The Annals of Probability
1989, Vol. 17, No. 4, 1277–1302

THE CORRELATION LENGTH FOR THE HIGH-DENSITY PHASE OF BERNOULLI PERCOLATION[1]

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We examine two standard types of connectivity functions in the high-density phase of nearest-neighbor Bernoulli (bond) percolation. We show that these two quantities decay exponentially at the same constant rate. The reciprocal of this constant defines therefore a correlation length. Unfortunately, we cannot prove that this correlation length is finite whenever $p > p_c$, although previous work established this result for p above a threshold which is conjectured to coincide with p_c. We examine also a third connectivity function and prove that it too decays exponentially with the same rate as the two standard connectivity functions. We establish various useful properties of our correlation length, such a semicontinuity as a function of bond density and convexity in its directional dependence. Finally, for bond percolation in two dimensions we show that the correlation length at bond density $p_1 > p_c = \frac{1}{2}$ is exactly half the correlation length at the subcritical bond density $p_2 = 1 - p_1 < p_c$. This sharpens some other exact results for two-dimensional percolation and is the precise analog of known results for the two-dimensional Ising model.

1. Introduction. We consider the Bernoulli bond percolation model at density p. (See Section 2 for notation, definitions and background material.) It is well known that in dimension 2 or higher this model has a phase transition at some value $p_c \in (0, 1)$. For $p > p_c$ there exists an infinite occupied cluster and we say that percolation occurs in this phase. This article is concerned with the connectivity functions in the percolating or high-density phase.

In the low-density phase (i.e., when $p < p_c$), there is a well-defined correlation length associated with the asymptotic behavior (as $|x - y| \to \infty$) of the connectivity function

$$(1.1) \qquad \tau(p, x, y) = P_p\{x \text{ and } y \text{ are connected by an occupied path}\}.$$

Received May 1988.

[1]J. T. Chayes and L. Chayes were supported in part by NSF Postdoctoral Fellowships. G. R. Grimmett participated in this research while visiting Cornell University with the aid of a travel grant from the Fulbright Commission. H. Kesten's and R. H. Schonmann's research was supported in part by the National Science Foundation through a grant to Cornell University. J. T. Chayes and R. H. Schonmann were partially supported by the U.S. Army Research Office through the Mathematical Sciences Institute at Cornell.

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AMS 1980 *subject classifications*. Primary 60K35; secondary 82A43.

Key words and phrases. Percolation, correlation length.

Indeed the limit

$$\frac{1}{\xi} = \frac{1}{\xi(p)} = -\lim_{n\to\infty}\frac{1}{n}\log\tau(p,\mathbf{0},ne_i) \tag{1.2}$$

exists, where e_i is the ith coordinate vector. In addition τ satisfies the bounds [Grimmett (1989), Sections 5.1 and 5.2]

$$K_1 n^{4(1-d)}\exp\left(-\frac{n}{\xi(p)}\right) \le \tau(p,\mathbf{0},ne_1) \le \exp\left(-\frac{n}{\xi(p)}\right) \tag{1.3}$$

for some $K_1 = K_1(p) > 0$, independent of n (d is the dimension). Thus, in first order, τ decays exactly exponentially—at least along a coordinate axis—at the rate $1/\xi(p)$. The quantity $\xi(p)$ is called the *correlation length*. For general directions one has [see (2.1) for $|x|$]

$$K_2|x|^{4d(1-d)}\exp\left(-\frac{1}{\xi(p)}\sum_1^d|x_i|\right) \le \tau(p,\mathbf{0},x) \le \exp\left(-\frac{|x|}{\xi(p)}\right). \tag{1.4}$$

Above the percolation threshold the limit in (1.2) still exists, but this provides no information since the limit is 0. In fact, since for $p > p_c$ there exists a unique infinite occupied cluster [Aizenman, Kesten and Newman (1987); see also Gandolfi, Grimmett and Russo (1988)], $\tau(p,\mathbf{0},x) \to P_\infty^2(p) > 0$ as $|x| \to \infty$, where $P_\infty(p)$ is the percolation probability. Nevertheless, one is interested in having a concrete notion of a correlation length even in the percolating regime. In fact the entire scaling theory is based on the notion that for each $p \ne p_c$ there is a single important length scale (called the correlation length) and that all quantities should be measured on this scale [cf. Fisher (1983) and Stauffer (1979)].

Most workers agree that there are two acceptable notions of a connectivity function above the percolation threshold:

$$\tau^f(x,y) = \tau^f(p,x,y) = P_p\{x \text{ and } y \text{ belong to the same } \textit{finite} \text{ occupied cluster}\} \tag{1.5}$$

and

$$\tilde\tau(x,y) = \tilde\tau(p,x,y) = \tau(p,x,y) - P_\infty^2(p). \tag{1.6}$$

The quantities in (1.5) and (1.6) have the advantages that $\tau^f(x,y) = \tilde\tau(x,y) = \tau(x,y)$ whenever P_∞ vanishes, and that for $p > p_c$ they tend to 0 as $|x-y| \to \infty$. [Later we shall introduce yet a third connectivity function which amounts to the difference of (1.5) and (1.6) and has a certain aesthetic appeal to the authors.]

Definitions (1.5) and (1.6) raise the following questions:

(0) Do these quantities actually decay exponentially when $P_\infty > 0$?
(i) Do limits exist in the sense of (1.2)?
(ii) Do bounds similar to (1.3) and (1.4) exist?
(iii) If the answers to (0) and (i) are affirmative and the limits are $1/\xi^f(p)$ and $1/\tilde\xi(p)$, respectively, are the "correlation lengths" ξ^f and $\tilde\xi$ the same?

The zeroth question has been resolved to a certain degree of satisfaction in Chayes, Chayes and Newman (1987); however, a technically stronger hypothesis than $P_\infty > 0$ was used. They assumed that $p > p_c(S_k)$ for some $k < \infty$, where $p_c(S_k)$ is the percolation threshold of the "slab" $S_k := Z^{d-1} \times \{0, 1, \dots, k\}$ (rather than of Z^d). In this article we shall prove (in Sections 3–5) that the limits of

$$-\frac{1}{|x|}\log \tau^f(\mathbf{0}, x) \quad \text{and} \quad -\frac{1}{|x|}\log \tilde{\tau}(\mathbf{0}, x)$$

exist when x moves out to ∞ in a fixed direction. The two limits are equal and when x moves out along a coordinate axis we denote the corresponding limit by $1/\xi^f(p)$. $\xi^f(p)$ seems to us to be the proper choice for correlation length, especially in view of the known results in dimension 2 [see Kesten (1987) and Section 7]. Unfortunately, we only have that $\xi^f(p) < \infty$ when $p > p_c(S_k)$ for some finite k, but many people believe that $p_c = \lim_{k \to \infty} p_c(S_k)$ so that $p > p_c(S_k)$ for some finite k would be equivalent to $p > p_c$ (see Note added in proof, at end). We will also obtain bounds corresponding to the upper bounds in (1.3) and (1.4), although not quite as sharp as those.

Although the above constitute the principal results of this article, we will also address some subsidiary issues:

(a) We establish that the inverse of the correlation length is convex as a function of direction and upper semicontinuous as a function of the bond density.

(b) As will be explicitly demonstrated in Section 6, the difference $\mathscr{C}(\mathbf{0}, x) := \tilde{\tau}(\mathbf{0}, x) - \tau^f(\mathbf{0}, x)$ is nonnegative. It turns out that $\mathscr{C}(\mathbf{0}, x)$ is the covariance of the indicators of the events that $\mathbf{0}$ and x belong to the infinite occupied cluster. Knowing that $\tau^f(\mathbf{0}, x)$ and $\tilde{\tau}(\mathbf{0}, x)$ decay at the same exponential rate, it is of interest to know whether or not their difference decays faster. Here we will show that this is not the case: The limit of $-|x|^{-1} \log \mathscr{C}(\mathbf{0}, x)$ agrees with the analogous limits for $\tilde{\tau}(\mathbf{0}, x)$ and $\tau^f(\mathbf{0}, x)$.

(c) In two dimensions all our results save the last are well known. We have for the square lattice that $p_c = \frac{1}{2}$ and several exact results relate the correlation length at $p > p_c$ and at $1 - p < p_c$. For example in Kesten (1987) it is shown that they are of the same order of magnitude. For the nearest-neighbor Ising ferromagnet [see, e.g., McCoy and Wu (1973), formulae 11.2.43 and 11.3.24] one knows that the correlation length above threshold is exactly half the correlation length of the dual model. For two-dimensional percolation we will establish this relation by direct methods in Section 7.

2. Definitions and notation. We consider the d-dimensional lattice Z^d. Lattice distances will be measured in the L^∞ norm; which is to say that for $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d) \in Z^d$ we take

$$|x - y| = \max\{|x_1 - y_1|, \dots, |x_d - y_d|\}. \tag{2.1}$$

Nearest-neighbor pairs of sites, that is, pairs $\{x, y\} \in Z^d$ with $\sum_i |x_i - y_i| = 1$, are called *bonds* or *edges*, and we shall use $\{x, y\}$ to denote the (unoriented) bond between x and y. The collection of all such bonds will be denoted by $\mathscr{B}_d$. A *path* is a sequence (finite or infinite) of bonds $b_1, b_2 \dots$ without repetitions, such that b_k and b_{k+1} have a common endpoint.

