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ON SERIES REPRESENTATIONS OF INFINITELY DIVISIBLE RANDOM VECTORS[1]

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General results on series representations, involving arrival times in a Poisson process, are established for infinitely divisible Banach space valued random vectors without Gaussian components. Applying these results, various generalizations of LePage's representation are obtained in a unified way. Certain conditionally Gaussian infinitely divisible random vectors are characterized and some problems related to a Gaussian randomization method are investigated.

1. Introduction. In this paper we study the convergence and limit distribution of the centered sums of the form

$$\sum_{j=1}^{n} H(\tau_j, \xi_j) - A_n, \tag{1.1}$$

in connection with series representations of infinitely divisible random vectors. Here $\{\tau_j\}$ is a sequence of arrival times in a Poisson process, $\{\xi_j\}$ is a sequence of i.i.d. random elements, which is independent of $\{\tau_j\}$, and H is a Banach space valued function. If these sums converge with $A_n = 0$, then their limit can be written as a series of dependent random vectors

$$\sum_{j=1}^{\infty} H(\tau_j, \xi_j). \tag{1.2}$$

Random elements of this type, usually with H taking values in a space of nonnegative functions, have been studied in the applied literature for quite a long time and they are known as "shot noise" (see, e.g., Vervaat [26] and references therein). Our results, in particular, yield conditions for the a.s. convergence of the series (1.2) in general Banach spaces.

Series representations involving arrival times in a Poisson process have been given by Ferguson and Klass [7], for real independent increment processes without Gaussian components and with positive jumps. Kallenberg [11] showed the uniform convergence in the Ferguson–Klass decomposition and Resnick [22] related the decomposition to the well-known Itô–Lévy representation of processes with independent increments. A series representation of Hilbert space valued stable random vectors, which generalizes and improves the Ferguson–

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Klass representation of one-dimensional stable random variables, has been established by LePage, Woodroofe and Zinn [16]. LePage [14] observed that symmetric stable random vectors can be represented as conditionally Gaussian. This important property has been generalized and extensively used by Marcus and Pisier [19] in their investigation of continuity of stable processes. Marcus and Pisier's work showed the significance of the series decompositions in the study of stable probability measures on general Banach spaces (see also [2] and [23]). A generalization of the Ferguson–Klass representation (one-dimensional, Lévy measure concentrated on positive reals) to the case of random vectors taking values in Banach spaces of cotype 2 is due to LePage [15]. Since this assumption on the geometry of Banach spaces is too restrictive for many interesting applications of the representation (e.g., for studying the continuity of stochastic processes), it is necessary to investigate series developments without any restrictions on the Banach spaces. The validity of the LePage representation for certain symmetric infinitely divisible random vectors in general Banach spaces has been recently shown in Marcus [18].

The main goal of the present paper is to give a simple and general scheme of deriving series representations of arbitrary Banach space valued infinitely divisible random vectors. Such representations can be regarded as particular cases of (1.1) and, in this respect, our approach is similar to the one in Vervaat [26], who obtained the Ferguson–Klass representation of positive infinitely divisible random variables as a special case of shot noise random variables. In Section 2 necessary and sufficient conditions for the a.s. convergence of centered sums (1.1) are given. Some important special cases of (1.1) are investigated in Section 3. The results of Sections 2 and 3 are applied in Section 4 to derive series representations of infinitely divisible random vectors. This approach enables us to obtain various series representations, which generalize those of LePage [15], in a unified way, while avoiding many obscuring details due to specific forms of the function H in concrete situations. In Section 5 an interesting subclass of infinitely divisible random vectors which can be represented as conditionally Gaussian is characterized. Certain problems regarding Gaussian randomization of H are studied in the conclusions of Sections 3 and 5.

Finally we would like to mention something about the methods in this paper. To determine the convergence in (1.1), we use a slight modification of the technique previously employed by Ferguson and Klass [7], who transformed certain dependent summand series into independent ones. The modification is that we associate with (1.1) a continuous time, independent increment, stochastic process, instead of the discrete time one, so that (1.1) is obtained by a random time substitution. This approach gives the results on the L^p-convergence immediately (see Corollary 2.5), and reveals a martingale structure of the decomposition [see Corollary 4.4(iv) and Theorem 3.1].

2. The convergence and limit distribution of centered sums (1.1). We recall and complete some notation that will be used throughout the paper. $\{\xi_j\}_{j=1}^{\infty}$ is a sequence of i.i.d. random elements taking values in a measurable space $(D, \mathscr{D})$ with the common distribution $\mathscr{L}(\xi_j) = \lambda$. By $\{N(t)\}_{t \geq 0}$ is denoted a Poisson process with parameter 1 and τ_j is the jth arrival time of $N(t)$, i.e., $\tau_j = \inf\{t > 0: N(t) = j\}$, $j = 1, 2, \dots$. $\{U_j\}_{j=1}^{\infty}$ stands for a sequence of i.i.d. uniform on $(0,1)$ random variables. We assume that $\{\xi_j\}_{j=1}^{\infty}$, $\{N(t)\}_{t\geq 0}$ and $\{U_j\}_{j=1}^{\infty}$ are defined on the same probability space $(\Omega, \mathscr{F}, P)$ and they are mutually independent.

In order to use the method of Ferguson and Klass [7] mentioned in the Introduction we shall need the following lemma which in the case $\mathscr{X} = \mathbf{R}$ can be deduced from Lemma 2 [7] and then easily extended to the case when $\mathscr{X}$ is a separable Banach space. Since this lemma constitutes the first important step of the method we shall give below a straightforward and different proof in a more general case.

LEMMA 2.1. *Let $(\mathscr{X}, \mathscr{B})$ be a measurable vector space and let $G: (0, \infty) \times D \to \mathscr{X}$ be a measurable map. Then the $\mathscr{X}$-valued stochastic process given by*

$$X(t) := \sum_{j=1}^{N(t)} G(\tau_j, \xi_j), \qquad t \geq 0,$$

has independent increments and

$$\mathscr{L}(X(t+s) - X(s)) = \mathscr{L}\left(\sum_{j=1}^{N(t)} G(s + tU_j, \xi_j)\right).$$

PROOF. Let $\mathscr{F}_t^{(1)} := \sigma(N(s): s \leq t)$ and $\mathscr{F}_k^{(2)} := \sigma(\xi_1, \dots, \xi_k)$. Put

$$\mathscr{F}_t := \left\{A \in \mathscr{F}: A \cap \{N(t) \leq k\} \in \mathscr{F}_t^{(1)} \vee \mathscr{F}_k^{(2)} \text{ for every } k \geq 1\right\}. \tag{2.1}$$

Then $\{\mathscr{F}_t\}_{t\geq 0}$ is an increasing filtration and $\{X(t)\}_{t\geq 0}$ is adapted to this filtration.

In order to prove that $\{X(t)\}_{t\geq 0}$ has independent increments it is enough to show that $\sigma(X(t+s) - X(s))$ and $\mathscr{F}_s$ are independent for every $t, s \geq 0$. Let $A \in \mathscr{F}_s$ and $B \in \mathscr{B}$. We get

$$\begin{aligned} &P\{X(t+s) - X(s) \in B, A\} \\ &\quad = \sum_{i,k\geq 0} P\{X(t+s) - X(s) \in B, N(s) = i, N(s+t) = i+k, A\} \\ &\quad = \sum_{i,k\geq 0} P\left\{\sum_{j=i+1}^{i+k} G(\tau_j, \xi_j) \in B, N(t+s) - N(s) = k, A_i\right\}, \end{aligned} \tag{2.2}$$

where $A_i := \{N(s) = i, A\} \in \mathscr{F}_s^{(1)} \vee \mathscr{F}_i^{(2)}$ by (2.1). Since we have on $\{N(s) = i\}$,

$$\sum_{j=i+1}^{i+k} G(\tau_j, \xi_j) = \sum_{j=i+1}^{i+k} G(s + \tau_{j-i}^{(1)}, \xi_j),$$

where $\tau_m^{(1)}$ is the mth arrival time in the Poisson process $N^{(1)}(u) = N(u+s) -$

$N(s), u \geq 0$, we conclude that the events A_i and $\{\sum_{j=i+1}^{i+k} G(\tau_j, \xi_j) \in B, N(t+s) - N(s) = k\}$ are independent. Therefore the last expression in (2.2) is equal to

$$\sum_{i, k \geq 0} P\left\{ \sum_{j=i+1}^{i+k} G\left(s + \tau_{j-i}^{(1)}, \xi_j\right) \in B, N^{(1)}(t) = k \right\} P(A_i)$$

$$= \sum_{i, k \geq 0} P\left\{ \sum_{m=1}^{k} G\left(s + \tau_m^{(1)}, \xi_m\right) \in B, N^{(1)}(t) = k \right\} P(A_i)$$

$$= \sum_{k \geq 0} P\left\{ \sum_{m=1}^{k} G(s + \tau_m, \xi_m) \in B, N(t) = k \right\} P(A)$$

$$= P\left\{ \sum_{j=1}^{N(t)} G\left(s + \tau_j, \xi_j\right) \in B \right\} P(A),$$

which proves the independence of $\sigma(X(t+s) - X(s))$ and $\mathscr{F}_s$ as well as the equality $\mathscr{L}(X(t+s) - X(s)) = \mathscr{L}(\sum_{j=1}^{N(t)} G(s + \tau_j, \xi_j))$.

