The Annals of Probability
1989, Vol. 17, No. 4, 1441–1453

ON INDEPENDENCE AND CONDITIONING ON WIENER SPACE

BY ALI SULEYMAN ÜSTÜNEL AND MOSHE ZAKAI[1]

Paris and Technion–Israel Institute of Technology

Let $I_p(f)$ and $I_q(g)$ be multiple Wiener–Itô integrals of order p and q, respectively. A characterization of independence of general random variables on Wiener space in the context of the stochastic calculus of variations is derived and a necessary and sufficient condition on the pair of kernels (f, g) is derived under which the random variables $I_p(f)$, $I_q(g)$ are independent.

1. Introduction. Let T denote the interval $[0, t_0]$ for some finite $t_0 > 0$ or $T = [0, \infty)$ and let W_t, $t \in T$, be the Wiener process on T. The purpose of this note is to consider the notion of independence of functionals of the Wiener process. More specifically, our purpose is to derive geometric conditions for independence, namely, conditions expressed in terms of notions of the Malliavin calculus. For simplicity we restrict the discussion to the case where T is an interval; however, the results presented here go over to the more general setup where the parameter space T is an atomless measure space and W is an orthogonal Gaussian measure on T and to an abstract Wiener space in which a one parameter "time" has been introduced through a self-adjoint operator with a continuous spectrum [13].

Let X, Y be L^2 functionals of W_t, $t \in T$, and let DX, DY be the H-derivatives of X and Y, respectively (cf., e.g. [8], [14], [15]) which are assumed to exist and belong to $L^2(\Omega \times T)$. The subsigma field induced by X is certainly different from the one induced by $(DX)_t$, $t \in T$, or by $\langle DX, DX\rangle_{L^2(T)}$. It is still natural to inquire whether there exists a relation between the independence of X and Y and the orthogonality of DX, DY, i.e., $\langle DX, DY\rangle_{L^2(T)} = 0$ a.s. Consider the following example: Let $f(t)$, $g(t)$ be two orthonormal L^2 kernels on T. Let $\tilde{f} = I_1(f)$ and $\tilde{g} = I_1(g)$. Then $\tilde{f}$ and $\tilde{g}$ are independent. Consider now the random variables

$$X = (\tilde{f})^2 + (\tilde{g})^2, \qquad Y = \tilde{f}/\sqrt{X}.$$

It is well known that X and Y are independent. Since $D\tilde{h} = h$, a straightforward calculation yields that $\langle DX, DY\rangle_{L^2(T)} = 0$ a.s. This special case is "singular" from the point of view of this paper since it does not seem to be an example to the results presented here [Proposition 5 applied to this case yields that, since X,Y are independent and X is, up to a constant, a multiple integral of order 2, $E(\langle DX, DY\rangle | Y) = 0$ a.s. but not that $\langle DX, DY\rangle = 0$ a.s.]. We do not know whether in general independence of X, Y implies the orthogonality of DX, DY;

Received November 1987; revised September 1988.
[1]Work supported by the Fund for the Promotion of Research at the Technion.
AMS 1980 *subject classifications*. Primary 60H07; secondary 60H05, 60J65.
Key words and phrases. Independence, Wiener chaos, multiple Wiener–Itô integrals, the Malliavin calculus.

however, as the following example shows, orthogonality of the H-derivatives does not imply, in general, independence. Let f and g be as in the previous example. Set

$$U = \min(\tilde{f}, \tilde{g}), \qquad V = \max(\tilde{f}, \tilde{g}).$$

Then $\langle DU, DV \rangle = 0$ a.s. since $\langle D\tilde{f}, D\tilde{g} \rangle = 0$ a.s. but U and V are obviously not independent. This does not eliminate the possibility of obtaining geometric characterizations for independence of Wiener functionals but obviously the conditions will have to be more complicated.

The next section is devoted to notation and preliminaries. In Sections 3 and 4 we consider functionals which are of the form $X = I_p(f_p)$ and $Y = I_q(g_q)$ where I_p and I_q are multiple Wiener–Itô integrals of order p and q, respectively, associated with the symmetric kernels f and g. In Section 3 we derive a necessary condition on the kernels f and g for the independence of X and Y. In Section 4 it is shown that this condition implies that $\langle DX, DY \rangle = 0$ a.s. The problem of independence of general L^2 Wiener functionals is considered in Section 5 and independence is characterized in terms of conditional expectations and the representation of Wiener functionals as stochastic integrals. The results of Section 5 are applied in Section 6 to show that the necessary conditions of Sections 3 and 4 are, in fact, sufficient.