The Bernoulli bond percolation model is defined by independently choosing each bond to be *occupied* (*vacant*) with probability p (respectively $q := 1 - p$). The corresponding product measure on the configurations of occupied and vacant bonds is denoted by P_p. E_p is expectation with respect to P_p. In most places, when no confusion is likely, we shall suppress the subscript p in P_p and E_p. A generic configuration is denoted by ω. If S_1 and S_2 are collections of sites in Z^d (and $B \subset \mathscr{B}_d$ a collection of bonds), then we say that S_1 is *connected* to S_2 (respectively, S_1 is connected to S_2 in B) if there exists an occupied path, that is, a path all of whose bonds are occupied, from a vertex in S_1 to a vertex in S_2 (respectively, an occupied path from S_1 to S_2 which is contained in B). We denote this event by $\{S_1 \leftrightarrow S_2\}$ (respectively, $\{S_1 \leftrightarrow S_2 \text{ in } B\}$). Maximal connected subsets (sets of which each pair of vertices is connected) will be called (*occupied*) *clusters*. The occupied cluster containing the vertex x will be denoted by $C(x) = C(x, \omega)$. If all bonds incident to x are vacant, then $C(x) = \{x\}$.

It is often convenient to visualize vacant bonds as occupied *dual* objects. For $d = 2$ the dual objects are bonds of the dual lattice $Z^2 + (\frac{1}{2}, \frac{1}{2})$. For $d = 3$ the dual objects can best be thought of as plaquettes, that is, as faces of unit cubes with corners on $Z^3 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ [see Aizenman, Chayes, Chayes, Fröhlich and Russo (1983)].

It is well known [Broadbent and Hammersley (1957), Hammersley (1959) and Harris (1960)] that if $d > 1$, then there is a so-called critical probability, or percolation threshold, p_c, strictly between 0 and 1 such that for $p > p_c$ there is with probability 1 an infinite occupied cluster, while for $p < p_c$, all occupied cluster are finite w.p.1. We write $P_\infty(p)$ for the so called *percolation probability*,

$$P_\infty(p) = P_p\{\mathbf{0} \text{ belongs to an infinite cluster}\}. \tag{2.2}$$

We write I_A for the indicator function of an event A, that is,

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

If ω_1, ω_2 are two configurations, then there is a natural partial order given by the definition

$$\omega_1 \prec \omega_2 \quad \text{if all occupied bonds in } \omega_1 \text{ are also occupied in } \omega_2. \tag{2.3}$$

Events are said to be *increasing* or *positive* (respectively, *decreasing* or *negative*) if their indicators are nondecreasing (respectively, nonincreasing) in this partial order. Harris (1960) [see also Fortuin, Kasteleyn and Ginibre (1971)] proved that if both A and B are increasing (or both decreasing) events, then

$$P_p\{A \cap B\} \geq P_p\{A\}P_p\{B\}. \tag{2.4}$$

We will refer to (2.4) as the Harris–FKG inequality. In van den Berg and Kesten (1985) and van den Berg and Fiebig (1987) an inequality going in the other direction was proved. We shall use the following special case:

$$(2.5)\quad \begin{aligned} &P_p\{\exists \text{ edge disjoint occupied paths from } S_1 \text{ to } S_2 \text{ and from } S_3 \text{ to } S_4\} \\ &\quad \le P_p\{S_1 \leftrightarrow S_2\}P_p\{S_3 \leftrightarrow S_4\}. \end{aligned}$$

Finally, some general notation: e_i is the ith coordinate vector, B_N is the cube $\{x\colon\ |x_i| \le N,\ 1 \le i \le d\}$, $[a_1, b_1] \times \cdots \times [a_d, b_d]$ or $\Pi[a_i, b_i]$ is the box $\{x\colon\ a_i \le x_i \le b_i,\ 1 \le i \le d\}$. Its *left face* and *right face* are the sets $\{a_1\} \times [a_2, b_2] \times \cdots \times [a_d, b_d]$ and $\{b_1\} \times [a_2, b_2] \times \cdots \times [a_d, b_d]$, respectively.

In Sections 4 and 5 we will have to consider $C(x)$, $\tau(p, x, y)$, and so forth, when x and/or y are not vertices of Z^d. In such a situation we define $C(x)$ as $C([x])$ and $\tau(p, x, y)$ as $\tau(p,[x],[y])$. Here $[a]$ denotes the largest integer $\le a$ when a is real, and $[x]$ denotes $([x_1],\dots,[x_d])$ when $x = (x_1, n, x_d)$. $K_1, K_2,\dots$ will denote finite strictly positive constants, whose specific values are of no significance for this article. Their values may depend on the dimension d and p, and K_i may even change from appearance to appearance.

3. The existence of ξ^f. To prove the existence of ξ^f and to give an upper bound for $\tau^f(\mathbf{0}, ne_1)$ we use a "subadditivity" argument. It turns out to be somewhat simpler to start with a subadditive-type inequality not for $-\log \tau^f(\mathbf{0}, ne_1)$, but for a closely related quantity, defined in terms of the diameter of $C(\mathbf{0})$, the cluster of the origin. We introduce this quantity now. Let $R_i(L_i)$ be the farthest reach of $C(\mathbf{0})$ in the positive (negative) ith coordinate direction:

$$R_i = R_i(\omega) = \sup\{x_i\colon x \in C(\mathbf{0}, \omega)\},$$
$$L_i = L_i(\omega) = \inf\{x_i\colon x \in C(\mathbf{0}, \omega)\}.$$

Note that R_i and L_i may be infinite. Since the origin $\mathbf{0}$ lies in $C(\mathbf{0})$ we have that

$$(3.1)\qquad -\infty \le L_i \le 0 \le R_i \le \infty.$$

The diameter of the cluster of the origin, $D(\omega)$, is defined by

$$D_i = D_i(\omega) = R_i(\omega) - L_i(\omega), \qquad D = D(\omega) = \max_i D_i(\omega),$$

and the event of principal interest is

$$T_n = \{D(\omega) = n\}.$$

In parts of the proof we shall want to specify which points in $C(\mathbf{0})$ are points of farthest reach. It is convenient to order the vertices in Z^d lexicographically. We will say that

$$(3.2)\qquad x \succ y \quad \text{if, for some } i,\ x_1 = y_1,\dots, x_i = y_i,\ x_{i+1} > y_{i+1}.$$

Using this order, we define

$$\begin{aligned} T_n^+(x) = \{&D = D_1 = n,\ x \in C(\mathbf{0}),\ R_1 = x_1 \text{ and } x \\ &\text{is the maximal point in } C(\mathbf{0}) \text{ with } x_1 = R_1\}, \\ T_n^-(x) = \{&D = D_1 = n,\ x \in C(\mathbf{0}),\ L_1 = x_1 \text{ and } x \\ &\text{is the maximal point in } C(\mathbf{0}) \text{ with } x_1 = L_1\}. \end{aligned}$$

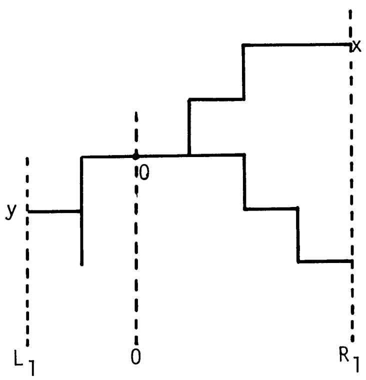

FIG. 1. *Illustration of a finite cluster with* $D = D_1 = 6$; T_6, $T_6^+(x)$ *and* $T_6^-(y)$ *occur in this example.*

$T_n^+(x)$ $[T_n^-(x)]$ occurs when the maximal width of $C(\mathbf{0})$ is n, occurs in the first coordinate direction and the farthest points to the right (left) in $C(\mathbf{0})$ lie in the hyperplane $\{y\colon\ y_1 = x_1\}$, while x is the maximal point among these points (see Figure 1). In particular, $T_n^{\pm}(x)$ can only happen for some x in B_n. Thus, by using symmetry in the coordinate directions we see that there must be some $x = x(n) = x(n, p)$ for which

$$P\{T_n^-(x)\} \geq \frac{1}{d(2n+1)^d} P\{T_n\}. \tag{3.3}$$

LEMMA 1. *For some* $K_1 = K_1(p)$ *which is bounded away from* 0 *on every interval* $[a, b]$ *with* $0 < a \leq b < 1$ *and all* p, $n \geq 1$, $m \geq 1$, *we have that*

$$P\{T_{n+m+2}\} \geq K_1 n^{-d} P\{T_n\} P\{T_m\}. \tag{3.4}$$

PROOF. We will construct a configuration in which T_{n+m+2} occurs, by "combining" a configuration in T_n and a translate of a configuration in $T_m^-(x(m))$ with $x(m)$ as in (3.3).

Assume that for some configuration ω and some y the event $T_n^+(y)$ occurs. For later reference we abbreviate this event by $E(y)$. Note that when $E(y)$ occurs, then $C(\mathbf{0})$ is contained in the box

$$B' := [y_1 - n, y_1] \times [-n, n] \times \cdots \times [-n, n],$$

$y \in C(\mathbf{0})$, the edge $\{y, y + e_1\}$ is vacant, and $C(\mathbf{0})$ has a point in the left face of B'. Assume further that ω is such that the translate by $y - x(m) + 2e_1$ of the event $T_m^-(x(m))$ occurs. We abbreviate this translated event by $F(y)$. Note that on $F(y)$ the point $y + 2e_1 = x(m) + (y - x(m) + 2e_1)$ belongs to the cluster $C(y - x(m) + 2e_1)$, and this cluster lies in the box

$$B'' := [y_1 + 2, y_1 + 2 + m] \times [y_2 - m, y_2 + m] \times \cdots \times [y_d - m, y_d + m].$$

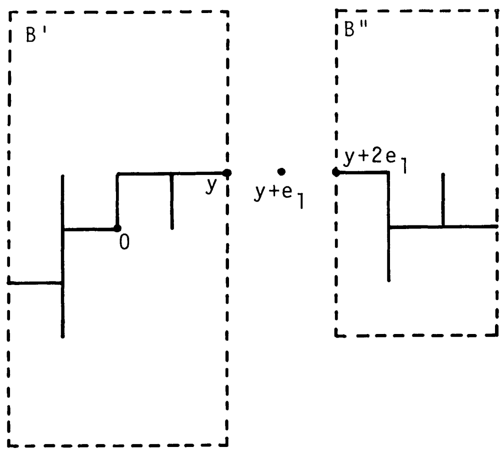

FIG. 2. *Illustration of a configuration* ω *in* $T_4^+(y)$ *and in the translate of* $T_3^-(x(3))$, *with the boxes* B', B''.

