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MARKOV PROPERTIES FOR POINT PROCESSES ON THE PLANE

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It is proved that for a wide class of point processes indexed by the positive quadrant of the plane, and for a class of compact sets in this quadrant, the germ σ-field is equal to the σ-field generated by the values of the process on the set. Therefore, there exists a large family of point processes in the plane (and among them the spatial Poisson process) which satisfy the sharp Markov property in the sense of P. Lévy. The strong Markov property with respect to stopping lines is also studied. Some examples are obtained by taking transformations of the probability measure.

0. Introduction. The problem of finding a good definition for a two-parameter (a multiparameter) stochastic process to be a Markov process has produced an important bibliography. In fact, the story began in 1948, when P. Lévy defined the Markov property in the following most natural way: We say that the process $\{X_z,\ z \in \mathbb{R}_+^2\}$ has the sharp Markov property with respect to A if the σ-fields $\sigma\{X_z,\ z \in A\}$ and $\sigma\{X_z,\ z \in A^c\}$ are conditionally independent given $\sigma\{X_z,\ z \in \partial A\}$. Lévy concluded by asking the following question: Do there exist nondegenerate two-parameter processes with the sharp Markov property with respect to a sufficiently large class of sets [9]? In 1976, J. B. Walsh showed that the Brownian sheet does not satisfy the sharp Markov property in the sense of Lévy even with respect to triangles ([19]; see also [23] for some generalizations). To obtain a broader class of processes with a Markov property, several other definitions were proposed. A weaker definition, called the "germ-field Markov property" was introduced by McKean [10] and Pitt [14] and was studied in the Gaussian case by several authors [1, 4, 7, 8, 16, 20]. In this definition, the σ-field $\sigma\{X_z,\ z \in \partial A\}$ is replaced by the larger germ σ-field of the boundary of A, that is, by $\bigcap_{\varepsilon>0}\sigma\{X_z,\ d(z, A) < \varepsilon\}$. Another type of definition was introduced by Nualart and Sanz [13], Guyon and Prum [6] and generalized by Korezlioglu, Lefort and Mazziotto [8]. However, in these definitions, the partial order induced by the cartesian coordinates plays a very important role and in the present paper we will not discuss them.

The last and most recent definition was proposed by Wong and Zakai in [22] and concerns processes parameterized by smooth curves in $\mathbb{R}_+^2$. Let γ denote a continuous finite nondecreasing or nonincreasing path and let $A(\gamma)$ and $B(\gamma)$

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denote, respectively, the vertical and horizontal shadow of γ. Let $\{X_z\}$ be an integrator process and set $X_\gamma = (X(A(\gamma)), X(B(\gamma)))$. X will be called γ-Markov if for every connected open set O with piecewise monotone boundary, $\mathscr{H}_O$ and $\mathscr{H}_{O^c}$ are conditionally independent given $\mathscr{H}_{\partial O}$, where $\mathscr{H}_O$ is the σ-algebra generated by X_γ where γ runs over all the continuous paths, which are either nonincreasing or nondecreasing and belong to $\overline{O}$.

Most of the papers on Markov random fields deal with the Gaussian case; the literature on the Markov property for point processes is very limited.

The first paper in this direction is a work of Carnal [4] (assembled by J. B. Walsh in 1980 after the death of Carnal), in which it is proved that the Poisson sheet satisfies the sharp Markov property relative to all bounded open relatively convex sets, and this property fails to be true for general unbounded open relatively convex sets. Moreover, quoting him: "We conjecture that the restriction to relatively convex sets is unnecessary, but our methods become cumbersome when the sets become complex." The next step was made by Russo [17], who extended the Carnal result in the following sense. Every two-parameter point process with independent increments satisfies the sharp Markov property relative to any finite union of rectangles whose sides are parallel to the axes.

In this paper we extend this result for point processes to a wide class $\mathscr{C}$ of compact sets. This extension of the Markov property for the Poisson process is based on a detailed analysis of the germ σ-field of a class of point processes. In particular, we show that for a set A of this class $\mathscr{C}$ and for an appropriate point process $N = \{N_z,\ z \in \mathbb{R}_+^2\}$, the σ-field $\mathscr{F}_A = \sigma\{N_z,\ z \in A\}$ is equal to the germ σ-field $\mathscr{G}_A = \bigcap_{\varepsilon > 0} \mathscr{F}_{A_\varepsilon}$ where $A_\varepsilon = \{z\colon\ d(z, A) < \varepsilon\}$. This class of sets contains the sets which are the boundary of an open and bounded set and which can be represented as a finite union of connected nonincreasing and nondecreasing curves.

The notion of Markov property with respect to nonincreasing or nondecreasing curves leads directly to the notion of a strong Markov process. This kind of property was first introduced for random fields by Evstigneev [5], but for a different kind of σ-field. In [22], Wong and Zakai conclude their work by defining the strong Markov property. Here, we prove that every strictly simple Markov point process whose intensity is absolutely continuous with respect to the Lebesgue measure, possesses the strong and sharp Markov property relative to all the bounded stopping lines. For optional increasing paths, the same results hold if the two sides of the optional increasing path are connected to one of the axes.

Before starting, let us mention that Markov point processes were studied by other authors (see for example [15]), but from a completely different point of view. The Markov property is defined there by considering the behavior of the process on the neighborhood of a single point.

Only the case of the plane will be considered in this paper, but clearly the same arguments hold for the $\mathbb{R}^n$-case.

In the next section we introduce all the tools and definitions related to a point process that will be needed later and the geometric structure of our classes of sets in the plane. Since we define general classes of sets, some theorems of measurability are needed, and therefore the selection theorem and a separability result are stated here. The main theorem is proved in Section 2. It gives sufficient conditions on sets and processes such that the germ σ-field is equal to the σ-field generated by the values of the process on the set. The main idea in proving this result is that the probability that there is a jump point on a deterministic set of measure zero, or even near to such a set, is very close to zero and therefore if we know the behavior of the point process on this set, we can find out the behavior of the process in any neighborhood of the set. Examples of sharp Markov processes are given in this section and different kinds of sets which satisfy the theorem are presented. Also, some counterexamples are mentioned, showing that our assumptions cannot be weakened.

Section 3 is devoted to the strong Markov property, that is the sharp Markov property relative to measurable and adapted random sets. We begin this section with a discussion of the different ways to define the σ-field associated with a measurable random set. The definition proposed here seems to be natural in the sense that it generalizes the case of a deterministic set. The main result is proved for bounded stopping lines, but it seems that this can be extended to more general bounded random sets.

In the last section, we consider transformations of probability measures via multiplicative functionals under which the Markov property is preserved. This section is similar to Section 3 of Wong and Zakai [22], and permits us to obtain many examples of Markov point processes which are transformations of the Poisson sheet.

1. Notation and preliminaries. The processes are indexed by points of $\mathbb{R}^2_+$, in which the partial order induced by the Cartesian coordinates is defined: Let $z = (s, t)$ and $z' = (s', t')$. Then $z \leq z'$ if $s \leq s'$ and $t \leq t'$, and $z < z'$ if $s < s'$ and $t < t'$.