In the proof of the second part of the lemma we shall use the well-known fact that the conditional distribution of $(\tau_1, \dots, \tau_{N(t)})$ given that $N(t) = k \geq 1$ is equal to the distribution of $(tU_{(1)}, \dots, tU_{(k)})$, where $U_{(j)}$ is the jth order statistic of $U_1, \dots, U_k$. We have, for every $B \in \mathscr{B}$,

$$P\{X(t+s) - X(s) \in B\} = P\left\{ \sum_{j=1}^{N(t)} G\left(s + \tau_j, \xi_j\right) \in B \right\}$$

$$= \sum_{k=0}^{\infty} P\left\{ \sum_{j=1}^{k} G\left(s + tU_{(j)}, \xi_j\right) \in B \right\} \frac{t^k}{k!} e^{-t}$$

$$= \sum_{k=0}^{\infty} P\left\{ \sum_{j=1}^{k} G\left(s + tU_j, \xi_j\right) \in B \right\} \frac{t^k}{k!} e^{-t}$$

$$= P\left\{ \sum_{j=1}^{N(t)} G\left(s + tU_j, \xi_j\right) \in B \right\},$$

which completes the proof. □

LEMMA 2.2. *If $(\mathscr{X}, \mathscr{A}) = (\mathbf{R}, \mathscr{B}_{\mathbf{R}})$, then, under the notation of Lemma 2.1,*

(i) $EX(t) = \int_0^t \int_D G(u, v) \lambda(dv)\, du$,

provided either one of the above quantities, on the left or right side, exists;

(ii) $E \exp[iX(t)] = \exp\{\int_0^t \int_D [e^{iG(u,v)} - 1] \lambda(dv)\, du\}$.

PROOF. By Lemma 2.1 and Wald's lemma we get

$$E[X(t)] = E\left[\sum_{j=1}^{N(t)} G(tU_j, \xi_j)\right] = tEG(tU_1, \xi_1) = \int_0^t \int_D G(u, v)\lambda(dv)\, du,$$

which gives (i). The proof of (ii) is similar. □

Further we shall need the following standard fact which follows, for instance, from Doob [5], Chapter 2, Theorem 2.3.

LEMMA 2.3. *Let $\{Y(t)\}_{t \geq 0}$ be a stochastic process with values in a separable metric space and whose sample paths are right-continuous. Then $\lim_{t\to\infty} Y(t, \omega)$ exists for a.e. ω if and only if for every increasing sequence $\{t_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} t_n = \infty$, the sequence $\{Y(t_n)\}_{n=1}^{\infty}$ converges a.s.*

To state and prove the main result of this section, we shall need some notation that will be also used throughout this paper. E will stand for a separable Banach space with the norm $\|\cdot\|$ and $B_r := \{x \in E: \|x\| \leq r\}$, $r \geq 0$ ($B_\infty := E$). The dual of E will be denoted by E' and $\langle x', x\rangle := x'(x)$, $x' \in E'$, $x \in E$.

We recall that a measure M on $\mathscr{B}_E$ with $M(\{0\}) = 0$ is said to be a Lévy measure if for every $x' \in E'$, $\int_E (\langle x', x\rangle^2 \wedge 1) M(dx) < \infty$ and for some (each) $r \in (0, \infty)$ the function ϕ_r defined by

$$\phi_r(x') := \exp\left\{\int_E \left[e^{i\langle x', x\rangle} - 1 - i\langle x', x\rangle I_{B_r}(x)\right] M(dx)\right\},$$

$x' \in E'$, is the characteristic function of a probability measure on E. The probability measure with characteristic function ϕ_r will be denoted by $c_r \operatorname{Pois}(M)$ (see de Acosta, Araujo and Giné [4]). If M is a Lévy measure and additionally $\int_{B_1^c} \|x\| M(dx) < \infty$ ($\int_{B_1} \|x\| M(dx) < \infty$, respectively), then we define $c_\infty \operatorname{Pois}(M)$ [$c_0 \operatorname{Pois}(M)$, respectively] as a probability measure with characteristic function ϕ_∞ (ϕ_0, respectively).

Now let $H: (0, \infty) \times D \to E$ be a Borel measurable map and define a measure F on $\mathscr{B}_E$ by

$$F(A) := \int_0^\infty \int_D I_{A \setminus \{0\}}(H(u, v))\lambda(dv)\, du, \qquad A \in \mathscr{B}_E. \tag{2.3}$$

Note that $F(\{0\}) = 0$. Put

$$A(t) := \int_0^t \int_D H(u, v) I_{B_1}(H(u, v))\lambda(dv)\, du, \qquad t \geq 0,$$

and

$$T_n := \sum_{j=1}^{n} H(\tau_j, \xi_j) - A(\tau_n), \qquad n = 1, 2, \dots .$$

Theorem 2.4. *The sequence of E-valued random vectors $\{T_n\}$ converges a.s. if and only if F is a Lévy measure on E. Further, if F is a Lévy measure and $T_\infty = \lim_{n\to\infty} T_n$, then*

$$\mathscr{L}(T_\infty) = c_1 \operatorname{Pois}(F).$$

Proof. Let

$$X(t) := \sum_{j=1}^{N(t)} H(\tau_j, \xi_j) - A(t), \qquad t \geq 0.$$

By Lemma 2.1 $\{X(t)\}_{t\geq 0}$ is an independent increment E-valued stochastic process with right-continuous sample paths. Using Lemma 2.2(ii), we get

$$\mathscr{L}(X(t)) = c_1 \operatorname{Pois}(F^{(t)}), \tag{2.4}$$

where

$$F^{(t)}(A) := \int_0^t \int_D I_{A\setminus\{0\}}(H(u,v))\lambda(dv)\,du, \qquad A \in \mathscr{B}_E \tag{2.5}$$

[note that $F^{(t)}(E) = t < \infty$].

Assume first that F is a Lévy measure. Since $F^{(t)} \nearrow F$ as $t \nearrow \infty$, we get

$$c_1 \operatorname{Pois}(F^{(t)}) \Rightarrow c_1 \operatorname{Pois}(F) \quad \text{as } t \nearrow \infty$$

(see de Acosta, Araujo and Giné [4], Theorem 1.6). Hence, by the Itô–Nisio theorem ([10], Theorem 3.1) and (2.4), $\{X(t_n)\}_{n=1}^{\infty}$ converges a.s. for each $t_1 < t_2 < \cdots < t_n \to \infty$. In view of Lemma 2.3, $X(\infty) := \lim_{t\to\infty} X(t)$ exists a.s. Clearly, $\mathscr{L}(X(\infty)) = c_1 \operatorname{Pois}(F)$. Now we notice that $T_n = X(\tau_n)$ and $\tau_n \to \infty$ a.s. Therefore $T_n \to T_\infty := X(\infty)$ a.s. as $n \to \infty$, which ends the proof of the sufficiency part of the theorem.

Now we prove the necessity. Assume that $\{T_n\}$ converges a.s. We have, for every t,

$$T_{N(t)+1} = X(t) + Y(t), \tag{2.6}$$

where

$$Y(t) = H(\tau_{N(t)+1}, \xi_{N(t)+1}) + A(t) - A(\tau_{N(t)+1}).$$

By the Markov property of $\{N(s)\}_{s\geq 0}$, the random vectors $X(t)$ and $Y(t)$ are independent for each t. Since $T_{N(t)+1} \to T_\infty$ a.s. as $t \to \infty$, by (2.6) $\{\mathscr{L}(X(t))\}_{t\geq 0}$ is relatively shift-compact. In view of (2.4) and [4], Theorem 1.6, F is a Lévy measure. The proof is complete. □

Corollary 2.5. *Let F be a Lévy measure and $\int_{B_1^c} \|x\|^p F(dx) < \infty$ for some $0 < p < \infty$. Then $T_n \to T_\infty$ a.s. and in L_E^p.*

Proof. Since $E\|X(\infty)\|^p < \infty$, $E \sup_{0\leq t<\infty} \|X(t)\|^p < \infty$ in Hoffmann-Jørgensen [9], Corollary 3.3. Hence

$$E \sup_n \|T_n\|^p = E \sup_n \|X(\tau_n)\|^p \leq E \sup_{0\leq t<\infty} \|X(t)\|^p < \infty,$$

which ends the proof. □

REMARK 2.6. Theorem 2.4, when specified to those Banach spaces for which a full characterization of Lévy measures is known, gives definite conditions in terms of the function H for the a.s. convergence of $\{T_n\}$. For example, if $E = \mathbf{R}^n$ or more general, if E is a separable Hilbert space, then $\int_0^\infty \int_D (1 \wedge \|H(u, v)\|^2)\lambda(dv)\,du < \infty$ is necessary and sufficient for the a.s. convergence of $\{T_n\}$. Similarly, if $E = l^p$, $2 \le p < \infty$, the conjunction of the following two conditions is equivalent to the a.s. convergence of $\{T_n\}$:

$$\int_0^\infty \int_D (1 \wedge \|H(u, v)\|^p)\lambda(dv)\,du < \infty$$

and

$$\sum_{j=1}^{\infty} \left[\int_0^\infty \int_D |\langle H(u, v), e_j\rangle|^2 I_{B_1}(H(u, v))\lambda(dv)\,du \right]^{p/2} < \infty,$$

where $\{e_j\}$ denotes the standard basis (see Giné, Mandrekar and Zinn [8], Theorem 4.2).