The cluster contains also a point in the right face of B'', and the edge $\{y + e_1, y + 2e_1\}$ is vacant (see Figure 2).

We now modify the configuration ω somewhat. We form a new configuration ω^y by making the edges $\{y, y + e_1\}$ and $\{y + e_1, y + 2e_1\}$ occupied, while we make the other $2d - 2$ edges incident to $y + e_1$ vacant. These are the edges

$$(3.5) \qquad \{y + e_1, y + e_1 \pm e_j\}, \qquad 2 \le j \le d.$$

In this new configuration the cluster of the origin, $C(\mathbf{0}, \omega^y)$, is contained in $B' \cup B'' \cup \{\{y, y + e_1\}, \{y + e_1, y + 2e_1\}\}$, and it contains points in the left face of B' and in the right face of B''. One easily checks that this implies $D(\omega^y) = n + m + 2$. Consequently,

$$(3.6) \qquad T_{n+m+2} \supset \bigcup_y \{\omega^y\colon \omega \in E(y) \cap F(y)\}.$$

To deduce an inequality for probabilities from (3.6), we observe that, when n and m are known, y can be uniquely reconstructed from ω^y, and that a given ω^y can arise from at most 2^{2d-2} configurations ω which differ only on the edges incident to $y + e_1$. Indeed, given ω^y, we find a farthest point to the left in $C(\mathbf{0}, \omega^y)$, with first coordinate $L_1(\omega^y)$. Then y is the unique maximal point [in the order of (3.2)] in $C(\mathbf{0}, \omega^y) \cap \{z\colon z_1 = L_1(\omega^y) + n\}$. Once we have found y we also can read off from ω^y what the state of any edge was in ω, except for the edges $\{y + e_1, y + e_1 \pm e_j\}$, $2 \le j \le d$. These are vacant in ω^y, regardless of their original state. For any $\omega \in E(y) \cap F(y)$, denote by $\tilde{\omega}^y$ the configuration obtained by making *all* edges incident to $y + e_1$ vacant. Thus ω^y differs from $\tilde{\omega}^y$ only in that it has the two edges $\{y, y + e_1\}, \{y + e_1, y + 2e_1\}$ occupied instead of vacant. From the above observations it follows that ω^y uniquely determines y

and $\tilde{\omega}^y$. Distinct ω^y come from distinct $\tilde{\omega}^y$. Moreover

$$P_p\{\omega^y \text{ occurs}\} = \left(\frac{p}{1-p}\right)^2 P\{\tilde{\omega}^y \text{ occurs}\}.$$

Combining the last paragraph with (3.6), we obtain

$$\begin{aligned} P\{T_{n+m+2}\} &\geq P\left\{\bigcup_y \{\omega^y \colon \tilde{\omega}^y \in E(y) \cap F(y)\}\right\} \\ &= \sum_y P\{\omega^y \colon \tilde{\omega}^y \in E(y) \cap F(y)\} \\ &= \left(\frac{p}{1-p}\right)^2 \sum_y P\{\tilde{\omega}^y \colon \omega \in E(y) \cap F(y)\}. \end{aligned} \tag{3.7}$$

Finally, to estimate the last probability, we observe that $E(y)$ depends only on the edges in B' or with one endpoint in B' and one outside. Similarly $F(y)$ depends only on the edges in B'' or with one endpoint in B'' and one outside. These two sets of edges are disjoint, hence independent, and both are independent of the set of edges in (3.5). It follows that

$$\begin{aligned} &\sum_y P\{\tilde{\omega}^y \colon \omega \in E(y) \cap F(y)\} \\ &\quad = (1-p)^{2d-2} \sum_y P\{E(y) \cap F(y)\} \\ &\quad = (1-p)^{2d-2} \sum_y P\{E(y)\} P\{F(y)\} \\ &\quad = (1-p)^{2d-2} \sum_y P\{E(y)\} P\{T_m^-(x(m))\} \text{ (by translation invariance)} \\ &\quad \geq (1-p)^{2d-2} \frac{1}{d(2n+1)^d} P\{T_m\} \sum_y P\{E(y)\} \quad \text{[by (3.3)]} \\ &\quad = (1-p)^{2d-2} \frac{1}{d(2n+1)^d} P\{T_m\} P\{T_n\}. \end{aligned}$$

In the last step we used the fact that

$$T_n = \bigcup_y T_n^+(y).$$

□

LEMMA 2.

$$\frac{1}{\xi^f(p)} := \lim_{n\to\infty} -\frac{1}{n} \log P\{T_n\} \tag{3.8}$$

exists, and

$$P\{T_n\} \leq K_0(p) n^d e^{-n/\xi^f(p)} \tag{3.9}$$

for some K_0 *which is uniformly bounded on each interval* $[a, b]$ *with* $0 < a \le b < 1$.

PROOF. The proof follows from (3.4) by standard manipulations with subadditive sequences [see Hammersley (1962)]. We therefore merely indicate some steps. Set

$$g(n) = g(n, p) = 4^{-d} K_1 n^{-d} P\{T_n\}. \tag{3.10}$$

We may assume without loss of generality that $m \ge n$ in (3.4), obtaining for $m \ge 3$,

$$g(n) g(m-2) \le 4^{-2d} K_1 m^{-d} P\{T_{n+m}\} \le g(n+m). \tag{3.11}$$

The existence of the limit in (3.8) now follows after simple modifications of the standard proof. [Note that we have to use $g(n) > 0$ for every $n \ge 1$].

Next (3.9) follows from (3.8) and (3.11):

$$\begin{aligned} \log g(n) &\le \frac{1}{2} \log g(2n+2) \le \cdots \\ &\le \frac{1}{2^k} \log g(2^k n + 2^{k+1} - 2) \\ &\to_{k \to \infty} (n+2) \lim_{m \to \infty} \frac{1}{m} \log g(m). \end{aligned} \tag{3.12}$$

□

THEOREM 3. *If* $\xi^f(p) = \infty$, *then*

$$\lim_{|x| \to \infty} -\frac{1}{|x|} \log \tau^f(p, \mathbf{0}, x) = \lim_{n \to \infty} -\frac{1}{n} \log P_p\{T_n\} = 0. \tag{3.13}$$

If $\xi^f(p) < \infty$, *then*

$$\lim_{n \to \infty} -\frac{1}{n} \log \tau^f(p, \mathbf{0}, ne_1) = \lim_{n \to \infty} -\frac{1}{n} \log P_p\{T_n\} = \frac{1}{\xi^f(p)} \tag{3.14}$$

and

$$\tau^f(p, \mathbf{0}, ne_1) \le K_2 \xi^f(p) \big(n + \xi^f(p)\big)^d e^{-n/\xi^f(p)} \tag{3.15}$$

for some $K_2 = K_2(p)$ *uniformly bounded on any interval* $[a, b] \subset (0, 1)$.

REMARK 1. In the next section (3.14) and (3.15) will be complemented by results on the asymptotic behavior of $\tau^f(p, \mathbf{0}, x)$ as x moves out to ∞ in a fixed direction.

PROOF. We use a variant of the argument in Lemma 1. Let D', D'', M and N be positive integers such that $D' \ge N$, $D'' \ge M$. We replace the events $E(y)$ and $F(y)$ in the proof of Lemma 1 by

$$\begin{aligned} E'(y) := \big\{ &y \in C(\mathbf{0}), C(\mathbf{0}) \subset [0, D'] \times [-N, N]^{d-1} \text{ and} \\ &y \text{ is maximal [in the ordering of (3.2)] in} \\ &C(\mathbf{0}) \cap \{z : z = D'\} \big\} \end{aligned}$$

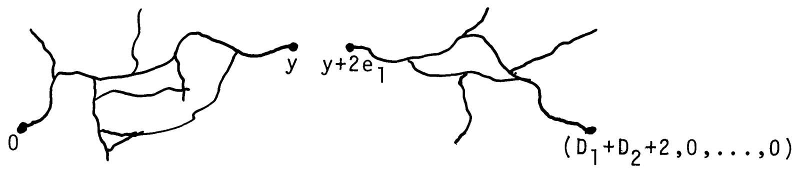

FIG. 3. *Two clusters for a configuration* ω *in* $E'(y) \cap F'(y)$.

and

$$F'(y) := \{C(y + 2e_1) \subset [D' + 2, D' + 2 + D''] \times [-M, M]^{d-1} \text{ and } C(y + 2e_1) \text{ contains the point } (D' + D'' + 2, 0, 0, \dots, 0)\},$$

respectively. For $E'(y)$ to be nonempty we need $y_1 = D'$. On E' we then have $D(\omega) = D_1(\omega) = D'$, and $\mathbf{0}$ (respectively, y) is one of the leftmost (respectively, rightmost) points of $C(\mathbf{0}, \omega)$. Similarly the diameter of $C(y + 2e_1) = D''$ on F', and $y + 2e_1$ [respectively, $(D' + D'' + 2, 0, \dots, 0)$] is one of the leftmost (respectively, rightmost) points of $C(y + 2e_1)$ (see Figure 3). We now construct a configuration ω^y by modifying ω as in Lemma 1. The cluster of the origin, $C(\mathbf{0}, \omega^y)$, in the configuration ω^y stretches from $\mathbf{0}$ to $(D' + D'' + 2, 0, \dots, 0)$, while it is contained in

$$[0, D' + D'' + 2] \times [-(N \vee M), (N \vee M)]^{d-1}$$

[$a \vee b$ stands for $\max(a, b)$]. The same arguments as in Lemma 1 (actually slightly simpler than those) show that for any y with $y_1 = D'$,

$$\begin{aligned} &P\{D = D' + D'' + 2, (D' + D'' + 2, 0, \dots, 0) \in C(\mathbf{0}) \text{ and} \\ &\quad C(\mathbf{0}) \subset [0, D' + D'' + 2] \times [-(N \vee M), (N \vee M)]^{d-1}\} \\ &\quad \geq p^2(1-p)^{2d-4} P\{E'(y)\} P\{F'(y)\}. \end{aligned} \tag{3.16}$$

