambda, m), Z_1(\lambda, m), Z_2(\lambda, m), \ldots$ be i.i.d. random variables with distribution function $F_{Z(\lambda, m)}$, that of X conditioned by $|X| \le d_m(\lambda)$; thus

$$dF_{Z(\lambda, m)}(x) = I(-d_m(\lambda) \le x \le d_m(\lambda))\, dF(x)/P(|X| \le d_m(\lambda)), \tag{5.1}$$

where F is the distribution function of X. We will also have need for the related random variables

$$\tilde{U}_n(\lambda, m) = Z_1(\lambda, m) + \cdots + Z_n(\lambda, m), \tag{5.2}$$
$$\tilde{T}_n(\lambda, m) = Z_1^2(\lambda, m) + \cdots + Z_n^2(\lambda, m).$$

The following lemma is a simple modification of Lemma 3.4 in [3], and provides proper emphasis for the conditioning interpretation. Lemma 3.4 of [3] is stated under the blanket assumption that $F(x)$ is continuous, but the proof remains the same even if F is not continuous.

LEMMA 5.1. *For any* $n, m \geq 1$, $\lambda > 0$ *and any bounded Borel function* $\psi: \mathbb{R}^n \to \mathbb{R}^1$,

$$\begin{aligned} &E\big(\psi(X_1, \dots, X_n) I(|X_j| \leq d_m(\lambda), 1 \leq j \leq n)\big) \\ &\quad = E\big(\psi(Z_1(\lambda, m), \dots, Z_n(\lambda, m))\big) \prod_{j=1}^{n} P\big(|X_j| \leq d_m(\lambda)\big). \end{aligned} \tag{5.3}$$

PROOF. Let $\phi(S_1, \dots, S_n) = \psi(S_1, S_2 - S_1, \dots, S_n - S_{n-1})$ and apply Lemma 3.4 of [3] to obtain (5.3). □

LEMMA 5.2. *Let* $\lambda > 0$, $\alpha_2(\lambda)$ *be as in* (3.11), *and suppose*

$$\Lambda > 1 + \alpha_2(\lambda). \tag{5.4}$$

Let $m \geq n$ *be such that* $m \sim n$ *as* $m, n \to \infty$. *Then for all n sufficiently large*

$$P\big(\tilde{T}_n(\lambda, m) > \Lambda E(\tilde{T}_n(\lambda, m))\big) \leq 2e^{-2L_2 n}. \tag{5.5}$$

PROOF. Applying Lemma 3.4 to the random variables $Y_j = Z_j^2(\lambda, m)$, $j \geq 1$, with $b = d_m^2(\lambda)$, $s = 2L_2 n$ and $v > 0$, we obtain

$$\begin{aligned} P\Big(\big|\tilde{T}_n(\lambda, m) - E(\tilde{T}_n(\lambda, m))\big| &> v e^v n d_m^2(\lambda) \tilde{K}\big(d_m^2(\lambda)\big)/2 \\ &\quad + 2L_2 n d_m^2(\lambda)/v\Big) \leq 2e^{-2L_2 n}, \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} \tilde{K}(x^2) &= E\big(Z_1^4(\lambda, m) I(Z_1^2(\lambda, m) \leq x^2)\big)/x^4 \\ &\leq E\big(Z_1^2(\lambda, m) I(|Z_1(\lambda, m)| \leq x)\big)/x^2. \end{aligned} \tag{5.7}$$

Thus

$$\begin{aligned} \tilde{K}\big(d_m^2(\lambda)\big) &\leq E\big(X^2 I(|X| \leq d_m(\lambda))\big)/\big(d_m^2(\lambda) P(|X| \leq d_m(\lambda))\big) \\ &= K\big(d_m(\lambda)\big)/P\big(|X| \leq d_m(\lambda)\big) \\ &= L_2 m/\big(\lambda m P(|X| \leq d_m(\lambda))\big). \end{aligned} \tag{5.8}$$

By setting $v = \lambda^{1/2}$ and combining (5.6) and (5.8), we obtain

$$P\Bigg(\big|\tilde{T}_n(\lambda, m) - E(\tilde{T}_n(\lambda, m))\big| > \frac{d_m^2(\lambda)L_2 m}{\lambda} \times\Big(\lambda^{1/2}e^{\lambda^{1/2}}n\big(2mP(|X| \le d_m(\lambda))\big)^{-1} + 2\lambda^{1/2}L_2 n(L_2 m)^{-1}\Big)\Bigg) \le 2e^{-2L_2 n}. \tag{5.9}$$

Next notice that

$$\begin{aligned} E(\tilde{T}_n(\lambda, m)) &= nE(Z_1^2(\lambda, m)) \\ &= nE(X^2 I(|X| \le d_m(\lambda)))/P(|X| \le d_m(\lambda)) \\ &= \frac{d_m^2(\lambda)L_2 m}{\lambda}(n/m)/P(|X| \le d_m(\lambda)). \end{aligned} \tag{5.10}$$

Now

$$\begin{aligned} &P(\tilde{T}_n(\lambda, m) > \Lambda E(\tilde{T}_n(\lambda, m))) \\ &\quad = P(\tilde{T}_n(\lambda, m) - E(\tilde{T}_n(\lambda, m)) > (\Lambda - 1)E(\tilde{T}_n(\lambda, m))) \\ &\quad \le P(|\tilde{T}_n(\lambda, m) - E(\tilde{T}_n(\lambda, m))| > (\Lambda - 1)E(\tilde{T}_n(\lambda, m))), \end{aligned}$$

and hence by (5.9)

$$P(\tilde{T}_n(\lambda, m) > \Lambda E(\tilde{T}_n(\lambda, m))) \le 2e^{-2L_2 n}, \tag{5.11}$$

provided

$$\begin{aligned} &(\Lambda - 1)E(\tilde{T}_n(\lambda, m)) \\ &\quad \ge \frac{L_2 m d_m^2(\lambda)}{\lambda}\Big(\lambda^{1/2}e^{\lambda^{1/2}}n\big(2mP(|X| \le d_m(\lambda))\big)^{-1} \\ &\qquad + 2\lambda^{1/2}L_2 n(L_2 m)^{-1}\Big). \end{aligned} \tag{5.12}$$