Consider a random measure N on $\mathbb{R}^2_+$, defined on some complete probability space $(\Omega, \mathscr{F}, P)$, and such that for every $\omega \in \Omega$, $N(\omega)$ is a finite or countable sum of Dirac measures on random and different points $Z_i(\omega)$, $i = 0, 1, \dots$. We also assume that the measure of the axes is zero and $N([0, z]) < \infty$ for any z in $\mathbb{R}^2_+$. Such a random measure defines a point process on the plane given by $N_z = N([0, z])$ (see [11] for a detailed study of point processes in the plane). Since N_z is an increasing process, it has limits in the four quadrants $Q_z^{++} = \{z' \colon z \leq z'\}$, $Q_z^{-+} = \{z' = (s', t) \colon s' < s,\ t \leq t'\}$, $Q_z^{--} = \{z' \colon z' < z\}$ and $Q_z^{+-} = \{z' = (s', t') \colon s \leq s',\ t' < t\}$. We denote these limits by N_z^{++}, N_z^{-+}, N_z^{--} and N_z^{+-}, respectively, and we suppose that the process N is right-continuous: $N_z = N_z^{++}$.

We fix $z_0 = (s_0, t_0)$. We are going to introduce a new point process $\hat{N}^{z_0}$ on $[0, z_0)$. Let $S_1, \dots, S_m$ be the jump points of the one-parameter process $\{N^{+-}(s, t_0), 0 \leq s < s_0\}$ and in the same way, we denote by $T_1, \dots, T_n$ the jump points of the one-parameter process $\{N^{-+}(s_0, t), 0 \leq t < t_0\}$. Then we define

$$\hat{N}^{z_0} = \sum_{i=1}^{m} \sum_{j=1}^{n} \delta_{(S_i, T_j)}. \tag{1.1}$$

Note that $\hat{N}^{z_0}$ is a point process on $[0, z_0)$ such that $\hat{N}^{z_0} \geq N$.

Suppose that the original process N is *strictly simple*, that is, $P\{$two jump points of N have the same first or second coordinate$\} = 0$. In that case, $n = m$ and $\hat{N}^{z_0}$ can also be defined by

$$\hat{N}^{z_0}(s, t) = N([0, s] \times [0, t_0))N([0, s_0) \times [0, t]). \tag{1.2}$$

DEFINITION 1.1. Let $\mathscr{E}$ be the class of Borel subsets A in $\mathbb{R}^2_+$ such that

$$\hat{N}^{z_0}(A) = 0 \quad \text{for any } z_0 \in \mathbb{R}^2_+,$$

where $\hat{N}^{z_0}$ is the point process defined by (1.1). The class $\mathscr{E}$ clearly depends on the point process N.

In particular, we will consider the family $\mathscr{E}$ of finite sets or the family of sets having zero Lebesgue measure.

Suppose that $E(N([0, z])) < \infty$ for any z in $\mathbb{R}^2_+$. Then we can introduce the intensity measure μ of N, given by $\mu(A) = E(N(A))$. The measure μ is a Radon measure on $\mathbb{R}^2_+$. Let $\hat{\mu}^{z_0}$ be the intensity measure of $\hat{N}^{z_0}$. Then the family $\mathscr{E}$ contains the class of sets of $\hat{\mu}^{z_0}$ zero measure. If $\hat{\mu}^{z_0}$ is absolutely continuous with respect to the Lebesgue measure for any z_0 in $\mathbb{R}^2_+$, then $\mathscr{E}$ contains the sets of zero Lebesgue measure.

PROPOSITION 1.2. *Suppose that the intensity measure μ is absolutely continuous with respect to a product measure $\mu_1 \times \mu_2$ and N is strictly simple. Then $\mathscr{E}$ contains the sets A such that $(\mu_1 \times \mu_2)(A) = 0$.*

PROOF. Let $A \subseteq \mathbb{R}^2_+$ be a Borel set such that $(\mu_1 \times \mu_2)(A) = 0$. Then using (1.2) we have

$$\hat{N}^{z_0}(A) = \int_{[0, z_0)} \mathbf{1}_A(s, t)N(ds \times [0, t_0))N([0, s_0) \times dt)$$

and

$$E\int_{[0, s_0)} \mathbf{1}_A(s, t)N(ds \times [0, t_0)) = E(N(A_t \times [0, t_0))) = \mu(A_t \times [0, t_0)) = 0$$

for all $t \in [0, t_0)$, μ_2-a.s., where $A_t = \{s \in [0, s_0)\colon (s, t) \in A\}$. Therefore, $\int_{[0, s_0)}\mathbf{1}_A(s, t)N(ds \times [0, t_0)) = 0$ a.s., for all $t \in [0, t_0)$, μ_2-a.s. and this implies $\hat{N}^{z_0}(A) = 0$. $\square$

Consider, for instance, the case of a Poisson process (cf. [11]). Then μ is the Lebesgue measure and $\hat{\mu}^{z_0}$ is the Lebesgue measure multiplied by the factor $1 + s_0 t_0$. In fact, we know that the Poisson process is strictly simple, and using the expression (1.2) we get $E(\hat{N}^{z_0}(s, t)) = E(N((0, s) \times [0, t_0))N([0, s_0) \times [0, t))) = st(1 + s_0 t_0)$. Consequently, in this case the class $\mathscr{E}$ contains all the sets of zero Lebesgue measure, by Proposition 1.2.

We denote by N^* the point process on $\mathbb{R}^2_+$ whose support is the set of points $z = (s, t)$ such that $N[([0, s] \times \{t\}) \cup (\{s\} \times [0, t])] > 0$. Note that the support of N^* includes the jump points of N but it may be larger. In fact, if we denote by $\{L'_n\}$ the stopping lines which are the boundaries of the sets $\{N_z \geq n\}$ as defined in [11], then the jump points of N^* are the minimal points of $\{L'_n\}$. Also, we have $N \leq N^* \leq \hat{N}^{z_0}$ on $[0, z_0)$, for any $z_0 \in \mathbb{R}_+^2$.

For any set $A \subset \mathbb{R}_+^2$ we introduce the following subsets of A where $B_\varepsilon(z) = \{z': d(z, z') < \varepsilon\}$:

1. We denote by $H_1^-(A)$ the set of points $z = (s, t) \in A$ such that there exists $\varepsilon > 0$ verifying $([0, s) \times \mathbb{R}_+) \cap A \cap B_\varepsilon(z) = \varnothing$.
2. We denote by $H_2^-(A)$ the set of points $z = (s, t) \in A$ such that there exists $\varepsilon > 0$ verifying $(\mathbb{R}_+ \times [0, t)) \cap A \cap B_\varepsilon(z) = \varnothing$.

   Set $H(A) = H_1^-(A) \cup H_2^-(A)$.

3. We denote by $H_1^+(A)$ the set of points $z = (s, t) \in A$ such that there exists $\varepsilon > 0$ verifying $((s, +\infty) \times \mathbb{R}_+) \cap A \cap B_\varepsilon(z) = \varnothing$.
4. We denote by $H_2^+(A)$ the set of points $z = (s, t) \in A$ such that there exists $\varepsilon > 0$ verifying $(\mathbb{R}_+ \times (t, +\infty)) \cap A \cap B_\varepsilon(z) = \varnothing$.
5. Let $G(A)$ be the set of points $z = (s, t) \in A$ such that for any $\varepsilon > 0$ we have

$$([0, s) \times [t, \infty)) \cap A \cap B_\varepsilon(z) \neq \varnothing$$

   and

$$([s, \infty) \times [0, t)) \cap A \cap B_\varepsilon(z) \neq \varnothing.$$

6. Let $F(A)$ be the set of points $z = (s, t) \in A$, $z \notin G(A)$ such that for any $\varepsilon > 0$ we have

$$[0, z) \cap A \cap B_\varepsilon(z) \neq \varnothing.$$

   Then $A = H(A) \cup G(A) \cup F(A)$ is a partition of the set A.