We first apply (3.16) with the following choices. Fix some n and take $N = M = n$ and take $x(n)$ as in (3.3). Note that on $T_n^-(x)$,

$$C(\mathbf{0}) = C(x) \subset [x_1, x_1 + n] \times [x_2 - n, x_2 + n] \times \cdots \times [x_d - n, x_d + n]$$

and $C(\mathbf{0})$ contains a point in the hyperplane $\{z\colon z_1 = x_1 + n\}$. By shifting this configuration by $-x$ we obtain a configuration in which

$$\begin{aligned} &C(-x) = C(\mathbf{0}) \subset [0, n] \times [-n, n]^{d-1} \\ &\text{and} \\ &C(\mathbf{0}) \text{ contains a point in the hyperplane } \{z\colon z_1 = n\}. \end{aligned} \tag{3.17}$$

Thus, the probability of the event in (3.17) is [by virtue of (3.3)] at least

$$P\{T_n^-(x(n))\} \geq \frac{1}{d(2n+1)^d} P\{T_n\}.$$

On the event in (3.17) there are at most $(2n+1)^{d-1}$ points of $C(\mathbf{0})$ in $\{z\colon z_1 = n\}$. Consequently, for $D' = n$, $N = n$ we can find a $y \in \{D'\} \times [-n, n]^{d-1}$ for which

$$P\{E'(y)\} \geq \frac{1}{d(2n+1)^{2d-1}} P\{T_n\}.$$

If we take also $D'' = n$, $M = n$, then by symmetry

$$P\{F'(y)\} = P\{E'(y)\}.$$

We therefore obtain from (3.16) for a suitable $K_3 = K_3(p)$,

$$\begin{aligned} P\big\{D = 2n+2, (2n+2, 0, \dots, 0) \in C(\mathbf{0}) \text{ and} \\ C(\mathbf{0}) \subset [0, 2n+2] \times [-n, n]^{d-1}\big\} \\ \geq K_3 n^{-4d+2} (P\{T_n\})^2. \end{aligned}$$

Repeating this argument [now with $D' = l(2n+2) + (l-1)2$, $y = (D', 0, \dots, 0)$, $D'' = 2n+2$], we obtain by induction on k,

$$\begin{aligned} P\big\{D = k(2n+2) + (k-1)2, (k(2n+2) + (k-1)2, 0, \dots, 0) \\ \in C(\mathbf{0}) \text{ and } C(\mathbf{0}) \subset [0, k(2n+2) + (k-1)2] \times [-n, n]^d\big\} \\ \geq \big(K_4 n^{-4d+2}\big)^k (P\{T_n\})^{2k}. \end{aligned} \tag{3.18}$$

In particular, for any $n \geq 1$, $k \geq 1$,

$$\tau^f(p, \mathbf{0}, (2kn + 4k - 2)e_1) \geq \big(K_4 n^{-4d+2}\big)^k (P\{T_n\})^{2k} \tag{3.19}$$

for some $K_4 > 0$ depending on p only (as long as $p \neq 0$ or 1). It is not hard to see from (3.19) that then also

$$\tau^f(p, \mathbf{0}, je_1) \geq K_5 \big(K_4 n^{-4d+2}\big)^k (P\{T_n\})^{2k} \quad \text{uniformly in } 2k(n+2) \leq j \leq (2k+2)(n+2), \tag{3.20}$$

where now K_5 depends on p and n, but not on k. One merely has to modify some edges incident to one of the vertices $(l, 0, \dots, 0)$ with $2kn + 4k - 2 \leq l \leq j + 1 \leq (2k+2)(n+2) + 1$.

From (3.20) we obtain (by first letting $k \to \infty$ and then $n \to \infty$) that

$$\begin{aligned} \limsup_{j \to \infty} -\frac{1}{j} \log \tau^f(p, \mathbf{0}, je_1) &\leq \lim_{n \to \infty} -\frac{1}{n} \log P\{T_n\} \\ &= \frac{1}{\xi^f(p)}. \end{aligned} \tag{3.21}$$

Since

$$\{\mathbf{0} \leftrightarrow ne_1, C(\mathbf{0}) \text{ finite}\} \subset \bigcup_{r \geq n} T_r$$

we also have, from (3.9)

$$\tau^f(p,\mathbf{0},ne_1) \le \sum_{r=n}^{\infty} P\{T_r\} \le K_0(p) \sum_{r=n}^{\infty} r^d e^{-r/\xi^f(p)}. \tag{3.22}$$

(3.21) and (3.22) imply (3.14) and (3.15) when $\xi^f(p) < \infty$. [Note that one can obtain (3.21), and hence (3.14), more directly from (3.18), as in Grimmett (1989), Theorem 6.33; (3.20) is useful for (3.23), though.]

If $\xi^f(p) = \infty$ we merely obtain

$$\lim_{n\to\infty} -\frac{1}{n}\tau^f(p,\mathbf{0},ne_i) = \lim_{n\to\infty} -\frac{1}{n}\tau^f(p,\mathbf{0},ne_1) = 0.$$

To obtain the full (3.13) in this case, we need once again a variant of the argument of Lemma 1. We merely give an indication by means of a figure, leaving the details to the reader. Let $\varepsilon > 0$ and $x = (x_1,\dots,x_d)$. We now wish to connect $\mathbf{0}$ to x by successively combining paths from $z^{(j)} := (x_1,\dots,x_j,0,\dots,0)$ to $z^{(j+1)} := (x_1,\dots,x_{j+1},0,\dots,0)$, $0 \le j \le d-1$. Choose n such that

$$\begin{aligned} P\big\{&D = k(2n+2)+(k-1)2, (k(2n+2)+(k-1)2,0,\dots,0) \\ &\in C(\mathbf{0}) \text{ and } C(\mathbf{0}) \subset [0, k(2n+2)+(k-1)2] \times [-n,n]^{d-1}\big\} \\ &\ge \exp(-\varepsilon kn), \qquad k \ge 1 \end{aligned}$$

[recall that we assume $\xi^f(p) = \infty$ now; cf. (3.18)]. As in (3.20) we have also that

$$\begin{aligned} P\big\{& je_1 \in C(\mathbf{0}) \text{ and } C(\mathbf{0}) \subset [0,j] \times [-n,n]^{d-1}\big\} \\ &\ge K_6 \exp(-\varepsilon j), \qquad j \ge 4n, \end{aligned} \tag{3.23}$$

for some $K_6 = K_6(p,n)$. For simplicity we take $d=2$, $x_1 \ge 4n$, $x_2 \ge 5n+2$. We now consider configurations for which

$$(x_1,0) \in C(\mathbf{0}) \quad \text{and} \quad C(\mathbf{0}) \subset [0,x_1] \times [-n,n] \tag{3.24}$$

as well as

$$\begin{aligned} &(x_1,x_2) \in C((x_1,n+2)) \quad \text{and} \\ &C((x_1,n+2)) \subset [x_1-n, x_1+n] \times [n+2, x_2] \end{aligned} \tag{3.25}$$

(see Figure 4). By modifications of such a configuration in the "small" box $[x_1-1, x_1+n] \times [0, n+2]$ we can now construct a new configuration $\tilde{\omega}$ in which $(x_1,0)$ and $(x_1,n+2)$ are connected, so that

$$(x_1,x_2) \in C(\mathbf{0},\tilde{\omega})$$

and

$$C(\mathbf{0},\tilde{\omega}) \subset [0,x_1+n] \times [-n,n] \cup [x_1-n, x_1+n] \times [-n, x_2].$$

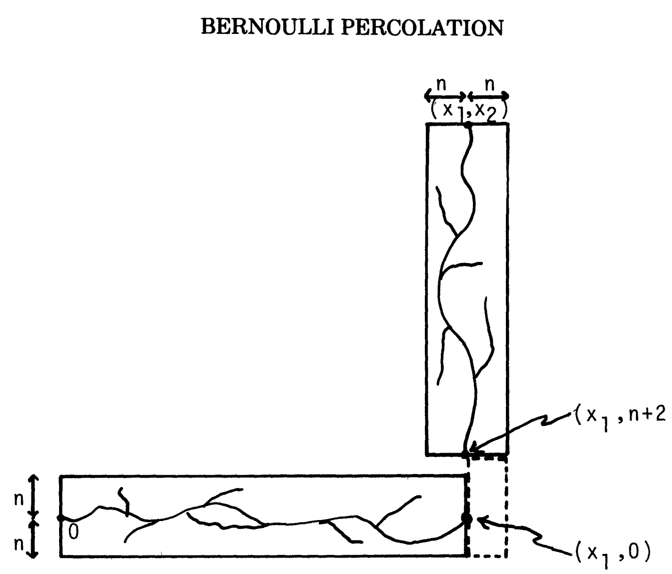

FIG. 4. *The clusters $C(\mathbf{0})$ and $C(x_1, n+2)$ are contained in the long horizontal and vertical rectangles, respectively. They can be connected by modifications in the dashed rectangle in the lower right corner.*

From this and (3.23) we obtain that

$$\tau^f(p, \mathbf{0}, x) \geq K_7(p, n) P\{\text{event in (3.24)}\} P\{\text{event in (3.25)}\}$$
$$\geq K_7 K_6^2 \exp(-\varepsilon|x_1| - \varepsilon|x_2|).$$

Since $\varepsilon > 0$ is arbitrary this leads to (3.13). □

We next list some of the properties of $\xi^f(p)$ as a function of p. Note that, for $p < p_c$, $\xi^f(p) = \xi(p)$ [as defined in (1.2)] since $C(\mathbf{0})$ is finite w.p.1 below the precolation threshold.