If $\lambda > 0$ is fixed and the strict inequality in (5.4) holds, then $m \ge n$ and $m \sim n$ as $m, n \to \infty$ together with (5.10) implies (5.12) for large n since $P(|X| \le d_m(\lambda)) \uparrow 1$ and $\alpha_2(\lambda) = (1/2)\lambda^{1/2}e^{\lambda^{1/2}} + 2\lambda^{1/2}$. Thus (5.11) is valid and the lemma is proved. □

LEMMA 5.3. *If $X \in \mathscr{F}(\theta)$ and $\lambda > 0$, then for all n sufficiently large and $m \ge n$,*

$$P\Big(\max_{1 \le j \le n} |X_j| \le d_m(\lambda)\Big) \ge \exp\{-(2\theta/\lambda)L_2 n\}. \tag{5.13}$$

PROOF. For all n sufficiently large and $m \geq n$,

$$\begin{aligned} P\Big(\max_{1 \leq j \leq n} |X_j| \leq d_m(\lambda)\Big) &= \big(1 - G(d_m(\lambda))\big)^n \\ &\geq \big(1 - G(d_n(\lambda))\big)^n \\ &\geq \exp\{-2nG(d_n(\lambda))\} \\ &\geq \exp\{-2n\theta K(d_n(\lambda))\} \\ &= \exp\{-(2\theta/\lambda)L_2 n\}. \end{aligned}$$ □

DEFINITION. For every $\lambda > 0$ and $m \geq 1$ set

$$\sigma_n^2(\lambda, m) = E\Big(\big(\tilde{U}_n(\lambda, m) - E(\tilde{U}_n(\lambda, m))\big)^2\Big). \tag{5.14}$$

LEMMA 5.4. *For* $\lambda > 0$, $\gamma > 0$ *and* $m \geq n \geq 1$ *there exist* $\pi(\gamma) > 0$ *and* $\varepsilon(\gamma) > 0$ *such that*

$$\begin{aligned} &P\Big(\tilde{U}_n(\lambda, m) - E(\tilde{U}_n(\lambda, m)) > \xi\big(2L_2 n\sigma_n^2(\lambda, m)\big)^{1/2}\Big) \\ &\quad \geq \exp\{-\xi^2(1+\gamma)L_2 n\}, \end{aligned} \tag{5.15}$$

provided both $\xi(2L_2n)^{1/2} \geq \varepsilon(\gamma)$ *and* $\xi(2L_2n)^{1/2}d_m(\lambda)/\sigma_n(\lambda, m) \leq \pi(\gamma)$.

PROOF. This is a simple application of Kolmogorov's exponential bound result in [10], Theorem 5.2.2. □

PROPOSITION 5.1. *Let* $X \in \mathscr{F}(\theta)$. *Suppose* $m \geq n$ *and* $m \sim n$ *as* $m, n \to \infty$. *Then for every* $\gamma > 0$, *there exists* $\pi(\gamma) > 0$ *such that for every* $\lambda > 0$, *every* $\Lambda > 1 + \alpha_2(\lambda)$, *every* $\delta < \pi(\gamma)/(2(1+\gamma)\lambda\Lambda)^{1/2}$ *and every* n *sufficiently large* (*depending on* γ, λ *and* δ),

$$\begin{aligned} &P\Big(S_n - E(\tilde{U}_n(\lambda, m)) > \delta(2L_2nV_n)^{1/2}\Big) \\ &\quad \geq \exp\{-(2\theta/\lambda)L_2n\} \\ &\qquad \times \Big\{\exp\{-\delta^2(1+\gamma)^2\Lambda L_2n\} - 2\exp\{-2L_2n\}\Big\}. \end{aligned} \tag{5.16}$$

PROOF. If $\delta > 0$, $\lambda > 0$, $m \geq n$ and

$$\begin{aligned} \psi(x_1, \dots, x_n) = I\Bigg(&(x_1, \dots, x_n) \colon x_1 + \cdots + x_n \\ &- E(\tilde{U}_n(\omega, m)) > \delta\bigg(2L_2n\sum_{i=1}^n x_i^2\bigg)^{1/2}\Bigg), \end{aligned}$$

then Lemma 5.1 implies

$$
\begin{aligned}
&P\Big(S_n - E\big(\tilde{U}_n(\lambda, m)\big) > \delta(2L_2nV_n)^{1/2}\Big)\\
&\quad \geq P\Big(S_n - E\big(\tilde{U}_n(\lambda, m)\big) > \delta(2L_2nV_n)^{1/2}, \max_{1\leq j\leq n}|X_j| \leq d_m(\lambda)\Big)\\
&\quad = P\Big(\tilde{U}_n(\lambda, m) - E\big(\tilde{U}_n(\lambda, m)\big) > \delta\big(2L_2n\tilde{T}_n(\lambda, m)\big)^{1/2}\Big)\\
&\qquad \times P\Big(\max_{1\leq j\leq n}|X_j| \leq d_m(\lambda)\Big).
\end{aligned}
\tag{5.17}
$$

Furthermore,

$$
\begin{aligned}
&P\Big(\tilde{U}_n(\lambda, m) - E\big(\tilde{U}_n(\lambda, m)\big) > \delta\big(2L_2n\tilde{T}_n(\lambda, m)\big)^{1/2}\Big)\\
&\quad \geq P\Big(\tilde{U}_n(\lambda, m) - E\big(\tilde{U}_n(\lambda, m)\big) > \delta\big(2L_2n\tilde{T}_n(\lambda, m)\big)^{1/2},\\
&\qquad\qquad \tilde{T}_n(\lambda, m) \leq \Lambda E\big(\tilde{T}_n(\lambda, m)\big)\Big)\\
&\quad \geq P\Big(\tilde{U}_n(\lambda, m) - E\big(\tilde{U}_n(\lambda, m)\big) > \delta\Lambda^{1/2}\big(2L_2nE\big(\tilde{T}_n(\lambda, m)\big)\big)^{1/2}\Big)\\
&\qquad - P\Big(\tilde{T}_n(\lambda, m) > \Lambda E\big(\tilde{T}_n(\lambda, m)\big)\Big).
\end{aligned}
\tag{5.18}
$$