DEFINITION 1.3. We denote by $\mathscr{C}$ the class of compact sets $A \subset \mathbb{R}_+^2$ verifying the following three conditions:

(i) $H_1^-(A) \cap H_2^-(A)$ is finite.
(ii) For any $z = (s, t) \in F(A)$, there exists $s' > s$ (or $t' > t$) such that $(s', t) \in G(A) \cup F(A)$ [or $(s, t') \in G(A) \cup F(A)$].
(iii) The sets $\pi_1(H_1^-(A) \cup H_1^+(A))$ and $\pi_2(H_2^-(A) \cup H_2^+(A))$ are countable, where $\pi_1(\ )$ and $\pi_2(\ )$ are the respective projections on the axes.

Let A be a set of the class $\mathscr{C}$. We will need the following countable and dense subset A_0 of A. Let A_1 be an arbitrary countable and dense subset of A. For any $s \in \pi_1(H_1^-(A) \cup H_1^+(A))$ we take a countable and dense subset $A_0(s)$ of the section $A(s)$ in such a way that any point of $A(s)$ can be approximated from above by points of $A_0(s)$. In the same way, for any $t \in \pi_2(H_2^-(A) \cup H_2^+(A))$ we take a countable and dense subset $A_0(t)$ of the section $A(t)$ in such a way that any point of $A(t)$ can be approximated from the right by points of $A_0(t)$. Then A_0 will be the union of A_1 and the set of points $\{s\} \times A_0(s)$, $A_0(t) \times \{t\}$ for all $s \in \pi_1(H_1^-(A) \cup H_1^+(A))$ and $t \in \pi_2(H_2^-(A) \cup H_2^-(A))$, respectively.

For any Borel subset A of $\mathbb{R}_+^2$ we denote by $\mathscr{F}_A$ the σ-field generated by the random variables $\{N_z, z \in A\}$ and the null sets of Ω. The germ σ-field of A is then defined by $\mathscr{G}_A = \bigcap_{\varepsilon > 0} \mathscr{F}_{A_\varepsilon}$, where A_ε represents the ε-neighborhood of A.

In the next section we will need the fact that the process N_z restricted to a given compact set $A \subset \mathbb{R}_+^2$ would be separable. In this sense the next result will be useful.

PROPOSITION 1.4. *Let A be a compact set belonging to $\mathscr{C}$ and such that $N(A) = 0$ a.s. Then the process $\{N_z, z \in A\}$, defined in the probability space $(\Omega, \mathscr{F}_A, P)$, is separable and admits A_0 as a separating set.*

PROOF. Let $z = (s, t) \in A$. We are going to consider several cases:

(i) Assume that for any $\varepsilon > 0$, $A_0 \cap B_\varepsilon(z) \cap [z, +\infty) \neq \varnothing$. In that case, N_z can be determined from the values of N on the points of $A_0 \cap [z, \infty)$.

(ii) If z belongs to $H_1^+(A)$ or $H_2^+(A)$, it can be approximated from the right or from above by points of A_0, by the definition of A_0.

(iii) Suppose $z \notin H_1^+(A) \cup H_2^+(A)$ and $A_0 \cap B_\delta(z) \cap [z, \infty) = \varnothing$ for some $\delta > 0$. Then for any $0 < \varepsilon < \delta$, we have

$$A_0 \cap B_\varepsilon(z) \cap ([0, s) \times (t, +\infty)) \neq \varnothing$$

and

$$A_0 \cap B_\varepsilon(z) \cap ((s, +\infty) \cap [0, t)) \neq \varnothing .$$

In this case we have

$$N_z = \max\left(\lim_{\substack{z' \in A_0 \\ z' \in Q_z^{+-}}} N_{z'}, \lim_{\substack{z' \in A_0 \\ z' \in Q_z^{-+}}} N_{z'} \right)$$

because $N(\{z\}) = 0$ a.s. □

DEFINITION 1.5. A mapping $\omega \to A(\omega)$ from a complete probability space $(\Omega, \mathscr{F}, P)$ into the Borel sets of $\mathbb{R}_+^2$ will be called a measurable random set if $\{U \cap A \neq \varnothing\} \in \mathscr{F}$ for any open subset U of $\mathbb{R}_+^2$.

We have the following result [18], that will be necessary in the next section.

PROPOSITION 1.6. *Suppose A is a closed-valued measurable random set. There exists a sequence of random variables $\{Z_n, n \geq 1\}$ valued in $\mathbb{R}_+^2$ such that $Z_n \in A$, a.s. for all n, and $\{Z_n, n \geq 1\}$ is a dense subset of A, a.s.*

2. The germ σ-field and the sharp Markov property. Let $\{N_z, z \in \mathbb{R}_+^2\}$ be a point process on $\mathbb{R}_+^2$. We introduce the following hypothesis.

(H) For any $z_0 = (s_0, t_0)$, the one-parameter point processes $\{N(s, t_0), 0 \leq s\}$ and $\{N(s_0, t), t \geq 0\}$ have no fixed point of discontinuity.

Recall that $\mathscr{E}$ and $\mathscr{C}$ are the families of sets given by Definitions 1.1 and 1.3, respectively.

Now we can state the main result of this section.

THEOREM 2.1. *Let N be a point process on $\mathbb{R}_+^2$ satisfying hypothesis* (H). *Then for any $A \in \mathscr{E} \cap \mathscr{C}$ we have $\mathscr{F}_A = \mathscr{G}_A$.*

PROOF. We fix a compact set $A \in \mathscr{E} \cap \mathscr{C}$ and suppose that $A \subset [0, z_0)$. We denote by $\mathscr{R}$ the support of the point process N^* on $[0, z_0)$, and by $\hat{\mathscr{R}}$ the support of $\hat{N}^{z_0}$. We define a random set $\mathscr{L} \subset [0, z_0)$ as follows. For any point $z = (s, t) \in \mathscr{R} \cup \{(0,0)\}$ we consider the horizontal segment $[s, s_0) \times \{t\}$ and the vertical segment $\{s\} \times [t, t_0)$, and by definition $\mathscr{L}$ will be the union of these segments. In other words, $\mathscr{L}$ is the union of the axes and the stopping lines associated with the point process N on $[0, z_0)$. Note that on each connected component of $[0, z_0) - \mathscr{L}$, the process N is constant, and its value is determined by the value of N on the lower boundary of this component.

Define the distances

$$\delta_1 = d(H_1^-(A) \cap H_2^-(A), \mathscr{L})$$

and

$$\delta_2 = d(A, \hat{\mathscr{R}}).$$

The fact that $A \in \mathscr{E}$ implies that $\delta_2 > 0$ a.s. and condition (i) in the definition of $\mathscr{C}$ and hypothesis (H) imply $\delta_1 > 0$ a.s. Set $\delta = \min(\delta_1, \delta_2)$. δ is a strictly positive random variable.