2. Notation and preliminaries. Some notation: $f = f_p = f(t_1, \dots, t_p)$ and $g = g_q = g(t_1, \dots, t_q)$ will denote L^2 kernels on T^p and T^q, respectively. The L^2 kernel on T^{p+q} which is the tensor product of f and g will be denoted by $f \otimes g$ [namely $(f \otimes g)(t_1, \dots, t_{p+q}) = f(t_1, \dots, t_p) \cdot g(t_{p+1}, \dots, t_{p+q})$]. The kernels f and g will be assumed to be symmetric throughout this note; note however, that $f \otimes g$ need not be symmetric. For a (not necessarily symmetric) kernel $h(t_1, \dots, t_k)$, $h^\wedge$ will denote the symmetrization of h. For $m \le \min(p, q)$, $f \otimes_m g$ will denote the contracted L^2 kernel on T^{p+q-2m} defined by

$$(1) \qquad f \otimes_m g = \int_{T^m} f(t_1, \dots, t_{p-m}, \sigma_1, \dots, \sigma_m) g(\sigma_1, \dots, \sigma_m, t_{p-m+1}, \dots, t_{p+q-2m})\, d\sigma_1 \cdots d\sigma_m$$

and in particular

$$(2) \qquad f \otimes_1 g = \int_T f(t_1, \dots, t_{p-1}, \sigma) g(\sigma, t_p, \dots, t_{p+q-2})\, d\sigma.$$

The multiple Wiener–Itô integrals $I_p(f)$ and $I_q(g)$ satisfy the product formula (for f and g symmetric) (cf., e.g., page 247 of [4] or Lemma 4.1 of [8])

$$(3) \qquad I_p(f) \cdot I_q(g) = \sum_{m=0}^{\min(p,q)} \frac{p!q!}{m!(p-m)!(q-m)!} I_{p+q-2m}(f \otimes_m g).$$

Also, for $h_k \in L^2(T^k)$, $E(I_k(h_k))^2 = k! \|h^\wedge\|^2_{L^2(T^k)}$ and are finite.

A pair of random variables X, Y possessing all moments will be said to be weakly independent if $EX^mY^n = EX^mEY^n$ for all integers m and n. This notion is introduced here for the following reason. If the moments $\mu_{m,n} = EX^mY^n$ satisfy a certain growth condition, then they determine uniquely the probability law of (X, Y) and in this case weak independence implies independence. However, as pointed out by McKean [3], page 202 (cf. also [2], Theorem 6.6 and page 119)

$$E\big(I_p(f)\big)^{2n} \le (p!)^{2n}((2n)!/2^n n!)^p \cdot \|f\|^{2n}_{L^2(T^p)}$$
$$\approx (p!)^{2n}(2^{n+1/2}n^n e^{-n})^p \cdot \|f\|^{2n}_{L^2(T^p)}.$$

Therefore

$$E|I_p(f)|^m \le \exp\left[\frac{pm}{2}\log m + O(m)\right]. \tag{4}$$

Consequently for $p \le 2$, a sufficient condition of Carleman for the uniqueness of the Hamburger moment problem ([9], Theorem 1.10) is satisfied. Consequently the moments $E(I_p(f))^m$, $m = 1, 2, \dots$, define uniquely the distribution function of the random variable $I_p(f)$. For $p \ge 3$ this sufficient condition is not satisfied and the uniqueness of the solution to the moment problem in this case is open. Similarly by Theorem 1.12 of [9], for $p, q \le 2$ the joint moments $\mu_{m,n}$ determine uniquely the joint probability distribution of $I_p(f)$, $I_q(g)$ and consequently for $p, q \le 2$ weak independence implies independence. For $p \ge 3$ or $q \ge 3$ the condition of Theorem 1.12 of [9] is not satisfied (cf. also [7]).

REMARK. As shown by Shigekawa [8], the random variable $I_P(f)$ possesses a probability density; however as the example $f_2(t_1, t_2) = f_1(t_1)f_1(t_2)$ shows $(I_2(f_2) = (I_1(f_1))^2 - \|f_1\|^2)$ therefore the density of $I_2(f_2)$ is not smooth.

3. A characterization of the pairs (f,g) for which $I_p(f), I_q(g)$ are weakly independent.

PROPOSITION 1. *A necessary and sufficient condition for the weak independence of $I_p(f)$ and $I_q(g)$ is*

$$f \otimes_1 g = 0 \tag{5}$$

a.e. Lebesgue on T^{p+q-2}, namely,

$$\|f \otimes_1 g\|_{L^2(T^{p+q-2})} = 0, \tag{5a}$$

where $f \otimes_1 g$ is as defined by (2) *and, since for $p, q \le 2$ weak independence is equivalent to independence,* (5) *is a necessary and sufficient condition for independence in this case.*

PROOF. Starting with the proof of necessity, since $I_p(f)$ and $I_q(g)$ are weakly independent with f and g symmetric,

$$\begin{aligned} E\big(I_p(f)I_q(g)\big)^2 &= p!\|f\|^2_{L^2(T^p)} \cdot q!\|g\|^2_{L^2(T^q)} \\ &= p!q!\|f \otimes g\|^2_{L^2(T^{p+q})}. \end{aligned} \tag{6}$$

On the other hand, by the product formula and the orthogonality between multiple integrals of different order,

$$(7)\qquad \begin{aligned} &E\big(I_p(f)I_q(g)\big)^2 \\ &\quad = \sum_{m=0}^{\min(p,q)} \left(m!\binom{p}{m}\binom{q}{m}\right)^2 (p+q-2m)!\cdot \|(f\otimes_m g)\hat{}\,\|^2_{L^2(T^{p+q-2m})}. \end{aligned}$$