PROPOSITION 4. (*i*) *$\xi^f(p) < \infty$ for $p < p_c$ or when there is percolation in a slab $S_k = Z^{d-1} \times \{0, 1, \ldots, k\}$ for some finite k.*
(ii) *$[\xi^f(\cdot)]^{-1}$ is upper semicontinuous on $(0,1)$, and continuous on $(0, p_c)$.*
(iii) *$\xi^f(p) \to 0$ when $p \downarrow 0$ or $p \uparrow 1$.*

REMARK 2. In a forthcoming article one of us (H.K.) will show that there actually is equivalence in (i). That is, for $p > p_c$, $\xi^f(p) < \infty$ if and only if there is percolation in some slab $S_k = Z^{d-1} \times \{0, 1, \ldots, k\}$. This shows that $\xi^f(p) < \infty$ except on the interval from p_c to $\lim_{k \to \infty} p_c(S_k)$ (endpoints possibly included). It is believed that $p_c = \lim_{k \to \infty} p_c(S_k)$ when $d > 2$ and that $\xi^f(p) = \infty$ only at $p = p_c$. (For $d = 2$ this is known to be the case.)

PROOF. The exponential decay of $\tau(p, \mathbf{0}, x)$, or equivalently the finiteness of $\xi(p)$, for $p < p_c$ was first proved in Hammersley (1957) [cf. also Aizenman and

Newman (1984), Proposition 5.1, and Kesten (1982), Theorem 5.1]; actually these references prove this for $p < p_T$ [where p_T is another critical probability, cf. Kesten (1982), page 52], but it is now known that $p_T = p_c$ [Menshikov (1986), Menshikov, Molchanov and Sidorenko (1986) and Aizenman and Barsky (1987)]. The fact that percolation in $Z^{d-1} \times \{0, 1, \dots, k\}$ implies $\xi^f(p) < \infty$ is proven in Chayes, Chayes and Newman (1987), Theorem 1; see also Grimmett (1989), Theorem 6.51. This completes (i).

We next point out that (iii) follows from simple Peierls arguments. For $p \downarrow 0$ one uses

$$\tau(p, \mathbf{0}, ne_1) \le \sum_{k=n}^{\infty} (\text{number of } n\text{-step self-avoiding walks from the origin}) p^n.$$

For $p \uparrow 1$ one uses

$$\tau^f(p, \mathbf{0}, ne_1) \le \sum_{k=n}^{\infty} \sum_{l=k}^{\infty} \left(\begin{array}{l}\text{number of edge sets which separate } \mathbf{0} \\ \text{and } ne_1 \text{ from } \infty \text{ but not from each} \\ \text{other, and which contain an edge} \\ \text{incident to } ke_1, \text{ and contain } l \text{ edges in} \\ \text{total}\end{array}\right)(1-p)^l.$$

The latter estimate is already used for $d = 2$ in Hammersley (1959), and for general d in Kunz and Souillard (1978).

Finally, the upper semicontinuity of $[\xi^f]^{-1}$ follows from the subadditivity property used in the proof of its existence. More specifically, the functions $g(n, \cdot)$ of (3.10) are continuous, since they depend only on the edges in the finite box B_n. Therefore [cf. Choquet (1966), Theorem 2.8.6]

$$h(p) := \inf\left\{ -\frac{1}{n+2} \log g(n, p) \colon n \ge 1 \right\}$$

is upper semicontinuous. But, by (3.12) and (3.8)

$$\begin{aligned} \frac{1}{\xi^f(p)} &\le \inf\left\{ -\frac{1}{n+2} \log g(h, p) \colon n \ge 1 \right\} \\ &\le \lim_{n \to \infty} -\frac{1}{n+2} \log g(n, p) = \frac{1}{\xi^f(p)}, \end{aligned}$$

so that $h(p) = [\xi^f(p)]^{-1}$.

For $p < p_c$ the continuity of $\xi(p) = \xi^f(p)$ is immediate from (1.3) and the continuity of $p \to \tau(p, \mathbf{0}, ne_1)$ for fixed n [Aizenman, Kesten and Newman (1987)]. If one wants to avoid using the (not entirely trivial) continuity of $\tau(\cdot, \mathbf{0}, ne_1)$ one can use the fact that also

$$\frac{1}{\xi(p)} = \lim_{n \to \infty} P_p\{\mathbf{0} \leftrightarrow \text{boundary of } B_n\},$$

together with upper and lower bounds for $P_p\{\mathbf{0} \leftrightarrow \text{boundary of } B_n\}$ analogous to (1.3) [cf. Grimmett (1989), Theorems 5.10 and 5.14]. □

4. Behavior of $\tau^f(p, 0, x)$ along general rays. In this section we consider the behavior of $\tau^f(p, \mathbf{0}, x)$ for general x, when $\xi^f(p) < \infty$. We will prove the following theorem.

THEOREM 5. *If $\xi^f(p) < \infty$, then there exists a convex function $\mu_p\colon \mathbb{R}^d \to [0, \infty)$ such that*

$$\lim_{n\to\infty} -\frac{1}{n}\log \tau^f(p, \mathbf{0}, nx) = \mu_p(x). \tag{4.1}$$

If $n \ge 1/|x|$, then

$$\tau^f(p, \mathbf{0}, nx) \le K_1(\xi^f(p) + 1)^{4d} n^d |x|^d e^{-n\mu_p(x)}, \tag{4.2}$$

where K_1 is a constant. Moreover, if $x \ne 0$, then

$$1/\xi^f(p) \le \mu_p(x)/|x| \le d/\xi^f(p). \tag{4.3}$$

REMARK 3. It is clear that for every $0 < p < 1$,

$$\mu_p(\mathbf{0}) = \lim_{n\to\infty} -\frac{1}{n}\log P\{C(\mathbf{0}) < \infty\} = 0.$$

REMARK 4. If $\xi^f = \infty$ the statements above are also true, by Theorem 3, with $\mu_p(x) = 0$ for every x. So, for $x \ne 0$, $\mu_p(x)$ is either always positive or always 0.

REMARK 5. (4.3) tells us that the behavior of $\mu_p(x)$ near p_c is the same for all $x \ne 0$. In particular, if their reciprocals diverge as $p \downarrow p_c$, they diverge with the same critical exponent.

REMARK 6. The proof of (4.1) given below can also be used to show that if $x_n \to x$, then

$$\lim_{n\to\infty} -\frac{1}{n}\log \tau^f(p, \mathbf{0}, nx_n) = \mu_p(x).$$

In order to prove Theorem 5, we will use an approach based on constructing two independent versions of the percolation model and then a third version which is coupled in a convenient fashion to the two previous ones. We will refer to this approach as the "duplication trick." Comparing it to the methods used in the previous sections, we can say that it relies less on symmetries and for this reason works when x is not on a coordinate axis. On the other hand the duplication trick, as used below, only works when we have a priori exponential decay of τ^f, that is, when we know that $\xi^f(p) < \infty$. Thus we need the combination of both methods to prove everything with the generality that we want to consider, in this article.

Next we introduce the three coupled versions of the percolation model. Let $\alpha = (\alpha_b,\ b \in \mathscr{B}_d)$ and $\beta = (\beta_b,\ b \in \mathscr{B}_d)$ be two independent product random fields of density p, each indexed by the set $\mathscr{B}_d$ of bonds of Z^d. As usual these random fields indicate which bonds are occupied. We will denote by $C_\alpha(\mathbf{0})$ [respectively, $C_\beta(\mathbf{0})$] the cluster of the origin in the configuration α (respectively, β). More generally, subscripts α and β (and also γ later on) will indicate objects corresponding to each one of these configurations. We shall use below also the notation

$$\overline{C}(\mathbf{0}) = \{b \in \mathscr{B}_d\text{: an endpoint of } b \text{ is connected to the origin by occupied bonds}\}.$$

Given α and β, we construct the third random field $\gamma = (\gamma_b,\ b \in \mathscr{B}_d)$ according to the following rules:

(a) If $|C_\alpha(\mathbf{0})| = \infty$, then $\gamma_b = \alpha_b$ for every $b \in \mathscr{B}_d$.
(b) If $|C_\alpha(\mathbf{0})| < \infty$, then $\gamma_b = \alpha_b$ for $b \in \overline{C}_\alpha(0)$ and $\gamma_b = \beta_b$ otherwise.

We claim that γ is also a density p product random field. The key idea [which appeared, e.g., in Aizenman and Newman (1984)] is that $\overline{C}(\mathbf{0})$ is "self-determined," that is, for every $A \subset \mathscr{B}_d$, the event $\{\overline{C}(\mathbf{0}) = A\}$ depends only on the states of occupancy of the bonds in A. [The event $\overline{C}(\mathbf{0}) = A$ can be viewed as a multidimensional generalization of the event that a Markov time takes a specified value.] With this in mind it should be easy for the reader to write down a formal proof of our claim and we omit it.

