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CONTINUITY PROPERTIES FOR RANDOM FIELDS

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Consider a random field on R^d, $d \geq 1$. A simple condition is given on the covariance function which ensures the existence of a version of the random field in which the realizations are everywhere continuous. The proof involves a rather delicate approximation of the random field by interpolating polynomials of suitably high order.

1. Introduction. Let $\{X(t)\colon t \in R^d\}$ be a real-valued stationary random field on R^d with mean 0, finite second moments and covariance function

$$\sigma(h) = E\{X(t)X(t+h)\}, \qquad h \in R^d.$$

If $\sigma(h)$ is m-times continuously differentiable with respect to h for $m \geq 0$, let

$$\sigma_m(h) = \sigma(h) - p_m(h), \tag{1.1}$$

where $p_m(h)$ is the polynomial of degree m given by the Taylor series expansion of $\sigma(h)$ about $h = 0$, up to order m

Since $\sigma(h)$ is an even function of h, $p_m(h)$ is an even polynomial. In particular if $m = 2k+1$ is odd, $p_{2k}(h) = p_{2k+1}(h)$. Further if $\sigma(h)$ is isotropic, depending only on $|h| = \{\Sigma h[l]^2\}^{1/2}$, we can write

$$p_{2k}(h) = p_{2k+1}(h) = \sum_{j=0}^{k} c_j |h|^{2j} \tag{1.2}$$

with $(-1)^j c_j > 0$. It is not possible for any of the c_j to equal 0 except for the degenerate random field with $\sigma(h) \equiv c_0$. [If $c_0 > 0$, then $\sigma(h)/c_0$ represents the characteristic function of an isotropic random vector $Y = (Y[1], \dots, Y[d])$, say, with moments $(-1)^j c_j / c_0 = E(Y[1]^{2j})$. None of these moments can vanish for $j \geq 1$ unless $\sigma(h) \equiv c_0$, that is, $P(Y = 0) = 1$. Further if $c_0 = 0$, then the elementary inequality $|\sigma(h)| \leq \sigma(0)$ implies $\sigma(h) \equiv 0$.]

The following theorem is the main result of this article.

THEOREM 1. *If $\sigma(h)$ is d-times continuously differentiable and*

$$|\sigma_d(h)| = O\left(r^d / |\log r|^{3+\gamma}\right) \quad \text{as } r = |h| \to 0 \tag{1.3}$$

for some $\gamma > 0$, then there exists a version of the d-dimensional random field $\{X(t)\colon t \in R^d\}$ with continuous realizations.

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REMARKS.

1. In $d = 1$ dimension, with $|\sigma_1(h)| = \sigma(0) - \sigma(h)$, this theorem is a classic result; see, for example, Cramér and Leadbetter (1967), pages 63–65.
2. In higher dimensions there has been some confusion in the literature over suitable conditions needed to ensure continuous realizations. Ripley (1981), pages 50 and 51, states incorrectly that the one-dimensional result will carry over directly to higher-dimensional random fields. Adler gives a sufficient condition due to Kozačenko and Jadrenko [cf. Adler (1981), page 47] which, when specialized to the covariance function, implies $\sigma(0) - \sigma(h) = o(|h|^{2d})$ as $h \to 0$. Hence this condition is useless because it implies $c_1 = 0$ in (1.2) even for the $d = 1$ dimensional case; only the trivial covariance function $\sigma(h) \equiv c_0$ satisfies this condition.
3. In practice, a simpler condition to check than (1.3) is

$$|\sigma_d(h)| = O(|h|^{d+\beta}) \quad \text{as } h \to 0 \tag{1.4}$$

for some $\beta > 0$. Clearly if (1.4) holds for any $\beta > 0$, then (1.3) holds for all $\gamma > 0$.
4. Condition (1.3) also holds if the mixed dth-order partial derivative of $\sigma(h)$ satisfies

$$|\partial^d \sigma(h)/\partial h[1] \dots \partial h[d]| = O(|\log r|^{-3-\gamma}) \quad \text{as } r = |h| \to 0. \tag{1.5}$$

