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UNIFORM DIMENSION RESULTS FOR THE BROWNIAN SHEET[1]

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We show that if $2N \le d$, then with probability 1 the Brownian sheet $W\colon R_+^N \to R^d$ satisfies $\forall$ Borel set E, $\dim W(E) = 2\dim E$.

1. Uniform dimension results for the Brownian sheet. Let $W\colon R_+^N \to R^d$ be a Brownian sheet. The Brownian sheet is a continuous process defined on R_+^N, where the finite-dimensional distributions are multivariate normal with means 0 and

$$E\left[W(\mathbf{t})_i W(\mathbf{s})_j\right] = \delta_{i,j} \prod_{i=1}^{N} \min(s_i, t_i)$$

with $\mathbf{t}(\mathbf{s}) = (t_1, t_2, \dots, t_N)((s_1, s_2, \dots, s_N))$. Orey and Pruitt (1973) proved:

1. Each fixed point in R^d is hit with probability 0 or 1, depending on whether $2N \le d$ or $2N > d$. This left open the question of whether every point of R^d was hit a.s. when $2N > d$. Rosen (1981) proved:
2. If $2N > d$, then a.s. $\forall x \in R^d$,

$$\dim\{\mathbf{t}\colon W(\mathbf{t}) = x\} = N - d/2.$$

In this paper, I wish to show that when $2N \le d$, another kind of dimensional regularity holds.

THEOREM 1. *Let $W\colon R_+^R \to R^d$ be a Brownian sheet, with $2N \le d$. With probability 1 for each Borel set $E \subset R_+^R$,* $\dim(W(E)) = 2\dim(E)$.

COMMENT. The Fourier analysis methods of Kahane (1968) show that for any time set E, a.s. $\dim(W(E)) = 2\dim(E)$.

Uniform dimension results were first obtained in Kaufman (1969) [see also Hawkes and Pruitt (1974)].

We state without proof analogous results for the Ornstein–Ühlenbeck process on Wiener space and as an application we sketch a proof of the following proposition.

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PROPOSITION 0. *Let* $\{O_s(\cdot): s \geq 0\}$ *be the Ornstein–Ühlenbeck process on d-dimensional Wiener space. Let* B *be a set in* R^d *of dimension* β *for* $\beta \in [d-4, d-2]$. *Then* $4 - d + \beta$ *is the supremum over* α *such that* $P[\dim\{x \in B: \exists\,(s,t) \text{ s.t. } O_s(t) = x\} > \alpha] > 0$.

If $d = 4$, *then* $P[\dim\{x \in B: \infty\,(s,t) \text{ s.t. } O_s(t) = x\} = \alpha] = 1$.

The following result is contained in Theorem 2.2 of Orey and Pruitt (1973): On every compact time set, W is α-Hölder continuous $\forall\,\alpha < \frac{1}{2}$. Given this result, it follows easily that

$$\forall \text{ Borel set } E \subset R_+^N, \quad \dim(W(E)) \leq 2\dim(E).$$

This gives us one side of the equality of our theorem; to complete the proof of the theorem, we have to show the converse inequality.

We now give a sketch of the proof.

HEURISTIC. Kaufman (1969) proved his uniform dimension result for planar Brownian motion by showing that a.s. on the time interval $[0,1]$, the time spent by Brownian motion inside each square of R^2, $[i/2^{n/2}, (i+1)/2^{n/2}) \times [j/2^{n/2}, (j+1)/2^{n/2})$ is contained in the union of n^4 dyadic time intervals $[k/2^n, (k+1)/2^n)$. This is the tactic we follow. This method shows that a.s. every set E in R^d satisfies

$$\dim\{W^{-1}(E)\} \leq \tfrac{1}{2}\dim\{E\}.$$

By the regularity properties of the Brownian sheet to prove Theorem 1 it will be sufficient to show the following theorem.

THEOREM 2. *Let* W *be a Brownian sheet from* R_+^N *to* R^d. *With probability* 1, *every Borel set* E *in* $[0,1]^d$ *satisfies*

$$\dim\{W^{-1}(E) \cap [1,2]^N\} \leq \tfrac{1}{2}\dim\{E\}.$$

Throughout this paper it will be a guiding principle that powers (rather than exponents) of n are irrelevant and that we may scatter them about quite liberally.

TERMINOLOGY. We will refer to squares or cubes of the form $[i/2^n, (i+1)/2^n] \times [i/2^n, (i+1)/2^n]$ as *dyadic squares of order n* or *dyadic cubes of order n*.

