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A SUFFICIENT CONDITION FOR TWO MARKOV SEMIGROUPS TO COMMUTE

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Let $\{P_t^{(k)}, t \geq 0\}$, $k = 1, 2$, be two Markov semigroups on $\hat{C}(E)$, the space of continuous functions on a separable, locally compact metric space E which tend to zero at infinity. In this article, we derive a sufficient condition for the two semigroups to commute, in the sense that for each $s \geq 0$, $t \geq 0$ and each $f \in \hat{C}(E)$, $P_s^{(1)} P_t^{(2)} f = P_t^{(2)} P_s^{(1)} f$.

We derive here a condition for two Feller–Markov semigroups of operators to commute; the result is a simple consequence of the following theorem of Ethier and Kurtz (1986).

Theorem 1. *Let E be a locally compact, separable metric space and let $\hat{C}(E)$ denote the Banach space of continuous real-valued functions on E that vanish at infinity, with the uniform norm. Suppose A is a linear operator in $\hat{C}(E)$ whose domain $\mathscr{D} \in \hat{C}(E)$ is a dense algebra and satisfies the following conditions*:

(i) (Maximum principle) *If $x \in E$, $f \in \mathscr{D}$ and $f(y) \leq f(x)$ for every $y \in E$, then $Af(x) \leq 0$.*

(ii) *The $D([0, \infty); E)$ martingale problem for A on $\mathscr{D}$ is well-posed*; *that is, for each $x \in E$, there exists a unique probability measure P on the Borel subsets of $D([0, \infty); E) = \{\theta\colon [0, \infty) \to E$, θ is right-continuous and has a left limit at every $t \in [0, \infty)\}$ endowed with the Skorokhod topology* [*see Ethier and Kurtz* (1986)], *such that $P(\theta(0) = x) = 1$ and*

$$X_f(t, \theta) = f(\theta(t)) - \int_0^t Af(\theta(s))\, ds$$

is a martingale under P.

If $\{\mu_t\colon t \geq 0\}$ and $\{\nu_t\colon t \geq 0\}$ are families of probability measures on $\mathscr{B}_E$, the Borel subsets of E, satisfying

$$\frac{d}{dt} \nu_t f = \nu_t A f$$

and

$$\frac{d}{dt} \mu_t f = \mu_t A f$$

for every $f \in \mathscr{D}$, then, if $\nu_0 = \mu_0$, it follows that $\mu_t = \nu_t$ for all $t \geq 0$.

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Before proving the proposition, we motivate it with the following example suggested to the author by T. Kurtz: $E = [0, \infty)$, $A = \{(f, \frac{1}{2}f''): f, f'' \in \hat{C}([0,\infty)), f''(0) = 0\}$, $B = \{(f, f'): f, f' \in \hat{C}([0,\infty))\}$ and

$$\mathscr{D} = \left\{ f: f, f', f'', f''' \in \hat{C}([0,\infty)), f''(0) = f'''(0) = 0 \right\}.$$

Clearly, for $f \in \mathscr{D}$, $ABf = BAf = \frac{1}{2}f'''$. It can be shown that $\mathscr{D}$ is a core for both A and B, so that the martingale problems for A and B are well-posed. The Markov semigroup generated by A corresponds to a Brownian motion in $(0, \infty)$ that is absorbed at the origin. B corresponds to uniform motion in $[0, \infty)$ to the right. One can easily check that their semigroups do not commute. The reason is that $\mathscr{D}$ is not an algebra.

The result on the commutativity of semigroups is the following proposition.

Proposition 1. *Let $\{P_t, t \geq 0\}$ be a Markov semigroup on $\hat{C}(E)$, that is, a semigroup of positive linear operators satisfying $\|P_t\| = 1$ for all $t \geq 0$, and let A: domain$(A) \to \hat{C}(E)$ be the generator of the semigroup. Let $\mathscr{D} \subseteq \hat{C}(E)$ be a dense algebra such that the $D([0,\infty); E)$ martingale problem for A on $\mathscr{D}$ is well-posed. Suppose that π: $\hat{C}(E) \to \hat{C}(E)$ is given by a probability transition function on $(E, \mathscr{B}_E)$, that is, $\pi g(x) = \int g(y)\pi(x, dy)$ and assume that*