5. For Gaussian random fields, continuity of the realizations follows under much milder conditions, for example, $\sigma(0) - \sigma(h) = O(1/|\log r|^{1+\varepsilon})$ for some $\varepsilon > 0$ with $r = |h|$ [Adler (1981), page 60].
6. Theorem 1 can also be extended to nonstationary processes with mean 0 and covariance function $\sigma(s, t) = E\{X(s)X(t)\}$. In this case suppose that for each t, $\sigma(t + h, t - h)$ is d-times continuously differentiable with respect to h. Let $p_d(h; t)$ denote the corresponding polynomial in h, for each t. Then continuity will follow if $\sup\{|\sigma(t - h, t + h) - p_d(h; t)|\}$ satisfies the bound in (1.2). Here the supremum is taken over t lying in each compact subset of R^d. The key step, as the reader is invited to verify, is to show that Lemma 2 below can be extended to the nonstationary situation.
7. The importance of the "irregular part" of $\sigma(h)$ in describing the behaviour of the random field has been noted by other authors, for example, Cramér and Leadbetter (1967), page 180, in the one-dimensional case, and Matheron (1971), pages 13 and 14. However, the question of continuity of the realizations does not seem to have been adequately addressed before.

The proof of Theorem 1 will occupy Sections 2–4. In Section 2 basic notation and interpolating polynomials are described. Properties of increments and their links with interpolating polynomials are given in Section 3. A suitably accurate sequence of approximations to $X(t)$ based on interpolating polynomials is constructed in Section 4. In Section 5 an example is given showing that the condition (1.3) in the theorem cannot be significantly relaxed.

2. Notation. The letters i, j, k, m, n will denote integers. In particular, $d \geq 1$ is the dimension of the random field $\{X(t): t \in R^d\}$.

A vector $t \in R^d$ will be called a *site*, and its components will be indicated using square brackets, $t = (t[1], \dots, t[d])$. The letters s and t will be used for sites in R^d.

Let $Z_+^d = \{u \in Z^d: u[l] \geq 0 \text{ for } l = 1, \dots, d\}$ denote the nonnegative integer lattice. The letters u, v, y, z will be used for elements of Z_+^d. Similarly let $L_d(k) = \{u \in Z_+^d: u[l] \leq k\}$ denote the cubic lattice in Z_+^d containing the $(k+1)^d$ sites for which each component takes values in $\{0, 1, \dots, k\}$.

Let $C_d = \{t \in R^d: 0 \leq t[l] \leq 1 \text{ for } l = 1, \dots, d\}$ denote the unit cube in R^d. For $n \geq 1$ we can partition C_d into 2^{dn} cells of the form $I_{n,y}$, where

$$I_{n,y} = 2^{-n}y + 2^{-n}C_d, \qquad y \in L_d(2^n - 1), \tag{2.1}$$

is the cube with lower corner at $2^{-n}y$ and side length 2^{-n}.

For $u \in Z_+^d$ two norms are useful:

$$\text{(i)} \quad \|u\|_1 = \sum u[l] \quad \text{and} \quad \text{(ii)} \quad \|u\|_\infty = \max\{u[l]\}.$$

Note that $L_d(k) = \{u \in Z_+^d: \|u\|_\infty \leq k\}$.

For sites $t \in R^d$, only one norm is of interest, $\|t\|_2 = \{\Sigma t[l]^2\}^{1/2}$, which we abbreviate as $\|t\|_2 = |t|$.

For $u \in Z_+^d$ and $t \in R^d$, define the monomial

$$t^u = \prod_{l=1}^{d} t[l]^{u[l]}.$$

Polynomials in t can be built up from linear combinations of such monomials. There are two convenient ways to specify the degree of such a polynomial. Say that a polynomial (with at least one nonzero highest-order coefficient)

$$\sum_{\|u\|_1 \leq k} a_u t^u$$

has *overall degree* k, whereas

$$\sum_{\|u\|_\infty \leq k} a_u t^u$$

has *individual degree* k. Thus the polynomial $p_m(h)$ in (1.1) has overall degree equal to m is m is even, and equal to $m - 1$ if m is odd.

