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AN ASYMPTOTIC EXPRESSION FOR THE PROBABILITY OF RUIN WITHIN FINITE TIME

BY THOMAS HÖGLUND

Royal Institute of Technology, Stockholm

We consider quantities such as the probability that a two-dimensional random walk crosses the ordinate y for the first time to the left of the abscissa x, and describe the asymptotic behaviour as x and y tend to ∞. The result is applied to the risk reserve process of insurance mathematics as well as to one-dimensional random walks.

1. Introduction. Consider a two-dimensional random walk $\Sigma_1^n Z_i$, where $Z_i = (X_i, Y_i)$, $i = 1, 2, \dots,$ are independent with a common distribution F. Define

$$\begin{aligned} N(x) &= \min\{n \geq 1;\, S_n > x\}, \\ u(x, y) &= \operatorname{prob}\big(N(x) < \infty,\, S_{N(x)} > x + a,\, T_{N(x)} \in y + B\big) \end{aligned} \tag{1.1}$$

when $x \geq 0$. Here $S_n = \Sigma_1^n X_i$, $T_n = \Sigma_1^n Y_i$, $a \geq 0$ and B is a regular set, a finite union of intervals, say.

von Bahr (1974) derived a renewal equation for u (when $a = 0$, $B = (-\infty, 0]$ and $Y_k \geq 0$) and then applied a renewal theorem to derive an asymptotic expression for u. The approximation is valid when x and y tend to infinity in the first quadrant in a neighbourhood of a certain line: $x/\alpha = y/\beta$. We shall here do about the same thing, but instead apply the renewal theorem of Höglund (1988) and obtain approximations which in some cases are valid when x and y tend to infinity in any other direction in the half-plane $x > 0$. It turns out that the problem is a truly two-dimensional one when $x/\alpha > y/\beta$ and an essentially one-dimensional one when $x/\alpha < y/\beta$. See also Stam (1971) for a related result valid in a neighbourhood of the line $x/EX = y/EY$. Approximations for $u(x, y)$ have applications in sequential analysis; see Siegmund (1985) and Woodroofe (1982).

A classical example is the risk reserve process

$$R(t) = x + ct - \sum_{k=1}^{P(t)} U_k \tag{1.2}$$

where $c > 0$, $P(t)$ is a Poisson process and $U_1, U_2, \dots$ are i.i.d. random variables, independent of the Poisson process. Ruin occurs before time y if

$$\inf\{R(t);\, 0 < t < y\} < 0. \tag{1.3}$$

The probability of this event equals $u(x, y)$ in the special case when $a = 0$, $B = (-\infty, 0)$, $X_k = U_k - c\tau_k$, $Y_k = \tau_k$. Here $\tau_1, \tau_2, \dots$ denotes the interarrival times of the Poisson process.

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The same remark applies to the more general case when the Poisson process is replaced by a renewal process, i.e., when $\tau_1, \tau_2, \dots$ are i.i.d. and positive, but not necessarily exponentially distributed. Thorin (1982) considered this case and gave the ruin probability a form suited for numerical calculations. See this paper and Asmussen (1984) for some of the history of the subject.

Arfwedson (1954 and 1955) considered the original risk reserve process and gave (in Section 14) approximations for $u(x, y)$. Recent results for this process are those of Asmussen (1984) and Martin-Löf (1986). Asmussen considered u in a neighbourhood of the same line as von Bahr, and his method made it possible to evaluate the approximation numerically in some cases. Martin-Löf gave upper bounds for u. The results of Arfwedson and Martin-Löf are valid for arbitrary x and y in the first quadrant, and they are as far as I know the only previous results that have this property.

2. Results. Define

$$\phi(\zeta) = \int e^{\zeta \cdot z} F(dz), \qquad F_\zeta(dz) = e^{\zeta \cdot z} F(dz)/\phi(\zeta) \tag{2.1}$$

for those $\zeta \in \mathbb{R}^2$ for which $\phi(\zeta) < \infty$ and let E_ζ denote expectation with respect to the probability measure F_ζ. Then

$$E_\zeta Z = (E_\zeta X, E_\zeta Y) = (\partial_1 \phi(\zeta), \partial_1 \phi(\zeta))/\phi(\zeta) \tag{2.2}$$

at least when ϕ is finite in a neighbourhood of ζ.

Put $\Delta = \{\zeta \in \mathbb{R}^2; \phi(\zeta) = 1\}$. Then $\Delta \neq \varnothing$ because $0 \in \Delta$. The function $\phi(\zeta)$ is convex and so is therefore also the set $\{\zeta; \phi(\zeta) \leq 1\}$. Its boundary coincides with Δ in regular cases. We are able to determine the asymptotic behaviour of $u(z)$ when z tends to infinity in the cone

$$K = \{tE_\zeta Z;\ t > 0,\ \zeta \in \Delta,\ E_\zeta X > 0\}, \tag{2.3}$$

provided F is sufficiently regular.