(i) $\pi(\mathscr{D}) \subseteq \mathscr{D}$,
(ii) *for each $f \in \mathscr{D}$, $A\pi f = \pi A f$.*

Then, for each $g \in \hat{C}(E)$ and each $t \geq 0$,

$$P_t \pi g = \pi P_t g.$$

Proof. Given an f in $\mathscr{D}$, πf is in $\mathscr{D} \subseteq$ domain(A), so

$$\frac{d}{dt} P_t \pi f(x) = P_t A \pi f(x) = P_t \pi A f(x),$$

where the last equality follows from the hypothesis. Integrating from 0 to t, we get

$$P_t \pi f(x) = \pi f(x) + \int_0^t P_t \pi A f(x)\, ds.$$

Similarly,

$$\pi P_t f(x) = \pi f(x) + \int_0^t \pi P_t A f(x)\, ds.$$

Defining $\mu_t f = P_t \pi f(x)$ and $\nu_t f = \pi P_t f(x)$, for $x \in E$, we see that the hypothesis of the theorem holds, so $P_t \pi f(x) = \mu_t f = \nu_t f = \pi P_t f(x)$. Since this is true for each $x \in E$, we conclude that

$$P_t \pi f = \pi P_t f, \qquad f \in \hat{C}(E),$$

which is what we wanted to prove. □

COROLLARY 1. *Let $\{P_t,\ t \geq 0\}$, A and $\mathscr{D}$ be as in the proposition. Suppose $\{Q_t,\ t \geq 0\}$ is another Markov semigroup on $\hat{C}(E)$ with generator B:* domain$(B) \to \hat{C}(E)$. *Suppose $B(\mathscr{D}) \subseteq$* domain(A) *and $A(\mathscr{D}) \subseteq$* domain(B) *and for large $\lambda > 0$, $R_\lambda^B(\mathscr{D}) \subseteq \mathscr{D}$, where R_λ^B is the resolvent of B. If $AB = BA$ on $\mathscr{D}$, then, for $g \in \hat{C}(E)$ and $t, s \geq 0$,*

$$P_t Q_s g = Q_s P_t g.$$

PROOF. Since λR_λ^B: $\hat{C}(E) \to \hat{C}(E)$ is a nonnegative linear operator sending the function 1 into itself, by the Riesz representation theorem on locally compact spaces it is given by a probability transition function.

Let $f \in \mathscr{D}$. Then, for large λ, $R_\lambda^B f \in \mathscr{D}$, and since $R_\lambda^B = (\lambda - B)^{-1}$ on $\hat{C}(E)$,

$$A(\lambda - B)^{-1} f = (\lambda - B)^{-1}(\lambda - B)A(\lambda - B)^{-1} f = (\lambda - B)^{-1} A f.$$

It follows from the proposition that

$$P_t \circ R_\lambda^B f = R_\lambda^B \circ P_t f, \qquad f \in \hat{C}(E).$$

Since $(I - tB/n)^{-n} = ((n/t)R_{n/t}^B)^n$, it follows that for $f \in \hat{C}(E)$, $s \geq 0$ and n a positive integer,

$$P_t \circ \left(\frac{n}{t} R_{n/t}^B\right)^n f = \left(\frac{n}{t} R_{n/t}^B\right)^n \circ P_t f.$$

Letting $n \to \infty$, we obtain

$$P_t \circ Q_s f = Q_s \circ P_t f$$

and the proof is complete. □

As an example, take $E = \mathbb{R}^n$ and let $\{P_t^{(k)},\ t \geq 0\}$, $k = 1, 2$, be two diffusion semigroups on $\hat{C}(\mathbb{R}^n)$ with generators A_k, $k = 1, 2$, that are given on C^2 by the differential operator

$$A_k = \sum_{i,j} a_k^{ij}(x) \frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_i b_k^i(x) \frac{\partial}{\partial x_i},$$