A useful class of polynomials is the class of interpolating polynomials; see, for example, Kunz (1957), pages 264–266. Fix $k \geq 0$, $s \in R^d$, $\lambda > 0$, and let $f(t)$ be a function defined on the cube $s + \lambda C_d$. The *interpolating polynomial for $f(t)$ of individual degree k* takes the form

$$q(t) = \sum_{\|u\|_\infty \leq k} A_u\{(t-s)/\lambda\}^u, \tag{2.2}$$

where

$$A_u = \sum_{\|v\|_\infty \leq k} a_{uv} f(s + \lambda v/k). \tag{2.3}$$

The coefficients a_{uv}, which depend on k and d, but not on the function $f(t)$ or on s or λ, are determined by the $(k+1)^d$ constraints

$$q(s + \lambda v/k) = f(s + \lambda v/k), \qquad v \in L_d(k).$$

If $k \geq 1$, it is easy to check that $q(t)$, restricted to a face of the cube $s + \lambda C_d$, represents an interpolating polynomial in R^{d-1}, passing through the $(k+1)^{d-1}$ sites of $s + \lambda L_d(k)$ lying in that face.

3. Interpolating polynomials and increments. This section contains the key results of the article. After defining the notion of an *increment*, we show that each of the coefficients in the difference of two interpolating polynomials is an increment, and that in the formula for the variance of a suitable increment, the covariance function $\sigma(\cdot)$ can be replaced by $\sigma_m(\cdot)$.

The use of interpolating polynomials in this context seems rather special. Indeed it is difficult to imagine any other method of approximation which will have similar properties. Increments also play an important role in the study of *intrinsic random fields* [Matheron (1971, 1973)].

DEFINITION. Let $f(t)$ be a function of $t \in R^d$, and let $k \geq 0$. A linear combination $\sum \alpha_i f(t_i)$ based on weights $\alpha_i \in R$ at sites $t_i \in R^d$, $i = 1, \dots, n$ is called a *kth-order increment for* $f(t)$ if for all $u \in Z_+^d$ with $\|u\|_1 \leq k$, we have

$$\sum_{i=1}^{n} \alpha_i t_i^u = 0.$$

That is, $\sum \alpha_i f(t_i) = 0$ whenever f is a polynomial in t of overall degree less than or equal to k.

LEMMA 1. *For $k \geq 0$ let $q_1(t)$ and $q_2(t)$ be the interpolating polynomials of individual degree k for a function $f(t)$, on two cubes, $s_1 + \lambda_1 C_d$ and $s_2 + \lambda_2 C_d$, respectively. Set $q(t) = q_1(t) - q_2(t)$. Then the coefficient of each power of t in $q(t)$ is a kth-order increment for $f(t)$, based on the sites used to define the interpolating polynomials.*

PROOF. Using (2.2), expand $q_1(t)$ in powers of t^u rather than $\{(t - s_1)/\lambda_1\}^u$; similarly for $q_2(t)$. Then we can express

$$q(t) = \sum_{\|u\|_\infty \leq k} B_u t^u,$$

where

$$B_u = \sum_{\|v\|_\infty \leq k} \left\{ b_{uv}^{(1)} f(s_1 + \lambda_1 v/k) + b_{uv}^{(2)} f(s_2 + \lambda_2 v/k) \right\}$$

for suitable constants $b_{uv}^{(1)}$ and $b_{uv}^{(2)}$. We wish to show each B_u is a kth-order increment for $f(t)$.

Suppose $f(t)$ is a polynomial of overall degree less than or equal to k. Since an interpolating polynomial reproduces such functions exactly (indeed it reproduces all polynomials with individual degree less than or equal to k), we have $q_1(t) = q_2(t) = f(t)$ for all $t \in R^d$. Hence $q(t) = 0$ for all t, which implies $B_u = 0$ for all u. Therefore, by the definition of a kth-order increment, Lemma 1 follows. □

LEMMA 2. *Suppose that $\sigma(h)$ is m-times continuously differentiable ($m \geq 0$) and $\sum \alpha_i X(t_i)$ is a kth-order increment for the random field $X(t)$, where k satisfies $2k \geq m$ if m is even and $2k \geq m - 1$ if m is odd. Then*

$$\begin{aligned} \operatorname{var}\left\{\sum \alpha_i X(t_i)\right\} &= \sum \alpha_i \alpha_j \sigma(t_i - t_j) \\ &= \sum \alpha_i \alpha_j \sigma_m(t_i - t_j). \end{aligned} \tag{3.1}$$

PROOF. The first line of (3.1) follows from elementary properties of the covariance function. It is the second line which needs proof.