As in Kaufman (1969), Theorem 2 will follow if we can show that when n is sufficiently large, every dyadic cube of order $n/2$, I, $\{W^{-1}(I)\} \cap [1,2]^N$ can be covered by n^r dyadic cubes of order n. Often we will treat large numbers as if they are integers (e.g., $\sqrt{6n}\,2^n$). It will hopefully be clear that this is merely a device to simplify notation.

2. Plan of the paper. First, I prove the result for the case $N = 2,\ d = 5$, then I make the suitable modifications to deal with the case $N = 2,\ d = 4$; finally, I illustrate how the proof in this case can be made to deal with the general case. The Brownian sheet in question is really the product of d independent coordinate processes. Accordingly, it is only necessary to deal with the case $2N = d$. In particular, each of the first two cases is rendered obsolete by the succeeding case. Nonetheless, I hope that this plan will make the ideas involved more plain.

CASE 1. $N = 2,\ d = 5$. Using the results of Orey and Pruitt (1973) or Fukishima (1984), we can and will assume that for all $(s, t) \in [1,2]^2$ and h small enough,

$$(*) \qquad |W(s,t) - W(s,t+h)| < \sqrt{6h \log 1/h}\,.$$

We now consider a five-dimensional Brownian motion $\{B(t)\colon t \geq 0\}$ and define the random variable

$$A(\omega) = \int_0^1 \frac{dt}{\left(\max\{|B(t)|, n^p 2^{-n/2}\}\right)^{5-2}}.$$

NOTE. A depends tacitly on the as yet unspecified parameter p.

LEMMA 1. *There exists a positive constant C such that $E[e^{CAn^p/2^{n/2}}] < 2$.*

PROOF. The expectation of A is less than

$$\int_0^1 \min\left\{\frac{C'}{t^{3/2}}, \frac{1}{(n^p 2^{-n/2})^3}\right\} dt = \int_0^{c'n^{2p}/2^n} \frac{1}{(n^p 2^{-n/2})^3}\,dt + \int_{c'n^{2p}/2^n}^1 \frac{C'}{t^{3/2}}\,dt$$
$$< K 2^{n/2} n^p,$$

for some C', c', K.

It is easy to see that for $s < t$,

$$E\left[\min\left\{\frac{1}{|B(t)|^3}, \frac{2^{3n/2}}{n^{3p}}\right\}\middle| B(u) 0 \leq u \leq s\right] \leq \min\left\{\frac{C}{|t-s|^{3/2}}, \frac{2^{3n/2}}{n^{3p}}\right\}$$

and so

$$\begin{aligned} E[A^n] \leq n! \int_0^1 &\min\left\{\frac{C}{t_1^{3/2}}, \frac{1}{(n^p 2^{-n/2})^3}\right\} dt_1 \\ &\times \int_{t_1}^1 \min\left\{\frac{C}{|t_2 - t_1|^{3/2}}, \frac{1}{(n^p 2^{-n/2})^3}\right\} dt_2 \cdots \\ &\times \int_{t_{n-1}}^1 \min\left\{\frac{C}{|t_n - t_{n-1}|^{3/2}}, \frac{1}{(n^p 2^{-n/2})^3}\right\} dt_n \\ \leq C n! (E[A])^n. \end{aligned}$$

By using the expansion for the exponential function, we obtain the result. □

COROLLARY 1. *Define the random variable*

$$A_1(\omega) = \int_1^2 \frac{dt}{\left(\max\{|W(1,t) - W(1,1)|, n^p 2^{-n/2}\}\right)^3}.$$

This variable satisfies

$$E\left[e^{CA_1 n^p/2^{n/2}}\right] < 2.$$

PROOF. The process $\{W(1+t,1) - W(1,1): t \geq 0\}$ is a Brownian motion. □

COROLLARY 2. *Define the variable $A_{1,s}$ to be*

$$\int_1^2 \frac{dt}{\left(\max\{|W(s,t) - W(s,1)|, n^p 2^{-n/2}\}\right)^3}.$$

Then $P[\sup_{s\geq 1} A_{1,s} > 2^{n/2} n^{q-p}] \leq e^{-Cn^q}$.

PROOF. By Fubini's theorem, the process $\{A_{1,s}: s \geq 1\}$ is a positive supermartingale; the result follows. □

Let us extend the definition of A by defining

$$A_{r,s}(\omega) = \int_1^2 \frac{dt}{\left(\max\{|W(s,t) - W(s,r)|, n^p 2^{-n/2}\}\right)^3}.$$

The following corollary is immediate.