where a_k: $x \mapsto (a_k^{ij}(x))$ and b_k: $x \mapsto (b_k^i(x))$ are C^2 functions from $\mathbb{R}^n$ into the $n \times n$ symmetric, positive semidefinite, real-valued matrices and into $\mathbb{R}^n$, respectively. For $\mathscr{D}$ take any dense subalgebra of $C^4 \cap \hat{C}(\mathbb{R}^n)$ such that A_k: $\mathscr{D} \to \hat{C}(R^n)$, $k = 1, 2$. (This holds, for instance, when the coefficients of the A_k are of polynomial growth and $\mathscr{D} = \mathscr{S}$, the space of rapidly decreasing functions.) Assume that

$$A_1 \circ A_2 f = A_2 \circ A_1 f, \qquad f \in \mathscr{D}.$$

Suppose also that the $C([0, \infty); \mathbb{R}^n)$ martingale problem for A_1 on $\mathscr{D}$ is well-posed (from which it follows that the $D([0, \infty); \mathbb{R}^n)$ martingale problem is well-posed too, since diffusions have continuous paths, almost surely). Then, if

$$R_\lambda^{A_2}(\mathscr{D}) = (\lambda - A_2)^{-1}(\mathscr{D}) \subseteq \mathscr{D},$$

it follows that

$$P_t^{(1)} \circ P_s^{(2)} f = P_s^{(2)} \circ P_t^{(1)} f, \qquad f \in \hat{C}(\mathbb{R}^n).$$

If the diffusions have C^∞ coefficients and the differential operator corresponding to $\lambda - A_2$ is hypoelliptic, a condition that is sometimes easy to verify, then we can let $\mathscr{D}$ be the space $C^\infty \cap \hat{C}(\mathbb{R}^n)$ and the invariance of $\mathscr{D}$ under the resolvent of B follows.

As a second example, we observe that the result applies to smooth diffusions on a compact C^∞ Riemannian manifold M: If A is a diffusion operator that maps smooth functions into smooth functions, we can take $\mathscr{D}$ to be the set of C^∞ functions on M.

To illustrate the usefulness of knowing when two semigroups commute, we mention two propositions whose hypotheses can sometimes be verified with the use of the result given above. The first proposition has been used, for instance, in Dudley and Stroock (1987).

PROPOSITION 2. *Suppose that* $\{P_t^{(1)}, t \geq 0\}$ *and* $\{P_t^{(2)}, t \geq 0\}$ *are contraction semigroups of bounded operators on a Banach space B and let* A_1 *and* A_2 *be their genertors, respectively. If the semigroups commute, then the map* $(s, t) \mapsto P_s^{(1)} \circ P_t^{(2)}$ *is strongly continuous, the set* $\mathscr{E} = \operatorname{dom}(A_1) \cap \operatorname{dom}(A_2)$ *is dense in B and, for all* $t \geq 0$ *and* $f \in B$,

$$P_t^{(1)} f - P_t^{(2)} f = \int_0^t P_{t-s}^{(2)} \circ P_s^{(2)} \circ (A_1 - A_2) f \, ds.$$

The second proposition is proved in Goldstein (1985) and gives a simple condition for the well-posedness of the abstract, time-dependent Cauchy problem.

PROPOSITION 3. *Suppose that for each* $t \geq 0$, $A(t)$ *generates a* C_0 *contraction semigroup on a Banach space B such that* $\exp(s_1 A(t_1))$ *and* $\exp(s_2 A(t_2))$ *commute for all nonnegative* s_1, s_2, t_1, t_2. *Let* $\mathscr{E} \subset \bigcap_{t \geq 0} \operatorname{dom}(A(t))$ *be a dense subspace of B such that the mapping* $t \mapsto A(t) f$ *is continuous for each* $f \in \mathscr{E}$. *Then there exists a strongly continuous family* $\{U(t, s): t \geq s \geq 0\}$ *of contraction evolution operators satisfying*

$$U(t, s) U(s, r) = U(t, r),$$
$$U(t, t) = I,$$

for $t \geq s \geq r \geq 0$ *and*

$$\frac{\partial}{\partial t} U(t, s) f = A(t) U(t, s) f = U(t, s) A(t) f,$$

$$\frac{\partial}{\partial s} U(t, s) f = -U(t, s) A(s) f = -A(s) U(t, s) f.$$

The function $u(t) = U(t,s)f$ is the unique solution of

$$\frac{d}{dt}u(t) = A(t)u(t), \qquad t \geq 0,$$
$$u(s) = f.$$

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