Proof of Theorem 5. We will use the construction above to show that given $0 < a < b < 1$ there exists $K_2 < \infty$ such that if $|x| \ge 1$ and $p \in [a, b]$,

$$\tau^f(p,\mathbf{0},x)\tau^f(p,x,y) \le K_2\big(\xi^f(p) + 1\big)^{2d}|x|^d\tau^f(p,\mathbf{0},y). \tag{4.4}$$

Observe that this inequality is trivial and useless if $\xi^f(p) = \infty$. For convenience of notation, we will prove (4.4) when $x, y \in \mathbb{Z}^d$, but the same proof works for general $x, y \in \mathbb{R}^d$. Recall the definition $B_n = \{x\colon |x_i| \le n,\ i = 1,\dots,d\}$. Let $K > 1$ be a constant. A look at Figure 5 should convince the reader that

$$\begin{aligned} &P\{\{x \in C(\mathbf{0}),\ C(\mathbf{0}) \subseteq B_{K|x|}\}_\alpha \cap \{y \in C(x),\ |C(x)| < \infty\}_\beta\} \\ &\quad\le P\{\{y \in C(\mathbf{0}),\ |C(\mathbf{0})| < \infty\}_\gamma \cup \{y \notin C(\mathbf{0}),\text{ but there is a vacant bond which has one endpoint in } B_{K|x|} \text{ such that if this bond were occupied, then the event } \{y \in C(\mathbf{0}),\ |C(\mathbf{0})| < \infty\} \text{ would occur}\}_\gamma\}. \end{aligned} \tag{4.5}$$

The subscripts α, β, γ indicate the configuration for which each event is considered. It is not hard to give a formal justification for the inclusion indicated in Figure 5. In fact, if the left-hand side occurs, then either $y \in C_\alpha(\mathbf{0})$, which implies $\{y \in C(\mathbf{0}),\ |C(\mathbf{0})| < \infty\}_\gamma$, or else (since x and y are connected in β) y must be connected in γ to $\overline{C}(\mathbf{0})$. In this last case there must (in γ) be two neighboring sites which are joined by a vacant bond $b \in \overline{C}(\mathbf{0})$, one of which is connected to $\mathbf{0}$ and the other to y. Since $C_\alpha(\mathbf{0}) \subset B_{K|x|}$, the site connected to $\mathbf{0}$ must belong to $B_{K|x|}$. Moreover, since $|C_\alpha(\mathbf{0})| < \infty$ and $|C_\beta(y)| < \infty$, making b occupied will preserve $|C_\gamma(\mathbf{0})| < \infty$.

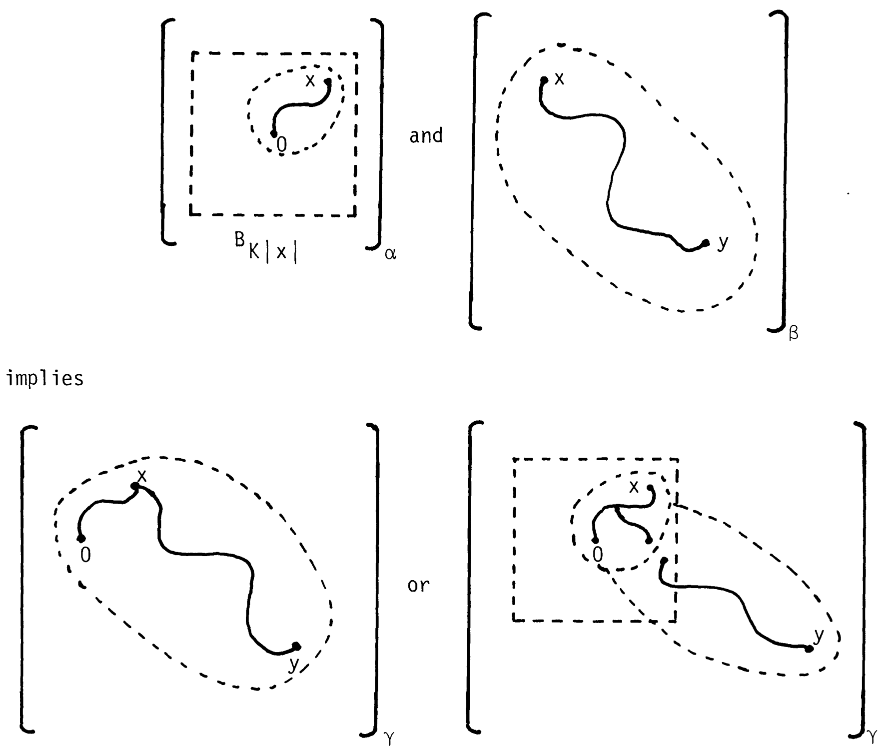

FIG. 5. *The solid curves are occupied paths. The dashed curves surrounding the occupied paths are vacant sets separating the occupied paths from* ∞.

Let $\tilde{B}_N$ be the set of bonds which have at least one endpoint in B_N. Then, by the independence of α and β and (4.5),

$$P\{x \in C(\mathbf{0}),\, C(\mathbf{0}) \subset B_{K|x|}\} P\{y \in C(x),\, |C(x)| < \infty\}$$

$$\leq P\{y \in C(\mathbf{0}),\, |C(\mathbf{0})| < \infty\}$$

$$+ \sum_{b \in \tilde{B}_{K|x|}} P\{b \text{ is vacant and } y \notin C(\mathbf{0}), \text{ but if } b \text{ were occupied, then } \{y \in C(\mathbf{0}),\, |C(\mathbf{0})| < \infty\} \text{ would occur}\}$$

$$\leq P\{y \in C(\mathbf{0}),\, |C(\mathbf{0})| < \infty\} + \sum_{b \in \tilde{B}_{K|x|}} \frac{1-p}{p} P\{b \text{ is occupied, } y \in C(\mathbf{0}),\, |C(\mathbf{0})| < \infty\} \tag{4.6}$$

$$\leq P\{y \in C(\mathbf{0}),\, |C(\mathbf{0})| < \infty\} + \frac{1-p}{p} \sum_{b \in \tilde{B}_{K|x|}} P\{y \in C(\mathbf{0}),\, |C(\mathbf{0})| < \infty\}$$

$$\leq K_3 K^d |x|^d P\{y \in C(\mathbf{0}),\, |C(\mathbf{0})| < \infty\}.$$

Here K_3 depends on p, but can be taken constant for $p \geq a$.

On the other hand

$$P\{x \in C(\mathbf{0}),\, C(\mathbf{0}) \subset B_{K|x|}\} \geq P\{x \in C(\mathbf{0}),\, |C(\mathbf{0})| < \infty\} \times \left(1 - \frac{P\{C(\mathbf{0}) \not\subset B_{K|x|},\, |C(\mathbf{0})| < \infty\}}{P\{x \in C(\mathbf{0}),\, |C(\mathbf{0})| < \infty\}}\right).$$

But it is easy to see that there exists a $c < \infty$ which can be chosen uniformly for $p \in [a, b]$ such that $P\{x \in C(\mathbf{0}),\ |C(\mathbf{0})| < \infty\} \geq e^{-c|x|}$. And if $\xi^f(p) < \infty$, by summing (3.9) over $n \geq K|x|$,

$$P\{C(\mathbf{0}) \not\subset B_{K|x|},\, |C(\mathbf{0})| < \infty\} \leq K_0(p)\xi^f(p)\left[K|x| + \xi^f(p)\right]^d \times \exp\left(-K|x|/\xi^f(p)\right).$$

Thus it is easy to check that there exists a constant $K_4 > 0$ such that if $K = K_4(\xi^f(p) + 1)^2$ and $|x| \geq 1$, then

$$P\{C(\mathbf{0}) \not\subset B_{K|x|},\, |C(\mathbf{0})| < \infty\} \leq \tfrac{1}{2}P\{x \in C(\mathbf{0}),\, |C(\mathbf{0})| < \infty\}.$$

Hence

$$P\{x \in C(\mathbf{0}),\, C(\mathbf{0}) \subset B_{K|x|}\} \geq \tfrac{1}{2}P\{x \in C(\mathbf{0}),\, |C(\mathbf{0})| < \infty\}. \tag{4.7}$$

Combining (4.6) and (4.7), we obtain (4.4).

Now we will use (4.4) to prove (4.1) and (4.2). For this purpose substitute mx for x and $(m + n)x$ for y. If $m \geq 1/|x|$, then

$$\tau^f(p, \mathbf{0}, mx)\tau^f(p, mx, (m+n)x) \leq K_2\left(\xi^f(p) + 1\right)^{2d} m^d|x|^d\tau^f(p, \mathbf{0}, (m+n)x).$$

We would like to say, using translation invariance, that $\tau^f(p, mx, (m + n)x)$ is equal to $\tau^f(p, \mathbf{0}, nx)$. Unfortunately, since mx and $(m + n)x$ may not belong to Z^d, this may be false. Nevertheless, the distance between $[nx]$ and $[(m + n)x] - [mx]$ is certainly not greater than 2 and so using (4.4) again and translation invariance, we obtain

$$\tau^f(p, mx, (m + n)x) \geq \left(K_2/(\xi^f(p) + 1)^{2d}\right)\tau^f(p, \mathbf{0}, nx).$$

And if $m \geq 1/|x|$,

$$\tau^f(p, \mathbf{0}, mx)\tau^f(p, \mathbf{0}, nx) \leq K_5\left(\xi^f(p) + 1\right)^{4d} m^d|x|^d\tau^f(p, \mathbf{0}, (m + n)x). \tag{4.8}$$

(4.1) and (4.2) follow from this inequality in the same way as in the proof of Lemma 2.