We are going to construct a countable covering of Ω by sets $\{H_{m,k}, m, k \geq 1\}$, such that on each $H_{m,k}$ the σ-fields $\mathscr{F}_A$ and $\mathscr{G}_A$ have the same trace. In order to show this property we will find, for any fixed m and k and $\omega \in H_{m,k}$, a countable covering $A \subset \bigcup_{j=1}^{\infty} B_j^{m,k}(\omega)$ by random sets verifying the following properties:

(i) For some $\varepsilon > 0$, $A_\varepsilon \subset \bigcup_{j=1}^{\infty} B_j^{m,k}(\omega)$, for all $\omega \in H_{m,k}$.
(ii) $B_j^{m,k}$ is a measurable set in the space $(H_{m,k}, \mathscr{F}_A|_{H_{m,k}}, P)$.
(iii) $B_j^{m,k}(\omega) \cap A_0 \neq \varnothing$.
(iv) N is constant on each $B_j^{m,k}(\omega)$.
(v) $B_j^{m,k}(\omega) \cap \hat{\mathscr{R}} = \varnothing$.

Properties (i)–(v) imply that $\mathscr{F}_A|_{H_{m,k}} = \mathscr{F}_{A_\varepsilon}|_{H_{m,k}}$ and, consequently, $\mathscr{F}_A|_{H_{m,k}} = \mathscr{G}_A|_{H_{m,k}}$. In fact, we have to show that $\{N_z = l\} \cap H_{m,k}$ belongs to $\mathscr{F}_A|_{H_{m,k}}$ for any fixed z in A_ε. We have

$$H_{m,k} \cap \{N_z = l\} = \bigcup_{j=1}^{\infty} \bigcup_{\xi \in A_0} \left(\{N_\xi = l\} \cap \left\{\xi \in B_j^{m,k}\right\} \cap \left\{z \in B_j^{m,k}\right\} \cap H_{m,k}\right),$$

which belongs to $\mathscr{F}_A|_{H_{m,k}}$ because $B_j^{m,k}$ is a measurable set in the space $(H_{m,k}, \mathscr{F}_A|_{H_{m,k}}, P)$.

Consider first the set $H_k = \{\delta > 1/k\}$ and denote by $\mathscr{F}_A^k$ the trace of the σ-field $\mathscr{F}_A$ on this set. Define

$$A' = A - \bigcup_{z \in H_1^-(A) \cap H_2^-(A)} B_{1/k}(z),$$

which is a compact subset of A.

We denote by $\mathscr{L}_H$ the union of the horizontal components of $\mathscr{L}$ and by $\mathscr{L}_V$ the union of the vertical ones. We claim that:

(I) $A' \cap \mathscr{L}_H$ and $A' \cap \mathscr{L}_V$ are random measurable compact sets in $(H_k, \mathscr{F}_A^k, P)$. Note that on $H_k = \{\delta > 1/k\}$ we have $A' \cap \mathscr{L} = (A' \cap \mathscr{L}_H) \cup (A' \cap \mathscr{L}_V)$ and $\{A' \cap \mathscr{L}_H) \cap (A' \cap \mathscr{L}_V) = \varnothing$, because $\delta_2 = d(A, \hat{\mathscr{R}}) > 1/k$.

PROOF OF CLAIM (I). In order to show (I) we have to prove that

$$H_k \cap \{U \cap A' \cap \mathscr{L}_H \neq \varnothing\} \in \mathscr{F}_A^k \tag{2.1}$$

and

$$H_k \cap \{U \cap A' \cap \mathscr{L}_V \neq \varnothing\} \in \mathscr{F}_A^k, \tag{2.2}$$

for any open subset U of $[0, z_0)$.

We are going to show (2.1), and the proof of (2.2) would be similar. Without any loss of generality we may assume that U is a ball of radius $r < 1/k$ centered at some point ξ of A'.

We introduce the sets

$$F_1 = \{U \cap A' \cap [(H_1^-(A) - H_2^-(A)) \cup G(A)] \cap \mathscr{L}_H \neq \varnothing\},$$
$$F_2 = \{U \cap A' \cap F(A) \cap \mathscr{L}_H \neq \varnothing\}$$

and

$$M = \{\exists\, \alpha \geq 1 \text{ such that } \forall\, n \geq 1, \text{there exist } z_1 = (s_1, t_1),\ z_2 = (s_2, t_2),$$
$$s_2 \geq s_1,\ t_2 < t_1, \text{ such that } z_1, z_2 \in A_0 \cap U,\ |z_1 - z_2| < 1/n,$$
$$|z_1 - \xi| \leq r - 1/\alpha,\ |z_2 - \xi| \leq r - 1/\alpha \text{ and } N_{z_1} > N_{z_2}\}.$$

Hypothesis (H) and property (iii) of $\mathscr{C}$ imply that

$$(H_2^-(A) - H_1^-(A)) \cap \mathscr{L}_H = \varnothing \quad \text{a.s.}$$

Consequently, we have

$$H_k \cap \{U \cap A' \cap \mathscr{L}_H \neq \varnothing\} = (H_k \cap F_1) \cup (H_k \cap F_2).$$

We have

$$F_1 \cap H_k \subset M \cap H_k. \tag{2.3}$$

In fact, suppose first that there exists some point $z \in U \cap A' \cap [H_1^-(A) - H_2^-(A)] \cap \mathscr{L}_H$. Then z can be approximated from above by points of A_0 because $z \in H_1^-(A)$; also it can be approximated in Q_z^{+-} by points of A_0. Therefore $z \in M$. If $z \in U \cap A' \cap G(A) \cap \mathscr{L}_H$, again the fact that $z \in G(A)$ implies that z can be approximated in Q_z^{+-} and Q_z^{-+} by points of A_0. This shows (2.3).

We claim that

$$M \cap H_k \subset \{U \cap A' \cap \mathscr{L}_H \neq \varnothing\} \cap H_k. \tag{2.4}$$

Indeed, on $H_k \cap M$ we have $U \cap A \cap \mathscr{L}_H \neq \varnothing$, and we know that $(A - A') \cap \mathscr{L}_H = \varnothing$ on H_k. So (2.4) is true. Now we can write

$$H_k \cap \{U \cup A' \cap \mathscr{L}_H \neq \varnothing\} = (M \cap H_k) \cup (M^c \cap F_2 \cap H_k).$$

Clearly $M \cap H_k$ belongs to $\mathscr{F}_A^k$. So, it remains to show that $F_2 \cap H_k$ belongs to $\mathscr{F}_A^k$.

We can write

$$\begin{aligned} H_k \cap F_2 = \{\omega\colon &\exists\ m \geq 2,\ \exists\ z_n,\ 1 \leq n \leq m \text{ such that } z_1, \dots, z_{m-1} \in F(A),\\ &z_m \in F(A) \cup G(A),\ z_1 \in U \cap A',\ z_n \text{ and } z_{n+1} \text{ have one coordinate in common for all } n,\ N_{z_n} > N_{z_n}^{--},\ 1 \leq n < m, \text{ and}\\ &\text{finally, } N_{z_m} = N_{z_m}^{--} \text{ if } z_m \in F(A) \text{ and } s_m = s_{m-1},\ N_{z_m} = N_{z_m}^{-+}\\ &\text{if } z_m \in G(A) \text{ and } s_m = s_{m-1}, \text{ or } N_{z_m} > N_{z_m}^{+-} \text{ if } z_m \in G(A) \text{ and}\\ &t_m = t_{m-1}\} \cap H_k. \end{aligned} \tag{2.5}$$