Let N denote the first strict ascending ladder index $N = \min\{n > 0;\ S_n > 0\}$ and let G and G_1 denote the smallest closed additive subgroups of $\mathbb{R}^2$, respectively $\mathbb{R}$, that contain the supports of the measures prob$(N < \infty,\ S_N \in dx,\ T_N \in dy)$, respectively prob$(N < \infty,\ S_N \in dx)$. Let $z = (x, y)$, $\zeta = (\xi, \eta)$ and put

$$\begin{aligned} C_\zeta(a, B) &= \int_{x > a} \int_{y \in B} P_\zeta(S_N \geq x) e^{-\zeta \cdot z} \sigma(dz)/E_\zeta S_N, \\ C_\zeta(a) &= \int_{x > a} P_\zeta(S_N \geq x) e^{-\xi x} \sigma_1(dz)/E_\zeta S_N \end{aligned} \tag{2.4}$$

provided ζ is such that $E_\zeta S_N < \infty$. Here P_ζ stands for the product measure on $\mathbb{R}^{\mathbf{Z}_+}$ determined by the cylinder set probabilities $P_\zeta(X_1 \in A_1, \dots, X_n \in A_n) = \int_{A_1} e^{\xi x} F(dx, \mathbb{R})/\phi(\zeta) \cdot \cdots \cdot \int_{A_n} e^{\xi x} F(dx, \mathbb{R})/\phi(\zeta)$, and E_ζ denotes expectation with respect to this measure. The measures σ and σ_1 are the Haar measures on G, respectively G_1, normed in such a way that $\sigma(tD_2)/l_2(tD_2) \to 1$ and

$\sigma_1(tD_1)/l_1(tD_1) \to 1$ as $t \to \infty$. Here l_j and D_j denote the Lebesgue measure, respectively the unit ball in $\mathbb{R}^j$, $j = 1, 2$.

THEOREM 2.1. *Assume that* $\dim G = 2$, *that* ϕ *is finite in neighbourhoods of the points* $\zeta = (\xi, \eta), (0, \eta)$ *and that* $\phi(\zeta) = 1$, $E_\zeta X > 0$. *Let* $z \in G$ *tend to infinity in such a way that* $z/|z| = E_\zeta Z/|E_\zeta Z|$.

(a) *If* $\int_B e^{-\eta t}\, dt < \infty$, *then* $C_\zeta(a, B) < \infty$ *and*

$$(2\pi Q(\zeta)x/E_\zeta X)^{1/2} e^{\zeta \cdot z} u(z) \to C_\zeta(a, B). \tag{2.5}$$

Here $Q(\zeta) = E_\zeta(Y - XE_\zeta Y/E_\zeta X)^2 > 0$.

(b) *If* $\int_{B^c} e^{-\eta t}\, dt < \infty$ *and if there is a* $\zeta_0 = (\kappa, 0)$ *with* $\phi(\zeta_0) = 1$ *and* $0 < E_{\zeta_0} X < \infty$, *then* $C_{\zeta_0}(a) < \infty$ *and*

$$e^{\zeta_0 \cdot z} u(z) \to C_{\zeta_0}(a). \tag{2.6}$$

PROOF. The theorem is a direct consequence of Proposition 3.2 and Lemma 3.3. □

In order to use the approximations of the theorem we thus have to solve the equation $z/|z| = E_\zeta Z/|E_\zeta Z|$, i.e., choose ζ so that $E_\zeta Z$ has the same direction as z. An alternative formulation of the theorem would be to state that the approximations hold uniformly when $z/|z|$ belongs to certain sets of directions, i.e., when ζ belongs to certain subsets of the plane. I have chosen to keep ζ fixed because this simplifies the formulation slightly. (Note that the ray $\{tE_\zeta Z, t > 0\}$ contains infinitely many $z \in G$ even when G is discrete, provided G and the ray have at least one point in common.) The approximations (2.5) and (2.6) hold even when z deviates $o(|z|^{1/2})$ from the ray.