COROLLARY 3. *For fixed r in* $[1,2]$,

$$P\left[\sup_{s\geq 1} A_{r,s} > 2 \cdot 2^{n/2} n^{q-p}\right] \leq 2e^{-Cn^q}.$$

By our assumption $(*)$, $|r - v| < 2^{-n}$ *implies* $|W(s,v) - W(s,r)| < 2^{-n/2}\sqrt{6n}$. *If p is greater than $\frac{1}{2}$, then for large n*

$$\begin{aligned}\max\{&|W(s,t) - W(s,r)|, n^p 2^{-n/2}\} \\ &\geq \max\{|W(s,t) - W(s,v)| - 2^{-n/2}\sqrt{6n}, n^p 2^{-n/2}\} \\ &\geq \tfrac{1}{2}\max\{|W(s,t) - W(s,v)|, n^p 2^{-n/2}\}\end{aligned}$$

and so if p is greater than $\frac{1}{2}$ and for each $v = 1 + i/2^n$, $A_{v,s} \leq 2^{n/2} n^{q-p}$, then $A_{r,s} \leq 2^{n/2} n^{q-p} (2)^3$.

From this we deduce the following proposition.

PROPOSITION 1. *For $p > \frac{1}{2}$,*

$$P\left[\sup_{s\geq 1,\, r\in[1,2]} A_{r,s} > 2 \cdot 2^3 2^{n/2} n^{q-p}\right] \leq 2 \cdot 2^n e^{-Cn^q}.$$

This probability will be very small if q is greater than 1.

Fix a dyadic cube of order n in $[0,1]^d$, I, centered at x. Consider the process

$$g_{x,s} = \int_1^2 \frac{dr}{(\max\{|W(s,r) - x|, n^p 2^{-n}\})^3}.$$

The integral of a bounded supermartingale is also a supermartingale and so $\{g_{x,s}\colon s \geq 1\}$ is a supermartingale.

Suppose that for some (s,t), $W(s,t) \in I$. Recall $(*)$; this assumption ensures that for h sufficiently small

$$|W(s, t+h) - x| < \sqrt{6h\log(1/h)} + 5 \cdot 2^{-n}$$

and consequently $g_{x,s} \geq n^{-3/2}2^{n/2}$ if n is sufficiently large. We now choose our values of p (to fully define $A_{r,s}$) and q to be $2.5 + \varepsilon$ and $1 + \varepsilon/2$, respectively. We know that for n large enough $A_{r,s}2^{n/2}n^{-(3/2+\varepsilon/2)}$. We shall in the following assume that this is so. If $W(s,t) \in I$, then'

$$E[g_{x,s+n^{5+2\varepsilon}2^{-n}}|F_s] \leq 2A_{t,s} \leq 2 \cdot 2^{n/2}n^{-(3/2+\varepsilon/2)},$$

where F_s is the σ-field generated by $\{W(r,t)\colon r \leq s\}$. If we define successive stopping times

$$T_1 = \inf\{s \geq 1\colon W(s,t) \in I \text{ for some } t \in [1,2]\},$$

$$T_{i+1} = \inf\{s \geq T_i + n^{5+2\varepsilon}2^{-n}\colon W(s,t) \in I \text{ for some } t \in [1,2]\},$$

then $\forall\, i$ by the supermartingale property of $g_{x,s}$,

$$n^{-3/2}2^{n/2}P[T_{i+1} < 2|F_{T_i}] \leq 2 \cdot 2^{n/2}n^{-(3/2+\varepsilon)}$$

or

$$P[T_{i+1} < 2|F_{T_i}] \leq Kn^{-\varepsilon/2}.$$

This easily yields

$$P[T_{n+1} < 2] \leq Kn^{n\varepsilon/3}.$$

So outside of a set of probability $2^{nd}n^{-n\varepsilon/3}$, $\forall\, n/2$ order dyadic cube in $[0,1]^d I$, the set $\{s\colon W(s,t) \in I \text{ for some } t \in [1,2]\}$ is contained in a set of $2n^{6+2\varepsilon}$ dyadic intervals of order n. By symmetry the same is true of the set $\{t\colon W(s,t) \in I$ for some $s \in [1,2]\}$. Therefore taking the most liberal of estimates we conclude by the Borel–Cantelli lemma that eventually every $n/2$ order dyadic cube in $[0,1]^d I$, the set $\{(s,t)\colon W(s,t) \in I\}$ is contained in a set of $4n^{12+4\varepsilon}$ dyadic squares of order n. This completes the proof of Theorem 2 in the case $N = 2$, $d = 5$.