Dropping terms with $m \geq 1$ yields

$$(8)\qquad E\big(I_p(f)\cdot I_q(g)\big)^2 \geq (p+q)!\|(f\otimes g)\hat{}\,\|^2_{L^2(T^{p+q})}.$$

Let Π denote the group of permutations of $(1,2,\dots,p+q)$ and for $\pi\in\Pi$ denote $\pi(1,2,\dots,p+q) = (\pi_1,\pi_2,\dots,\pi_{p+q})$. Then

$$(9)\qquad \begin{aligned} \|(f\otimes g)\hat{}\,\|^2_{L^2(T^{p+q})} &= \left\| \frac{1}{(p+q)!} \sum_{\pi\in\Pi} f\big(t_{\pi_1},\dots,t_{\pi_p}\big) g\big(t_{\pi_{p+1}},\dots,t_{\pi_{p+q}}\big) \right\|^2_{L^2(T^{p+q})} \\ &= \sum_{\mu,\rho\in\Pi} \big((p+q)!\big)^{-2} \lambda_{\mu,\rho}, \end{aligned}$$

where

$$(10)\qquad \begin{aligned} \lambda_{\mu,\rho} = \int_{T^{p+q}} & f\big(t_{\mu_1},\dots,t_{\mu_p}\big) g\big(t_{\mu_{p+1}},\dots,t_{\mu_{p+q}}\big) f\big(t_{\rho_1},\dots,t_{\rho_p}\big) \\ & g\big(t_{\rho_{p+1}},\dots,t_{\rho_{p+q}}\big)\, dt_1 \cdots dt_{p+q}. \end{aligned}$$

We want to show that $\lambda_{\mu,\rho} \geq 0$ for all $\mu,\rho\in\Pi$. Assume, now, that $p \leq q$ and that $(\mu_1,\dots,\mu_p)$ and $(\rho_1,\dots,\rho_p)$ have k elements in common. Then, since f and g are symmetric,

$$(11)\qquad \begin{aligned} \lambda_{\mu,\rho} = \int_{T^{p+q}} & f(t_1,\dots,t_k,s_1,\dots,s_{p-k}) g\big(t_{k+1},\dots,t_p,\sigma_1,\dots,\sigma_{q-(p-k)}\big) \\ & \times f(t_1,\dots,t_k,t_{k+1},\dots,t_p) g\big(s_1,\dots,s_{p-k},\sigma_1,\dots,\sigma_{q-(p-k)}\big) \\ & \times dt_1 \cdots dt_p\, d\sigma_1 \cdots d\sigma_{q-(p-k)}\, ds_1 \cdots ds_{p-k} \\ = \|f\otimes_{p-k} & g\|^2_{L^2(T^{q-p+2k})}. \end{aligned}$$

Hence, $\lambda_{\mu,\rho} \geq 0$. Note that for $k = p$, $\lambda_{\mu,\rho} = \|f\otimes g\|^2_{L^2(T^{p+q})}$. Substituting (11) and (9) into (8) yields

$$(12)\qquad EI_p^2(f)I_q^2(g) \geq p!q!\|f\otimes g\|^2_{L^2(T^{p+q})} + \sum_{k=0}^{p-1} c_k \|f\otimes_{p-k} g\|^2_{L^2(T^{q-p+2k})}$$

with $c_k > 0$. Comparing this with (6) yields

$$(13)\qquad \|f\otimes_m g\|^2_{L^2(T^{p+q-2m})} = 0$$

for all $1 \leq m \leq \min(p,q)$ and in particular $m = 1$ yields (5).

Turning to the converse direction, in order to prove that $f \otimes_1 g = 0$ a.e. $L^2(T^{p+q-2})$ implies weak independence, it is necessary to show that

$$E\{(I_p(f))^{k_1}(I_q(g))^{k_2}\} = EI_p^{k_1}(f) \cdot EI_q^{k_2}(g) \tag{14}$$

for all integers k_1, k_2. From the multiplication formula (3) it follows that

$$\begin{aligned} I_p^{k_1}(f) &= \sum_{r=1}^{pk_1} I_r(a_r) + EI_p^{k_1}(f), \\ I_q^{k_2}(g) &= \sum_{s=1}^{qk_2} I_s(b_2) + EI_q^{k_2}(g). \end{aligned} \tag{15}$$