Since we are assuming $E(X) = 0$ when $E(X^2) < \infty$,

$$
\big(E(XI(|X| \leq b))\big)^2 = o\big(E\big(X^2I(|X| \leq b)\big)\big)
$$

as $b \to \infty$. Thus it is easy to see that as $n, m \to \infty$ with $m \sim n$,

$$
\begin{aligned}
\sigma_n^2(\lambda, m) &= nE\Big(\big(Z_1(\lambda, m) - E(Z_1(\lambda, m))\big)^2\Big)\\
&\sim nE\big(X^2I(|X| \leq d_m(\lambda))\big)\\
&\sim L_2md_m^2(\lambda)/\lambda\\
&\sim E\big(\tilde{T}_n(\lambda, m)\big)
\end{aligned}
\tag{5.19}
$$

by (5.10). Next observe that

$$
\delta\big((1+\gamma)\Lambda 2L_2n\big)^{1/2}d_m(\lambda)/\sigma_n(\lambda, m) \sim \delta\big(2(1+\gamma)\Lambda\lambda\big)^{1/2}
$$

as $n, m \to \infty$ with $n \sim m$. Since $\delta < \pi(\gamma)(2(1+\gamma)\Lambda\lambda)^{-1/2}$, (5.19) and Lemma 5.4 then imply that for any $\gamma > 0$ if m, n are sufficiently large

$$
\begin{aligned}
&P\Big(\tilde{U}_n(\lambda, m) - E\big(\tilde{U}_m(\lambda, m)\big) > \delta\Lambda^{1/2}\big(2L_2nE\big(\tilde{T}_n(\lambda, m)\big)\big)^{1/2}\Big)\\
&\quad \geq P\Big(\tilde{U}_n(\lambda, m) - E\big(\tilde{U}_n(\lambda, m)\big) > \delta(1+\gamma)^{1/2}\Lambda^{1/2}\big(2L_2n\sigma_n^2(\lambda, m)\big)^{1/2}\Big)\\
&\quad \geq \exp\big\{-\delta^2\Lambda(1+\gamma)^2L_2n\big\}.
\end{aligned}
\tag{5.20}
$$

Combining (5.17), (5.18) and (5.20) along with Lemma 5.2 and Lemma 5.3, we obtain for $m \sim n$ and n sufficiently large that

$$\begin{aligned} &P\big(S_n - E(\tilde{U}_n(\lambda, m)) > \delta(2L_2 nV_n)^{1/2}\big) \\ (5.21)\qquad &\geq \exp\{-(2\theta/\lambda)L_2 n\} \\ &\times \big\{\exp\{-\delta^2\Lambda(1+\gamma)^2 L_2 n\} - 2\exp\{-2L_2 n\}\big\}, \end{aligned}$$

provided $\delta < \pi(\gamma)(2(1+\gamma)\lambda\Lambda)^{-1/2}$. Hence (5.16) holds and the proposition is proved. □

Lemma 5.5. *Let $X \in \mathscr{F}(\theta)$. Then for all λ sufficiently large there exists a $\delta = \delta(\lambda) > 0$ such that for $m \geq n$, $m \sim n$ and n sufficiently large*

$$(5.22)\qquad P\big(S_n - E(\tilde{U}_n(\lambda, m)) > \delta(2L_2 nV_n)^{1/2}\big) \geq \exp\{-(1/2)L_2 n\}.$$

Proof. Set $\gamma = 1$ in Proposition 5.1 and let $\lambda = 16\theta \vee 4\pi^2(\gamma)$, where $\pi(\gamma) = \pi(1)$ is given as in Proposition 5.1. Then for

$$\delta = \delta(\lambda) = \pi(\gamma)/\big(2((1+\gamma)\lambda\Lambda)^{1/2}\big),$$

(5.22) follows from (5.16) since

$$\frac{2\theta}{\lambda} \leq \frac{1}{8} \quad \text{and} \quad \delta^2(1+\gamma)^2\Lambda = \frac{\pi^2(\gamma)(1+\gamma)^2\Lambda}{4(1+\gamma)\lambda\Lambda} \leq \frac{(1+\gamma)}{16} = \frac{1}{8}.$$

The extra term $2\exp\{-2L_2 n\}$ in (5.16) is small enough to be absorbed into the right-hand side of (5.22). □

Lemma 5.6. *Let $X \in \mathscr{F}(\theta)$. Then for every $\gamma \in (0, 1/2)$ there exists a $\theta_0 > 0$ such that for all $\theta \in (0, \theta_0)$ if $\lambda = \theta^{1/2}$, then*

$$\begin{aligned} &P\big(S_n - E(\tilde{U}_n(\lambda, m)) > (1-2\gamma)^{1/2}(2L_2 nV_n)^{1/2}\big) \\ (5.23)\qquad &\geq \exp\Big\{-L_2 n\big\{2\theta^{1/2} + \big(1-\gamma^2\big)^2\big\}\Big\} \end{aligned}$$

as $m, n \to \infty$ with $m \geq n$ and $m \sim n$.

Proof. Fix $\gamma \in (0, 1/2)$. Choose $\theta_0 > 0$ sufficiently small so that with $\lambda = \theta^{1/2}$ and $\theta \in (0, \theta_0)$ we can find Λ satisfying

(i) $\Lambda > 1 + \alpha_2(\lambda)$,
(ii) $\Lambda < (1-\gamma)^2/(1-2\gamma)$ and
(iii) $\pi(\gamma)/(2(1+\gamma)\lambda\Lambda)^{1/2} \geq 1$.

Then we can take $\delta = (1-2\gamma)^{1/2}$ in (5.16) and obtain (5.23) since again the extra term $2\exp\{-2L_2 n\}$ in (5.16) is small enough to be absorbed into the right-hand side of (5.23). □

Another useful lemma is the following.