CASE 2. $N = 2$, $d = 4$. We now treat the case $d = 4$. First consider the random variable

$$A = \int_0^1 \frac{1}{(\max\{2^{-n}n^p, |B(t)|^2\})}\,dt.$$

Some messy but elementary calculations show that

$$E[A] \text{ is of the order } \int_0^1 \min\left\{\frac{C}{t}, \frac{2^n}{n^p}\right\} dt = \int_0^{Cn^p/2^n} \frac{2^n}{n^p} dt + \int_{Cn^p/2^n}^1 \frac{C}{t} dt$$

$$= C(\log_e 2)n + O(\log(n)).$$

So $E[A]$ is of order n. Otherwise the arguments of Section 1 can be used to show the following proposition.

PROPOSITION 2. *Define the random variable*

$$A_{r,s}(\omega) = \int_1^2 \frac{dt}{\left(\max\{|W(s,t) - W(s,r)|, 2^{-n/2}\}\right)^2}.$$

Then for arbitrary fixed $\varepsilon > 0$, *outside a set with probability majorized by* $Kn4^n e^{-n^{1+\varepsilon}}$,

$$\sup_{(r,s)\in[1,2]^2} A_{r,s} < n^{2+\varepsilon}.$$

Therefore by the first Borel–Cantelli lemma we can deduce that for all n large enough

$$\sup_{(r,s)\in[1,2]^2} A_{r,s} < n^{2+\varepsilon}.$$

The major problem in extending Theorem 2 to $N = 2$, $d = 4$, is that for a fixed dyadic cube I of center x, the event $W(s,t) \in I$ cannot guarantee that

$$g_{x,s}(\omega) = \int_1^2 \frac{dr}{\left(\max\{|x - W(s,r)|, 2^{-n/2}\}\right)^2}$$

is of greater order than n. Indeed it does not (easily) imply that $g_{x,s}$ is of order n. In the succeeding paragraphs assumption $(*)$ will be in force.

LEMMA 2. *Fix dyadic cube I of order n/2, with center x. Assume* $\exists\, t \in [1,2]$ *with* $W(s,t) \in I$ *and that*

$$\sup_{r\in[1,2]} A_{r,s} < n^{2+\varepsilon}.$$

Define the random variable

$$N_s = \#\{\textit{dyadic intervals } D, \textit{ of order } n \textit{ in } [s,2] \textit{ s.t. } W(s',t) \in I \textit{ for some } (s',t) \in D \times [1,2]\}.$$

There exists a constant C such that $E[N_s|F_s] \le Cn^{2.5+\varepsilon}$.

RECALL. F_s denotes the σ-field generated by $\{W(u, v): u \leq s\}$.

In particular if ε is sufficiently small

$$P[N_s > n^3] < c/n^{1/3},$$

for some c.

PROOF OF LEMMA 2. Divide up the time interval $[1,2]$ of coordinate t into $\sqrt{6n}\,2^n$ intervals of equal length, $J_1, J_2, \ldots, J_{\sqrt{6n}2^n}$. By assumption for each J_i, we may choose an element t_i ($\in J_i$) such that

$$\sum_{i=1} \frac{1}{\left(\max\{|x - W(s, t_i)|, 2^{-n/2}\}\right)^2} \leq 2 \cdot 2^n \sqrt{6n}\, n^{2+\varepsilon}.$$

Define the random variable

$$N_{i,s} = \#\{\text{dyadic intervals } D, \text{ of order } n \text{ in } [s,2] \text{ s.t. } W(s',t) \in I \text{ for some } (s',t) \in D \times J_i\}.$$

Let us now expand I about x by a factor of 2 to obtain a new cube I'. If for some $t \in J_i$, $W(r, t) \in I$, then by assumption $(*)$ we must have $W(r, t_i) \in I'$. It follows that N_i is $\leq$ the number the dyadic intervals containing s' such that $W(s', t_i) \in I'$. But this has expected value $< C\min(1, 2^{-n}/|W(s, t_i) - x|^2)$ for some constant C. We deduce that

$$E[N_s] \leq \sum_i \min\{C2^{-n}/|W(s, t_i) - x|^2, 1\} \leq C\sqrt{6n}\, n^{2+\varepsilon}. \qquad \square$$