Let $F_2' \cap H_k$ be the right-hand side of (2.5). First note that $F_2' \cap H_k \subset F_2 \cap H_k$. Indeed, $N_{z_m} > N_{z_m}^{--}$ implies that $z_n \in \mathscr{L}$ for $1 \leq n < m$. Furthermore, $\mathscr{L}_V \cap \mathscr{L}_H \cap A = \varnothing$ on $\{\delta > 1/k\}$. Then $N_{z_m} > N_{z_m}^{+-}$ and $s_{m-1} = s_m$ imply that $z_m \in \mathscr{L}_H$, $z_{m-1} \in \mathscr{L}_H$, and, therefore, $z_n \in \mathscr{L}_H$ for $1 \leq n < m$. On the other hand, $N_{z_m} = N_{z_m}^{-+}$ and $t_{m-1} = t_m$ imply that $z_m \notin \mathscr{L}_V$, so $z_{m-1} \notin \mathscr{L}_V$ and we have again that $z_m \in \mathscr{L}_H$ for $1 \leq n < m$. Finally, $N_{z_m} = N_{z_m}^{--}$ for $z_m \in F(A)$ and $s_m = s_{m-1}$ also imply that $z_m \notin \mathscr{L}_V$. Conversely, suppose that $z \in U \cap A' \cap F(A) \cap \mathscr{L}_H$. Applying condition (ii) of the definition of the class $\mathscr{C}$ we can find a finite sequence $z_1, \dots, z_m$ such that $z_1 = z, z_1, \dots, z_{m-1}$ belong to $F(A)$, $z_n \in \mathscr{L}_H$ for $1 \leq n < m$, and z_m verifies one of the following conditions:

$$\begin{aligned} &z_m \in F(A), \quad s_m = s_{m-1} \text{ and } N_{z_m} = N_{z_m}^{--},\\ &z_m \in G(A), \quad s_m = s_{m-1} \text{ and } N_{z_m} = N_{z_m}^{-+},\\ &z_m \in G(A), \quad t_m = t_{m-1} \text{ and } N_{z_m} > N_{z_m}^{+-}. \end{aligned}$$

If none of these possibilities happens to be true for z_m, we have necessarily that $z_m \in F(A)$ and $N_{z_m}^{--} > N_{z_m}$. It is not possible to have an infinite sequence of points $z_m \in F(A)$ verifying $N_{z_m}^{--} > N_{z_m}$. So we must stop at some instant and this shows that $F_2' \cap H_k \supset F_2 \cap H_k$.

We can finally show that $F_2' \cap H_k \in \mathscr{F}_A^k$ by expressing the set F_2' in terms of the values of the process N_z on points $z \in A_0$ by means of Proposition 1.4. Note that for $z \in F(A)$, N_z^{--} is $\mathscr{F}_A$-measurable and for $z \in G(A)$, N_z^{+-} and N_z^{-+} are $\mathscr{F}_A$-measurable.

(II) By the selection theorem (Proposition 1.6), the measurability of the sets $A' \cap \mathscr{L}_V$ in the space $(H_k, \mathscr{F}_A^k, P)$ implies the existence of two sequences of random variables in this space: $\{X_n, n \geq 1\}$, $\{Y_n, n \geq 1\}$ such that $X_n \in A' \cap$

$\mathscr{L}_H$, $Y_n \in A' \cap \mathscr{L}_V$ a.s. and the sets $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are dense a.s. in $A' \cap \mathscr{L}_H$ and $A' \cap \mathscr{L}_V$, respectively.

Consider the balls $B_{1/2k}(X_n)$, $B_{1/2k}(Y_n)$. Now we are going to introduce a first family of open sets which will satisfy conditions (ii)–(v) (with H_k replacing $H_{m,k}$).

Let $\mathscr{B}_k$ be the family of the following sets:

The balls $B_{1/2k}(z)$, $z \in H_1^-(A) \cap H_2^-(A)$, and the sets

$$B_k^{1,X} = B_{1/2k}(X_n) \cap \{(s,t): t < X_n^2\}, \qquad B_k^{2,X} = B_{1/2k}(X_n) \cap \{(s,t): t \geq X_n^2\},$$

$$B_k^{1,Y} = B_{1/2k}(Y_n) \cap \{(s,t): s < Y_n^1\}, \qquad B_k^{2,X} = B_{1/2k}(Y_n) \cap \{(s,t): s \geq Y_n^1\}$$

verify properties (ii)–(iv):

If $z \in H_1^-(A) \cap H_2^-(A)$ and $\omega \in H_k$, the ball $B_{1/2k}(z)$ has an empty intersection with $\mathscr{L}$, because $\delta_1 > 1/k$. Consequently, N is constant on each of these balls. On the other hand, they are deterministic and intersect A_0. For the other type of random sets, their measurability in $(H_k, \mathscr{F}_A^k, P)$ follows from the measurability of X_n and Y_n. Also, N must be constant on each one of these sets (on H_k) because each ball $B_{1/2k}(X_n)$ contains a unique horizontal segment of $\mathscr{L}_H$ passing through X_n and $B_{1/2k}(X_n) \cap \mathscr{L}_V = \varnothing$. Similarly, each ball $B_{1/2k}(Y_n)$ contains a unique vertical segment of $\mathscr{L}_V$ passing through Y_n and $B_{1/2k}(Y_n) \cap \mathscr{L}_H = \varnothing$. It is also immediate that $B_k^{2,X} \cap A_0 \neq \varnothing$ and $B_k^{2,Y} \cap A_0 \neq \varnothing$. Finally, the definition of the set A_0 shows that $B_k^{1,X} \cap A_0 \neq \varnothing$ and $B_k^{1,Y} \cap A_0 \neq \varnothing$.

(III) Set

$$\begin{aligned} A'' &= A' - \left(\bigcup_{n=1}^{\infty} B_{1/2k}(X_n) \cup \bigcup_{n=1}^{\infty} B_{1/2k}(Y_n) \right) \\ &= A - \bigcup_{B \in \mathscr{B}_k} B. \end{aligned}$$

A'' is a measurable compact set in $(\{\delta > 1/k\}, \mathscr{F}_A^k, P)$. We have that $\eta = d(A'', \mathscr{L}) > 0$. Set $H_{m,k} = \{\delta > 1/k, \eta > 1/m\} \subset H_k$.

Again applying the selection theorem (Proposition 1.6) we can find a sequence of random variables $\{Z_n, n \geq 1\}$ in the space $(H_{m,k}, \mathscr{F}_A|_{H_{m,k}}, P)$ such that $Z_n \in A''$ a.s. and the set $\{Z_n, n \geq 1\}$ is dense in A''. Now, let $\mathscr{B}_{m,k}$ be the family of balls $B_{1/2m}(Z_n)$, $n \geq 1$. These balls again verify properties (ii)–(v). Finally, $\mathscr{B}_k \cup \mathscr{B}_{m,k}$ will be a family of random sets satisfying properties (i)–(v) with $\varepsilon < \min(1/2m, 1/2k)$. □

REMARKS. We have already seen that $\mathscr{E}$ contains the sets of $\hat{\mu}^{z_0}$ measure zero, or the sets of $\mu_1 \times \mu_2$ measure zero if $\mu \ll \mu_1 \times \mu_2$ and N is strictly simple. Therefore, we have

COROLLARY 2.2. *Let N be a strictly simple point process on $\mathbb{R}_+^2$ verifying condition* (H) *and with an intensity $\mu \ll \mu_1 \times \mu_2$. Then $\mathscr{F}_A = \mathscr{G}_A$ for any set $A \in \mathscr{C}$ of measure $\mu_1 \times \mu_2$ zero.*

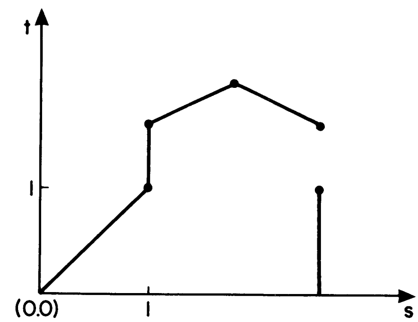

FIG. 1.