Consequently it suffices to show that

$$E(I_r(a_r)I_r(b_r)) = 0$$

for all $1 \le r \le \min(qk_2, pk_1)$. Let ϕ be an L^2 kernel on T^k and let i, j be two distinct integers $1 \le i, j \le k$. Assume that ϕ is of trace class for the i and j variables (this is automatically true if $\phi = f \otimes g$ with t_i in f and t_j in g). Define the L^2 kernel on T^{k-2}:

$$c(i, j)\phi = \int_T \phi\left(t_1, \ldots, \underset{i}{\theta}, \ldots, \underset{j}{\theta}, \ldots, t_{k-2}\right) d\theta;$$

repeating this procedure we define the multiple contractions $c(i_2 j_2)(c(i_1, j_1)\phi)$, etc. Let $f^{\otimes k_1}$ denote the k_1th order tensor product $f \otimes \cdots \otimes f$. Then [by (3)] the kernels a_r in (15) are obtained by multiple contractions on $\phi = f^{\otimes k_1}$, similarly b_r are obtained by multiple contractions on $\Psi = g^{\otimes k_2}$. Consequently and because $f \otimes_1 g = 0$ a.e. the scalar product,

$$\int_{T^r} a_r(t_1, \ldots, t_r) b_r(t_1, \ldots, t_r)\, dt_1 \cdots dt_r$$

must vanish; the same holds for the scalar product of $a_r(t_1, \ldots, t_2)$ with $b_r(t_{\pi_1}, \ldots, t_{\pi_r})$ for any permutation π of $(1, \ldots, r)$, which completes the proof. □

From Proposition 1 we have the following corollaries.

COROLLARY A. *The Wick product of two elements of the Wiener chaos is defined by*

$$I_p(f) : I_q(g) = I_{p+q}(f \otimes g).$$

Under the assumptions of Proposition 1,

$$I_p(f) : I_q(g) = I_p(f) \cdot I_q(g).$$

The proof follows directly from (5), (13) and the product formula.

For the case $p = q = 2$ and for kernels which are of trace class it is well known (cf., e.g., page 204 of [3]) that the characteristic function of $I_2(f)$ is given by

$$E \exp i\alpha I_2(f_2) = \left[\det(I - i\alpha f_2)\right]^{-1/2} \cdot \exp\left(-\tfrac{1}{2} i\alpha \operatorname{trace} f_2\right),$$

where det denotes the Fredholm determinant (the product of the eigenvalues of the symmetric operator taking into account multiplicities [10]). It holds in general that $\det(I + A)\det(I + B) = \det(I + A + B + AB)$ (cf., e.g., Theorem 3.5 of [10]). From Proposition 1 we have the nonprobabilistic result regarding Fredholm determinants:

COROLLARY B. *Let f, g be $L^2(T^2)$ symmetric kernels of trace class. Then*

$$\det(I + \alpha f + \beta g) = \det(I + \alpha f)\det(I + \beta g) \tag{16}$$

for any (α, β) in a sufficiently small neighborhood of $(0,0)$ if and only if $f \otimes_1 g = g \otimes_1 f = 0$ (as an operator).

4. A necessary condition for weak independence in terms of H-derivatives. Let $DI(f)$ denote the H-derivative of $I(f)$ (cf., e.g., [8], [11], [14] and [15]). Consider now the pth order kernel f as a kernel of order $(p-1)$ parameterized by one of the variables, i.e., $f^{(\theta)} = f(t_1, \dots, t_{p-1}, \theta)$ and denote by $I_{p-1}(f^{(\theta)})$ the collection of random variables belonging to the $(p-1)$th chaos parametrized by θ. Then

$$DI_p(f) = pI_{p-1}(f^{(\theta)})$$

and

$$\langle DI_p(f), DI_q(g)\rangle_{L^2(T)} = pq\int_T I_{p-1}(f^{(\theta)}) I_{q-1}(g^{(\theta)})\, d\theta.$$

PROPOSITION 2. *A necessary condition for the weak independence of $I_p(f)$ and $I_q(g)$ is*

$$\langle DI_p(f), DI_q(g)\rangle_{L^2(T)} = 0 \quad a.s. \tag{17}$$

PROOF. Let $Q = \langle DI_p(f), DI_q(g)\rangle$. We show that $f \otimes_1 g = 0$ a.e. implies that $EQ^2 = 0$ or

$$0 = p_2 q^2 \int_{T^2} E\left\{I_{p-1}(f^{(\theta)}) I_{q-1}(g^{(\theta)}) I_{p-1}(f^{(\psi)}) I_{q-1}(g^{(\psi)})\right\} d\psi\, d\theta.$$

By the multiplication formula,

$$\begin{aligned} I_{p-1}(f^{(\theta)}) I_{q-1}(g^{(\theta)}) &= I_{p+q-2}\big(f(t_1, \dots, t_{p-1}, \theta) g(t_p, \dots, t_{p+q-2}, \theta)\big) \\ &\quad + \sum_{m=1}^{\min(p,q)-1} \alpha_m I_{p+q-2-2m}\big(f^{(\theta)} \otimes_m g^{(\theta)}\big). \end{aligned} \tag{18}$$