A typical term of the polynomial $p_m(h)$ in (1.1) is proportional to h^u with $\|u\|_1 \leq m$ if is even and $\|u\|_1 \leq m - 1$ if m is odd. In either case $\|u\|_1 \leq 2k$. Expanding $(t_i - t_j)^u$ into powers of t_i and t_j yields a sum in which a typical term is proportional to $t_i^v t_j^{u-v}$, where $0 \leq v[l] \leq u[l]$, $l = 1, \dots, d$. Since $\|u\|_1 \leq 2k$, note that either $\|v\|_1 \leq k$ or $\|u - v\|_1 \leq k$. Hence

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j t_i^v t_j^{u-v} = \left\{\sum \alpha_i t_i^v\right\}\left\{\sum \alpha_j t_j^{u-v}\right\} = 0$$

from which it follows that $\sum \alpha_i \alpha_j p_m(t_i - t_j) = 0$. Therefore $\sigma(t_i - t_j)$ can be replaced by $\sigma_m(t_i - t_j)$ in (3.1). □

4. Approximation of the random field. In this section we shall approximate the random field $X(t)$ on the unit cube, $t \in C_d$, by a sequence of continuous functions $\{X_n(t)\}$ which, with probability 1, converges uniformly on C_d to a function $X_\infty(t)$, where $X_\infty(t)$ equals $X(t)$ at a countably dense set of sites in C_d. This limiting function $X_\infty(t)$ defines the desired continuous version of the random field on C_d. Further, once we have a continuous version on C_d, it is straightforward to construct a continuous version on all of R^d.

Fix $k \geq 1$. (A suitable choice for k will be made below.) For $n \geq 1$, the function $X_n(t)$ is constructed as follows. On the cubic cell $I_{n,y}$ in (2.1), set $X_n(t)$ to be the interpolating polynomial of individual degree k for the random field $X(t)$. In particular, $X_n(2^{-n}y) = X(2^{-n}y)$ for all $y \in L_d(2^n)$. From the last paragraph of Section 2, we see that on the face between two adjacent cells, $X_n(t)$ reduces to the same $(d-1)$-dimensional interpolating polynomial through the same $(k+1)^{d-1}$ sites. Therefore $X_n(t)$ is continuous (though not differentiable) across the boundaries of adjacent cells, and so $X_n(t)$ is continuous throughout C_d.

The next lemma shows that $X_{n+1}(t)$ and $X_n(t)$ are likely to be close together on each of the cells $I_{n+1,z}$. Suppose $\sigma(h)$ is m-times continuously differentiable and that $k \geq 1$ satisfies $2k \geq m$ if m is even and $2k \geq m - 1$ if m is odd. For

$r > 0$, set

$$\sigma_m^*(r) = \sup\{|\sigma_m(h)|: |h| \le d^{1/2}r\}.$$

Also, define

$$W_{n+1,z} = \sup\{|X_{n+1}(t) - X_n(t)|: t \in I_{n+1,z}\}$$

for $z \in L_d(2^{n+1} - 1)$.

LEMMA 3. *For each* $z \in L_d(2^{n+1} - 1)$,

$$EW_{n+1,z}^2 \le c\sigma_m^*(2^{-n}) \tag{4.1}$$

for some constant c not depending on n.