LEMMA 5.7. *Let $W_1, W_2, \dots, W_l$ be i.i.d. random variables. Then for any* $0 \le r \le l-1$,

$$P\left(\sum_{j=1}^{l} I(W_j < W_1) \ge l - r\right) \le r/l. \tag{5.24}$$

REMARK. The strict inequality $W_j < W_1$ in (5.24) is essential, as easy examples show.

PROOF. Let F be the distribution function of W_1 and let $F^{-1}(x) = \inf\{y: F(y) > x\}$ for $x \in [0,1)$. Then F^{-1} is right-continuous and if $U_1, \dots, U_l$ are i.i.d. uniform on $[0,1]$, then

$$F^{-1}(U_1), \dots, F^{-1}(U_l)$$

have the same joint law as $W_1, \dots, W_l$. Thus assume $W_1, \dots, W_l$ are given by $F^{-1}(U_1), \dots, F^{-1}(U_l)$. Since $W_1 > W_j$ implies $U_1 > U_j$,

$$P\left(\sum_{j=1}^{l} I(W_j < W_1) \ge l - r\right) \le P\left(\sum_{j=1}^{l} I(U_j < U_1) \ge l - r\right) = r/l. \qquad \square$$

PROPOSITION 5.2. *If $n_k = [e^{k^q}]$ where $q > 1$, then*

$$\lim_k V_{n_{k+1}}/V_{n_k} = \infty \quad w.p.1. \tag{5.25}$$

PROOF. An easy application of the Borel–Cantelli lemma yields (5.25) if for every $M > 0$,

$$\sum_k P\left(V_{n_{k+1}} < MV_{n_k}\right) < \infty. \tag{5.26}$$

Fix $M > 0$ and without loss of generality assume M is an integer. Then

$$V_{n_{k+1}} \ge W_1 + \cdots + W_{l_k},$$

where

$$l_k = [n_{k+1}/n_k] \tag{i}$$

and

$$W_j = V_{jn_k} - V_{(j-1)n_k} \quad \text{for } j = 1, \dots, l_k. \tag{ii}$$

Now $\lim_k l_k = +\infty$ and

$$\begin{aligned} P\left(V_{n_{k+1}} < MV_{n_k}\right) &\le P\left((W_1 + \cdots + W_{l_k}) < MW_1\right) \\ &\le P\left(\sum_{j=1}^{l_k} I(W_j \ge W_1) < M\right) \\ &= P\left(\sum_{j=1}^{l_k} I(W_j < W_1) \ge l_k - M\right) \\ &\le M/l_k \end{aligned}$$

by Lemma 5.7. Hence, since

$$l_k \sim e^{(k+1)^q - k^q} \geq \exp\{qk^{q-1}\} \tag{5.27}$$

for large k (with $q > 1$), (5.26) follows immediately. Thus (5.25) holds and the proposition is proved. □

We now prove a technical result which enables us to complete the proof of Theorem 4 with a simple application of the Borel–Cantelli lemma.

PROPOSITION 5.3. *Let $X \in \mathscr{F}(\theta)$ and set $n_k = [e^{k^q}]$ where $q > 1$. Then for all $\lambda > 0$,*

$$\begin{aligned} &\limsup_n \frac{S_n - nE(XI(|X| \leq d_n(\lambda)))}{(2L_2 nV_n)^{1/2}} \\ &\quad \geq \limsup_k \Big((S_{n_{k+1}} - S_{n_k}) - (n_{k+1} - n_k) E\big(XI(|X| \leq d_{n_{k+1}}(\lambda))\big) \\ &\qquad\qquad \times \big(P(|X| \leq d_{n_{k+1}}(\lambda))\big)^{-1} \Big) \\ &\qquad \div \big(2L_2(n_{k+1} - n_k)(V_{n_{k+1}} - V_{n_k})\big)^{1/2}. \end{aligned} \tag{5.28}$$

PROOF. Write

$$\begin{aligned} &S_{n_{k+1}} - n_{k+1} E\big(XI(|X| \leq d_{n_{k+1}}(\lambda))\big) \\ &\quad = (S_{n_{k+1}} - S_{n_k}) - (n_{k+1} - n_k) E\big(XI(|X| \leq d_{n_{k+1}}(\lambda))\big) \\ &\qquad + S_{n_k} - n_k E\big(XI(|X| \leq d_{n_k}(\lambda))\big) \\ &\qquad - n_k E\big(XI(d_{n_k}(\lambda) < |X| \leq d_{n_{k+1}}(\lambda))\big). \end{aligned} \tag{5.29}$$

By the upper-bound portion of Theorem 4 already established [see (4.10)] and Proposition 5.2

$$\limsup_k \frac{|S_{n_k} - n_k E\big(XI(|X| \leq d_{n_k}(\lambda))\big)|}{(2L_2 n_{k+1} V_{n_{k+1}})^{1/2}} = 0 \quad \text{a.s.} \tag{5.30}$$

for all $\lambda > 0$. Next by (3.16), for all $\lambda > 0$ there is a $\xi = \xi(\lambda) > 0$ such that $\liminf V_n/\beta_n^2(\lambda) \geq \xi$. Hence

$$\begin{aligned} &\limsup_k \frac{n_k E\big(|X| I(d_{n_k}(\lambda) < |X| \leq d_{n_{k+1}}(\lambda))\big)}{(2L_2 n_{k+1} V_{n_{k+1}})^{1/2}} \\ &\quad \leq \limsup_k \frac{n_k\big(E\big(X^2 I(|X| \leq d_{n_{k+1}}(\lambda))\big)\big)^{1/2} \big(G(d_{n_k}(\lambda))\big)^{1/2}}{\big(2\xi\beta_{n_{k+1}}^2(\lambda) L_2 n_{k+1}\big)^{1/2}} \\ &\quad = \limsup_k (n_k/(2\xi n_{k+1}))^{1/2} \big(n_k G(d_{n_k}(\lambda))/L_2 n_{k+1}\big)^{1/2} \\ &\quad \leq \limsup_k (\theta n_k/(2\xi\lambda n_{k+1}))^{1/2} = 0. \end{aligned} \tag{5.31}$$