From now on $\{\varepsilon_n\}$ will stand for a sequence of i.i.d. random variables, independent of the other random sequences defined in this paper, and such that $P\{\varepsilon_n = 1\} = P\{\varepsilon_n = -1\} = \frac{1}{2}$. Put

$$\tilde{S}_n := \sum_{j=1}^{n} \varepsilon_j H(\tau_j, \xi_j).$$

Further, let $\tilde{F}$ denote a symmetrization of a measure F given by

$$\tilde{F}(A) := 2^{-1}[F(A) + F(-A)], \qquad A \in \mathscr{B}_E.$$

PROPOSITION 2.7. *Under the above notation, the following are equivalent:*

(i) T_n *converges a.s. to* T_∞ *and* $\mathscr{L}(T_\infty) = c_1 \operatorname{Pois}(F)$.
(ii) $\tilde{S}_n$ *converges a.s. to* $\tilde{S}_\infty$ *and* $\mathscr{L}(\tilde{S}_\infty) = c_1 \operatorname{Pois}(\tilde{F})$.
(iii) $\{\mathscr{L}(\tilde{S}_n)\colon n \ge 1\}$ *is uniformly tight.*

PROOF. Let $\tilde{D} := \{-1, 1\} \times D$, $\tilde{\xi}_j := (\varepsilon_j, \xi_j)$, $\tilde{\lambda} := \mathscr{L}(\tilde{\xi}_j) = [2^{-1}(\delta_1 + \delta_{-1})] \otimes \lambda$, $\tilde{H}(u, \tilde{v}) := \varepsilon H(u, v)$ for every $\tilde{v} = (\varepsilon, v) \in \tilde{D}$. Then we can write

$$\tilde{S}_n = \sum_{j=1}^{n} \tilde{H}(\tau_j, \tilde{\xi}_j).$$

Since

$$\tilde{A}(t) := \int_0^t \int_{\tilde{D}} \tilde{H}(u, \tilde{v}) I_{B_1}(\tilde{H}(u, \tilde{v}))\tilde{\lambda}(d\tilde{v})\,du = 0,$$

for every $t > 0$, by Theorem 2.4, $\tilde{S}_n$ converges a.s. if and only if

$$G(A) := \int_0^\infty \int_{\tilde{D}} I_{A \setminus \{0\}}(\tilde{H}(u, \tilde{v}))\tilde{\lambda}(d\tilde{v})\,du$$
$$= 2^{-1}[F(A) + F(-A)] = \tilde{F}(A), \qquad A \in \mathscr{B}_E,$$

is a Lévy measure on E. Using the fact that $\tilde{F}$ is a Lévy measure if and only if F is too, and again Theorem 2.4, we prove the equivalence of (i) and (ii).

It remains to show that (iii) implies (ii). Indeed, there exists a σ-compact convex symmetric set $K \subset E$ such that $\sup_n P\{\tilde{S}_n \notin K\} = 0$. Using Lévy's inequality, we get

$$EP\left[\tilde{S}_n \notin K \text{ for some } n \geq 1 \mid \{\tau_j\}\right] = 0.$$

Hence, for almost all realizations of $\{\tau_j\}$, the sequence of conditional distributions of $\tilde{S}_n$, given $\{\tau_j\}$, is uniformly tight. Conditionally, $\tilde{S}_n$ are sums of independent symmetric random vectors, hence by the Itô–Nisio theorem ([10], Theorem 4.1) $\tilde{S}_n$ converges a.s. for almost every (fixed) realization of $\{\tau_j\}$. Now Fubini's theorem implies that $\tilde{S}_n$ converges a.s., which completes the proof. □

Proposition 2.7 shows the equivalence of the following two statements: (a) *a measure F, defined by* (2.3), *is a Lévy measure*, and (b) *the series $\sum_{j=1}^{\infty} \varepsilon_j H(\tau_j, \xi_j)$ of dependent sign-invariant random vectors converges*. The next proposition proves, under some additional hypotheses on H, that the dependent components τ_j in (b) can be replaced by integers j and the equivalence of (a) and (b) still holds. This reduces the problem of determining whether F is a Lévy measure to a more tractable one, when a certain series of *independent* symmetric summands converges. This idea was first presented in Marcus and Pisier [19], Remark 3.15, in the context of series expansions of stable random vectors and recently extended by Marcus [18] to the case of the so-called ξ-radial random vectors.

PROPOSITION 2.8. *Suppose that function H admits a factorization*

$$H(u, v) = K(u, v)B(v),$$

where $K: (0, \infty) \times D \to R$, $B: D \to E$ are measurable functions such that, for each $v \in D$, $|K(u, v)|$ is a nonincreasing function of $u > 0$. Then the following are equivalent:

(i) *A measure F, given by* (2.3), *is a Lévy measure on E.*
(ii) $\sum_{j=1}^{\infty} \varepsilon_j H(j, \xi_j)$ *converges a.s.*

PROOF. First notice that F is a Lévy measure if and only if, for each (some) $a > 0$, aF is a Lévy measure. In view of Theorem 2.4 and Proposition 2.7 the latter condition is equivalent to: For each (some) $a > 0$ series

$$\sum_{j=1}^{\infty} \varepsilon_j H\left(\tau_j a^{-1}, \xi_j\right) \tag{2.7}$$

converges a.s. Now, by the strong law of large numbers, $\tau_j/j \to 1$ a.s., so that $P\{2^{-1}\tau_j \leq j \leq 2\tau_j \text{ eventually}\} = 1$. Hence, with probability 1,

$$|K\left(2^{-1}\tau_j, \xi_j\right)| \leq |K\left(j, \xi_j\right)| \leq |K\left(2\tau_j, \xi_j\right)|$$

eventually. Using the contraction principle conditionally, for a fixed realization of $\{\tau_j\}$, we prove that the convergence in (2.7) with $a = 2^{-1}$ implies (ii) which, in turn, implies the convergence in (2.7) with $a = 2$. The proof is complete. □

3. Convergence and centering in (1.1) in some special cases. In this section we shall discuss some interesting modifications in (1.1) which are possible when F satisfies certain additional hypotheses.

THEOREM 3.1. *Assume that* F, *defined by* (2.3), *is a Lévy measure on* E *such that* $\int_{B_1^c} \|x\|^p F(dx) < \infty$ *for some* $p \geq 1$. *Let*

$$C(t) := \int_0^t \int_D H(u, v) \lambda(dv)\, du, \qquad t \geq 0.$$

Then:

(i) $M_n := \sum_{j=1}^n H(\tau_j, \xi_j) - C(\tau_n)$, $n \geq 1$, *is a martingale with respect to* $\sigma(\tau_1, \dots, \tau_n, \xi_1, \dots, \xi_n)$.

(ii) $M_n \to M_\infty$ *a.s. and in* L_E^p *as* $n \to \infty$.

(iii) $\mathscr{L}(M_\infty) = c_\infty \operatorname{Pois}(F)$.

PROOF. First note that $C(t)$ is well defined as a Bochner integral. Indeed,

$$\int_0^t \int_D \|H(u, v)\| \lambda(dv)\, du \leq t + \int_0^t \int_D \|H(u, v)\| I_{B_1^c}(H(u, v)) \lambda(dv)\, du$$
$$\leq t + \int_{B_1^c} \|x\|^p F(dx) < \infty.$$

Put $X_1(t) := \sum_{j=1}^{N(t)} H(\tau_j, \xi_j) - C(t) = X(t) + A(t) - C(t)$, where $X(t)$ is defined in the proof of Theorem 2.4. In the proofs of Theorem 2.4 and Corollary 2.5 we have shown that $X(t) \to X(\infty)$ a.s. $t \to \infty$, $\mathscr{L}(X(\infty)) = c_1 \operatorname{Pois}(F)$ and $E \sup_{0 \leq t < \infty} \|X(t)\|^p < \infty$. Since

$$A(t) - C(t) = -\int_0^t \int_D H(u, v) I_{B_1^c}(H(u, v)) \lambda(dv)\, du \to -\int_{B_1^c} x\, dF(x),$$

as $t \to \infty$, we conclude that

$$X_1(t) \to X_1(\infty) \quad \text{a.s. as } t \to \infty,$$
$$\mathscr{L}(X_1(\infty)) = c_\infty \operatorname{Pois}(F)$$

and

$$E \sup_{0 \leq t < \infty} \|X_1(t)\|^p < \infty. \tag{3.1}$$

By Lemmas 2.1 and 2.2, $\{X_1(t)\}_{t \geq 0}$ is an independent increment process with right-continuous sample paths and $EX_1(t) = 0$. Moreover, $\{X_1(t)\}_{t \geq 0}$ is adapted to the filtration $\{\mathscr{F}_t\}_{t \geq 0}$ defined by (2.1) and $X_1(t + s) - X_1(s)$ is independent of $\mathscr{F}_s$. Hence $\{X_1(t), \mathscr{F}_t\}_{t \geq 0}$ is a martingale. Using (3.1) and the optional sampling theorem, we conclude that

$$M_n = X_1(\tau_n), \qquad n \geq 1,$$

forms a martingale with respect to $\mathscr{F}_{\tau_n} \supset \sigma(\tau_1, \ldots, \tau_n, \xi_1, \ldots, \xi_n)$ and clearly $M_n \to M_\infty := X_1(\infty)$ a.s. and in L_E^p. The proof is complete. □