Note that $\int_B e^{-\eta t}\, dt < \infty$ for all η if B is bounded, for all $\eta > 0$ if $\inf B > 0$ and for all $\eta < 0$ if $\sup B < \infty$. If B is bounded, then the case $\int_{B^c} < \infty$ never occurs, but if $B = (-\infty, 0]$, say, then the first approximation holds when $\eta < 0$ and the second when $\eta > 0$. What happens when $\eta = 0$? The approximation of von Bahr mentioned in the introduction is

$$u(z) \sim C_{\zeta_0}(0)\exp(-\kappa x)\Phi(w), \tag{2.7}$$

valid when

$$w = (y - xE_{\zeta_0}Y/E_{\zeta_0}X)(Q(\zeta_0)x/E_{\zeta_0}X)^{-1/2} \tag{2.8}$$

is bounded. Here

$$\Phi(w) = (2\pi)^{-1/2}\int_{-\infty}^{w} \exp(-\tfrac{1}{2}t^2)\, dt. \tag{2.9}$$

The function $\xi \to \phi(\xi, 0)$ is convex and $E_{\xi,0}X$ and $\partial_1\phi(\xi, 0)$ have the same sign. Therefore if $E_{0,0}X > 0$, then $\kappa = 0$ and if $E_{0,0}X < 0$, then there is a possibility that there is a $\kappa > 0$ such that $\phi(\kappa, 0) = 1$ and then we must necessarily have $E_{\kappa,0}X > 0$. Finally if $E_{0,0}X = 0$, then $\phi(\xi, 0) > 1$ for all $\xi \neq 0$.

It follows from the definition (2.4) that $C_\zeta(a) \le C_\zeta(0)$ and that $C_\zeta(0) < 1$, $= 1$ or > 1 according as $\xi > 0$, $= 0$ or < 0. If G has the special form $G = G_1 \times G_2$, then

$$C_\zeta(a, B) = C_\zeta(a)\int_{y \in B} e^{-\eta y}\sigma_2(dy). \tag{2.10}$$

Here σ_2 is the Haar measure on G_2 normed as σ_1 above. (That is, σ_2 equals the Lebesgue measure if $G_2 = \mathbb{R}$ and it equals h times the counting measure if $G_2 = h\mathbb{Z}$.) The constant $C_\zeta(0)$ can be expressed in terms of the quantities

$$I(\xi) = \begin{cases} \xi h/(e^{\xi h} - 1) & \text{if } G_1 = h\mathbb{Z} \text{ and } \xi \ne 0, \\ 1 & \text{if } G_1 = R \text{ or } \xi = 0, \end{cases} \tag{2.11}$$

$$\begin{aligned} \Sigma^+(\zeta) &= \sum_{n=1}^{\infty} n^{-1} \int_{x>0}\int e^{\zeta \cdot z} F^{n*}(dz), \\ \Sigma^-(\zeta) &= \sum_{n=1}^{\infty} n^{-1} \int_{x\le 0}\int e^{\zeta \cdot z} F^{n*}(dz). \end{aligned} \tag{2.12}$$

It will be shown at the end of the paper that

$$C_\zeta(0)/I(\xi) = \begin{cases} (1 - \phi(0,\eta))(\xi E_\zeta X)^{-1} e^{\Sigma^-(0,\eta) - \Sigma^-(\zeta)} & \text{if } \xi < 0, \\ 1 & \text{if } \xi = 0, \\ (\xi E_\zeta X)^{-1} e^{-\Sigma^+(0,\eta) - \Sigma^-(\zeta)} & \text{if } \xi > 0. \end{cases} \tag{2.13}$$

Note that

$$\begin{aligned} T_{N(x)} - y &= T_{N(x)} - N(x)E_\zeta Y + \big(N(x) - x/E_\zeta X\big)E_\zeta Y, \\ x/N(x) &< S_{N(x)}/N(x) \le x/N(x) + \varepsilon(N(x)), \end{aligned} \tag{2.14}$$

where $\varepsilon(n) = \max_{1 \le k \le n} X_k n^{-1/2} \to 0$ in probability as $n \to \infty$, since X has finite second moment. Therefore by Anscombe's central limit theorem,

$$\big(T_{N(x)} - y\big)/\big(Q(\zeta)x/E_\zeta X\big)^{1/2} \tag{2.15}$$

is asymptotically $N(0,1)$ under P_ζ. This is one of several comments made by a referee which have improved the paper.

An interesting special case is when prob$(Y = 1) = 1$, $a = 0$ and $B = (-\infty, 0]$. Let $\chi(\xi) = E\exp(\xi X)$ and let $\mu(\xi)$ and $\tau^2(\xi)$ be the expectation and variance of the conjugate distribution, i.e.,

$$\begin{aligned} \mu(\xi) &= E(X\exp(\xi X)/\chi(\xi)) = \chi'(\xi)/\chi(\xi), \\ \tau^2(\xi) &= E\big((X - \mu(\xi))^2 \exp(\xi X)/\chi(\xi)\big) \\ &= \chi''(\xi)/\chi(\xi) - (\chi'(\xi)/\chi(\xi))^2. \end{aligned} \tag{2.16}$$

Then $\phi(\xi, \eta) = \chi(\xi)e^{\eta}$, $E_{\xi,\eta}Z = (\mu(\xi), 1)$ and $Q(\xi, \eta) = \tau^2(\xi)/\mu(\xi)^2$.