PROOF. Each cell $I_{n,y}$, $y \in L_d(2^n - 1)$, at level n of the construction is split into 2^d cells at level $n + 1$. One such cell is $I_{n+1,z}$, with $z = 2y$. (The other cells are obtained by adding 1 to a proper subset of the components of z.) For $t \in I_{n+1,z}$, we have from (2.2) and (2.3)

$$\begin{aligned} X_{n+1}(t) - X_n(t) &= \sum_u \Big\{\sum_v a_{uv} X(t_v)\Big\}\{2^{n+1}(t - 2^{-n-1}z)\}^u \\ &\quad - \sum_u \Big\{\sum_v a_{uv} X(s_v)\Big\}\{2^n(t - 2^{-n}y)\}^u \\ &= \sum_u \Big\{\sum_v a_{uv} X(t_v) - 2^{-\|u\|_1} a_{uv} X(s_v)\Big\}\{2^{n+1}(t - 2^{-n-1}z)\}^u \\ &= \sum_u B_u \{2^{n+1}(t - 2^{-n-1}z)\}^u \quad \text{say,} \end{aligned}$$

where the sums are over $u, v \in L_d(k)$. Here $t_v = 2^{-n-1}z + 2^{-n-1}v/k$ and $s_v = 2^{-n}y + 2^{-n}v/k$.

For any real numbers $\alpha_1, \dots, \alpha_N$, recall the elementary inequality $(\Sigma\alpha_i)^2 \le N\Sigma\alpha_i^2$. Also note that $\{2^{n+1}(t - 2^{-n-1}z)\}^u$ lies between 0 and 1 for each $t \in I_{n+1,z}$. Thus

$$W_{n+1,z}^2 \le (k+1)^d \sum_u B_u^2.$$

Now from Lemma 1, each B_u is a kth-order increment for $X(t)$. Using Lemma 2 to calculate EB_u^2 and noting that the distance between any two points in $I_{n,y}$ is bounded by $d^{1/2}2^{-n}$ yields the statement of the lemma with

$$c = 2(k+1)^{2d}\max\{a_{uv}^2: u, v \in L_d(k)\}.$$

A similar calculation yields the same bound for each of the other $2^d - 1$ cells at level $n + 1$ lying within $I_{n,y}$. □

We now have the necessary bounds to complete the proof of Theorem 1 in the standard manner [see, e.g., Cramér and Leadbetter (1967), pages 63–65]. The

Markov inequality in (4.1) tells us that

$$P(|W_{n+1,z}| \geq \varepsilon_n) \leq c\sigma_m^*(2^{-n})/\varepsilon_n^2$$

for any numbers $\varepsilon_n > 0$, and since there are $2^{d(n+1)}$ cells at level $n+1$,

$$P(\sup\{|X_{n+1}(t) - \dot{X}_n(t)|: t \in C_d\} > \varepsilon_n) \leq 2^{d(n+1)} c\sigma_m^*(2^{-n})/\varepsilon_n^2 \tag{4.2}$$
$$= \delta_n \quad \text{say.}$$

Next we make use of the assumptions in Theorem 1. Suppose (1.3) holds and suppose $k \geq [d/2]$ ($k \geq 1$ in $d = 1$ dimension). Then we can let $m = d$ in Lemma 3 and (4.2). Choose $\varepsilon_n = n^{-(1+\gamma/4)}$. Then $\sum \varepsilon_n < \infty$ and

$$\sum \delta_n = \sum O\big(2^{dn} 2^{-dn} / [n^{3+\gamma} \varepsilon_n^2]\big)$$
$$= \sum O(n^{-1-\gamma/2}) < \infty.$$

The key point of assumption (1.3) is to ensure that $\sigma_m^*(2^{-n})$ is small enough to balance the factor $2^{d(n+1)}$ in (4.2).

Hence by the first Borel–Cantelli lemma, with probability 1, $\{X_n(t),\ t \in C_d\}$ forms a uniformly Cauchy sequence and converges to a limiting continuous random field $X_\infty(t)$. Further if t is a rational vector of the form $t = 2^{-n_0}u$, $u \in L_d(2^{n_0})$, for some $n_0 \geq 1$, then $X_n(t) = X(t)$ for $n \geq n_0$ and so $X_\infty(t) = X(t)$ for this value of t.

Note that the smallest suitable degree for the interpolating polynomials is $k = 1$ in dimensions $d = 1, 2, 3$ and $k = [d/2]$ in higher dimensions.