THEOREM 3.2. *Assume that F, defined by* (2.3), *is a Lévy measure such that* $\int_{B_1} \|x\| F(dx) < \infty$. *Then*

$$\sum_{j=1}^{\infty} \|H(\tau_j, \xi_j)\| < \infty \quad a.s.$$

and

$$\mathscr{L}\left(\sum_{j=1}^{\infty} H(\tau_j, \xi_j) \right) = c_0 \operatorname{Pois}(F).$$

PROOF. Since

$$\int_0^\infty \int_D \|H(u, v)\| I_{B_1}(H(u, v)) \lambda(dv)\, du = \int_{B_1} \|x\| F(dx) < \infty,$$

it follows by the dominated convergence theorem that

$$A(\tau_n) \to \int_{B_1} x F(dx) \quad \text{a.s. as } n \to \infty.$$

Applying Theorem 2.4 twice, for positive real and vector valued random variables, we complete the proof. □

Random centers $A_n = A(\tau_n)$ in (1.1) provide a fine connection between the centered sums and the associated compound Poisson process. Random centers $A_n = C(\tau_n)$ are also necessary for the martingale property in Theorem 3.1. Nevertheless, it is an interesting question whether random centers can be replaced by nonrandom ones with the a.s. convergence still holding. We cannot answer this question in its full generality but under certain additional conditions the answer is yes. To proceed with this question, we begin with a lemma that is a special case of Lemma 4 in Ferguson and Klass [7]. We shall give below a short proof of this lemma and also indicate that our method can be easily extended to obtain a new and short proof of Lemma 4 in [7].

LEMMA 3.3. *Let g be a nonincreasing square integrable function defined on* $(0, \infty)$. *Then*

$$\int_n^{\tau_n} g(u)\, du \to 0 \quad a.s. \text{ as } n \to \infty.$$

PROOF. We have

$$\left| \int_n^{\tau_n} g(u)\, dt \right| \le g(\tau_n \wedge n) |\tau_n - n|, \tag{3.2}$$

by the monotonicity of g. Further, by the strong law of large numbers we have with probability 1,

$$g(\tau_n \wedge n) \le g\left(\frac{n}{2}\right) \quad \text{eventually.} \tag{3.3}$$

Using the Hájek–Rényi–Chow inequality ([3], page 243) we get, for every $\varepsilon > 0$,

$$P\left\{ \max_{m\le k\le n} g\left(\frac{k}{2}\right)|\tau_k - k| \ge \varepsilon \right\} \le \varepsilon^{-2}\left(\sum_{k=m+1}^{n} g^2\left(\frac{k}{2}\right) + mg^2\left(\frac{m}{2}\right)\right) \to 0,$$

as $m, n \to \infty$. Thus $g(n/2)|\tau_n - n| \to 0$ a.s., which combined with (3.2) and (3.3) completes the proof. □

THEOREM 3.4. *Assume that F, defined by* (2.3), *is a Lévy measure on E such that $\int_E(\|x\|^2 \wedge 1)F(dx) < \infty$. Suppose that, for each $v \in D$, $\|H(u,v)\|$ is a nonincreasing function of $u \in (0,\infty)$. Then*

$$\sum_{j=1}^{n} H(\tau_j,\xi_j) - A(n) \to T_\infty,$$

where T_∞ is specified in Theorem 2.4.

PROOF. Let

$$V_n = A(\tau_n) - A(n) = \int_n^{\tau_n}\int_D H(u,v) I_{B_1}(H(u,v))\lambda(dv)\,du$$

and

$$g(u) = \left\{\int_D (\|H(u,v)\|^2 \wedge 1)F(dx) < \infty\right\}^{1/2}.$$

g is nonincreasing,

$$\int_0^\infty g^2(u)\,du = \int_E (\|x\|^2 \wedge 1)F(dx) < \infty$$

and we have

$$\|V_n\| \le \left|\int_n^{\tau_n}\int_D (\|H(u,v)\| \wedge 1)\lambda(dv)\,du\right|$$
$$\le \left|\int_n^{\tau_n} g(u)\,du\right|$$

by Jensen's inequality. Applying Lemma 3.3, we get $V_n \to 0$ a.s. Theorem 2.4 completes the proof. □

In the rest of this section we study random series of the form

$$\sum_{j=1}^{\infty} \eta_j H(\tau_j, \xi_j), \tag{3.4}$$

where $\{\eta_j\}$ is a (nonzero) sequence of i.i.d. symmetric random variables, independent of the other random sequences defined so far. Series of this form in the case when $\{\eta_j\}$ is a Rademacher sequence have been investigated in the last part of Section 2. The other case of importance is when $\{\eta_j\}$ is a Gaussian sequence and we devote Section 5 to study of this case. Here we shall establish some general results on the convergence and distribution of (3.4).

PROPOSITION 3.5. *The following are equivalent:*

(i) *Series* (3.4) *converges a.s.*

(ii) F_η *defined by* $F_\eta(A) = E[F(A\eta_1^{-1})]$, $A \in \mathscr{B}_E$, *is a Lévy measure, where* F *is defined by* (2.3) $[F(A \cdot 0^{-1}) = 0$, *by convention*$]$.

Further,

$$\mathscr{L}\left(\sum_{j=1}^{\infty} \eta_j H(\tau_j, \xi_j) \right) = c_1 \operatorname{Pois}(F_\eta).$$

PROOF. A proof of this proposition is analogous to the first part of the proof of Proposition 2.7; one replaces (ε_j, ξ_j) by (η_j, ξ_j), $\{-1, 1\} \times D$ by $\mathbf{R} \times D$, etc. □

COROLLARY 3.6. (a) *If series* (3.4) *converges, then* F *is a Lévy measure.*

(b) *If* $\eta_j = \gamma_j$ *are standard normal random variables and* $\sum_{j=1}^{\infty} \gamma_j H(\tau_j, \xi_j)$ *converges a.s., then*

$$\hat{\mathscr{L}}\left(\sum_{j=1}^{\infty} \gamma_j H(\tau_j, \xi_j) \right)(x') = \exp\left\{ \int_E \left[\exp(-\langle x', x\rangle^2/2) - 1 \right] \tilde{F}(dx) \right\},$$

where $\tilde{F}$ *is the symmetrization of* F.

(c) *If* $c_1 \operatorname{Pois}(F)$ *is a p-stable probability measure, then* $F_\eta(A) = c^p \tilde{F}(A)$ *for every* $A \in \mathscr{B}_E$, *where* $c = (E|\eta_1|^p)^{1/p}$. *Thus, series* (3.4) *converges a.s. if and only if* $E|\eta_1|^p < \infty$ *and, further,*

$$\mathscr{L}\left(\sum_{j=1}^{\infty} \eta_j H(\tau_j, \xi_j) \right) = \mathscr{L}\left(c \sum_{j=1}^{\infty} \varepsilon_j H(\tau_j, \xi_j) \right). \tag{3.5}$$

(d) *Equality* (3.5) *characterizes Lévy measures* $\tilde{F}$ *corresponding to p-stable probability measures.*

PROOF. (a) By the contraction principle the convergence in (3.4) implies the convergence of the corresponding series with Rademacher coefficients, $\sum_1^\infty \varepsilon_j H(\tau_j, \xi_j)$. Therefore, by Proposition 2.7, F is a Lévy measure.

(b) Follows by a direct computation.

(c) Note that the symmetric Lévy measure $\tilde{F}$, which corresponds to a p-stable probability measure, has the property: $\tilde{F}(Au^{-1}) = u^p \tilde{F}(A)$ for every $A \in \mathscr{B}_E$ and $u > 0$. Hence for all $A \in \mathscr{B}_E$,

$$F_\eta(A) = E\left[\tilde{F}(A|\eta_1|^{-1}) \right] = E|\eta_1|^p \tilde{F}(A) = \tilde{F}(Ac^{-1}),$$

and (3.5) follows.

(d) If η_j are such that $P\{|\eta_j| = 1\} = t = 1 - P\{\eta_j = 0\}$, then the left side in (3.5) is equal to $c_1 \operatorname{Pois}(t\tilde{F})$ and the right side is $c_1 \operatorname{Pois}(\tilde{F}_t)$, where $\tilde{F}_t(A) = \tilde{F}(At^{-1/p})$, $A \in \mathscr{B}_E$. Since $t \in (0,1)$ is arbitrary, $c_1 \operatorname{Pois}(\tilde{F})$ is p-stable. The proof is complete. □

REMARK. The fact given in Corollary 3.6(c) was first observed by LePage, Woodroote and Zinn [16] and LePage [14] and was established for some special forms of H.