Furthermore $e^{-\eta} = \chi(\xi)$ and hence $\int_B e^{-\eta t}\,dt < \infty$ if and only if $\chi(\xi) > 1$, and $\int_{B^c} e^{-\eta t}\,dt < \infty$ if and only if $\chi(\xi) < 1$. Assume that $\mu(\xi) > 0$ and that there is a

κ such that $\chi(\kappa) = 1$ and $\mu(\kappa) \geq 0$. Then $\mu(\kappa) = 0$ implies $\kappa = 0$. Furthermore, $\chi(\xi) > 1$ is equivalent to $\mu(\xi) > \mu(\kappa)$ and $\chi(\xi) < 1$ to $\mu(\xi) < \mu(\kappa)$.

The constants (2.4) take the form

$$C_\xi = \int_{x>0}\int_{y\leq 0} P_\xi(S_N \geq x)e^{-\xi x}\chi(\xi)^y\sigma(dx, dy)/E_\xi S_N, \tag{2.17}$$

$$C = \int_{x>0} P_\kappa(S_N \geq x)e^{-\kappa x}\sigma_1(dx)/E_\kappa S_N.$$

Here P_ξ stands for the product measure on $\mathbb{R}^{\mathbb{Z}_+}$ determined by the cylinder set probabilities

$$P_\xi(X_1 \in A_1, \ldots, X_n \in A_n) = \int_{A_1} e^{\xi x}F_1(dx)/\chi(\xi) \cdot \cdots \cdot \int_{A_n} e^{\xi x}F_1(dx)/\chi(\xi)$$

and E_ξ denotes expectation with respect to this measure. Here F_1 is the original distribution of X.

The observation that if X is not concentrated at one point, then $\dim G = 2$ now gives

COROLLARY 2.2. *Let $X_1, X_2, \ldots$ be independent (one-dimensional) random variables with a common distribution and put $S_k = X_1 + \cdots + X_k$. Assume that X_1 is not concentrated at one point, that χ is finite in a neighbourhood of the point ξ and that $\mu(\xi) > 0$. Assume further that there is a κ such that $\chi(\kappa) = 1$ and $0 \leq \mu(\kappa) < \infty$. If $(x, n) \in G$ tends to infinity in such a way that $x/n = \mu(\xi)$, then*

$$P\Big(\max_{k\leq n} S_k > x\Big) \sim \begin{cases} C_\xi\mu(\xi)\big(2\pi n\tau^2(\xi)\big)^{-1/2}\chi(\xi)^n e^{-\xi x} & \text{if } x/n > \mu(\kappa), \\ Ce^{-\kappa x} & \text{if } x/n < \mu(\kappa). \end{cases} \tag{2.18}$$

Compare (2.18) with

$$P(S_n > x) \sim \int_{y>0} e^{-\xi y}\sigma_1(dy)\big(2\pi n\tau^2(\xi)\big)^{-1/2}\chi(\xi)^n e^{-\xi x} \tag{2.19}$$

valid when $x/n = \mu(\xi) > \mu(0)$. [See Blackwell and Hodges (1959) for the case $G_1 = \mathbb{Z}$, Bahadur and Ranga Rao (1960) for the case $G_1 = \mathbb{R}$ or Höglund (1979) for a unified treatment.] The two probabilities are thus of the same order of magnitude when $x/n > \mu(\kappa)$.

EXAMPLE. Let $P(X = 1) = p$ and $P(X = -1) = 1 - p = q$. Then $G = \{(x, y) \in \mathbb{Z}^2;\ x - y \text{ is even}\}$, $S_N = 1$ identically, and $\chi(\xi) = pe^{\xi} + qe^{-\xi}$. Further calculations show that the approximations to the right in (2.18) in this case take the form

$$(2.20)\qquad \begin{cases} \left((p/\hat p)^{\hat p}(q/\hat q)^{\hat q}\right)^n (2\pi n\hat p\hat q)^{-1/2}\dfrac{p\hat q(\hat p - q)}{(pq-\hat p\hat q)} & \text{if } x/n > |p-q|,\\ (\min(1,p/q))^{x+1} & \text{if } 0 < x/n < |p-q|. \end{cases}$$

Here $\hat p = (1 + x/n)/2$ and $\hat q = (1 - x/n)/2$.

The approximation (2.18) can also be expressed in another way. Let s denote the convex function

$$(2.21)\qquad s(t) = \sup_\xi \{\xi t - \log\chi(\xi)\}.$$

Then $s(\mu(\xi)) = \xi\mu(\xi) - \log\chi(\xi)$ and hence

$$(2.22)\qquad \chi(\xi)^n e^{-\xi x} = e^{-ns(x/n)}$$

when $x/n = \mu(\xi)$. Furthermore, $s(t) \geq \kappa t$ for all t with equality if and only if $t = \mu(\kappa)$, and hence $ns(x/n) \geq \kappa x$ with equality only if $x/n = \mu(\kappa)$.