Furthermore, for all $\lambda > 0$,

$$
\begin{aligned}
&(n_{k+1} - n_k)E\big(XI(|X| \le d_{n_{k+1}}(\lambda))\big)\big(P(|X| \le d_{n_{k+1}}(\lambda))\big)^{-1} \\
&\quad -(n_{k+1} - n_k)E\big(XI(|X| \le d_{n_{k+1}}(\lambda))\big) \\
&\quad = (n_{k+1} - n_k)E\big(XI(|X| \le d_{n_{k+1}}(\lambda))\big) \\
&\qquad \times G\big(d_{n_{k+1}}(\lambda)\big)/P\big(|X| \le d_{n_{k+1}}(\lambda)\big),
\end{aligned} \tag{5.32}
$$

while

$$
\lambda m G\big(d_m(\lambda)\big)/L_2 m = G\big(d_m(\lambda)\big)/K\big(d_m(\lambda)\big) < \theta
$$

for all m sufficiently large. Applying (3.16) as above, we obtain for all $\lambda > 0$,

$$
\begin{aligned}
&\limsup_k (n_{k+1} - n_k) \frac{E\big(|X| I(|X| \le d_{n_{k+1}}(\lambda))\big) G\big(d_{n_{k+1}}(\lambda)\big)}{P\big(|X| \le d_{n_{k+1}}(\lambda)\big)\big(2L_2 n_{k+1} V_{n_{k+1}}\big)^{1/2}} \\
&\quad \le \frac{\theta}{(2\xi\lambda)^{1/2}} \limsup_k E\big(|X| I(|X| \le d_{n_{k+1}}(\lambda))\big)/d_{n_{k+1}}(\lambda) = 0
\end{aligned} \tag{5.33}
$$

by the dominated convergence theorem. Finally, by Proposition 5.2

$$
2L_2(n_{k+1} - n_k)\big(V_{n_{k+1}} - V_{n_k}\big) \sim 2L_2 n_{k+1} V_{n_{k+1}};
$$

thus (5.28) follows from (5.29)–(5.33). □

PROOF OF (II) AND THE REMAINDER OF (III) OF THEOREM 4. Set $n_k = [e^{k^q}]$ with $q \in (1, 3/2)$. Since $X_1, X_2, \dots$ are i.i.d., by applying Lemma 5.5 with $n = n_{k+1} - n_k$ and $m = n_{k+1}$, it follows that for all $\lambda > 0$ sufficiently large there is a $\delta = \delta(\lambda) > 0$ such that for k sufficiently large

$$
\begin{aligned}
&P\Bigg\{ \big(S_{n_{k+1}} - S_{n_k}\big) - \frac{(n_{k+1} - n_k)E\big(XI(|X| \le d_{n_{k+1}}(\lambda))\big)}{P\big(|X| \le d_{n_{k+1}}(\lambda)\big)} \\
&\qquad\qquad > \delta\big(2L_2(n_{k+1} - n_k)\big(V_{n_{k+1}} - V_{n_k}\big)\big)^{1/2} \Bigg\} \\
&\quad \ge \exp\left\{ -\left(\frac{1}{2}\right) L_2(n_{k+1} - n_k) \right\}.
\end{aligned} \tag{5.34}
$$

Since the events in (5.34) are independent, and the corresponding probabilities sum to $+\infty$ (recall $q/2 < 1$), the Borel–Cantelli lemma implies the right-hand term in (5.28) is greater than or equal to $\delta = \delta(\lambda) > 0$ a.s. Hence (II) of Theorem 4 follows from Proposition 5.3.

To complete the proof of Theorem 4, it suffices to show that the right-hand term in (5.28) can be made arbitrarily close to 1 as $\theta \to 0$ with $\lambda = \theta^{1/2}$.

To verify this, fix $\gamma \in (0, 1/2)$ and take $q \in (1, 3/2)$ such that $q(1-\gamma^2) < 1$. Again set $n_k = [e^{k^q}]$. Choose $\theta_0 > 0$ as in Lemma 5.6 such that in addition $2\theta_0^{1/2} + (1-\gamma^2)^2 < (1-\gamma^2)$. Let $\theta \in (0, \theta_0)$ and set $\lambda = \theta^{1/2}$. Then with $n = n_{k+1} - n_k$ and $m = n_{k+1}$, (5.23) and the Borel–Cantelli lemma imply for k sufficiently large that the right-hand term in (5.28) is greater than or equal to $(1-2\gamma)^{1/2}$. Since $\gamma > 0$ is arbitrary this implies

$$\lim_{\substack{\theta \downarrow 0 \\ \lambda = \theta^{1/2}}} \inf_{X \in \mathscr{F}(\theta)} C_1(X, \lambda) = 1. \tag{5.35}$$

Combining (4.20) and (5.35) yields (2.9) and Theorem 4 is proved. □

6. Proof of Theorems 2, 3 and 1. Since $X \in D(\alpha)$ implies

$$\lim_{x \to \infty} G(x)/K(x) = (2-\alpha)/\alpha < \infty, \tag{6.1}$$

Theorem 4 implies $k_2(X, \lambda) = C_1(X, \lambda)$ is a finite nonnegative number for each $\lambda > 0$, and is strictly positive if λ is sufficiently large. Applying Theorem 4 with $X, X_1, X_2, \dots$ replaced by $-X, -X_1, -X_2, \dots$ yields

$$\limsup_n \frac{(-S_n) - E(-XI(|X| \le d_n(\lambda)))}{(2L_2 n V_n)^{1/2}} = k_2(-X, \lambda) \quad \text{a.s.}, \tag{6.2}$$

where $k_2(-X, \lambda) = C_1(-X, \lambda)$ is also a finite nonnegative number for each $\lambda > 0$ and is strictly positive if λ is sufficiently large. Hence, setting $k_1(X, \lambda) = k_2(-X, \lambda)$, we have

$$\liminf_n = \frac{S_n - nE(XI(|X| \le d_n(x)))}{(2L_2 n V_n)^{1/2}} = -k_1(X, \lambda) \quad \text{a.s.} \tag{6.3}$$