The following lemma shows, in particular, that in general Banach spaces $\sum_1^\infty \gamma_j H(\tau_j, \xi_j)$ may diverge while $\sum_1^\infty \varepsilon_j H(\tau_j, \xi_j)$ converges a.s., where the γ_j's are $N(0,1)$ random variables. This clearly indicates certain limitations for applying Gaussian randomization techniques in order to determine the convergence in (1.1) (see [18] and also Theorem 5.8 in Section 5 for positive results in this direction).

LEMMA 3.7. *For any unbounded random variable η there is a Lévy measure F on $E = C[0,1]$ such that F_η is not a Lévy measure* ($F_\eta(A) := E[F(A\eta^{-1})]$, $A \in \mathscr{B}_E$).

PROOF. Let $q(t)$ be the right-continuous $(1-t)$-quantile of $|\eta|$ and put $h(t) = 1/q(t)$, $0 < t < 1$. Then, for every symmetric measure F, we have

$$\begin{aligned} F_\eta(\{x: \|x\| \geq 1\}) &= E\left[F\left(\{x: \|x\| \geq |\eta|^{-1}\}\right)\right] \\ &= \int_{(0,1)} F\left(\left\{x: \|x\| \geq [q(t)]^{-1}\right\}\right) dt \qquad (3.6)\\ &= \int_{(0,1)} F(\{x: \|x\| \geq h(t)\})\, dt. \end{aligned}$$

We also observe that h is nondecreasing, $h(t) > 0$ for all $0 < t < 1$, and since η is unbounded, $h(0+) = 0$.

We now construct a Lévy measure on $C[0,1]$ in a similar way as in [1], page 140. For every $k \geq 2$ and $2^{k-1} \leq n < 2^k$, define $x_n \in C[0,1]$ by

$$x_n(s) = \begin{cases} 0 & \text{if } s \notin [2^{-n}, 2^{-n+1}], \\ h(k^{-1}) & \text{if } s = 3 \cdot 2^{-n}, \\ \text{linear} & \text{if } s \in [2^{-n}, 3 \cdot 2^{-n}] \text{ or } s \in [3 \cdot 2^{-n}, 2^{-n+1}]. \end{cases}$$

Then, by the argument in [1], page 141,

$$F := \sum_{n=2}^{\infty} n^{-1}\left(\delta_{x_n} + \delta_{-x_n}\right)$$

is a Lévy measure. Using (3.6), we get

$$\begin{aligned} F_\eta(\{x: \|x\| \geq 1\}) &\geq \sum_{k=2}^{\infty} \int_{((k+1)^{-1}, k^{-1})} F(\{x: \|x\| \geq h(t)\})\, dt \\ &\geq \sum_{k=2}^{\infty} [k(k+1)]^{-1} F\big(\{x: \|x\| \geq h(k^{-1})\}\big) \\ &\geq \sum_{k=2}^{\infty} [k(k+1)]^{-1} \sum_{n=2}^{2^k - 1} 2n^{-1} = +\infty. \end{aligned}$$

Thus F_η is not a Lévy measure. □

4. Series representations of infinitely divisible random vectors. In the previous sections we studied the convergence in (1.1) for a general function H. Now, assuming that a measure F is given, we shall construct some examples of H which satisfy equality (2.3).

Let F be a Borel measure on E with $F(\{0\}) = 0$. We say that F admits a *polar decomposition with respect to a Borel set* D, $0 \notin D \subset E$, if

$$F(A) = \int_D \int_{(0,\infty)} I_A(tx) \rho(dt, x) \lambda(dx), \qquad A \in B_E, \tag{4.1}$$

where $\{\rho(\cdot, x)\}_{x \in D}$ is a measurable family of Borel measures on $(0, \infty)$ and λ is a Borel probability measure on D.

EXAMPLE 4.1. Let F be a σ-finite Borel measure on E with $F(\{0\}) = 0$ and let λ be an arbitrary Borel probability measure on E such that F is absolutely continuous with respect to λ. Put $f(x) := (dF/d\lambda)(x)$. Then, for every $A \in \mathscr{B}_E$,

$$F(A) = \int_A f\, d\lambda = \int_{E \setminus \{0\}} \int_{(0,\infty)} I_A(tx) \delta_1(dx) f(x) \lambda(dx).$$

This is an example of a polar decomposition of F with respect to $D = E \setminus \{0\}$, and with $\rho(dt, x) := \delta_1(dt) f(x)$.

Polar decompositions with respect to the unit sphere of Lévy measures on Hilbert spaces and their applications to stochastic integral representations of infinitely divisible processes were studied by Rajput and Rosinski [21]. We shall show here that arbitrary Lévy measures on Banach spaces admit polar decompositions with respect to the unit sphere $\partial B_1 := \{x \in E: \|x\| = 1\}$. This fact seems to be known but we cannot find appropriate references. We include its proof for the sake of completeness.

PROPOSITION 4.2. *Let F be a Borel measure on E such that $F(\{0\}) = 0$ and $F(B_r^c) < \infty$ for every $r > 0$. Then F admits a polar decomposition with respect to ∂B_1.*

PROOF. If $F \equiv 0$, then (4.1) holds trivially with $\rho(\cdot, \cdot) \equiv 0$ and an arbitrary λ. Therefore we may assume that $0 < F(E) \le \infty$. Let $\Phi: E \setminus \{0\} \to (0, \infty) \times \partial B_1$ be defined by $\Phi(x) := (\|x\|, x/\|x\|)$ and put $G := F_0 \circ \Phi^{-1}$, where $F_0 := F_{|E \setminus \{0\}}$. Since

$$G((r, \infty) \times \partial B_1) = F(B_r^c) < \infty \quad \text{for every } r > 0,$$

G is σ-finite and there exists a Borel function $g: (0, \infty) \to (0, \infty)$ such that

$$G_1(dt, dx) := g(t) G(dt, dx)$$

is a probability measure on $(0, \infty) \times \partial B_1$. Define

$$\lambda(B) := G_1((0, \infty) \times B), \qquad B \in \mathscr{B}_{\partial B_1}.$$

Using the well-known fact on the existence of regular conditional probabilities, we infer that there exists a measurable family $\{\nu(\cdot, x)\}_{x \in \partial B_1}$ of probability measures on $(0, \infty)$ such that, for every $C \in \mathscr{B}_{(0, \infty) \times \partial B_1}$,

$$G_1(C) = \int_{\partial B_1} \int_{(0, \infty)} I_C(t, x) \nu(dt, x) \lambda(dx).$$

Hence

$$G(C) = \int_{\partial B_1} \int_{(0, \infty)} I_C(t, x) [g(t)]^{-1} \nu(dt, x) \lambda(dx).$$

Therefore, for every $A \in \mathscr{B}_E$,

$$\begin{aligned} F(A) &= F_0(A \setminus \{0\}) = G(\Phi(A \setminus \{0\})) \\ &= \int_{\partial B_1} \int_{(0, \infty)} I_A(tx) [g(t)]^{-1} \nu(dt, x) \lambda(dx), \end{aligned}$$

so that (4.1) is fulfilled with $\rho(dt, x) = \nu(dt, x)/g(t)$. □

PROPOSITION 4.3. *Let F be a Borel measure on E satisfying* (4.1). *Let, for each $v \in D$,*

$$R(u, v) := \inf\{t > 0: \rho((t, \infty), v) \le u\}, \qquad u > 0, \tag{4.2}$$

be the right-continuous inverse of the function $t \to \rho((t, \infty), v)$. Then the function H defined by

$$H(u, v) := R(u, v) v$$

satisfies (2.3).

PROOF. For every $A \in \mathscr{B}_E$ we have

$$\begin{aligned} \int_0^\infty \int_D I_{A \setminus \{0\}}(R(u, v) v) \lambda(dv)\, du &= \int_D \left[\int_0^\infty I_{A \setminus \{0\}}(R(u, v) v)\, du \right] \lambda(dv) \\ &= \int_D \left[\int_0^\infty I_{A \setminus \{0\}}(tv) \rho(dt, v) \right] \lambda(dv) = F(A), \end{aligned}$$

where we utilized the fact that $\mathrm{Leb}(\{u \geq 0: R(u, v) \in (t, \infty)\}) = \rho((t, \infty), v)$, $t \geq 0$. □

The results of Sections 2 and 3 when specified to the case $H(u, v) = R(u, v)v$ give the following generalizations of LePage's result ([15], Theorem 2).