If we instead apply the theorem to the same situation as in Corollary 2.2, but with $B = \{0\}$ we obtain the variant

$$(2.23)\qquad P(S_1 \leq x,\ldots,S_{n-1}\leq x, S_n > x) \sim \tilde C_\xi \mu(\xi)\left(2\pi n\tau^2(\xi)\right)^{-1/2}\chi(\xi)^n e^{-\xi x}$$

as x and n tend to ∞ in such a way that $(x, n) \in G$ and $x/n = \mu(\xi) > 0$. Here

$$(2.24)\qquad \tilde C_\xi = \int_{x>0} P_\xi(S_N \geq x)e^{-\xi x}\sigma(dx,\{0\})/E_\xi S_N.$$

Note that the constant (2.24) equals zero in the example above. This is as it should be because the expression to the left in (2.23) equals zero when $(x, n) \in G$.

Another special case is the risk reserve process described in the introduction. Put $T(x) = \inf\{t \geq 0;\ R(t) < 0\}$. Then $\text{prob}(T(x) < \infty,\ T(x) \in y + B) = u(x, y)$ when $a = 0$.

Let $\omega(\xi) = E\exp(\xi U)$. Then $\phi(\xi,\eta) = \omega(\xi)\rho/(\rho + c\xi - \eta)$ where ρ is the intensity of the Poisson process, and hence $\phi(\xi,\eta) = 1$ if and only if $\eta = c\xi - \rho(\omega(\xi) - 1)$. If this is the case, then

$$(2.25)\qquad \begin{aligned} E_\xi Z &= \left(\rho\omega(\xi)\right)^{-1}\left(\rho\omega'(\xi) - c, 1\right),\\ Q(\zeta) &= \omega''(\xi)/\left(\omega(\xi)(\rho\omega'(\xi)-c)^2\right). \end{aligned}$$

Furthermore, $E_\xi X > 0$ implies that $\dim G = 2$. In this context κ is known as Lundberg's constant, which is usually denoted by R, and it is convenient to express the conditions in terms of the function $g(\xi) = \rho(\omega(\xi) - 1) - c\xi$.

Note that $-\eta = g(\xi) > 0$ if and only if $g'(\xi) > g'(R)$ under the condition $g'(\xi) > 0$, since g is convex. Some further calculations therefore yield.

COROLLARY 2.3. *The following holds for the risk reserve process. Assume that ω is finite in a neighbourhood of the point ξ and that $g'(\xi) > 0$. Assume* *further that there is a* $R \geq 0$ *such that* $g(R) = 0$ *and* $g'(R) > 0$. *If* x *and* $y \to \infty$ *in such a way that* $(x, y) \in G$ *and* $x/y = g'(\xi)$, *then*

$$(2.26) \qquad P(T(x) < y) \sim \begin{cases} C_\xi^* y^{-1/2} \exp(-yr(\xi)) & \text{if } x/y > g'(R), \\ C^* \exp(-Rx) & \text{if } x/y < g'(R), \end{cases}$$

$$(2.27) \qquad P(y < T(x) < \infty) \sim \begin{cases} C^* \exp(-Rx) & \text{if } x/y > g'(R), \\ C_\xi^* y^{-1/2} \exp(-yr(\xi)) & \text{if } x/y < g'(R). \end{cases}$$

Here $r(\xi) = -g(\xi) + \xi g'(\xi) > 0$ *for* $\xi > 0$.

In this case there are simple explicit expressions for the constants when U is positive. Namely, $C^* = -g'(0)/g'(R)$ [see Feller (1971), Chapter XI, for example] and if $G = \mathbb{R}^2$ and if there is a $\xi' < \xi$ satisfying $g(\xi') = g(\xi)$, then [Arfwedson (1955)]

$$(2.28) \qquad C_\delta^* \xi = \frac{\xi - \xi'}{|\xi\xi'|\sqrt{2\pi g''(\xi)}}.$$

There is a similar expression for the constant when U is discrete.

Let

$$(2.29) \qquad s^*(t) = \sup_\xi \{\xi t - g(\xi)\}.$$

Then $s^*(t)$ is convex, $s^*(g'(\xi)) = r(\xi)$ and $s^*(t) \geq Rt$ for all t with equality only if $t = g'(R)$. Therefore, $yr(\xi) = ys^*(x/y) \geq xR$ with equality only if $x/y = g'(R)$.