If $f_p \otimes_1 g_q = 0$ a.e., then $f^{(\theta)} \otimes_m g^{(\theta)} = 0$ a.e. for all $m \geq 1$ and all terms in the sum from $m = 1$ to $m = \min(p, q) - 1$ will vanish. Consequently

$$EQ^2 = p^2q^2\int_{T^2} E\{I_{p+q-2}(f^{(\theta)} \otimes g^{(\theta)})I_{p+q-2}(f^{(\psi)} \otimes g^{(\psi)})\}\, d\psi\, d\theta;$$

the kernels $f^{(\theta)} \otimes g^{(\theta)}$ are not necessarily symmetric, but a straightforward calculation yields that $(f \otimes_1 g) = 0$ implies $EQ^2 = 0$ which is (17). Note that, if (17) holds, then by (18),

$$\int_{T^2} E\{I_{p+q-2-2m}(f^{(\theta)} \otimes_m g^{(\theta)})I_{p+q-2-2m}(f^{(\psi)} \otimes_m g^{(\psi)})\}\, d\theta\, d\psi = 0$$

and in particular

$$0 = \int_{T^2} E\{I_{p+q-2}(f^{(\theta)} \otimes g^{(\theta)})I_{p+q-2}(f^{(\psi)} \otimes g^{(\psi)})\}\, d\psi\, d\theta. \tag{19}$$

By a Fubini type identity [cf. (30)], (19) is equivalent to

$$E\big(I_{p+q-2}(f \otimes_1 g)\big)^2 = 0,$$

from which $(f \otimes_1 g)^\wedge = 0$ a.e. follows directly; however, this does not imply $f \otimes_1 g = 0$. □

5. A characterization of independence. In this section we consider the case in which at least one of the random variables is a general random variable on Wiener space. $\mathbb{D}_{2,1}$ will be used to denote the collection of L^2 functionals X which possess a square integrable H-derivative, namely $E\|DX\|^2_{L^2(T)} < \infty$, and $\mathbb{D}_{2,k}$ will denote the collection of L^2 functionals possessing a square integrable kth H-derivative (cf., e.g., [5], [8], [14], and [15]). For any L^2 functional Y on the Wiener space we have the (nonunique) representation $Y = EY + \delta U_Y$, where δ denotes the dual to D which is the Skorohod integral (cf. [5], [6] and [12]). The integrand U_Y may be chosen to be adapted, in which case δU_Y coincides with the Itô integral (cf., e.g., [1], [6] and [12]). Another possibility is as follows: L^{-1}, the inverse of the Ornstein–Uhlenbeck operator L, is well defined on the class of zero mean L^2 functionals and $L \cdot L^{-1}Y = Y - EY$. Since $L = \delta D$, it follows that $Y = EY + \delta(DL^{-1}[Y - EY])$; hence we may set $U_Y = DL^{-1}Y$. Recall that the class $\mathbb{D}_{2,1}$ denotes the collection of L^2 functionals Z which possess a square integrable H-derivative DZ.

PROPOSITION 3. *Let $X \in \mathbb{D}_{2,1}$ and $Y \in L^2$. Then X and Y are independent if and only if for every C_b^2 function η: $\mathbb{R} \to \mathbb{R}$ [or just for every η of the form $\eta(x) = \exp i\alpha x$, $x \in \mathbb{R}$, with $\alpha \in \mathbb{R}$],*

$$E\{\langle U_{\eta(Y)}, DX\rangle \mid X\} = 0 \quad a.s., \tag{20}$$

where $U_{\eta(Y)}$ is defined through $\eta(Y) = \delta U_{\eta(Y)} + E\eta(Y)$ or

$$E\{\langle DL^{-1}[\eta(Y) - E\eta(Y)], DX\rangle \mid X\} = 0 \quad a.s. \tag{20a}$$

or

$$E\{\langle (I + L)^{-1}D\eta(Y), DX\rangle \mid X\} = 0. \tag{20b}$$

PROOF. If X and Y are independent, then $E(\delta U_{\eta(Y)} \cdot \rho(X)) = 0$ for every ρ: $\mathbb{R} \to \mathbb{R}$ which is bounded and smooth. Therefore, applying the integration by parts formula (cf., e.g. [5]),

$$\begin{aligned} 0 &= E\langle U_{\eta(Y)}, D\rho(X)\rangle \\ &= E\rho'(X)\langle U_{\eta(Y)}, DX\rangle; \end{aligned}$$

(20) follows since ρ is arbitrary. Equation (20a) follows since we may choose $U_{\eta(Y)} = DL^{-1}\eta(Y)$ and (20b) follows from (20a) by the commutation relation $DL^{-1} = (L+1)^{-1}D$. To prove the converse note that the same argument works in the converse direction. □

The following extension of the previous proposition will be needed for the proof of Proposition 7 in the next section.