In order to apply Proposition 2.1 to prove clustering in (2.2), it suffices to show

$$\lim_n \frac{nE(XI(|X| \le d_n(\lambda))) - (n-1)E(XI(|X| \le d_{n-1}(\lambda)))}{\left(2L_2 n\beta_n^2(\lambda)\right)^{1/2}} = 0 \quad \text{a.s.} \tag{6.4}$$

Now the absolute value of the numerator in (6.4) is dominated by the quantity

$$nE(|X|I(d_{n-1}(\lambda) < |X| \le d_n(\lambda))) + E(|X|I(|X| \le d_n(\lambda))). \tag{6.5}$$

Clearly,

$$E(|X|I(|X| \le d_n(\lambda))) \le d_n(\lambda) = o\left(\left(2L_2 n\beta_n^2(\lambda)\right)^{1/2}\right),$$

while

$$\limsup_n nE\big(|X|I(d_{n-1}(\lambda) < |X| \le d_n(\lambda))\big)/\big(2L_2n\beta_n^2(\lambda)\big)^{1/2}$$

$$\le \limsup_n nd_n(\lambda)\big[G(d_{n-1}(\lambda)) - G(d_n(\lambda))\big]/\big((2/\lambda)^{1/2}L_2nd_n(\lambda)\big)$$

$$= \limsup_n (\lambda/2)^{1/2}nG(d_{n-1}(\lambda))\left[1 - \frac{G(d_n(\lambda))}{G(d_{n-1}(\lambda))}\right](L_2n)^{-1}$$

$$(6.6)\qquad = \limsup_n \left(\frac{2-\alpha}{\alpha}\right)(2\lambda)^{-1/2}\left[1 - \frac{G(d_n(\lambda))}{G(d_{n-1}(\lambda))}\right]$$

$$= \begin{cases} 0 & \text{if } \alpha = 2, \\ \limsup_n \left(\frac{2-\alpha}{\alpha}\right)(2\lambda)^{-1/2}\left[1 - \frac{K(d_n(\lambda))}{K(d_{n-1}(\lambda))}\right] & \text{if } 0 < \alpha < 2 \end{cases}$$

$$= \begin{cases} 0 & \text{if } \alpha = 2, \\ \limsup_n \left(\frac{2-\alpha}{\alpha}\right)(2\lambda)^{-1/2}\left[1 - \frac{(n-1)L_2n}{nL_2(n-1)}\right] = 0 & \text{if } 0 < \alpha < 2 \end{cases}$$

$$= 0.$$

Thus (2.2) holds and for the remainder of Theorem 2 we simply apply Theorem 4(III) since $\alpha \uparrow 2$ implies θ can be made arbitrarily small. Thus Theorem 2 is proved. □

It now suffices to prove Theorem 3, since Theorem 1 is an immediate consequence of (2.6) and (2.7), and that $\alpha = 2$ allows θ to be taken arbitrarily small.

The main step of the proof of Theorem 3 is the following proposition.

PROPOSITION 6.1. *If $X \in \mathscr{F}(\theta)$ with $\theta < 1$, then $E|X| < \infty$ and*

$$(6.7)\qquad \limsup_n \frac{S_n - nE(X)}{(2L_2nV_n)^{1/2}} = c(X) \quad a.s.,$$

where $0 < c(X) < \infty$ is a constant such that

$$(6.8)\qquad \lim_{\theta \downarrow 0} \sup_{X \in \mathscr{F}(\theta)} c(X) = \lim_{\theta \downarrow 0} \inf_{X \in \mathscr{F}(\theta)} c(X) = 1.$$

PROOF. That $E|X| < \infty$ is known from [4]. Furthermore, by the Kolmogorov zero–one law $c(X)$ is a constant with probability 1. We will prove that $c(X) < \infty$ and (6.8) holds via the following three lemmas. That $c(X) > 0$ for all $\theta < 1$ then follows in Lemma 6.7. □

LEMMA 6.1. *If* $E|X| < \infty$ *and*

$$N(x) = E(|X|I(|X| > x))/x, \qquad x > 0,$$

then

$$\int_x^\infty K(y)\,dy = xN(x) + xK(x), \qquad x > 0.$$

PROOF. Integrate by parts. □

LEMMA 6.2. *If* $X \in \mathscr{F}(\theta)$ *with* $\theta < 1$ *and* $p = 2/(1+\theta)$ *as in* (3.3), *then for all large* x,

$$N(x) \le K(x)\left(\frac{2+\theta-p}{p-1}\right). \tag{6.9}$$

PROOF. For x sufficiently large

$$\int_x^\infty K(y)\,dy \le \int_x^\infty Q(y)\,dy \le x^p Q(x)\int_x^\infty y^{-p}\,dy,$$

since $y^p Q(y)$ decreases eventually ([8], Lemma 2.4). Thus for sufficiently large x,

$$\int_x^\infty K(y)\,dy \le xQ(x)/(p-1),$$

since $p > 1$ when $\theta < 1$. Applying Lemma 6.1, we thus have for all x sufficiently large that

$$\begin{aligned} N(x) &= x^{-1}\int_x^\infty K(y)\,dy - K(x) \\ &\le Q(x)/(p-1) - K(x) \\ &\le K(x)(2+\theta-p)/(p-1). \end{aligned}$$ □

LEMMA 6.3. *If* $X \in \mathscr{F}(\theta)$ *with* $\theta < 1$, *then*

$$\limsup_n \frac{S_n - nE(X)}{(2L_2 nV_n)^{1/2}} = c(X) < \infty \quad a.s. \tag{6.10}$$

Furthermore, (6.8) *holds.*

PROOF. By Theorem 4(I), for any $\lambda > 0$,

$$|c(X) - C_1(X,\lambda)| \le \limsup_n nE(|X|I(|X| > d_n(\lambda)))/(2L_2 nV_n)^{1/2}. \tag{6.11}$$