The corollary can be modified to hold for the more general risk reserve process where the Poisson process is replaced by a renewal process, i.e., when $\tau_1, \tau_2, \dots$ are independent and positive random variables with a common distribution. Write ψ for the inverse of the monotonic function $\eta \to Ee^{\eta\tau}$. The corollary holds for this process as well, provided we replace g by $g(\xi) = -c\xi - \psi(1/\omega(\xi))$ and add the restriction that $1/\omega(\xi)$ belongs to the interior of the domain of ψ.

3. Proofs. Let $N_1, N_2, \dots$ stand for the successive strict ascending ladder indices, i.e., $N_k = \infty$ if $N_{k-1} = \infty$ and otherwise N_k is the first index $n > N_{k-1}$ for which $S_n > S_{N_{k-1}}$, and $N_k = \infty$ if such an index does not exist. Here $N_0 = 0$. Define for $k \geq 1$, $\overline{X}_k = S_{N_k} - S_{N_{k-1}}$ and $\overline{Y}_k = T_{N_k} - T_{N_{k-1}}$ if $N_k < \infty$ and let $\overline{S}_n$, $\overline{T}_n$ stand for the corresponding partial sums. Then $(\overline{X}_k, \overline{Y}_k)$, $k = 1, \dots, n$, are independent and identically distributed, given $N_n < \infty$.

Write N for N_1 and L for the possibly defective distribution of (S_N, T_N),

$$(3.1) \qquad L(dx, dy) = \text{prob}(N < \infty, S_N \in dx, T_N \in dy),$$

and let E_n stand for the event

$$(3.2) \qquad E_n = \{N_n = N(x) < \infty, S_{N_n} > x + a, T_{N_n} \in y + B\}.$$

Then $E_1, E_2, \dots$ are disjoint sets whose union equals

$$\{N(x) < \infty,\ S_{N(x)} > x + a,\ T_{N(x)} \in y + B\}, \tag{3.3}$$

and furthermore,

$$E_n = \{\bar{S}_{n-1} \le x,\ \bar{S}_n > x + a,\ \bar{T}_n \in y + B,\ N_n < \infty\}. \tag{3.4}$$

Therefore

$$\operatorname{prob}(E_n) = \iint_{s \le x} L((x - s + a, \infty), y - t + B) L^{(n-1)^*}(ds, dt) \tag{3.5}$$

for $x \ge 0$. If we define $f(x, y) = L((x + a, \infty), y + B)$ when $x \ge 0$ and $u(x, y) = f(x, y) = 0$ when $x < 0$, we therefore conclude

$$u(x, y) = \sum_{n=0}^{\infty} L^{n^*} * f(x, y) \tag{3.6}$$

for all real x and y. Here L^{0^*} is the Dirac measure concentrated at zero.

Write $u_B(x, y)$ and $f_B(x, y)$ instead of $u(x, y)$ and $f(x, y)$ in order to show the dependence on B. Also write $u_1(x) = u_{\mathbb{R}}(x, y)$, $f_1(x) = f_{\mathbb{R}}(x, y)$, and $L_1(dx) = L(dx, \mathbb{R})$. We shall use the following identities which we collect in a lemma.

LEMMA 3.1. *The ruin probabilities satisfy*

$$u_B(x, y) = \sum_{n=0}^{\infty} L^{n*} * f_B(x, y), \qquad u_1(x) = \sum_{n=0}^{\infty} L^{n*}{}_1 * f_1(x), \tag{3.7}$$

$$u_1(x) = u_B(x, y) + u_{B^c}(x, y) \tag{3.8}$$

for all x, y *and* B.

Let $\zeta = (\xi, \eta)$ and $z = (x, y)$. Define

$$\lambda(\zeta) = \int \exp(\zeta \cdot z) L(dz), \qquad L_\zeta(dz) = \exp(\zeta \cdot z) L(dz)/\lambda(\zeta) \tag{3.9}$$

and put

$$Q^L(\zeta) = E_\zeta\big(T_N - S_N E_\zeta T_N / E_\zeta S_N\big)^2 \tag{3.10}$$

whenever $\int |z|^2 \exp(\zeta \cdot z) L(dz) < \infty$. Note that the condition $\dim G = 2$ implies that $Q^L(\zeta) > 0$.

PROPOSITION 3.2. *Assume that* $\dim G = 2$, *that* λ *is finite in neighbourhoods of the points* $\zeta = (\xi, \eta)$ *and* $(0, \eta)$ *and that* $\lambda(\zeta) = 1$. *Let* $z \in G$ *tend to infinity in such a way that* $z/|z| = E_\zeta(S_N, T_N)/|E_\zeta(S_N, T_N)|$.

