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CORRECTION

RECURRENCE, TRANSIENCE AND BOUNDED HARMONIC FUNCTIONS FOR DIFFUSIONS IN THE PLANE

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There is an error in the statement of Theorem 1.3 in the case $k = 1$. The function $G(\theta)$ has not been defined correctly in this case. As deduced heuristically in Remark 1 following this theorem, the function $G(\theta)$ should be defined (up to a scalar multiple) in all cases ($k < 1$, $k = 1$ or $k > 1$) as the invariant probability density for L_θ, which depends on k and is defined in the remark. The expression given for G in the case $k = 1$ [which actually reduces to $\exp(\int_0^\theta (2\gamma_2/e_2)(s)\,ds)$ since $e_5 = e_2'$] is incorrect as it is not periodic. The correct expression is

$$G(\theta) = \exp\left(\int_0^\theta \frac{2\gamma_2}{e_2}(s)\,ds\right)\left[\int_0^\theta \frac{ds}{e_2(s)}\exp\left(-\int_0^s \frac{2\gamma_2}{e_2}(z)\,dz\right) + \exp\left(\int_0^{2\pi}\frac{2\gamma_2}{e_2}(z)\,dz\right) \times \int_\theta^{2\pi}\frac{ds}{e_2(s)}\exp\left(-\int_0^s \frac{2\gamma_2}{e_2}(z)\,dz\right)\right]. \tag{1}$$

G as defined in (1) is periodic, positive and satisfies $\tilde{L}_\theta G = 0$, where $\tilde{L}_\theta$ is the adjoint of L_θ. We also note in passing that the observation $e_5 = e_2'$ simplifies the expression for G in the case $k > 1$:

$$G(\theta) = 1, \qquad k > 1. \tag{2}$$

The original proof in the case $k = 1$ breaks down because of this lack of periodicity. The function $V(\theta, m) = \int_0^\theta g(s, m)\,ds$, defined after (4.11), is not periodic, that is, $g(\theta, m)$ is not a gradient when considered as a function on the circle. Thus the Rayleigh–Ritz formula (4.11) does not lead to the correct answer. Indeed, as g is not a gradient, the operator $A_m \equiv -(d^2/d\theta^2 + g(d/d\theta) + h)$ defined in (4.10) with $g = g(\theta, m)$ and $h = h(\theta, m)$ defined above (4.10) is not self-adjoint on $L^2(S)$.

The proof in the case $k = 1$ goes as follows. Let $\lambda_0(m)$ denote the smallest eigenvalue of A_m and let $\rho_0(\cdot, m)$ denote the corresponding eigenfunction which

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is positive by the Krein–Rutman theorem. We have

$$(3)\qquad A_m\rho_0(\cdot, m) = \lambda_0(m)\rho_0(\cdot, m).$$

Note that $\lambda_0(0) = 0$ and $\rho_0(\cdot, 0)$ = constant greater than 0. Let $\tilde{\rho}_0 = e_2G$. Since G is the lead eigenfunction with eigenvalue zero for $\tilde{L}_\theta$ and since $L_\theta = (-e_2/2)A_0$, it is easy to deduce that $\tilde{\rho}_0$ is the lead eigenfunction with eigenvalue zero for $\tilde{A}_0$, the adjoint of A_0. Since $e_2 > 0$ (by ellipticity) and $G > 0$, we have $\tilde{\rho}_0 > 0$ (which of course also follows from Krein–Rutman).

Multiplying (3) by $\tilde{\rho}_0$ and integrating over S gives

$$(4)\qquad \lambda_0(m)\int_0^{2\pi}\tilde{\rho}_0(\theta)\rho_0(\theta, m)\,d\theta = \int_0^{2\pi}\tilde{\rho}_0(\theta)A_m\rho_0(\theta, m)\,d\theta.$$

But $h(\theta, m)$ is of the form $mh_0(\theta) + m^2h_1(\theta)$ where, in particular,

$$(5)\qquad h_0 = \begin{cases} \dfrac{e_4 - e_1 + 2\gamma_1}{e_2}, & \text{if } \delta = 1, \\[2ex] \dfrac{e_4 - e_1}{e_2}, & \text{if } \delta > 1. \end{cases}$$

Thus from (4.10) and the definitions of $g(\cdot, m)$ and $h(\cdot, m)$, we may write $A_m = A_0 - m(e_3/e_2)d/d\theta - (mh_0 + m^2h_1)$. Substituting this into the right-hand side of (4) gives

$$(6)\qquad \begin{aligned} &\lambda_0(m)\int_0^{2\pi}\tilde{\rho}_0(\theta)\rho_0(\theta, m)\,d\theta \\ &\quad= \int_0^{2\pi}\tilde{\rho}_0(\theta)A_0\rho_0(\theta, m)\,d\theta \\ &\qquad - m\int_0^{2\pi}\tilde{\rho}_0(\theta)\bigg(\frac{e_3}{e_2}(\theta)\rho_0'(\theta, m) \\ &\qquad\qquad + (h_0 + mh_1)(\theta)\rho_0(\theta, m)\bigg)\,d\theta. \end{aligned}$$

The first term on the right-hand side of (6) is zero since $\tilde{A}_0\tilde{\rho}_0 = 0$. Now $\lambda_0(m)$ and $\phi_0(\theta, m)$ are analytic in m, as can be deduced, for example, from Crandall and Rabinowitz [(1973), Lemma 1.3]. Thus $\lambda_0(m) = m\lambda_1 + O(m^2)$ and $\rho_0(\theta, m) = \rho_0(\theta) + m\rho_1(\theta) + O(m^2)$. Equating terms of order m in (6) yields

$$\lambda_1\int_0^{2\pi}\tilde{\rho}_0\rho_0\,d\theta = -\int_0^{2\pi}\tilde{\rho}_0\left(\frac{e_3}{e_2}\rho_0' + h_0\rho_0\right)d\theta = -\int_0^{2\pi}\tilde{\rho}_0h_0\rho_0\,d\theta,$$

where the second equality follows since ρ_0 = constant. We thus obtain

$$\lambda_1 = \frac{-\displaystyle\int_0^{2\pi}\tilde{\rho}_0h_0\,d\theta}{\displaystyle\int_0^{2\pi}\tilde{\rho}_0\,d\theta}.$$

However, since $\tilde{\rho}_0 = e_2 G$, it follows from (5) and the definition of H in the statement of Theorem 1.3 that $H(\theta)G(\theta) = \tilde{\rho}_0(\theta)h_0(\theta)$. Thus,

$$\lambda_1 = \frac{-\int_0^{2\pi} G(\theta)H(\theta)\,d(\theta)}{\int_0^{2\pi} \tilde{\rho}_0(\theta)\,d\theta}$$

and the proof proceeds as in the original.

Acknowledgment. The proof given above was suggested by a referee and is simpler than the one originally proposed by the author.

REFERENCE

CRANDALL, M. G. and RABINOWITZ, P. H. (1973). Birfurcation, perturbation of simple eigenvalues and linearized stability. *Arch. Rational Mech. Anal.* **52** 161–180.

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