By (3.16) and (3.11), with $\lambda_0 = \min(\lambda/2, 1/64)$, we have

$$\begin{aligned} \liminf_n V_n/\beta_n^2(\lambda) &\ge (1-\alpha_2(\lambda_0))(1+\theta)^{-4/p}(\lambda/\lambda_0)^{(p-2)/p} \\ &\ge (1/2)(1+\theta)^{-4/p}2^{(p-2)/p} \\ &= (1+\theta)^{-4/p}2^{-2/p}, \end{aligned} \tag{6.12}$$

where $p = 2/(1+\theta)$. Since $\beta_n^2(\lambda) = L_2 n d_n^2(\lambda)/\lambda$, Lemma 6.2 and (6.12) imply

$$
\begin{aligned}
&\limsup_n nE\big(|X| I(|X| > d_n(\lambda))\big)/(2L_2 n V_n)^{1/2} \\
&\quad \le \limsup_n \frac{n d_n(\lambda) N(d_n(\lambda))}{\big(2(1+\theta)^{-4/p}2^{-2/p}/\lambda\big)^{1/2} L_2 n d_n(\lambda)} \\
&\quad \le \limsup_n \frac{nK(d_n(\lambda))(2+\theta-p)\lambda^{1/2}}{L_2 n \big(2(1+\theta)^{-4/p}2^{-2/p}\big)^{1/2}(p-1)} \qquad (6.13)\\
&\quad = (2+\theta-p)\Big/\Big\{(p-1)\big(2\lambda(1+\theta)^{-4/p}2^{-2/p}\big)^{1/2}\Big\}.
\end{aligned}
$$

Combining Theorem 4(I), (6.11) and (6.13), we see $c(X) < \infty$. Furthermore, by (6.11) for all $\lambda > 0$,

$$c(X) = C_1(X, \lambda) + \varepsilon(X, \lambda),$$

where

$$|\varepsilon(X, \lambda)| \le \limsup_n nE\big(|X| I(|X| \ge d_n(\lambda))\big)/(2L_2 n V_n)^{1/2}.$$

Now (6.13) with $p = 2/(1+\theta)$ implies

$$\lim_{\substack{\theta \downarrow 0 \\ \lambda = \theta^{1/2}}} |\varepsilon(X, \lambda)| \le \lim_{\theta \downarrow 0} \frac{\theta^2 + 3\theta}{(2\theta)^{1/2}(1+\theta)^{-2/p}2^{-(1+\theta)/2}(1-\theta)} = 0.$$

Combining this with Theorem 4(III) gives (6.8). □

To complete the proof of Proposition 6.1, it now suffices to prove the lower bound, that is, $c(X) > 0$ for all $\theta < 1$. Using the notation of the above lemma, we have for all $\lambda > 0$,

$$c(X) = C_1(X, \lambda) + \varepsilon(X, \lambda).$$

Unfortunately, our estimates on $C_1(X, \cdot)$ and $\varepsilon(X, \cdot)$ are not good enough to show that $C_1(X, \lambda) + \varepsilon(X, \lambda) > 0$ for even one value of λ. Thus we proceed to verify $c(X) > 0$ directly.

LEMMA 6.4. *If $X \in \mathscr{F}(\theta)$ with $\theta < 1$, then $E|X| < \infty$ and there exists $c > 0$ such that*

$$P(S_n - nE(X) \ge d(n)) \ge c \qquad (6.14)$$

for all n sufficiently large. Furthermore, it is also the case that there is a $\hat{c} > 0$ such that

$$\inf_{n \ge 1} P(S_n - nE(X) > 0) > \hat{c}. \qquad (6.15)$$

PROOF. The inequality in (6.14) is essentially Theorem 3 of [4] because $X \in \mathscr{F}$ implies the constants $d(n)$ in (6.14) are equivalent (asymptotically) to the constants a_n used in [4]. The inequality in (6.15) now holds because X nondegenerate implies $P(S_n - nE(X) > 0) > 0$ for each n, and for some $c > 0$,

$$P(S_n - nE(X) > 0) \geq P(S_n - nE(X) \geq d(n)) \geq c,$$

provided n is sufficiently large. □

LEMMA 6.5. *If $X \in \mathscr{F}(\theta)$, then $\{V_n/d^2(n): n \geq 1\}$ is bounded in probability, that is, tight.*

PROOF. First observe that

$$\begin{aligned} &P\big(V_n \geq Md^2(n)\big) \\ (6.16) \qquad &\leq P\left(\sum_{i=1}^{n} X_i^2 I(|X_i| \leq \xi d(n)) \geq Md^2(n)\right) + nG(\xi d(n)) \\ &\leq n\xi^2 K(\xi d(n))/M + nG(\xi d(n)). \end{aligned}$$

Recalling $x^p Q(x)$ eventually decreases for $p = 2/(1+\theta)$, it follows that for $\xi > 1$ and n large (independent of $\xi > 1$)

$$\begin{aligned} G(\xi d(n)) &\leq Q(\xi d(n)) \\ &\leq \xi^{-p} Q(d(n)) \\ &\leq \xi^{-p}(1+\theta) K(d(n)). \end{aligned}$$

Similarly, $K(\xi d(n)) \leq \xi^{-p}(1+\theta)K(d(n))$, and hence for large n independent of $\xi > 1$,

$$P\big(V_n \geq Md^2(n)\big) \leq \xi^{2-p}(1+\theta)/M + \xi^{-p}(1+\theta).$$

Thus given $\varepsilon > 0$ choose ξ large enough that $(1+\theta)\xi^{-p} < \varepsilon/2$ and then M large enough that $\xi^{2-p}(1+\theta)/M < \varepsilon/2$. For this choice of M and n large

$$P\big(V_n \geq Md^2(n)\big) < \varepsilon,$$

which proves the lemma. □

LEMMA 6.6. *Assume $X \in \mathscr{F}(\theta)$ with $\theta < 1$. Then for $\delta > 0$ sufficiently small and all n sufficiently large*

$$(6.17) \qquad P\big(S_n - nE(X) \geq \delta(2L_2 n V_n)^{1/2}\big) \geq \exp(-(1/2)L_2 n).$$