As a consequence of this corollary, the equality $\mathscr{F}_A = \mathscr{G}_A$ is true for any set $A \in \mathscr{C}$ of zero Lebesgue measure, if N is the Poisson plane process.

Suppose that A is a compact subset of $\mathbb{R}_+^2$ which can be expressed as a finite union of nonincreasing and nondecreasing lines. Then, for A to belong to the class $\mathscr{C}$, only the condition (ii) needs to be satisfied. Therefore we have

COROLLARY 2.3. *Let N be a strictly simple point process on $\mathbb{R}_+^2$ with an intensity μ absolutely continuous with respect to the Lebesgue measure. Then $\mathscr{G}_A = \mathscr{F}_A$ for any set A which is a finite union of nonincreasing and nondecreasing lines and verifies* (ii).

This condition (ii) cannot be removed, as is shown by the following example. Let N be the Poisson plane process. Then, for $A = \{(s, s), 0 \le s \le 1\}$ we have $\mathscr{F}_A \not\subseteq \mathscr{G}_A$, because $\{N(D_1) = 1\} \cap \{N(D_2) = 0\}$ belongs to $\mathscr{G}_A$ but not to $\mathscr{F}_A$, where $D_1 = \{(s, t) \in [0,1]^2\colon\ s \ge t\}$ and $D_2 = \{(s, t) \in [0,1]^2\colon\ s \le t\}$. This example has been inspired by the paper of Carnal [4]. Here condition (ii) is not satisfied. However, if A is the set given by Figure 1, for example, property (ii) holds and $\mathscr{F}_A = \mathscr{G}_A$.

Note that (ii) is always satisfied for a closed curve without double points, which is a finite union of increasing or decreasing lines.

We can apply the preceding results to study the Markov property for a point process on the plane.

DEFINITION 2.4. Let N be a point process on $\mathbb{R}_+^2$. We say that N has the *sharp Markov property* (SMP) relative to a set $A \subset \mathbb{R}_+^2$ if $\mathscr{F}_A \perp \mathscr{F}_{A^c}/\mathscr{F}(\partial A)$. We say that N has the *Markov property* (MP) relative to $A \subset \mathbb{R}_+^2$ if $\mathscr{F}_A \perp \mathscr{F}_{A^c}/\mathscr{G}(\partial A)$.

Then, as an immediate consequence of Theorem 2.1 we deduce the following general result.

PROPOSITION 2.5. *Let N be a strictly simple point process on $\mathbb{R}_+^2$ verifying hypothesis* (H) *and such that $\mu \ll \mu_1\mu_2$. If N has the MP with respect to a set A* *such that A or A^c is bounded and $\partial A \in \mathscr{C}$ and $(\mu_1 \times \mu_2)(\partial A) = 0$, then it also has the SMP relative to A.*

If we assume that this intensity μ of N is absolutely continuous with respect to the Lebesgue measure, then (H) is satisfied, and the MP with respect to a set A whose boundary is a finite union of increasing or decreasing lines and verifies (ii), implies the SMP relative to this set. In particular, using the results of Russo [17], we deduce that any point process with independent increments which is strictly simple and with an absolutely continuous intensity has the SMP relative to any open connected and bounded set with a piecewise decreasing or increasing boundary. In the case of a Poisson process this improves the results of Carnal [4], where the SMP was proved for open bounded and relatively convex sets, and those of Russo [17], where the SMP was established for finite union of rectangles.

3. Markov property for stopping lines. We shall say that L is a decreasing line if L is the image of a continuous function $\gamma\colon [0,1] \to \mathbb{R}_+^2$ such that $\gamma(0)$ belongs to the y-axis, $\gamma(1)$ belongs to the x-axis, $\gamma((0,1)) \subset (0,\infty)^2$ and γ is decreasing, that means $\theta_1 \le \theta_2$ implies $\gamma_1(\theta_1) \le \gamma_1(\theta_2)$ and $\gamma_2(\theta_1) \ge \gamma_2(\theta_2)$, where $\gamma(\theta_i) = (\gamma_1(\theta_i), \gamma_2(\theta_i))$. To every decreasing line L we associate the set $D(L) = \{z \in \mathbb{R}_+^2\colon$ there exists $z' \in L$ such that $z \le z'\}$. The relation $z \le L$ means that $z \in D(L)$ and $z \ge L$ means that $z \ge z'$ for some $z' \in L$. If L_1 and L_2 are two decreasing lines, the relation $L_1 \le L_2$ means that $D(L_1) \subset D(L_2)$ and $[L_1, L_2]$ will denote the set $\{z\colon L_1 \le z \le L_2\}$. We denote by $\mathscr{S}$ the set of all the decreasing lines.

Suppose that an increasing, right-continuous and complete family of σ-fields $\{\mathscr{F}_z,\ z \in \mathbb{R}_+^2\}$ is given. Then a random decreasing line $L\colon \Omega \to \mathscr{S}$ is called a *stopping line* if for every $z \in \mathbb{R}_+^2$, $\{\omega\colon z \le L(\omega)\} \in \mathscr{F}_z$ ([3], [11], [21]). For any $l \in \mathscr{S}$ we set $\mathscr{F}_{D(l)} = \bigvee_{z \le l} \mathscr{F}_z$ and we also define $\mathscr{F}_{D(l)}^+ = \bigcap_{\varepsilon > 0} \mathscr{F}_{D(l)_\varepsilon}$, that means $\mathscr{F}_{D(l)}^+ = \mathscr{G}_{D(l)}$. We do not know, in general, if the family $\{\mathscr{F}_{D(l)},\ l \in \mathscr{S}\}$ is right-continuous. As in the one-parameter case, to every stopping line L we can associate the σ-field $\mathscr{F}_{D(L)}^* = \{A \in \mathscr{F}\colon A \cap \{L \le l\} \in \mathscr{F}_{D(l)}^+$ for all $l \in \mathscr{S}\}$. The family of σ-fields $\{\mathscr{F}_{D(L)}^*,\ L$ stopping line$\}$ is increasing and right-continuous. That means $\mathscr{F}_{D(L)}^* = \bigcap_n \mathscr{F}_{D(L_n)}^*$ if $L_n \downarrow L$. These facts are proved by the same arguments as in the one-parameter case. Note that for σ-fields generated by the Brownian sheet, the right-continuity was proved in [3] for a very similar filtration.

In the sequel we will assume that $\mathscr{F}_z$ is the natural filtration of a point process $N = \{N_z,\ z \in \mathbb{R}_+^2\}$. That means $\mathscr{F}_z = \mathscr{F}_{[0,z]}$ for any $z \in \mathbb{R}_+^2$, and therefore the above definition of $\mathscr{F}_{D(l)} = \bigvee_{z \le l} \mathscr{F}_z$ is consistent with the one given in the previous section, i.e., $\mathscr{F}_{D(l)} = \sigma\{N_z,\ z \le l\} \vee \mathscr{N}$, where $\mathscr{N}$ denotes the family of null sets. Lemma 3.2 will imply the right-continuity of the family of σ-fields $\{\mathscr{F}_{D(l)},\ l \in \mathscr{S}\}$. Notice that here we have $\mathscr{F}_{D(l)} = \mathscr{G}_{D(l)}$ without any of the supplementary assumption required in the previous section. Notice also that if the filtration is generated by the Brownian sheet, then the right-continuity of $\mathscr{F}_{D(l)}$ was proved in [21].

