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ON THE EXTINCTION OF MEASURE-VALUED CRITICAL BRANCHING BROWNIAN MOTION

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We show that the diameter of the support of a measure-valued critical branching Brownian motion tends to zero almost surely at the time of extinction.

1. Introduction. The following type of measure-valued process $(X_t)_{t\geq 0}$, which is the limit of certain critical branching Brownian motions, has been studied by several authors since 1968 [2]. It is an $M_F(R^d)$-valued Markov process whose transition measures are characterized through their Laplace transforms as

$$E^{X_0}[\exp(-\langle\phi, X_t\rangle)] = \exp(-\langle u(t,\cdot), X_0\rangle), \qquad \phi \in C_b(R^d)_+. \tag{1.1}$$

Here, $M_F(R^d)$ denotes the space of finite Borel measures endowed with the weak topology, E^{X_0} is the expectation with respect to the probability P^{X_0}, the law of $(X_t)_{t\geq 0}$ which has a deterministic initial measure X_0, $C_b(R^d)_+$ denotes the set of bounded continuous nonnegative real-valued functions on R^d, $\langle\phi,\mu\rangle$ is an abbreviation of $\int_{R^d}\phi\, d\mu$ and u is the solution of

$$\dot{u}(t,x) = \Delta u(t,x) - u^2(t,x),$$
$$u(0,x) = \phi(x),$$

where Δ is the Laplacian $\sum_{i=1}^d \partial^2/\partial x_i^2$ and $\dot{u}$ is an abbreviation of $(\partial/\partial t)u$.

In this paper we will show that the diameter of the support of $(X_t)_{t\geq 0}$, with a suitable initial measure X_0, tends to zero almost surely at the time of extinction. The following notation will be used: $B(x_0; r) \equiv \{x \in R^d: |x - x_0| < r\}$, $\partial B(x_0; r) \equiv \{x \in R^d: |x - x_0| = r\}$ and $[\overline{B}(x_0; r)]^c \equiv \{x \in R^d: |x - x_0| > r\}$ for fixed $r > 0$ and $x_0 \in R^d$; supp(ν) and diam(supp ν) are the respective abbreviations of the support and the diameter of the support of a measure ν; $\rho_t \equiv \text{diam}(\text{supp } X_t)$ and $\xi \equiv \inf\{t > 0: X_t \text{ is extinct}\}$.

2. Main result.

Theorem 2.1. *Let μ be a finite Borel measure with compact support and let $X_0 = \mu$. Then*

$$\lim_{t\downarrow 0} \rho_{\xi - t} = 0, \qquad P^{X_0=\mu}\text{-}a.s. \tag{2.1}$$

Proof. We divide the proof into six steps as follows:

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STEP 1. $P^{X_0=\mu}(\sup_{0\le t<\infty}\rho_t<\infty)=1$. Let us quote a result by Iscoe [1], which concerns the range of $(X_t)_{t\ge 0}$ globally in space and time when the finite initial measure X_0 has compact support: If $X_0=\nu$ with $\operatorname{supp}(\nu)\subset B(x_0;r)$ and if $r\ge r_0$, then

$$P^{X_0=\nu}\big(X_\cdot \text{ ever charges } [\overline{B}(x_0;r)]^c\big) = 1-\exp\big(-r^{-2}\langle v(r^{-1}[\cdot - x_0]),\nu\rangle\big), \tag{2.2}$$

where v is the unique positive (radial) solution of the singular elliptic boundary value problem

$$\begin{aligned} \Delta v(x) &= v^2(x), && x\in B(0;1),\\ v(x) &\to \infty, && \text{as } x\to \partial B(0;1). \end{aligned} \tag{2.3}$$

Since the right-hand side of (2.2) tends to 0 as $r\to\infty$, it is easy to see that $\sup_{0\le t<\infty}\rho_t<\infty$, $P^{X_0=\mu}$-a.s.

STEP 2. For each $\delta>0$, there exists a constant $K=K(d)$ such that if $\nu\in M_F(R^d)$ and $\operatorname{supp}(\nu)\subset B(x_0;\delta/4)$, then

$$P^{X_0=\nu}\Big(\limsup_{t\uparrow\xi}\rho_t>\delta\Big)\le K\nu(R^d)\delta^{-2}. \tag{2.4}$$

This follows from the continuity of ν on $B(0;1)$ [see (2.3)] and

$$\begin{aligned} P^{\nu}\Big(\limsup_{t\uparrow\xi}\rho_t>\delta\Big) &\le P^{\nu}\big(X_\cdot \text{ ever charges } [\overline{B}(x_0;\delta/2)]^c\big)\\ &= 1-\exp\big(-4\delta^{-2}\langle v(2\delta^{-1}[\cdot - x_0]),\nu\rangle\big) \quad \text{[by (2.2)]}\\ &\le 4\delta^{-2}\langle v(2\delta^{-1}[\cdot - x_0]),\nu\rangle\\ &\qquad [\text{since } 1-\exp(-\alpha)\le\alpha \text{ for } \alpha\ge 0]\\ &\le \Big[4\max_{|x|<1/2} v(x)\Big]\nu(R^d)\delta^{-2}. \end{aligned}$$