(a) *If* $\int_B e^{-\eta t}\, dt < \infty$, *then* $C_\zeta(a, B) < \infty$, $Q^L(\zeta) > 0$ *and*

$$\big(2\pi Q^L(\zeta) x / E_\zeta S_N\big)^{1/2} e^{\zeta \cdot z} u(z) \to C_\zeta(a, B). \tag{3.11}$$

(b) *If* $\int_{B^c} e^{-\eta t}\,dt < \infty$ *and if there is a* $\zeta_0 = (\kappa, 0)$ *with* $\lambda(\zeta_0) = 1$ *and* $E_{\zeta_0} S_N < \infty$, *then* $C_{\zeta_0}(a) < \infty$ *and*

$$e^{\zeta_0 \cdot z} u(z) \to C_{\zeta_0}(a). \tag{3.12}$$

It will be seen in the proof that the assumption that λ is finite in neighbourhoods of the above points is unnecessarily strong, but it will suffice for our purposes, because this is what we get from Lemma 3.3.

Note that $E_\theta S_N > 0$ for all θ, and hence that z lies in the half-plane $x > 0$.

PROOF OF PROPOSITION 3.2. It follows from Theorem 1.4 of Höglund (1988) that

$$E_\zeta S_N \big(2\pi Q^L(\zeta) x / E_\zeta S_N\big)^{1/2} e^{\zeta \cdot z} \sum_{n=0}^{\infty} L^{n*} * f(z) \to \int e^{\zeta \cdot u} f(u) \sigma(du) \tag{3.13}$$

if $\int |u|^2 e^{\zeta \cdot u} L(du) < \infty$ and if $(1 + |z|^2) e^{\zeta \cdot z} |f(z)|$ is directly Riemann integrable with respect to σ.

Let $z = (x, y)$ and $u = (s, t)$. We have

$$\int e^{\zeta \cdot z} f(z) \sigma(dz) = \int_{x \geq 0} e^{\zeta \cdot z} \sigma(dz) \int_{s - x > a} \int_{t - y \in B} L(ds, dt). \tag{3.14}$$

If we make the substitution $z \to u - z$, $u \to u$ and use the fact that $\sigma(u - dz) = \sigma(dz)$ for $u \in G$ we see that this integral equals

$$C_\zeta(a, B) E_\zeta S_N. \tag{3.15}$$

An interchange of the order of integration and the same substitution as above show that

$$\begin{aligned} &\int (1 + |z|^2) e^{\zeta \cdot z} |f(z)| \sigma(dz) \\ &\quad \leq \int e^{\zeta \cdot u} L(du) \iint_{\substack{0 < x \leq s \\ y \in B}} (1 + |u - z|^2) e^{-\zeta \cdot z} \sigma(dz). \end{aligned} \tag{3.16}$$

If B has a nonempty interior, then the inner integral is dominated by a constant times

$$\begin{aligned} &\iint_{\substack{0 < x \leq s \\ y \in B}} (1 + |u|^2 + |z|^2) e^{-\zeta \cdot z}\, dz \\ &\quad \leq (1 + |u|^2)(1 + e^{-\xi s}) \int_{y \in B} (1 + |y|^2) e^{-\eta y}\, dy \end{aligned} \tag{3.17}$$

if $\zeta \neq 0$ and if $\xi = 0$, then the factor $(1 + e^{-\xi s})$ has to be replaced by s. The case when B is a one-point set can be treated similarly.

The function $(1 + |z|^2)e^{\zeta \cdot z}f(z)$ is thus integrable and it is directly Riemann integrable because when B is an interval, $f(x, y)$ is a difference of two functions that are monotonic in each of the variables x and y.

We have thus shown (a).

Assume that $\int_{B^c} e^{-\eta t}\, dt < \infty$. We can then apply the just proved approximation to $u_{B^c}(z)$. By one-dimensional renewal theory [see Feller (1971), page 349]

$$e^{\xi_0 x} u_1(x) \to C_{\zeta_0}(a). \tag{3.18}$$

The remainder of the proposition now follows from (3.8). □

LEMMA 3.3. *Assume that* ϕ *is finite in a neighbourhood of* $\zeta = (\xi, \eta)$, $\phi(\zeta) = 1$ *and* $E_\zeta X > 0$.

(a) *If* ϕ *is finite in a neighbourhood of the point* (α, η), *then also* λ *is finite in a neighbourhood of* (α, η).

(b) $\lambda(\zeta) = 1$, $E_\zeta N < \infty$, $E_\zeta(S_N, T_N) = E_\zeta N E_\zeta Z$ *and* $Q^L(\zeta) = E_\zeta N Q(\zeta)$.