COROLLARY 4.4. *Let μ be an infinitely divisible probability measure on E without Gaussian component, i.e.,*

$$\mu = \delta_a \circ c_1 \operatorname{Pois}(F), \tag{4.3}$$

where $a \in E$ and F is a Lévy measure. Assume that F admits a polar decomposition (4.1) *and let R be defined by* (4.2). *Put $S_n = \sum_{j=1}^{n} R(\tau_j, \xi_j)\xi_j$ and*

$$A(t) = \int_0^t \int_D R(u, v) v I_{B_1}(R(u, v)v) \lambda(dv)\, du, \qquad t \geq 0.$$

Then:

(i) *$S_n - A(\tau_n)$ converges a.s., as $n \to \infty$, and $\mathscr{L}(\lim[S_n - A(\tau_n) + a]) = \mu$. If $\int_E \|x\|^p \mu(dx) < \infty$ for some $p > 0$, then the convergence holds also in the L_E^p-norm.*

(ii) *If $\int_E (\|x\|^2 \wedge 1) F(dx) < \infty$, then $S_n - A(n)$ converges a.s., as $n \to \infty$, and $\mathscr{L}(\lim[S_n - A(n) + a]) = \mu$.*

(iii) *If $\int_{B_1} \|x\| F(dx) < \infty$, then $\sum_1^\infty \|R(\tau_j, \xi_j)\xi_j\| < \infty$ a.s., and $\mathscr{L}(\lim S_n + a_0) = \mu$, where $a_0 = a - \int_{B_1} x F(dx)$. In addition, S_n converges in L_E^p provided $\int_E \|x\|^p \mu(dx) < \infty$ for some $p > 0$.*

(iv) *If $\int_E \|x\|^p \mu(dx) < \infty$ for some $p \geq 1$, then $M_n = S_n - C(\tau_n)$ is a martingale with respect to $\sigma(\tau_1, \dots, \tau_n, \xi_1, \dots, \xi_n)$, M_n converges a.s. and in L_E^p as $n \to \infty$, and $\mathscr{L}(\lim M_n + a_1) = \mu$, where $a_1 = a + \int_{B_1^c} x F(dx)$ and*

$$C(t) = \int_0^t \int_D R(u, v) v \lambda(dv)\, du, \qquad t \geq 0.$$

(v) *If μ is symmetric, then $\tilde{S}_n = \sum_{j=1}^{n} \varepsilon_j R(\tau_j, \xi_j)\xi_j$ converges a.s. as $n \to \infty$ and $\mathscr{L}(\lim \tilde{S}_n) = \mu$. In addition, $\tilde{S}_n$ converges in L_E^p provided $\int_E \|x\|^p \mu(dx) < \infty$, for some $p > 0$.*

PROOF. Indeed, by Proposition 4.2 the equality (2.3) is satisfied. Thus, (i) follows from Theorem 2.4 and Corollary 2.5, (ii) is a consequence of Theorem 3.4, Theorem 3.2 justifies the first part of (iii) and the second part follows from Corollary 2.5 and the observation that $\|A(\tau_n)\|$ is uniformly bounded by $\int_{B_1} \|x\| F(dx)$, (iv) is a corollary to Theorem 3.1 and (v) follows from Proposition 2.7 and Corollary 2.5. The proof is complete. □

A few comments are now in order. First note that Corollary 4.4(i) and (ii) generalize LePage ([15], Theorem 2), by removing the restriction concerning the geometry of Banach space E and in our case D is an arbitrary Borel set. This makes the representation useful, e.g., in investigation of general infinitely divisible processes with sample paths in arbitrary Banach spaces. The results on the L_E^p-convergence and the martingale development given in (iv) are also new. Further, one may consider more general polar decompositions of Lévy measures, for example, by replacing tx in (4.1) with $t^B(x) := \exp[(\log t)B](x)$, where B is a given linear operator. Then $H(u, v) := [R(u, v)]^B v$ satisfies (2.3), and Corollary 4.4 holds with obvious modifications. This in particular proves the validity of LePage's representation for operator stable random vectors in general Banach spaces ([15], Theorem 1). Finally, we note that the centering constants in LePage [15], Theorem 2, are erroneous. They should be asymptotically equal to $A(n)$.

Applying Proposition 2.8, we get immediately

COROLLARY 4.5. *Let F be a Borel measure on E which admits a polar decomposition* (4.1). *Then F is a Lévy measure if and only if $\sum_{j=1}^{\infty} \varepsilon_j R(j, \xi_j)\xi_j$ converges a.s.*

EXAMPLE 4.6. Consider the polar decomposition given in Example 4.1. Then $R(u, v) = I(f(v) > u)$ and $A(t) = \int_{B_1}(t \wedge f(x))x\lambda(dx)$. By Corollary 4.4(i) and (v),

$$\sum_{j=1}^{n} I(f(\xi_j) > \tau_j)\xi_j - A(\tau_n) \to T_\infty \quad \text{a.s.}$$

and

$$\sum_{j=1}^{n} \varepsilon_j I(f(\xi_j) > \tau_j)\xi_j \to \tilde{S}_\infty \quad \text{a.s.},$$

as $n \to \infty$, provided F is a Lévy measure [$\mathscr{L}(T_\infty) = c_1 \operatorname{Pois}(F)$ and $\mathscr{L}(\tilde{S}_\infty) = c_1 \operatorname{Pois}(\tilde{F})$]. Moreover, by Corollary 4.5, F is a Lévy measure if and only if

$$\sum_{j=1}^{\infty} \varepsilon_j I(f(\xi_j) > j)\xi_j \quad \text{converges a.s.}$$

Series developments given in the example above provide a simple argument for zero–one laws for infinitely divisible probability measures; a detailed proof will appear elsewhere.

The representation of μ becomes simpler when a polar decomposition of F is of product type for some D, i.e.,

$$F(A) = \int_D \int_{(0,\infty)} I_A(tx)\rho(dt)\lambda(dx) \tag{4.4}$$

for all $A \in \mathscr{B}_E$. In this case, $\rho(\cdot, x) \equiv \rho(\cdot)$ is the same Lévy measure for all x's.

LEMMA 4.7. *Let F be a Lévy measure on E which satisfies* (4.4), *where D is bounded. Then $\int_E(\|x\|^2 \wedge 1)F(dx) < \infty$.*

PROOF. Put $d := \sup\{\|x\|: x \in D\} < \infty$. We have

$$\int_E (\|x\|^2 \wedge 1) F(dx) = \int_D \int_{(0,\infty)} (\|tx\|^2 \wedge 1) \rho(dt) \lambda(dx)$$

$$\leq \int_{(0,\infty)} (d^2 t^2 \wedge 1) \rho(dt) < \infty. \qquad \square$$

The above lemma and Corollary 4.4(ii) give the following.

COROLLARY 4.8. *Let μ be given by* (4.3) *and let F admit decomposition* (4.4) *with D bounded. Define*

$$R(u) := \inf\{t > 0: \rho((t,\infty)) \leq u\}, \qquad u > 0,$$

as the right-continuous inverse of the function $t \to \rho((t,\infty))$. Then

$$\sum_{j=1}^{n} R(\tau_j)\xi_j - b_n + a \to T_\infty \quad a.s., \text{ as } n \to \infty,$$

and $\mathscr{L}(T_\infty) = \mu$, where

$$b_n = \int_0^n \left[R(u) \int_D v I_{B_1}(R(u)v) \lambda(dv) \right] du.$$

EXAMPLE 4.9 (General stable distributions). Let μ be a p-stable probability measure on E, $0 < p < 2$. By the Lévy spectral representation theorem there exist a finite Borel measure σ on ∂B_1 and $x_0 \in E$ such that the characteristic function $\hat{\mu}$ of μ can be written as

$$\hat{\mu}(x') = \exp\left\{ -\int_{\partial B_1} |\langle x', x\rangle|^p \sigma(dx) + iQ_p(\sigma, x') + i\langle x', x_0\rangle \right\}, \tag{4.5}$$

where

$$Q_p(\sigma, x') = \begin{cases} \tan(\pi p/2) \int_{\partial B_1} |\langle x', x\rangle|^p \operatorname{sign}\langle x', x\rangle \sigma(dx), & p \neq 1, \\ -2/\pi \int_{\partial B_1} \langle x', x\rangle \ln|\langle x', x\rangle| \sigma(dx), & p = 1 \end{cases}$$

(for this and further facts concerning stable measures we refer the reader to Linde [17], Chapter 6.3). In order to obtain the series representation of μ we write μ in the form (4.3). Put $m := [\sigma(\partial B_1)]^{1/p}$. Elementary computations give

$$a = \begin{cases} x_0 - (c_p(p-1))^{-1} m^p \bar{x}_\sigma, & p \neq 1, \\ x_0 - 2(1-\gamma)/\pi m \bar{x}_\sigma, & p = 1, \end{cases}$$

where $c_p = \cos(\pi p/2)\Gamma(-p)$, $p \neq 1$, $c_1 = \pi/2$, γ denotes Euler's constant and

$$\bar{x}_\sigma := m^{-p}\int_{\partial B_1} x\sigma(dx).$$

Further, we can represent the Lévy measure F of μ as

$$F(A) = c_p^{-1}\int_{\partial B_1}\int_{(0,\infty)} I_A(tx)t^{-1-p}\,dt\,\sigma(dx)$$
$$= \int_{\partial B_1}\int_{(0,\infty)} I_A(tx)\rho(dt)\lambda(dx),$$

where $\rho(dt) := c_p^{-1}m^p t^{1-p}\,dt$, $\lambda(dx) := m^{-p}\sigma(dx)$. Therefore the assumptions of Corollary 4.8 are satisfied, and we compute

$$R(u) = d_p m u^{-1/p},$$

where $d_p = (pc_p)^{-1/p}$, and, for $n \geq d_p^p m^p$,

$$b_n = \begin{cases} p/(p-1)\left[d_p m n^{1-1/p} - d_p^p m^p\right]\bar{x}_\sigma, & p \neq 1, \\ 2/\pi\left[\ln n - \ln(2/\pi m)\right] m\bar{x}_\sigma, & p = 1. \end{cases}$$

Under the above notation, using Corollaries 4.8, 4.4(iii) and (iv), we obtain

COROLLARY 4.10. *Let μ be a p-stable probability measure on E with the characteristic function given by* (4.5), $0 < p < 2$. *Let*

$$V_n := d_p m\left\{\sum_{j=1}^n \tau_j^{-1/p}\xi_j - k(n)\bar{x}_\sigma\right\} + x_0,$$

where

$$k(t) := \begin{cases} (1-1/p)^{-1}t^{1-1/p}, & 1 < p < 2, \\ \ln t + 1 - \gamma - \ln(d_1 m), & p = 1, \\ 0, & 0 < p < 1. \end{cases}$$

Then $V = \lim_{n\to\infty} V_n$ exists a.s. and $\mathscr{L}(V) = \mu$. Further, for $1 < p < 2$, put

$$M_n := d_p m\left\{\sum_{j=1}^n \tau_j^{-1/p}\xi_j - k(\tau_n)\bar{x}_\sigma\right\} + x_0.$$

Then M_n is a martingale with respect to $\sigma(\tau_1,\dots,\tau_n,\xi_1,\dots,\xi_n)$, $n \geq 1$, $M = \lim_{n\to\infty} M_n$ exists a.s. and in L_E^q for every $0 < q < p$ and $\mathscr{L}(M) = \mu$.