STEP 3. $P^{X_0=\mu}(\xi<\infty)=1$. Insert $\phi\equiv\theta$ in (1.1), note that the corresponding $u(t,x)=\theta/[1+\theta t]$ (independent of x) and let $\theta\to\infty$:

$$\begin{aligned} P^{\mu}(\xi\le t) = P^{\mu}\big(X_t(R^d)=0\big) &= \lim_{\theta\to\infty} E^{\mu}[\exp(-\langle\theta, X_t\rangle)]\\ &= \lim_{\theta\to\infty}\exp(-\langle\theta/[1+\theta t],\mu\rangle) = \exp\big(-\mu(R^d)/t\big). \end{aligned}$$

Since the last term above tends to 1 as $t\to\infty$, we have $\xi<\infty$, $P^{X_0=\mu}$-a.s.

STEP 4. For each $\varepsilon\in(0,\mu(R^d))$, $\tau_\varepsilon\equiv\inf\{t>0: X_t(R^d)=\varepsilon\}$ is (by a routine argument) a stopping time. Thus, due to the strong Markov property of $(X_t)_{t\ge 0}$, we have $P^{X_0=\mu}(\limsup_{t\uparrow\xi}\rho_t>\delta)=E^{\mu}(P^{X_{\tau_\varepsilon}}(\limsup_{t\uparrow\xi}\rho_t>\delta))$.

STEP 5. $P^{X_{\tau_\varepsilon}(\omega)}(\limsup_{t\uparrow\xi}\rho_t>\delta)\le K\varepsilon\delta^{-2}$ for $P^{X_0=\mu}$-a.s. ω, where K is the constant defined in Step 2.

From Step 1, we have that for $P^{X_0=\mu}$-a.s. ω, there is a positive real number, say, $N(\omega)$ such that $\mathrm{supp}(X_{\tau_\varepsilon}(\omega)) \subset B(0; N(\omega))$. Let us chop up the ball $B(0; N(\omega))$ into small and nonoverlapping pieces, say, I_i, $i = 1, \dots, k$. Each piece has diameter less than $\delta/4$; let J be the set $\{j \in \{1, \dots, k\}: X_{\tau_\varepsilon}(\omega)(I_j) > 0\}$.

Denote $(X_t^{(j)})_{t\geq 0}$ for the process $(X_t)_{t\geq 0}$ with $X_0 = D_j$, where D_j is the measure $X_{\tau_\varepsilon}(\omega)$ restricted to I_j, $j \in J$. By (1.1), $(X_t)_{t\geq 0}$ is a multiplicative process. This implies that the distributions of $(X_t)_{t\geq 0}$ and $(\sum_{j\in J} X_t^{(j)})_{t\geq 0}$ are identical in law, where $(X_t^{(j)})_{t\geq 0}$ are independent $M_F(R^d)$.

To apply (2.4), let $\hat{\tau}_j$ be the time of death for $(X_t^{(j)}(R^d))_{t\geq 0}$, $j \in J$. Since J is a finite set and since the $\hat{\tau}_j$'s above are independent continuous random variables (see Step 3), it follows that P(at least two $\hat{\tau}_j$'s have the same value) $= 0$ and so $P(\max_{k\in J}\hat{\tau}_k$ is attained for only one $j) = 1$. This implies that $E_j \equiv \{\max_{k\in J}\hat{\tau}_k = \hat{\tau}_j\}$, $j \in J$, can be considered as mutually disjoint events with $\sum_{j\in J} P(E_j) = 1$.

Note that $P(E_j) > 0$ for each $j \in J$. After defining $\hat{\tau}_0 \equiv \max_{j\in J}\hat{\tau}_j$, $\varepsilon_j \equiv D_j(R^d)$, $\hat{\rho}_t^{(j)} \equiv \mathrm{diam}(\mathrm{supp}\, X_t^{(j)})$ and $\hat{\rho}_t \equiv \mathrm{diam}(\mathrm{supp} \sum_{j\in J} X_t^{(j)})$, we have

$$\begin{aligned} P^{X_{\tau_\varepsilon}(\omega)}\Big(\limsup_{t\uparrow\xi} \rho_t > \delta\Big) &= P\Big(\limsup_{t\uparrow\hat{\tau}_0} \hat{\rho}_t > \delta\Big) \\ &= \sum_{j\in J}\Big[P\Big(\limsup_{t\uparrow\hat{\tau}_0} \hat{\rho}_t > \delta \,|\, \hat{\tau}_t = \hat{\tau}_j\Big) \cdot P\big(E_j\big)\Big] \\ &\leq \sum_{j\in J} P^{D_j}\Big(\limsup_{t\uparrow\hat{\tau}_j} \hat{\rho}_t^{(j)} > \delta\Big) \\ &\leq \sum_{j\in J} K\varepsilon_j\delta^{-2} \quad \text{[by Step 2]} \\ &= K\varepsilon\delta^{-2}. \end{aligned}$$

STEP 6. (2.1) then can be easily verified through Steps 4 and 5. □

Edwin Perkins has informed us that he has independently proved Theorem 2.1.

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