(c) $E_\zeta N = e^{\Sigma^-(\zeta)}$ *and if* $\phi(0, \eta) < \infty$, *then*

$$1 - \lambda(0, \eta) = \begin{cases} (1 - \phi(0, \eta))e^{\Sigma^-(0, \eta)} & \text{if } \phi(\alpha, \eta) < 1 \text{ for some } \alpha \le 0, \\ e^{-\Sigma^+(0, \eta)} & \text{if } \phi(\alpha, \eta) < 1 \text{ for some } \alpha \ge 0. \end{cases} \tag{3.19}$$

Note that the second identity in (b) (Walds equation) implies that $E_\zeta(S_N, T_N)$ and $E_\zeta Z$ have the same direction. It is easy to verify directly that the two expressions to the right in (c) are identical if $\phi(0, \eta) < 1$.

PROOF. Put $H_n(dz) = \int \cdots_E \int F(dz_1) \cdot \cdots \cdot F(dz_n)$ where

$$E = \{(z_1, \dots, z_n);\ x_1 + \cdots + x_k \le 0, 1 \le k \le n - 1,$$
$$x_1 + \cdots + x_n \in dx,\ y_1 + \cdots + y_n \in dy\}$$

and define

$$\begin{aligned} \lambda_r(\zeta) &= \sum_1^\infty r^n \iint_{x > 0} \exp(\zeta \cdot z) H_n(dz), \\ \gamma_r(\zeta) &= 1 + \sum_1^\infty r^n \iint_{x \le 0} \exp(\zeta \cdot z) H_n(dz). \end{aligned} \tag{3.20}$$

Then $\gamma_1(\zeta) = E_\zeta N$ when $\phi(\zeta) = 1$ and $\lambda_1(\theta) = \lambda(\theta)$ for all θ.

Let $\theta = (\alpha, \beta)$. Then

$$\gamma_1(\theta) \le 1 + \sum_{n=1}^\infty \iint_{x \le 0} e^{\alpha' x + \beta y} F^{n*}(dx, dy) \le \sum_{n=0}^\infty \phi(\alpha', \beta)^n \tag{3.21}$$

if $\alpha' \le \alpha$ and hence $\gamma_1(\theta) < \infty$ if $\phi(\alpha', \beta) < 1$ for some $\alpha' \le \alpha$.

Let $\phi(\zeta) = 1$. Then $\lambda(\theta) = E_\zeta[\exp((\alpha - \xi)S_N + (\beta - \eta)T_N)]$ and hence $\lambda(\theta) < \infty$ if and only if

$$(3.22)\qquad \iint_{s>0} e^{(\alpha-\xi)s+(\beta-\eta)t} P_\zeta(S_N > s,\, T_N \in dt)\, ds < \infty,$$

provided $\alpha \neq \xi$, which we assume.

If $N = n$, $S_N > s$ and $T_N \in I$, then $N > n - 1$, $X_n > s$ and $T_{n-1} + Y_n \in I$. Therefore

$$(3.23)\qquad \begin{aligned} &P_\zeta(N = n,\, S_N > s,\, T_N \in I) \\ &\quad < \int P_\zeta(N > n-1,\, T_{n-1} \in dt) P(X > s,\, Y \in I - t) \end{aligned}$$

since the event $\{N > n - 1\}$ and the variable T_{n-1} only depend on $Z_1, \dots, Z_{n-1}$ and therefore are independent of Z_n. Summing over n and integrating we therefore see that $\lambda(\theta) < \infty$ if

$$(3.24)\qquad \iint_{s>0} e^{(\alpha-\xi)s+(\beta-\eta)t} P_\zeta(X > s,\, Y \in dt)\, ds\, \gamma_1(\xi, \beta) < \infty.$$

As we have seen this is the case if $\phi(\alpha, \beta) < \infty$ and $\phi(\xi', \beta) < 1$ for some $\xi' \leq \xi$. The latter is the case when β is sufficiently close to η since $\phi(\xi, \eta) = 1$ and $\partial_1\phi(\xi, \eta) = E_{\xi,\eta}X > 0$. This proves (a).

The last identity in (b) is essentially the identity (6.10) of von Bahr (1974). The remainder of (b) and (c) are either well known or obvious modifications of the argument in Chapter XVIII in Feller (1971) or von Bahr (1974). □

PROOF OF (2.13). The identity

$$(3.25)\qquad C_\zeta(0) = I(\xi)(1 - \lambda(0, \eta))/(\xi E_\zeta S_N) \quad \text{if } \xi \neq 0, \; = 1 \text{ if } \xi = 0$$

follows from an interchange of the order of integration and the identity

$$(3.26)\qquad \int_{s \geq x > 0} e^{-\xi x}\sigma_1(dx) = I(\xi)(1 - e^{-\xi s})/\xi \quad \text{if } \xi \neq 0, \; = s \text{ if } \xi = 0.$$

The identities (2.13) now follow from Lemma 3.3. □

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DEPARTMENT OF MATHEMATICS
ROYAL INSTITUTE OF TECHNOLOGY
S-100 44 STOCKHOLM
SWEDEN