PROPOSITION 4. *Let $X \in \mathbb{D}_{2,1}$ and $Y \in L^2$, $Z \in L^2$. Then X is independent of the pair (Y, Z) if and only if for every real α, β,*

$$E\{\langle (I+L)^{-1}De^{i(\alpha Y+\beta Z)}, DX\rangle \mid X\} = 0 \quad a.s. \tag{21}$$

PROOF. The proof follows along the same lines as that of the previous proposition: Let $\rho(\cdot)$ be as before. Then

$$\begin{aligned} &E\big(LL^{-1}(e^{i(\alpha Y+\beta Z)} - Ee^{i(\alpha Y+\beta Z)})\rho(X)\big) \\ &\quad = E\{\langle DL^{-1}(e^{i(\alpha Y+\beta Z)} - Ee^{i(\alpha Y+\beta Z)}), \rho'(X)DX\rangle\}, \end{aligned} \tag{22}$$

from which the result follows by the commutation relation $DL^{-1} = (I+L)^{-1}D$. □

PROPOSITION 5. *Let $G = I_q(g)$ and $X \in \mathbb{D}_{2,k}$. Assume that G and K are independent. Then*

$$E\big(\langle D^kG, D^kX\rangle_{L^2(T^k)} \mid X\big) = 0 \quad a.s. \tag{23}$$

PROOF. For $k > q$, $D^kG = 0$ and (21) is trivially true. Consider now the case $k = 1$. Since $\delta DG = qG$ and $EG = 0$, it follows from the integration by parts formula that for any C_b^1, η: $\mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} 0 &= E(\eta(X) \cdot G) \\ &= \frac{1}{q}E(\eta(X) \cdot \delta DG) \\ &= \frac{1}{q}E\langle D\eta(X), DG\rangle \\ &= \frac{1}{q}E\{\eta'(X)\langle DX, DG\rangle\}, \end{aligned}$$

which proves the result since η is arbitrary. For $1 < k \le q$ the proof follows along the same line with δD replaced by $\delta^k D^k$ and the generalized integration by parts formula (Proposition 2.7 of [6]). □

6. A characterization of the independence of $I_p(f)$ and $I_q(g)$. In this section we apply the results of the previous section to show that the conditions of Propositions 1 or 2 imply independence.

Let $f \in L^2(T^p)$, S_f will denote the following subspace of $L^2(T)$:

$$S_f = \mathrm{Span}\left\{ \int_{T^{p-1}} f(t, \underline{t}_{p-1}) \phi(\underline{t}_{p-1})\, d\underline{t}_{p-1}, \phi(\underline{t}_{p-1}) \in L^2(T^{p-1}) \right\}.$$

THEOREM 6. *Let f_p and g_q be symmetric L^2 kernels on T^p and T^q, respectively. The following are equivalent*:

(a) *$I_p(f)$ and $I_q(g)$ are independent random variables.*
(b) *$f_p \otimes_1 g_q = 0$ a.e. Lebesgue on T^{p+q-2}.*
(c) *$S_f \perp S_g$.*
(d) *The subsigma fields $\sigma\{I_1(h), h \in S_f\}$ and $\sigma\{I_1(f), h \in S_g\}$ are independent.*

Note that it follows directly from (d) that if $S_f = S_g$, then any multiple integral $I_m(h_m)$ which is independent of $I_p(f)$ is also independent of $I_q(g)$.

PROOF. In view of Propositions 1 and 2 it suffices to prove that (b) implies (a). In fact, we will show that assumption (b) implies that for $F = I_p(f)$, $G = I_q(g)$,

$$\langle DF, (1+L)^{-1} De^{i\alpha G}\rangle_{L^2(T)} = 0 \quad \text{a.s.} \tag{24}$$

It will then follow directly from Proposition 3 [(20b)] that F and G are independent. Fix α and denote the chaos decomposition of $i\alpha \exp i\alpha G$ by

$$i\alpha \exp i\alpha G = \sum_0^\infty I_n(h_n);$$

set

$$Q_N = \sum_0^N I_n(h_n),$$

where the h_n are symmetric kernels on T^n. In order to prove (24) it suffices to prove that for a subsequence $N_1, N_2, \dots$ with $N_i \to \infty$ as $i \to \infty$,

$$\langle DF, (1+L)^{-1}(Q_{N_i} DG)\rangle \to_{i\to\infty} \langle DF, (1+L)^{-1} i\alpha e^{i\alpha G} DG\rangle \quad \text{a.s.} \tag{25}$$

and that for all N,

$$\langle DF, (1+L)^{-1}(Q_N DG)\rangle = 0 \quad \text{a.s.} \tag{26}$$

We start by proving (25). Note first that by the Schwarz inequality, the following holds a.s.:

$$|\langle DF, (1+L)^{-1}\{(Q_N - i\alpha e^{i\alpha G})DG\}\rangle| \le \|DF\|_{L^2(T)} \cdot \|(1+L)^{-1}(Q_N - i\alpha e^{i\alpha G})DG\|_{L^2(T)}.$$

Consider the last term; taking expectations we have [15]

$$\begin{aligned} U_N &= E^{2/3}\left\|(1+L)^{-1}\left((Q_N - i\alpha e^{i\alpha G})DG\right)\right\|_{L^2(T)}^{3/2} \\ &= \|(Q_N - i\alpha e^{i\alpha G})DG\|_{\mathbb{D}_{3/2,-2}(T)} \\ &\le \|(Q_N - i\alpha e^{i\alpha G})DG\|_{\mathbb{D}_{3/2,0}(T)}, \end{aligned} \tag{27}$$

where $\mathbb{D}_{p,s}$ are the Sobolev spaces defined in ([14], page 25). Consequently, by the Meyer–Hölder inequality (Proposition 1.10 of [14] with $p = 2$, $s = 6$, $r = \frac{3}{2}$),

$$U_N \le K\|Q_N - i\alpha e^{i\alpha G}\|_{\mathbb{D}_{2,0}} \cdot \|DG\|_{\mathbb{D}_{6,o}(T)}, \tag{28}$$

which proves (25) since $\|Q_N - i\alpha e^{I\alpha G}\|_{\mathbb{D}_{2,0}} \to 0$ as $N \to \infty$.