5. Example. We now give an example to show that the conditions in Theorem 1 cannot be significantly weakened. Let $\phi(h)$, $h \in R^d$, be a rotationally symmetric function of h, both absolutely integrable and square integrable. Let $\{T_j\colon j \geq 1\}$ be a labelling of the events from a realization of a Poisson process in R^d, of intensity 1. A convenient random field for our purposes is defined by

$$X(t) = \sum_{j=1}^{\infty} \phi(t - T_j), \qquad t \in R^d.$$

It is straightforward to show that $\{X(t)\}$ is stationary with mean $\int \phi(t)\,dt$ and covariance function

$$\sigma(h) = (\phi * \phi)(h)$$
$$= \int \phi(t)\phi(h - t)\,dt,$$

where * denotes convolution. Also, if $\phi(h)$ is the Fourier transform of a rotationally symmetric function $f(\omega)$, $\omega \in R^d$,

$$\phi(h) = \int \exp(ih \cdot \omega) f(\omega)\,d\omega, \tag{5.1}$$

then $f(\omega)$ is the Fourier transform of $(2\pi)^{-d}\phi(h)$ and $\sigma(h)$ is the Fourier transform of $(2\pi)^d f^2(\omega)$.

If we write $\phi^{\#}(r) = \phi(h)$ for $r = |h|$, and $f^{\#}(\rho) = f(\omega)$ for $\rho = |\omega|$, then (5.1) can be simplified using polar coordinates to

$$\phi^{\#}(r) = (2\pi)^{d/2} r^{1-d/2} \int_0^\infty \rho^{d/2} J_{d/2-1}(r\rho) f^{\#}(\rho)\, d\rho$$

[see, e.g., Sneddon (1972), page 82], where $J_\nu(r)$ and $K_\nu(r)$ below denote the usual Bessel functions.

For our purposes a convenient family of choices for f and ϕ is

$$f_\nu(\omega) = 1/(1 + |\omega|^2)^{\nu + d/2},$$

$$\phi_\nu(h) = \frac{\pi^{d/2}|h|^\nu}{2^{\nu-1}\Gamma(\nu + d/2)} K_\nu(|h|)$$

[Abramowitz and Stegun (1972), page 488, formula 11.4.44].

It is easily checked that $f_\nu(\omega)$ and $\phi_\nu(h)$ are square integrable and $\phi_\nu(h)$ is integrable if and only if $\nu > -d/4$. Further, since $f_\nu^2(\omega) = f_\mu(\omega)$ with $\mu = 2\nu + d/2$, we see that

$$\sigma_\nu(h) = (\phi_\nu^* \phi_\nu)(h) = (2\pi)^d \phi_\mu(h).$$

Note that $\mu > 0$ when $\nu > -d/4$.

Elementary properties of the Bessel function $K_\nu(r)$ yield the following properties for functions $\phi_\nu(h)$ as $h \to 0$ [Abramowitz and Stegun (1972), page 375]:

1. If $\nu < 0$, $\phi_\nu(h) = O(|h|^{2\nu}) \to \infty$ as $h \to 0$.
2. If $\nu > 0$ is not an integer, then we can express

$$\phi_\nu(h) = \sum_{j<\nu} a_{j,\nu}|h|^{2j} + O(|h|^{2\nu}).$$

3. If $\nu = m \geq 0$ is an integer, then we can express

$$\phi_\nu(h) = \sum_{j<m} a_{j,\nu}|h|^{2j} + O(|h|^{2m}|\log|h||).$$

We can now link the behaviour of $X(t)$ with the conditions of Theorem 1. For $-d/4 < \nu \leq 0$ ($0 < \mu \leq d/2$), $X(t)$ will have discontinuous realizations [since $\phi_\nu(h) \to \infty$ as $|h| \to 0$] and the conditions of Theorem 1 will not be met. On the other hand, for $\nu > 0$ ($\mu > d/2$), $X(t)$ will have continuous realizations and the conditions of Theorem 1 will be met [take $0 < \beta \leq \min(1, 2\mu - d)$ in (1.4)]. For this model the realizations of $X(t)$ will look like landscapes of randomly placed volcanoes, each with an infinite peak ($-d/4 < \nu \leq 0$) or a finite peak ($\nu > 0$).

Therefore the conditions in Theorem 1 cannot be significantly relaxed; in particular, it is not possible to replace r^d by any smaller power of r in (1.3).

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