DEFINITION 3.1. Let A be a random set, that means $\{\mathbf{1}_A(z),\ z \in \mathbb{R}_+^2\}$ is a stochastic process. Then, we define $\mathscr{F}_A$ as the σ-field generated by the *random* variables $N_z\mathbf{1}_A(z), \mathbf{1}_A(z)$, $z \in \mathbb{R}_+^2$, and the null sets of Ω. We shall see below that again this definition is consistent with the definition of $\mathscr{F}^*_{D(L)}$ when L is a stopping line, i.e., $\mathscr{F}^*_{D(L)} = \mathscr{F}_{D(L)}$.

LEMMA 3.2. *Let L_n be a decreasing sequence of stopping lines converging to the stopping line L. Then, in the sense of Definition* 3.1,

$$\bigcap_n \mathscr{F}_{D(L_n)} = \mathscr{F}_{D(L)} \quad \text{and} \quad \bigcap_n \mathscr{F}_{[L, L_n]} = \mathscr{F}_L.$$

PROOF. First we will show that $\bigcap_n \mathscr{F}_{D(L_n)}$ is included in $\mathscr{F}_{D(L)}$. It suffices to see that the random variables of the form $N_{z_i^n}\mathbf{1}_{\{z_i^n \le L_n\}}, \mathbf{1}_{\{z_i^n \le L_n\}}$, $i \ge 1$, are $\mathscr{F}_{D(L)}$-measurable. The sets $H_n = \{N([L, L_n]) = N(L)\}$ increase to Ω. Let δ be the distance between L and the support of $N|_{D(L)^c}$. δ is a strictly positive random variable. Let $H_{n,k} = \{d(L, L_n) < 1/k < \delta\}$. We have that $\omega \in H_{n,k}$ for $n \ge n_0(k, \omega)$. The sets $H_{n,k} = \{d(L, L_n) < 1/k < \delta)\} \subset H_n$ also increase to Ω when $n \uparrow \infty$ and $k \uparrow \infty$. Consequently it suffices to show that on every set $H_{n,k}$, the random variables $N_{z_i^n}\mathbf{1}_{\{z_i^n \le L_n\}}$ and $\mathbf{1}_{\{z_i^n \le L_n\}}$ coincide with an $F_{D(L)}$-measurable random variable. On $H_{n,k}$ we have

$$N_{z_i^n}\mathbf{1}_{\{z_i^n \le L_n\}} = N\big(D(L) \cap [0, z_1^n]\big)\mathbf{1}_{\{z_i^n \le L_n\}}.$$

The random variable $N(D(L) \cap [0, z_i^n])$ is $\mathscr{F}_{D(L)}$-measurable. This follows from the definition of $\mathscr{F}_{D(L)}$. Note that $\{z_i^n \le L_n\} \in \mathscr{F}_{z_i^n}$ and $\mathbf{1}_{\{z_i^n \le L_n\}}$ is a function Φ of a countable collection of random variables $\{N_\xi,\ \xi \le z_i^n\}$. Then

$$\begin{aligned}\mathbf{1}_{H_{n,k}}\mathbf{1}_{\{z_i^n \le L_n\}} &= \mathbf{1}_{\{z_i^n \le L_n\}}\mathbf{1}_{\{d(z_i^n, L) < 1/k < \delta\}}\mathbf{1}_{H_{n,k}}\mathbf{1}_{\{z_i^n > L\}} + \mathbf{1}_{H_{n,k}}\mathbf{1}_{\{z_i^n \le L\}} \\ &= \Phi\big(N\big(D(L) \cap [0, \xi]\big), \xi \le z_i^n\big)\mathbf{1}_{\{d(z_i^n, L) < 1/k\}}\mathbf{1}_{H_{n,k}}\mathbf{1}_{\{z_i^n > L\}} \\ &\quad + \mathbf{1}_{H_{n,k}}\mathbf{1}_{\{z_i^n \le L\}}.\end{aligned}$$

Then $\mathbf{1}_{\{z_i^n \le L_n\}}$ is $\mathscr{F}_{D(L)}$-measurable on each $H_{m,k}$. This completes the proof of the inclusion $\bigcap_n \mathscr{F}_{D(L_n)} \subset \mathscr{F}_{D(L)}$.

The proof of the inclusion $\bigcap_n \mathscr{F}_{[L, L_n]} \subset \mathscr{F}_L$ is similar, but proceeds by replacing $N(D(L) \cap [0, \xi])$ by $N(L \cap [0, \xi])$. □

LEMMA 3.3. *For any stopping line L, we have*

$$\mathscr{F}_{D(L)} = \big\{A \in \mathscr{F}\colon A \cap \{L \le l\} \in \mathscr{F}_{D(l)},\ \textit{for any}\ l \in \mathscr{S}\big\}.$$

PROOF. As before, set $\mathscr{F}^*_{D(L)} = \{A \in \mathscr{F}\colon A \cap \{L \le l\} \in \mathscr{F}_{D(l)}$, for any $l \in \mathscr{S}\}$. The families of σ-fields $\mathscr{F}^*_{D(L)}$ and $\mathscr{F}_{D(L)}$ are right-continuous. On the other hand, using dyadic approximations (see [3]) we can approximate any stopping line by a decreasing sequence of stopping lines L_n such that for any n, L_n is a stepped line determined by points of the form $(i2^{-n}, j2^{-n})$, $i, j \in \mathbb{N}$. Consequently, it suffices to check that $\mathscr{F}_{D(L)} = \mathscr{F}^*_{D(L)}$ when L takes a countable number of possible configurations.

The inclusion $\mathscr{F}_{D(L)} \subset \mathscr{F}^*_{D(L)}$ is easy to prove, for an arbitrary stopping line L. In fact, we have to show that the generators of $\mathscr{F}_{D(L)}$ belong to $\mathscr{F}^*_{D(L)}$. If $A = \{z \le L\}$, then $\{z \le L\} \cap \{L \le l\} = \{z \le l\} \cap \{z \le L\} \cap \{L \le l\}$ and $\{z \le L\} \in \mathscr{F}_z \subset \mathscr{F}_{D(L)}$ if $z \le l$ and $\{L \le l\} \in \mathscr{F}_{D(L)}$ because

$$\{L \not\le l\} = \bigcup_{\substack{z \in \mathbb{Q}^2_+ \\ z \not\le l,\ d(z,l) < \varepsilon}} \{z \le L\} \in \mathscr{F}_{D(L)_\varepsilon} \quad \text{for all } \varepsilon > 0.$$

If $A = \{N_z = k\} \cap \{z \le L\}$, then $\{N_z = k\} \cap \{z \le L\} \cap \{L \le l\}$ also belongs to $\mathscr{F}_{D(L)}$ by the same argument.