$\mu_p(x)$ depends of course on the direction and on the size of x. The size of x influences μ_p in a trivial way since μ_p clearly has the homogeneity property

$$\mu_p(ax) = a\mu_p(x).$$

The convexity of $\mu_p(\cdot)$ follows easily from (4.4). Indeed, given $x, y \in \mathbb{R}^d$ and $n \geq 1/|x|$, we have, analogously to (4.8),

$$\tau^f(p,\mathbf{0},nx)\tau^f(p,\mathbf{0},ny)$$
$$\leq K_5(\xi^f(p)+1)^{4d} n^d |x|^d \tau(p,\mathbf{0},n(x+y)).$$

Taking logarithms, dividing by n and letting $n \to \infty$ yields

$$\mu_p(x+y) \leq \mu_p(x) + \mu_p(y). \tag{4.9}$$

In particular, if $0 \leq a \leq 1$,

$$\mu_p(ax+(1-a)y) \leq \mu_p(ax) + \mu_p((1-a)y) = a\mu_p(x) + (1-a)\mu_p(y).$$

To prove the lower bound in (4.3), observe that by (4.9) and symmetry, if $x = (x_1, \dots, x_d)$, then

$$2|x_1|/\xi^f(p) = 2|x_1|\mu_p(e_1) = \mu_p(2|x_1|e_1)$$
$$\leq \mu_p((|x_1|, x_2, \dots, x_d)) + \mu_p((|x_1|, -x_2, \dots, -x_d))$$
$$= 2\mu_p(x).$$

The upper bound in (4.3) follows easily from (4.9), using induction on the number of coordinates of x which are different from 0. □

5. Equivalence with another definition of correlation length. We turn now to the correlation length defined in terms of the quantity

$$\tilde{\tau}(p,\mathbf{0},x) = P(x \in C(\mathbf{0})) - P_\infty^2(p).$$

We will prove that the correlation length (in every direction) defined by using $\tilde{\tau}$ is identical to the one obtained using τ^f.

THEOREM 6. *For every* $x \in \mathbb{R}^d$,

$$\lim_{n\to\infty} -\frac{1}{n}\log \tilde{\tau}(p,\mathbf{0},nx) = \mu_p(x). \tag{5.1}$$

In particular,

$$\lim_{n\to\infty} -\frac{1}{n}\log \tilde{\tau}(p,\mathbf{0},ne_1) = 1/\xi^f(p).$$

Proof. We start by relating τ^f and $\tilde{\tau}$ to another quantity of interest. Define

$$\mathscr{C}(p,x,y) = P\{|C(x)| = \infty, |C(y)| = \infty\} - P_\infty^2(p).$$

$\mathscr{C}(p,x,y)$ is the covariance of the indicator functions of the events that x belongs to the infinite cluster and that y belongs to the infinite cluster. Now, using uniqueness of the infinite cluster [Aizenman, Kesten and Newman (1987) or Gandolfi, Grimmett and Russo (1988)],

$$\begin{aligned}\tilde{\tau}(p,x,y) &= P\{y \in C(x), |C(x)| < \infty\} \\ &\quad + P\{y \in C(x), |C(x)| = \infty\} - P_\infty^2(p) \\ &= \tau^f(p,x,y) + P\{|C(x)| = |C(y)| = \infty\} - P_\infty^2(p) \\ &= \tau^f(p,x,y) + \mathscr{C}(p,x,y).\end{aligned} \tag{5.2}$$

By the Harris–FKG inequality, $\mathscr{C}(p,x,y) \ge 0$. Hence

$$\tilde{\tau}(p,x,y) \ge \tau^f(p,x,y). \tag{5.3}$$

Therefore

$$\limsup_{n\to\infty} -\frac{1}{n}\log\tilde{\tau}(p,\mathbf{0},nx) \le \mu_p(x). \tag{5.4}$$

So, in case $\xi^f(p) = \infty$, (5.1) follows immediately from (3.13), with $\mu_p(x) = 0$.
If $\xi^f(p) < \infty$, we will show that

$$\liminf_{n\to\infty} -\frac{1}{n}\log\mathscr{C}(p,\mathbf{0},nx) \ge \mu_p(x). \tag{5.5}$$

Combining (5.5) with (5.2) and (5.4) yields (5.1). To prove (5.5), we observe that

$$\begin{aligned}\mathscr{C}(p,\mathbf{0},nx) &= P\{|C(\mathbf{0})| < \infty, |C(nx)| < \infty\} - (1 - P_\infty(p))^2 \\ &= P\{|C(\mathbf{0})| < \infty, |C(nx)| < \infty, \bar{C}(\mathbf{0}) \cap \bar{C}(nx) = \phi\} \\ &\quad + P\{|C(\mathbf{0})| < \infty, |C(nx)| < \infty, nx \in C(\mathbf{0})\} \\ &\quad + P\{|C(\mathbf{0})| < \infty, |C(nx)| < \infty, nx \notin C(\mathbf{0}), \bar{C}(\mathbf{0}) \cap \bar{C}(nx) \ne \phi\} \\ &\quad - (1 - P_\infty(p))^2.\end{aligned}$$

Now, by the van den Berg–Kesten [van den Berg and Kesten (1985)] inequality, the first term in the right-hand side is smaller than the last one. Also if the event in the third term happens, then there is a bond b which is vacant and has one of its endpoints connected to $\mathbf{0}$ and the other to nx. So for every K,

$$\begin{aligned}\mathscr{C}(p,\mathbf{0},nx) &\le \tau^f(p,\mathbf{0},nx) + P\{|C(\mathbf{0})| < \infty, C(\mathbf{0}) \cap (B_{Kn})^c \ne \phi\} \\ &\quad + \sum_{b \in \tilde{B}_{Kn}} P\{b \text{ is vacant, but if } b \text{ were occupied, then the event } \{nx \in C(\mathbf{0}), |C(\mathbf{0})| < \infty\} \text{ would occur}\} \\ &\le \tau^f(p,\mathbf{0},nx) + P\{|C(\mathbf{0})| < \infty, C(\mathbf{0}) \cap (B_{Kn})^c \ne \varnothing\} \\ &\quad + (1-p)p^{-1}|\tilde{B}_{Kn}|P\{nx \in C(\mathbf{0}), |C(\mathbf{0})| < \infty\}.\end{aligned}$$

Now we use Lemma 2 to bound the second term and observe that the probability which appears in the last term is again $\tau^f(p, \mathbf{0}, nx)$. We have that there exists a constant $K_1 = K_1(p) < \infty$ such that, for large n,

$$(5.6) \qquad \mathscr{C}(p, \mathbf{0}, nx) \le K_1 K^d n^d \tau^f(p, \mathbf{0}, nx) + \exp\big(-Kn/(2\xi^f(p))\big).$$

Since K is arbitrary, (5.6) together with (4.1) implies (5.5). □

6. The covariance. (5.5) motivates us to ask whether $\mathscr{C}(p, \mathbf{0}, nx)$ decays faster than $\tau^f(p, \mathbf{0}, nx)$ and $\tilde{\tau}(p, \mathbf{0}, nx)$. We will prove that in fact it decays to first order with the same rate, when $P_\infty(p) > 0$.

THEOREM 7. *For every* $x, y \in \mathbb{R}^d$,

$$(6.1) \qquad \mathscr{C}(p, x, y) \ge P_\infty(p)\tau^f(p, x, y).$$

If $P_\infty(p) > 0$, *then*

$$(6.2) \qquad \lim_n -\frac{1}{n}\log \mathscr{C}(p, \mathbf{0}, nx) = \mu_p(x).$$

It is clear that (6.2) follows from (5.5) and (6.1). It is interesting to observe that when $P_\infty(p) = 0$, then $\mathscr{C}(p, \mathbf{0}, nx) = 0$ and the limit in (6.2) is in fact $+\infty$. Hence (6.2) holds for $p > p_c$, but we expect it to fail even at p_c, where one believes that $P_\infty(p) = 0$ but $\xi^f(p) = \infty$. (In two dimensions these facts have indeed been proven.)

PROOF OF (6.1). Let $\mathscr{F}$ be the family of finite subsets of Z^d. For $S \in \mathscr{F}$, let $E(y, S)$ be the event that y is connected to infinitely many sites using only bonds which do not touch S. Now

$$\begin{aligned}
\mathscr{C}(p, x, y) &= P\{|C(x)| < \infty\}P\{|C(y)| = \infty\} - P\{|C(x)| < \infty, |C(y)| = \infty\}\\
&= \sum_{S \in \mathscr{F}} \big(P\{C(x) = S\}P\{|C(y)| = \infty\}\\
&\qquad - P\{C(x) = S, |C(y)| = \infty\}\big)\\
&= \sum_{S \in \mathscr{F}} P\{C(x) = S\}\big(P\{|C(y)| = \infty\} - P\{E(y, S)\}\big)\\
&\ge \sum_{S \in \mathscr{F}:\, y \in S} P\{C(x) = S\}\big(P\{|C(y)| = \infty\} - P\{E(y, S)\}\big)\\
&= \sum_{S \in \mathscr{F}:\, y \in S} P\{C(x) = S\}P_\infty(p) = P_\infty(p)\tau^f(p, x, y).
\end{aligned}$$

In the next to last step we used the fact that if $y \in S$, then $E(y, S)$ cannot happen. □

7. Two-dimensional duality relation. Here we consider bond percolation on Z^2, and prove that for $p > p_c = \frac{1}{2}$, it is the case that $\xi^f(p) = \frac{1}{2}\xi(1 - p)$. The explicitness of this result stems in large part from the self-duality of the model.

However, even for a larger class of two-dimensional percolation models, a proof along the lines of this section would equate ξ^f with one-half the correlation length of the *dual* model.