Turning to the proof of (26), let h_n be a symmetric n-kernel on T^n, i.e., a symmetric L^2 function on T^n. By $h_{n-1}^{(t)}$ we denote the symmetric $(n-1)$ kernel obtained from h_n by considering t as a parameter, i.e., $h_{n-1}^{(\theta)}(t_1, \dots, t_{n-1}) = h_n(t_1, \dots, t_{n-1}, \theta)$, and then

$$DI_n(h_n) = nI_{n-1}(h_{n-1}^{(t)}). \tag{29}$$

Note the Fubini type result

$$\int_T I_{p-1}(f_{p-1}^{(\theta)})\, d\theta = I_{p-1}\left(\int_T f_{p-1}^{(\theta)}\, d\theta\right). \tag{30}$$

[This result follows by the following argument: Assume that the Lebesgue measure of T is finite, say μ. Then it follows from the isometry property of the multiple integral that

$$E\left(\int_T\left\{I_{p-1}(f_{p-1}^{(\theta)}) - I_{p-1}\left(\frac{1}{\mu}\int_T f_{p-1}^{(\lambda)}\, d\lambda\right)\right\} d\theta\right)^2 = 0$$

and (30) follows.]

Returning to (26),

$$Q_N DG = \sum_1^N I_n(h_n) q I_{q-1}(g_{q-1}^{(t)}).$$

By the product formula (3), for every $\theta \in T$, each term in the right-hand side is a finite sum of multiple Wiener–Itô integrals with kernels of the form

$$\begin{aligned} \eta_{n+q-2m-1}^{(t)} &= \int_{T^m} h_n(t_1, \dots, \rho_1, \dots, \rho_m) g_{q-1}^{(t)}(\dots, \rho_1, \dots, \rho_m)\, d\rho_1 \cdots d\rho_m \\ &= h_n \otimes_m g_{q-1}^{(t)} \end{aligned} \tag{31}$$

with $m \leq n$, $m \leq q - 1$. Since applying $(1 + L)^{-1}$ to a multiple Wiener–Itô integral of order $r + 1$ is $(1 + r)^{-1}$ times this integral, it follows that $(1 + L)^{-1}Q_N DG$ is a finite sum of multiple Wiener–Itô integrals of the form $(n + q - 2m)^{-1} I_{n+q-2m-1}(\eta^{(t)})$. Since $DF = pI_{p-1}(f_{p-1}^{(t)})$, in order to prove (26) we have to consider the scalar product of $I_{p-1}(f_{p-1}^{(t)})$ with $I_{n+q-2m+1}(\eta^{(t)})$:

$$\int_T I_{p-1}\big(f_{p-1}^{(\theta)}\big) I_{n+q-2m-1}\big(\eta^{(\theta)}\big)\, d\theta. \tag{32}$$

Applying the product formula again will yield a finite sum of multiple integrals with kernels of the type

$$z^\theta = f_{p-1}^{(\theta)}(t_1, \dots) \otimes_{m_1} h_n(t_j, \dots) \otimes_{m_2} g^{(\theta)}(t_k, \dots). \tag{33}$$

Finally we have to integrate with respect to θ. Note that while evaluating the multiple integral of the kernel z^θ we may have to symmetrize it with respect to the variables of the kernel but no symmetrization is necessary with respect to the θ parameter. By (30) we may interchange the order of the integrations, integrating first with respect to θ and then performing the Wiener–Itô integration. Because of the assumption $f \otimes_1 g = 0$ a.e. it follows that $\int_T z^\theta = 0$ proving (26). (c) and (d) follow immediately from (b) which completes the proof. □

From Theorem 6 we also obtain the following two propositions.

PROPOSITION 7. *Assume that f, η and ξ are symmetric L^2 kernels on T^p, T^q and T^m, respectively. Further assume that $f \otimes_1 \eta = 0$ and $f \otimes_1 \xi = 0$ a.e. Lebesgue on T^{p+q-2} and T^{p+m-2}, respectively. Then $I_p(f)$ is independent of the pair $(I_g(\eta), I_m(\xi))$, i.e., in this case pairwise independence implies mutual independence.*

PROOF. The proof follows along the same lines as that of Theorem 6 and is based on Proposition 4 instead of Proposition 3. Except for this the proof is the same and is therefore omitted. □

REMARK. Note that in the proof of Theorem 6, the fact that $\int_T z^\theta \, d\theta = 0$, where z^θ is as given by (33), follows from the fact that $f \otimes_1 g = 0$ and this does not depend on the h_n which appears in (33).