PROOF. Let $N = 4\log(2/c)$ where $c > 0$ is such that (6.14) holds. Let $\tau(N,n) = [Nn/L_2 n]$ and $\rho(N,n) = [n/\tau(N,n)] + 1$, where $[\cdot]$ is the greatest integer function. Then $\rho(N, n) \sim (L_2 n)/N$ and we define for $i = 1, \dots, \rho(N, n)$,

$$\begin{aligned} R_i &= \left(S_{i\tau(N,n)} - i\tau(N,n)E(X)\right) \\ &\qquad -\left(S_{(i-1)\tau(N,n)} - (i-1)\tau(N,n)E(X)\right), \qquad (6.18) \\ W_i &= V_{i\tau(N,n)} - V_{(i-1)\tau(N,n)}. \end{aligned}$$

Letting $k(N, n) = (\rho(N, n) - 1)\tau(N, n)$, it follows that $k(N, n) \le n$ and for large n,

$$\begin{aligned} &P\left(S_n - nE(X) \ge \delta(2L_2 n V_n)^{1/2}\right) \\ &\quad \ge P\left(S_n - nE(X) \ge \delta(2L_2 n V_n)^{1/2},\right. \\ &\qquad\quad \left. W_i \le Md^2(n/L_2 n), 1 \le i \le \rho(N, n)\right) \\ &\quad \ge P\left(S_n - nE(X) \ge \delta\left(2(L_2 n)\rho(N, n)Md^2(n/L_2 n)\right)^{1/2},\right. \\ &\qquad\quad \left. W_i \le Md^2(n/L_2 n), 1 \le i \le \rho(N, n)\right) \qquad (6.19) \\ &\quad \ge P\left\{R_i \ge 2\delta(MN)^{1/2} d(n/L_2 n),\right. \\ &\qquad\quad W_i \le Md^2(n/L_2 n), 1 \le i < \rho(N, n) \\ &\qquad\quad \left. \text{and } S_n - S_{k(N,n)} - E\left(S_n - S_{k(N,n)}\right) > 0, W_{\rho(N,n)} \le Md^2(n/L_2 n)\right\} \\ &\quad \ge P\left(R_1 \ge 2\delta(MN)^{1/2} d(n/L_2 n), W_1 \le Md^2(n/L_2 n)\right)^{\rho(N,n)} \Gamma_n, \end{aligned}$$

where $\Gamma_n = P(S_n - S_{k(N,n)} - E(S_n - S_{k(N,n)}) > 0$ and $W_{\rho(N,n)} \le Md^2(n/L_2 n))$. With c as in (6.14) and $\hat{c}$ as in (6.15), apply (3.2) and Lemma 6.5 (since N is a fixed finite number) to choose M large enough so that

$$\begin{aligned} P\left(W_1 \le Md^2(n/L_2 n)\right) &= P\left(V_{[Nn/L_2 n]} \le Md^2(n/L_2 n)\right) \\ &\ge \max\{1 - c/2, 1 - \hat{c}/2\}. \end{aligned}$$

For this choice of M take $\delta = (1/2)(MN)^{-1/2}$. Then by (6.14)

$$P\left(R_1 \ge 2\delta(MN)^{1/2} d(n/L_2 n)\right) \ge P\left(R_1 \ge d([Nn/L_2 n])\right) \ge c \qquad (6.20)$$

for all large n since $N > 1$. Furthermore, $\Gamma_n \ge \hat{c}/2$ for all large n. Thus for n sufficiently large (6.15), (6.19) and (6.20) combine to yield

$$\begin{aligned} P\left(S_n - nE(X) \ge \delta(2L_2 n V_n)^{1/2}\right) &\ge (\hat{c}/2)(c/2)^{\rho(N,n)} \\ &= (\hat{c}/2)\exp\{-(N/4)\rho(N, n)\} \\ &\ge \exp\{-1/2L_2 n\} \end{aligned}$$

and (6.17) holds. □

LEMMA 6.7. *If $X \in \mathscr{F}(\theta)$ with $\theta < 1$, then*

$$\limsup_n \frac{S_n - nE(X)}{(2L_2 n V_n)^{1/2}} > 0 \quad a.s. \tag{6.21}$$

PROOF. Let $n_k = [e^{k^q}]$ where $q \in (1,2)$ and write

$$\begin{aligned} S_{n_{k+1}} - n_{k+1}E(X) &= \left(S_{n_{k+1}} - S_{n_k}\right) \\ &\quad - (n_{k+1} - n_k)E(X) + S_{n_k} - n_k E(X). \end{aligned} \tag{6.22}$$

Applying Lemma 6.3 to X and $-X$, we have

$$\limsup_n \frac{|S_n - nE(X)|}{(2L_2 n V_n)^{1/2}} < \infty \quad \text{a.s.} \tag{6.23}$$

Together with Proposition 5.2, this implies that

$$\limsup_k \frac{|S_{n_k} - n_k E(X)|}{\left(2L_2 n_{k+1} V_{n_{k+1}}\right)^{1/2}} = 0 \quad \text{a.s.} \tag{6.24}$$

Furthermore, if $\delta > 0$ is as in Lemma 6.6 and

$$\begin{aligned} E_k = \Big\{ &\left(S_{n_{k+1}} - S_{n_k}\right) - (n_{k+1} - n_k)E(X) \\ &> \delta\left(2L_2(n_{k+1} - n_k)\left(V_{n_{k+1}} - V_{n_k}\right)\right)^{1/2} \Big\}, \end{aligned} \tag{6.25}$$

then for all large k,

$$P(E_k) \geq (k+1)^{-q/2}. \tag{6.26}$$

Since the events $\{E_k\}$ are independent and $1 < q < 2$, (6.26) and the Borel–Cantelli lemma easily imply

$$\limsup_k \frac{\left(S_{n_{k+1}} - S_{n_k}\right) - (n_{k+1} - n_k)E(X)}{\left(2L_2(n_{k+1} - n_k)\left(V_{n_{k+1}} - V_{n_k}\right)\right)^{1/2}} \geq \delta \quad \text{a.s.} \tag{6.27}$$

Hence (6.22), (6.24) and (6.27) and Proposition 5.2 together yield (6.21), and Lemma 6.7 is proved. □

Thus Proposition 6.1 holds. The final step in the proof of Theorem 3 is to see that clustering occurs; this is now an immediate consequence of Proposition 2.1. □

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