For bond percolation on Z^2, the dual problem deals with bond percolation on $Z^2 + (\frac{1}{2}, \frac{1}{2})$. Each bond of Z^2 intersects exactly one bond of $Z^2 + (\frac{1}{2}, \frac{1}{2})$ and vice versa. We declare a dual bond of $Z^2 + (\frac{1}{2}, \frac{1}{2})$ to be occupied (vacant) if the unique bond of Z^2 which it intersects is vacant (occupied). Thus, when the probability of a bond being occupied in the original problem on Z^2 equals p, then the corresponding probability for the dual problem is $1 - p$. It is well known [cf. Hammersley (1959) and Kesten (1982), Corollary 2.2] that

$$\{\mathbf{0} \leftrightarrow ne_1 \text{ but } C(\mathbf{0}) \text{ finite}\} \tag{7.1}$$

occurs if and only if

$$\{\mathbf{0} \leftrightarrow ne_1 \text{ and there exists an occupied } \textit{dual} \text{ circuit surrounding } \mathbf{0} \text{ and } ne_1\} \tag{7.2}$$

occurs. The intuitive reason for $\xi^f(p) = \frac{1}{2}\xi(1-p)$ when $p > p_c$ is now clear. For the event in (7.2) to occur, the dual occupied circuit must contain at least two disjoint paths of diameter at least n. Since $1 - p < p_c$, the dual process is in the nonpercolating phase, and it turns out that the probability of two such paths existing will in first order be $\exp(-2n/\xi(1-p))$. As we shall see below the extra cost of the connection between $\mathbf{0}$ and ne_1 is insignificant.

THEOREM 8. *For bond percolation on* Z^2,

$$\xi^f(p) = \tfrac{1}{2}\xi(1-p) \quad \textit{when } p > p_c. \tag{7.3}$$

PROOF. First we show that

$$\tau^f(p, \mathbf{0}, ne_1) \le \sum_{r=n+1}^{\infty} r\{\tau(1-p, \mathbf{0}, re_1)\}^2. \tag{7.4}$$

By virtue of (1.3) and the finiteness of $\xi(1-p)$ [cf. Proposition 4(i)] the right-hand side of (7.4) is at most

$$(n+2)\left[1 - \exp\left(-\frac{2}{\xi(1-p)}\right)\right]^{-2} \exp\left(-\frac{2n}{\xi(1-p)}\right).$$

Thus, by (3.14), (7.4) will imply

$$\xi^f(p) \le \tfrac{1}{2}\xi(1-p). \tag{7.5}$$

Now (7.4) follows quickly from the intuitive argument given above, and the inequality (2.5). A dual circuit surrounding 0 and ne_1 must intersect the first coordinate axis in (at least) two points $(-k - \frac{1}{2}, 0)$ and $(n + l + \frac{1}{2}, 0)$, $k, l \ge 0$. By applying (2.5) to the dual process we see that the probability of (7.2) is at

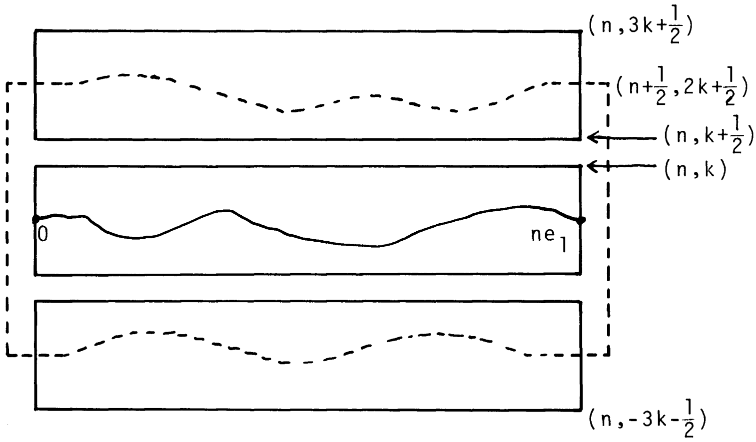

FIG. 6. *The occupied path in the middle rectangle (drawn solidly) guarantees the occurrence of A_4. The dashed paths are occupied dual paths which guarantee A_1–A_3.*

most

$$\sum_{k,l\geq 0} P_p\{\exists \text{ two edge disjoint dual occupied paths from } \left(-k-\tfrac{1}{2},0\right) \text{ to } \left(n+l+\tfrac{1}{2},0\right)\}$$
$$\leq \sum_{k,l\geq 0} \{\tau(1-p,\mathbf{0},(n+l+k+1)e_1)\}^2$$
$$\leq \sum_{r=n+1}^{\infty} r\{\tau(1-p,\mathbf{0},re_1)\}^2.$$

This proves (7.4) and therefore (7.5) also. To find a lower bound for the probability of (7.2), we note that (7.2) occurs when, for some fixed k, the following four events A_1–A_4 occur (see Figure 6):

$$A_1 = \left\{\exists \text{ occupied dual connection in } \left[-\tfrac{1}{2}, n+\tfrac{1}{2}\right] \times \left[k+\tfrac{1}{2}, 3k+\tfrac{1}{2}\right] \text{ from } \left(-\tfrac{1}{2}, 2k+\tfrac{1}{2}\right) \text{ to } \left(n+\tfrac{1}{2}, 2k+\tfrac{1}{2}\right)\right\},$$

$$A_2 = \left\{\exists \text{ occupied dual connection in } \left[-\tfrac{1}{2}, n+\tfrac{1}{2}\right] \times \left[-3k-\tfrac{1}{2}, -k-\tfrac{1}{2}\right] \text{ from } \left(-\tfrac{1}{2}, -2k-\tfrac{1}{2}\right) \text{ to } \left(n+\tfrac{1}{2}, -2k-\tfrac{1}{2}\right)\right\},$$

$$A_3 = \left\{\text{all the dual edges } \left\{\left(-\tfrac{1}{2}, j-\tfrac{1}{2}\right), \left(-\tfrac{1}{2}, j+\tfrac{1}{2}\right)\right\} \text{ and } \left\{\left(n+\tfrac{1}{2}, j-\tfrac{1}{2}\right), \left(n+\tfrac{1}{2}, j+\tfrac{1}{2}\right), -2k \leq j \leq 2k\right\} \text{ are occupied}\right\},$$

$$A_4 = \{\mathbf{0} \leftrightarrow ne_1 \text{ in } [0,n] \times [-k,k] \text{ (on the original lattice)}\}.$$

The events $A_1 \cap A_2 \cap A_3$ and A_4 depend on disjoint edge sets and are therefore independent. A_1, A_2 and A_3 are decreasing events to which we can apply the Harris–FKG inequality. The probability of A_3 equals $(1-p)^{8k+2}$, and the probabilities of A_1 and A_3 are equal. These observations yield

$$\tau^f(p, \mathbf{0}, ne_1) \geq P_p\{A_1 \cap A_2 \cap A_3 \cap A_4\} \\ \geq (1-p)^{8k+2}\big[P_p\{A_1\}\big]^2 P_p\{A_4\}. \tag{7.6}$$

Now for every fixed $\varepsilon > 0$ it is possible to find a k such that for all large n,

$$P_p\{A_1\} \geq \exp\left[-(n+1)\left(\frac{1}{\xi(1-p)} + \varepsilon\right)\right], \tag{7.7}$$

$$P_p\{A_4\} \geq \exp(-\varepsilon n). \tag{7.8}$$

(7.7) and (7.8) have been used elsewhere, and the reader can find related results and proofs in Chayes and Chayes (1986a), Chayes and Chayes (1986b) and Grimmett (1989), Section 9.5. Suffice it here to say that for given $\varepsilon > 0$ we can first pick N, then $k > N$, such that

$$P_{1-p}\{\mathbf{0} \leftrightarrow Ne_1\} = \tau(1-p, \mathbf{0}, Ne_1) \\ \geq \exp\left(-N\left(\frac{1}{\xi(1-p)} + \frac{\varepsilon}{3}\right)\right) \quad \text{[cf. (1.2)]},$$

$$P_{1-p}\{\mathbf{0} \leftrightarrow Ne_1 \text{ in } B_k\} \geq \exp\left\{-N\left(\frac{1}{\xi(1-p)} + \frac{2\varepsilon}{3}\right)\right\}. \tag{7.9}$$

Then we observe that A_1 occurs if there are dual connections from $(-\frac{1}{2}, 2k+\frac{1}{2})$ to $(k-\frac{1}{2}, 2k+\frac{1}{2})$ in $[-\frac{1}{2}, k-\frac{1}{2}] \times [k+\frac{1}{2}, 3k+\frac{1}{2}]$, from $(k+(j-1)N-\frac{1}{2}, 2k+\frac{1}{2})$ to $(k+jN-\frac{1}{2}, 2k+\frac{1}{2})$ in $(k+(j-1)N-\frac{1}{2}, 2k+\frac{1}{2}) + B_k$ for $j = 1, \dots, j_0 := [(n-2k)/N]$, and finally from $(k+j_0N-\frac{1}{2}, 2k+\frac{1}{2})$ to $(n+\frac{1}{2}, 2k+\frac{1}{2})$ in $[k+j_0N-\frac{1}{2}, n+\frac{1}{2}] \times [k+\frac{1}{2}, 3k+\frac{1}{2}]$. All these are decreasing events, and by the Harris–FKG inequality the probability of all these events occurring is at least

$$K_8(p, N, k)\big[P_{1-p}\{0 \leftrightarrow Ne_1 \text{ in } B_k\}\big]^{j_0} \\ \geq \exp\left(-n\left(\frac{1}{\xi(1-p)} + \varepsilon\right)\right) \quad \text{for large } n \ \big[\text{by (7.9)}\big].$$

This proves (7.7), and (7.8) is proved similarly. (7.6)–(7.8) together imply that

$$\xi^f(p) \geq \tfrac{1}{2}\xi(1-p). \qquad \square$$

Note added in proof. G. Grimmett and J. M. Marstrand have recently proven that $(*)$ $p_c = \lim_{k\to\infty} p_c(S_k)$. In particular, $\xi^f(p) < \infty$ for all $p > p_c$. Several comments (such as Remark 2) in this article are superseded by $(*)$.

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