EXAMPLE 4.11 (Symmetric semistable measures). We recall that an infinitely divisible measure μ on E is said to be a (r, p)-semistable probability measure $(0 < r < 1, 0 < p < 2)$ if

$$\mu^{*r} = \left(r^{1/p}\circ\mu\right) * \delta_{x_0} \quad \text{for some } x_0 \in E.$$

Here, the measure $a \circ \mu$ is defined by $(a\circ\mu)(B) = \mu(a^{-1}B)$, $B \in \mathscr{B}_E$, $a \neq 0$.

The spectral representation of the characteristic function of semistable measures was obtained independently by Krakowiak [13] and by Rajput and Rama-Murthy [20], which, in the symmetric case, reduces to

$$\hat{\mu}(x') = \exp\left\{ \sum_{n=-\infty}^{\infty} r^{-n} \int_{\Delta} \left[\cos\left(r^{n/p}\langle x', x\rangle\right) - 1\right] \sigma(dx) \right\}, \tag{4.6}$$

where σ is a finite symmetric measure on $\Delta := \{x \in E: r^{1/p} < \|x\| \le 1\}$. Since

$$\hat{\mu}(x') = \exp\left\{ \int_{\Delta} \int_{(0,\infty)} [\cos\langle x', tx\rangle - 1] \nu(dt) \sigma(dx) \right\},$$

where ν is a discrete measure concentrated on the set $\{r^{n/p}: n \in \mathbb{Z}\}$, such that $\nu(\{r^{n/p}\}) = r^{-n}$, $n \in \mathbb{Z}$, we conclude that (4.4) is satisfied with $\lambda(dx) := \sigma^{-1}(\Delta)\sigma(dx)$ and $\rho(dt) := \sigma(\Delta)\nu(dt)$. Now by elementary computations we obtain

$$R(u) = \left[(1/r - 1)\sigma^{-1}(\Delta)u\right]_r^{-1/p},$$

where $[t]_r := r^k$ if $r^k \le t < r^{k-1}$. In view of Corollary 4.4(v) we get that

$$\sum_{j=1}^{n} \varepsilon_j \left[(1/r - 1)\sigma^{-1}(\Delta)\tau_j\right]_r^{-1/p} \xi_j \to S \quad \text{a.s.} \tag{4.7}$$

and in L_E^q, for every $0 \le q < p$ and $\mathscr{L}(S) = \mu$. We have obtained a series representation of semistable random vectors in the symmetric case.

Now we note that the multipliers in (4.7) are bounded on both sides, up to a constant multiplier, by $\sigma^{1/p}(\Delta)\tau_j^{-1/p}$, because $rt < [t]_r \le t$, $t > 0$. Further, a p-stable limit is obtained in (4.7) when one replaces $[(1/r - 1)\sigma^{-1}(\Delta)\tau_j]_r$ by $(1/r - 1)\sigma^{-1}(\Delta)\tau_j$. This, in conjunction with the contraction principle, explains why the moment properties of stable and semistable distributions, are so closely related. Using a different method of stochastic integral, this observation was also justified in Rosinski [24], pages 67–68, and comparisons of moments of stable and semistable measures were given.

5. Conditionally Gaussian infinitely divisible random vectors. Now we return to some problems considered in the last part of Section 3. Let $\{\gamma_j\}$ be a sequence of i.i.d. standard normal random variables, which is independent of other random sequences defined in this paper. Let F be a Borel measure which admits a decomposition (4.1) and consider the series

$$\sum_{j=1}^{\infty} \gamma_j R(\tau_j, \xi_j)\xi_j, \tag{5.1}$$

where R, τ_j, ξ_j are as in Corollary 4.4. If this series converges a.s., then its distribution is infinitely divisible without Gaussian component, while, for each fixed realization of $\{\tau_j\}$ and $\{\xi_j\}$, series (5.1) represents a Gaussian random vector in E. Proposition 3.5 shows that neither the convergence nor distribution of the above series depends on the decomposition (4.1). Specifically, series (5.1) converges a.s. if and only if F_γ is a Lévy measure, where

$$F_\gamma(A) = E\left[\tilde{F}(A|\gamma_1|^{-1})\right] = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} F(Au^{-1})e^{-u^2/2}\,du,$$

and further, $\mathscr{L}(\sum_1^\infty \gamma_j R(\tau_j, \xi_j)\xi_j) = c_1 \operatorname{Pois}(F_\gamma)$. In particular, F has to be a Lévy measure provided series (5.1) converges.

We shall say that a measure μ on E *is of type* G if there exists a σ-finite measure F on E such that $\mu = c_1 \operatorname{Pois}(F_\gamma)$. From the above discussion it follows that every measure of type G has a series representation of the form (5.1). Thus measures of type G are mixtures (of a special kind) of Gaussian probability measures on E.

REMARK. The notion of a measure of type G has been introduced by Marcus [18], but his definition is more restrictive than ours. Namely, in [18] it is assumed that F admits a "product-type" polar decomposition (4.4), so that R in (5.1) depends only on the τ_j's. We shall characterize this subclass of measures of type G in Corollary 5.3.

THEOREM 5.1. *A measure μ is of type G if and only if its characteristic function can be written in the form*

$$\hat{\mu}(x') = \exp\left\{-\int_E \psi(\langle x', x\rangle^2)V(dx)\right\}, \tag{5.2}$$

where V is a Borel measure on E and $\psi\colon [0,\infty) \to [0,\infty)$ has completely monotone derivative [*i.e.*, $(-1)^{n-1}[d^n\psi/ds^n] \geq 0$ *for all* $s > 0$, $n = 1, 2, \dots$] *and* $\psi(0) = 0$.

PROOF. The necessity follows by Corollary 3.6(b). Namely, one takes $V = \tilde{F}$ and $\psi(s) := 1 - \exp(-s/2)$. To prove the sufficiency, note that by Feller [6], Chapter 13.7, Theorems 1–2, there exists a Borel measure ν on $(0,\infty)$ with $\int_{(0,\infty)}(1 \wedge u^{-1})\nu(du) < \infty$ such that

$$\psi(s) = \int_{(0,\infty)} [1 - e^{-su}]u^{-1}\nu(du), \qquad s \geq 0.$$

Define

$$\rho(B) := \int_{(0,\infty)} I_B(\sqrt{2u})u^{-1}\nu(du),$$

$$F(A) := \int_{E\setminus\{0\}}\int_{(0,\infty)} I_A(tx)\rho(dt)V(dx),$$

$A \in \mathscr{B}_E$, $B \in \mathscr{B}_{(0,\infty)}$. Then we have

$$\begin{aligned}\hat{\mu}(x') &= \exp\left\{-\int_{E\setminus\{0\}}\int_{(0,\infty)}\left[1 - e^{-\langle x', x\rangle^2 u}\right]u^{-1}\nu(du)V(dx)\right\}\\ &= \exp\left\{\int_{E\setminus\{0\}}\int_{(0,\infty)}\left[e^{-\langle x', x\rangle^2 t^2/2} - 1\right]\rho(dt)V(dx)\right\}\\ &= \exp\left\{\int_E E\left[\cos(\langle x', y\rangle\gamma_1) - 1\right]F(dy)\right\}\\ &= \exp\left\{\int_E\left[\cos\langle x', x\rangle - 1\right]F_\gamma(dx)\right\}.\end{aligned}$$

Hence F_γ is a Lévy measure and $\mu = c_1\,\mathrm{Pois}(F_\gamma)$, which completes the proof. □

EXAMPLE 5.2. (i) If μ is a symmetric p-stable probability measure, then the usual representation of $\hat{\mu}$,

$$\hat{\mu}(x') = \exp\left\{-\int_{\partial B_1}|\langle x', x\rangle|^p\sigma(dx)\right\}, \tag{5.3}$$

is of the form (5.2) with $\psi(s) = s^{p/2}$ and $V = \sigma$. Thus the fact that p-stable random vectors can be represented as conditionally Gaussian follows immediately from Theorem 5.1.