PROPOSITION 8. *Let X, Y be L^2 random variables on the Wiener space with the decomposition*

$$X = \sum_1^\infty I_n(f_n), \qquad Y = \sum_1^\infty I_n(g_n).$$

If $f_n \otimes_1 g_m = 0$ a.e. Lebesgue on T^{m+n-2} for all $m, n \geq 1$, then X and Y are independent, in particular if $X \in \mathbb{D}_{2,1}$ and if for every $m \geq 1$,

$$\|DX \otimes_1 g_m\|_{L^2(T^{m-1})} = 0 \quad a.s.$$

Then X and Y are independent.

PROOF. Note first that convergence in probability and independence commute, i.e., if $X_n \to_P X$ and $Y_n \to_P Y$ and for all n, X_n and Y_n are independent, then

$$\begin{aligned} E \exp i(\alpha X + \beta Y) &= \lim_{n\to\infty} E \exp i(\alpha X_n + \beta Y_n) \\ &= \lim_{n\to\infty} (E \exp i\alpha X_n \cdot E \exp i\beta Y_n) \\ &= E \exp i\alpha X \cdot E \exp i\beta Y \end{aligned}$$

and consequently X and Y are independent. Therefore it suffices to prove the result for

$$X = \sum_1^N I_n(f_n), \qquad Y = \sum_1^M I_m(g_m). \tag{34}$$

Consequently by Proposition 3, (20b) it suffices to prove that for all n,

$$\langle (1+L)^{-1} De^{i\alpha Y}, DI_n(f_n)\rangle = 0 \quad \text{a.s.}$$

It therefore suffices to show that

$$\langle (1+L)^{-1} i\alpha e^{i\alpha Y} + DI_m(g_m), DI_n(f_n)\rangle = 0$$

for all $1 \le m \le M$, $1 \le n$. The rest of the proof is the same as that of Theorem 6 (cf. the remark following Proposition 7) and is therefore omitted. □

Acknowledgments. We wish to thank Robert Adler, Haya Kaspi and David Nualart for pointing out some errors in an earlier version of this paper.

REFERENCES

[1] CLARK, J. M. C. (1970). The representation of functionals of Brownian motion by stochastic integrals. *Ann. Math. Statist.* **41** 1282–1295. Correction **42** (1971) 1778.

[2] MAJOR, P. (1981). *Multiple Wiener–Itô Integrals. Lecture Notes in Math.* **849.** Springer, New York.

[3] MCKEAN, H. P. (1973). Wiener's theory of nonlinear noise. In *Stochastic Differential Equations* (J. B. Keller and H. P. McKean, eds.). *Proc. SIAM–AMS* **6** 191–201. Amer. Math. Soc., Providence, R.I.

[4] MEYER, P.-A. (1986). Eléments de probabilités quantiques. *Séminaire de Probabilités XX. Lecture Notes in Math.* **1204**. Springer, New York.

[5] NUALART, D. and ZAKAI, M. (1986). Generalized stochastic integrals and the Malliavin calculus. *Probab. Theory Related Fields* **73** 255–280.

[6] NUALART, D. and ZAKAI, M. (1987). Generalized multiple stochastic integrals and the representation of Wiener functionals. *Stochastics* **23** 311–330.

[7] NUALART, D., ÜSTÜNEL, A. S. and ZAKAI, M. (1989). On the moments of a multiple Wiener–Itô integral and the space induced by the polynomials of the integral. *Stochastics* **25** 233–240.

[8] SHIGEKAWA, I. (1980). Derivatives of Wiener functionals and absolute continuity of induced measures. *J. Math. Kyoto Univ.* **20** 263–289.

[9] SHOHAT, J. A. and TAMARKIN, J. D. (1943). *The Problem of Moments.* Amer. Math. Soc., New York.

[10] SIMON, B. (1979). *Trace Ideals and Their Applications. London Math. Soc. Lecture Note Ser.* **35.** Cambridge Univ. Press, Cambridge.

[11] STROOCK, D. W. (1987). Homogeneous chaos revisited. *Séminaire de Probabilités XXI. Lecture Notes in Math.* **1247**. Springer, New York.
[12] ÜSTÜNEL, A. S. (1987). Representation of the distributions on Wiener space and stochastic calculus of variations. *J. Funct. Anal.* **70** 126–139.
[13] ÜSTÜNEL, A. S. (1987). Construction du calcul stochastique sur un espace de Wiener abstrait. *C. R. Acad. Sci. Paris Sér. I Math.* **305** 279–282.
[14] WATANABE, S. (1984). *Lectures on Stochastic Differential Equations and Malliavin Calculus.* Springer, Berlin.
[15] ZAKAI, M. (1985). The Malliavin calculus. *Acta Appl. Math.* **3** 175–207.

2 BD. AUGUSTE BLANQUI
75013 PARIS
FRANCE

DEPARTMENT OF ELECTRICAL ENGINEERING
TECHNION–ISRAEL INSTITUTE OF TECHNOLOGY
HAIFA 32000
ISRAEL