We define the variable

$$N_{s,r} = \#\{\text{dyadic intervals } D, \text{ of order } n \text{ in } [s,r] \text{ s.t. } W(s',t) \in I \text{ for some } (s',t) \in D \times [1,2]\}.$$

Define successively the stopping times

$$T_1 = \inf\{1 + i/2^n \geq 1: W(s,t) \in I \text{ for some } (s,t) \in [1 + (i-1)/2^n, 1 + i/2^n] \times [1,2]\}$$

and for i greater than 1,

$$T_i = \inf\left\{s \geq T_{i-1}: N_{T_{i-1},s} > n^3\right\}.$$

We know that for n sufficiently large,

$$\sup_{(s,r)\in[1,2]^2} A_{r,s} < n^{2+\varepsilon}.$$

So throughout the following we shall assume that this is the case and so if ε is sufficiently small,

$$\forall i, \quad P\left[T_i < 2 \mid F_{T_{i-1}}\right] < cn^{-1/3}.$$

Therefore if we define N_I to be the number of dyadic intervals D of order n in

$[1,2]$ such that $W(s,t) \in I$ for some $(s,t) \in D \times [1,2]$, then

$$P[N_I > n^4] \le P[T_n \le 2] \le (cn^{-1/3})^{n-1}.$$

This implies (as in Section 1) that outside a set of probability $2(cn^{-1/3})^{n-1}$, the time spent in I is covered by $n^4 \times n^4$ dyadic squares. Now there are $2^{n/2} \times 2^{n/2} \times 2^{n/2} \times 2^{n/2}$ dyadic cubes of order $n/2$ in $[0,1]^4$; therefore the chance that there is a dyadic cube of order $n/2$, J, such that the time spent in J by $\{W(s,t): (s,t) \in [1,2]^2\}$ cannot be covered by n^8 dyadic cubes of order n, is less than

$$2^{2n}(cn^{-1/3})^{n-1}.$$

Since these terms are summable in n we can invoke the Borel–Cantelli lemma to complete the proof.

3. In extending the proof of Section 2 to the general case $2N = d$. We note that the argument goes through essentially as before once we have shown that the random variable

$$A = \int_{[1,2]^{N-1}} \frac{1}{\max\{|W(\mathbf{t}) - W(\mathbf{1})^{d-2}, (2^{-n})^{(d-2)/2}\}}\, d\mathbf{t}$$

satisfies $E[A^r] \le r!(mn)^r$ for some constant m not depending on n. But this we can do by using the inequalities of Rosen (1981). These show that given $n-1$ time points $\{t_1, t_2, \dots, t_{n-1}\}$ in $[1,2]^{N-1}$ the conditional distribution of $W(t_n)$ given the values of W at the other t_{n-1} is Gaussian with componentwise variance greater than

$$\min\{c|t_n - t_i| 1 \le i \le n-1\}.$$

This allows us to conclude that

$$E[A^r] \le r!(cmn)^r,$$

where m is

$$\sup_{\mathbf{t} \in [1,2]^{N-1}} \frac{1}{n} \int_{[1,2]^{N-1}} \frac{1}{\max\{|\mathbf{t} - \mathbf{s}|^{d-2}, (2^{-n})^{(d-2)/2}\}}\, d\mathbf{s}.$$

4. An application. Using essentially the same arguments as in the first two sections, we can prove the following theorem.

THEOREM 3. *Let $\{O_s(\cdot): s \ge 0\}$ be an Ornstein–Uhlenbeck process on d-dimensional Wiener space. If d is greater than or equal to 4, then a.s.*

$$\forall \text{ Borel sets } E \text{ in } R^2_+, \quad \dim\{O_s(t): (s,t) \in E\} = 2 \dim E.$$

Using this and ideas from Hawkes (1971), we can easily deduce the following proposition.

PROPOSITION 3. *Let B be a Borel set in R^d with dimension equal to $\beta \in [d-4, d-2]$. Then the supremum of the α such that*

$$\dim\{x: x \in B, x = O_s(t) \text{ for some } (s,t)\} \geq \alpha$$

with positive probability, is equal to $4 - d + \beta$.

If $d = 4$, then with probability 1,

$$\dim\{x: x \in B, x = O_s(t) \text{ for some } (s,t)\} = 4 - d + \beta.$$

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