(ii) Consider a probability measure μ whose characteristic function can be written in the form

$$\hat{\mu}(x') = \exp\left\{-\int_E \log(1 + \langle x', x\rangle^2)V(dx)\right\}, \tag{5.4}$$

where V is a measure on E. If V is discrete, say, $V = \sum_1^n p_j\delta_{x_j}$, $p_j > 0$, then μ is the distribution of $\sum_1^n(\zeta_j - \zeta_j')x_j$, where $\zeta_1, \zeta_1', \dots, \zeta_n, \zeta_n'$ are independent random variables such that ζ_j and ζ_j' have the same gamma distribution with parameters $(p_j, 1)$. We shall call probability measures satisfying (5.4) *symmetric gamma distributions on E*. Since $\psi(s) := \log(1 + s)$ has completely monotone derivative, clearly symmetric gamma distributions are of type G.

COROLLARY 5.3. *A probability measure μ is of the form $\mu = c_1\,Pois(F_\gamma)$ for some F which admits a "product-type" polar decomposition* (4.4) *if and only if $\hat{\mu}$ can be written in the form* (5.2) *where V is a probability measure.*

PROOF. By Corollary 3.6(b) and (4.4) we get

$$\hat{\mu}(x') = \exp\left\{-\int_D\left(\int_{(0,\infty)}\left[1 - \exp(-\langle x', x\rangle^2 t^2/2)\right]\rho(dt)\right)\tilde{\lambda}(dx)\right\}.$$

This proves the necessity since $\psi(s) := \int_{(0,\infty)}[1 - \exp(-st^2/2)]\rho(dt)$ has completely monotone derivative [one can interchange differentiation and integration since $\int_{(0,\infty)}[1 \wedge u^2]\rho(du) < \infty]$ and $V := \tilde{\lambda}$ is a probability measure. A proof of the sufficiency is the same as in Theorem 5.1. □

COROLLARY 5.4. *A probability measure μ on $\mathbf{R}$ is of type G if and only if*

$$\hat{\mu}(s) = \exp(-\psi(s^2)), \qquad s \in \mathbf{R}, \tag{5.5}$$

for some $\psi: [0,\infty) \to [0,\infty)$ with completely monotone derivative and $\psi(0) = 0$.

PROOF. From Corollary 3.6(b) $\hat{\mu}$ satisfies (5.5) with

$$\psi(s) := \int_{\mathbf{R}} [1 - \exp(-st^2/2)] \tilde{F}(dt),$$

which proves the necessity. Sufficiency follows from Theorem 5.1. □

REMARK 5.5. Marcus [18], Lemma 2.6, showed that every measure μ of type G on $\mathbf{R}$ is a variance mixture of the normal distribution, i.e.,

$$\mu = \mathscr{L}(\gamma\eta), \tag{5.6}$$

where γ and η are independent and γ is $\mathscr{N}(0,1)$. Moreover, η^2 is infinitely divisible. Using Corollary 5.4 and Feller [6], Chapter 13.7, Theorem 1, it is easy to prove the following: *a measure μ on $\mathbf{R}$ is of type G if and only if μ is of the form* (5.6) *where η^2 is infinitely divisible*. Note that the assumption that η^2 is infinitely divisible is crucial for the converse to hold. Indeed, Kelker [12] gave an example of an infinitely divisible distribution μ which satisfies (5.6) and such that η^2 is not infinitely divisible. Since $\mathscr{L}(\eta^2)$ is completely determined by μ in (5.6), it follows that there are infinitely divisible distributions which are variance mixtures of the normal distribution and, at the same time, they are not of type G. This disproves the conjecture in Marcus [18], Remark 2.7.

REMARK 5.6. The right side in (5.2) represents a characteristic function of a certain cylindrical measure on E, for any choice of ψ and V, provided V is a measure on E such that $\int_E \psi(\langle x', x\rangle^2)V(dx) < \infty$ for every $x' \in E'$, and ψ has completely monotone derivative with $\psi(0) = 0$. This fact can be used to produce a number of examples of measures of type G on $\mathbf{R}^n$, and on general Banach spaces (where an additional problem of extension of cylindrical measures must be dealt with). The above fact can be also used to give an appropriate definition of stochastic processes of type G.

Lemma 3.6 proves that there are Lévy measures F on $E = C[0,1]$ for which series (5.1) diverges. On the other hand, if the Banach space E is of finite cotype, then it follows, from a more general fact (see, e.g., [1], Theorem 8.19), that series (5.1) converges a.s., provided F is a Lévy measure. In the theorem below we shall give a condition on a Lévy measure F which guarantees the convergence in (5.1) without any hypothesis on E. The next lemma will be used for this purpose but it also has an independent interest. In particular, it generalizes the contraction principles for certain stochastic integrals and ξ-radial processes (see Rosinski [24], Theorem 4.3.3, and Marcus [18], Lemma 7.3, respectively).

LEMMA 5.7 (Contraction principle for Lévy measures). *Let F_i be Borel measures on E with $F_i(\{0\}) = 0$, $i = 1, 2$. Suppose that F_1 is a Lévy measure and there are constants c, k, $r_0 > 0$ such that*

$$(5.7)\qquad F_2(\{x: \|x\| > r, x/\|x\| \in B\}) \le cF_1(\{x: \|x\| > kr, x/\|x\| \in B\}),$$

for every $0 < r \le r_0$ and $B \in \mathscr{B}_{\partial B_1}$. Then F_2 is also a Lévy measure.

PROOF. Since cF_1 is also a Lévy measure we may (and do) assume that $c = 1$. Further, by (5.7), $F_2(B_{r_0}^c) < \infty$, therefore it is enough to show that $F_3 := F_{2|B_{r_0}}$ is a Lévy measure. Inequality (5.7) yields

$$(5.8)\qquad F_3(\{x: \|x\| > r, x/\|x\| \in B\}) \le F_1(\{x: \|x\| > r, x/\|x\| \in B\}),$$

for all $r > 0$ and $B \in \mathscr{B}_{\partial B_1}$. Let

$$F_i(A) = \int_{\partial B_1} \int_{(0,\infty)} I_A(tx)\rho_i(dt, x)\lambda_i(dx), \qquad A \in \mathscr{B}_E,$$

be polar decompositions of F_i, $i = 1, 3$ (see Proposition 4.2). Put $\lambda_0 := (\lambda_1 + \lambda_3)$ and $\rho_i^*(dt, x) := \rho_i(dt, x)[d\lambda_i/d\lambda_0](x)$, $i = 1, 3$. Then we have

$$F_i(A) = \int_{\partial B_1} \int_{(0,\infty)} I_A(tx)\rho_i^*(dt, x)\lambda_0(dx),$$

for all $A \in \mathscr{B}_E$, $i = 1, 3$. Using (5.8), we get, for λ_0-almost all $x \in \partial B_1$,

$$\rho_3^*((r, \infty), x) \le \rho_1^*((r, \infty), x),$$

for all $r > 0$. Let R_i be defined by (4.2) with ρ replaced by ρ_i^*, $i = 1, 3$. The above inequality yields, for λ_0-almost all $x \in \partial B_1$,

$$(5.9)\qquad R_3(u, x) \le R_1(u, x) \quad \text{for all } u > 0.$$

By Corollary 4.4(v), $\sum_1^\infty \varepsilon_j R_1(\tau_j, \xi_j)\xi_j$ converges a.s., where the ξ_j's have common distribution λ_0. Using (5.9) and the contraction principle, $\sum_1^\infty \varepsilon_j R_3(\tau_j, \xi_j)\xi_j$ converges a.s. which implies, by Proposition 2.7, that F_3 is a Lévy measure. The proof is complete. □

THEOREM 5.8. *Let F be a Lévy measure which admits decomposition* (4.1) *such that*

$$\rho((2^{-1}t, \infty), x) \le c\rho((t, \infty), x),$$

for all $0 < t \le r_0\|x\|^{-1}$, $x \in D$ and some constants c, $r_0 > 0$. Then series (5.1) *converges a.s.*

PROOF. The condition for ρ implies

$$\rho((s^{-1}t, \infty), x) \le c(s^q \vee 1)\rho((t, \infty), x),$$

for all $s > 0$, $0 < t \le r_0\|x\|^{-1}$ and $x \in D$, where $q = \log_2 c$. Without loss of generality we may assume that F is symmetric. Let $0 < r \le r_0$, $B \in \mathscr{B}_{\partial B_1}$ and put $A = \{x: \|x\| > r, x/\|x\| \in B\}$. We get

$$F_\gamma(A) = E\left[F(A|\gamma_1|^{-1})\right] = E\int_D \rho\big((|\gamma_1|^{-1}r\|x\|^{-1}, \infty), x\big) I_B(x/\|x\|)\lambda(dx)$$

$$\leq E\int_D c\big[|\gamma_1|^q \vee 1\big]\rho\big((r\|x\|^{-1}, \infty), x\big) I_B(x/\|x\|)\lambda(dx) = c_1 F(A),$$

where $c_1 = cE[|\gamma_1|^q \vee 1] < \infty$. Lemma 5.5 completes the proof. □

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