To show the reverse inclusion, let $G \in \mathscr{F}^*_{D(L)}$ and suppose that L takes only a countable family of configurations $\{l_i,\ i \ge 1\} \subset \mathscr{S}$. Then, $G = \bigcup_{i=1}^\infty (G \cap \{L = l_i\})$ and for any $i \ge 1$, we have $G \cap \{L = l_i\} = G \cap \{L = l_i\} \cap \{L \le l_i\} \in \mathscr{F}_{D(l_i)}$ because $G \cap \{L \le l_i\}$ and $\{L = l_i\}$ belongs to $\mathscr{F}_{D(l_i)}$. Indeed,

$$\{L = l_i\} = \left(\bigcup_{\substack{z \in \mathbb{Q}^2_+ \\ z \le l_i,\ d(z,l_i) < \varepsilon}} \{z \not\le L\} \right) \cap \left(\bigcap_{z \in l_i \cap \mathbb{Q}^2_+} \{z \le L\} \right) \in \mathscr{F}_{(l_i)_\varepsilon} \qquad \forall\, \varepsilon > 0.$$

So, $\mathbf{1}_{G \cap \{L = l_i\}}$ is a function of $N_z \mathbf{1}_{\{z \le l_i\}} = N_z \mathbf{1}_{\{z \le L\}}$ and of $\mathbf{1}_{\{L = l_i\}}$ and those random variables are $\mathscr{F}_{D(L)}$-measurable. □

Theorem 3.4. *Let N be a point process on the plane verifying the SMP relative to the sets $\{D(l),\ l \in \mathscr{S}\}$. Then, for any stopping line L we have $\mathscr{F}_{D(L)} \perp \mathscr{F}_{\overline{D(L)^c}} / \mathscr{F}_L$.*

Remark. From Theorem 2.1 it follows that the SMP relative to the sets $\{D(l),\ l \in \mathscr{S}\}$ is equivalent to the MP under hypothesis (H′).

Proof. Fix $z_1, \dots, z_k$ in $\mathbb{R}^2_+$ and let $f\colon \mathbb{R}^{2k} \to \mathbb{R}$ be a continuous and bounded function. We want to show that

$$\begin{aligned} &E\left[f\left(N_{z_1}\mathbf{1}_{\{z_1 \ge L\}}, \mathbf{1}_{\{z_1 \ge L\}}, \dots, N_{z_k}\mathbf{1}_{\{z_k \ge L\}}, \mathbf{1}_{\{z_k \ge L\}}\right) / \mathscr{F}_{D(L)}\right] \\ &\quad = E\left[f\left(N_{z_1}\mathbf{1}_{\{z_1 \ge L\}}, \mathbf{1}_{\{z_1 \ge L\}}, \dots, N_{z_k}\mathbf{1}_{\{z_k \ge L\}}, \mathbf{1}_{\{z_k \ge L\}}\right) / \mathscr{F}_L\right]. \end{aligned}$$

Notice that $\mathbf{1}_{\{z_i \ge L\}} = \mathbf{1}_{\{z_i > L\}} + \mathbf{1}_{\{z_i \in L\}}$ and $N_{z_i}\mathbf{1}_{\{z_i \in L\}}, \mathbf{1}_{\{z_i \in L\}}$ are $\mathscr{F}_L$-measurable. Consequently, it suffices to show that

$$\begin{aligned} &E\left[f\left(N_{z_1}\mathbf{1}_{\{z_1 > L\}}, \mathbf{1}_{\{z_1 > L\}}, \dots, N_{z_k}\mathbf{1}_{\{z_k \ge L\}}, \mathbf{1}_{\{z_k \ge L\}}\right) / \mathscr{F}_{D(L)}\right] \\ &\quad = E\left[f\left(N_{z_1}\mathbf{1}_{\{z_1 > L\}}, \mathbf{1}_{\{z_1 > L\}}, \dots, N_{z_k}\mathbf{1}_{\{z_k \ge L\}}, \mathbf{1}_{\{z_k \ge L\}}\right) / \mathscr{F}_L\right]. \end{aligned} \tag{3.1}$$

Assume first that the stopping line L takes only a countable number of possible configurations $\{l_i,\ i \ge 1\} \subset \mathscr{S}$. Then, the left-hand side of (3.1) can be written as

$$\sum_{i=1}^{\infty} E\left[\mathbf{1}_{\{L = l_i\}} f\left(N_{z_1}\mathbf{1}_{\{z_1 > l_i\}}, \mathbf{1}_{\{z_1 > l_i\}}, \dots, N_{z_k}\mathbf{1}_{\{z_k > l_i\}}, \mathbf{1}_{\{z_k > l_i\}}\right) / \mathscr{F}_{D(L)}\right]. \tag{3.2}$$

We have already seen that the set $\{L = l_i\}$ belongs to $\mathscr{F}_{D(L)} \cap \mathscr{F}_{D(l_i)}$ and on this set the σ-fields $\mathscr{F}_{D(L)}$ and $\mathscr{F}_{D(l_i)}$ have the same trace. Therefore, (3.2) is equal to

$$\sum_{i=1}^{\infty} E\left[\mathbf{1}_{\{L=l_i\}} f\left(N_{z_1}\mathbf{1}_{\{z_1>l_i\}}, \mathbf{1}_{\{z_1>l_i\}}, \dots, N_{z_k}\mathbf{1}_{\{z_k>l_i\}}, \mathbf{1}_{\{z_k>l_i\}}\right)/\mathscr{F}_{D(l_i)}\right]$$

$$= \sum_{i=1}^{\infty} \mathbf{1}_{\{L=l_i\}} E\left[f\left(N_{z_1}\mathbf{1}_{\{z_1>l_i\}}, \mathbf{1}_{\{z_1>l_i\}}, \dots, N_{z_k}\mathbf{1}_{\{z_k>l_i\}}, \mathbf{1}_{\{z_k>l_i\}}\right)/\mathscr{F}_{l_i}\right],$$

which is $\mathscr{F}_L$-measurable. In fact, each term can be expressed as

$$\mathbf{1}_{\{L=l_i\}}\Phi_i\left(N_{z_j^i}\mathbf{1}_{\{z_j^i \in L\}},\ j \geq 1\right),$$

where Φ_i are measurable and bounded functions.

In the general case, we consider the sequence of stepped stopping lines $L_n \downarrow L$ introduced in the proof of Lemma 3.3. Then, using Lemma 3.2 we have

$$E\left[f\left(N_{z_1}\mathbf{1}_{\{z_1>L\}}, \mathbf{1}_{\{z_1>L\}}, \dots, N_{z_k}\mathbf{1}_{\{z_k>L\}}, \mathbf{1}_{\{z_k>L\}}\right)/\mathscr{F}_{D(L)}\right]$$

$$= \lim_n E\left[f\left(N_{z_1}\mathbf{1}_{\{z_1>L_n\}}, \mathbf{1}_{\{z_1>L_n\}}, \dots, N_{z_k}\mathbf{1}_{\{z_k>L_n\}}, \mathbf{1}_{\{z_k>L_n\}}\right)/\mathscr{F}_{D(L_n)}\right]$$

$$= \lim_n E\left[f\left(N_{z_1}\mathbf{1}_{\{z_1>L_n\}}, \mathbf{1}_{\{z_1>L_n\}}, \dots, N_{z_k}\mathbf{1}_{\{z_k>L_n\}}, \mathbf{1}_{\{z_k>L_n\}}\right)/\mathscr{F}_{[L, L_n]}\right]$$

$$= E\left[f\left(N_{z_1}\mathbf{1}_{\{z_1>L\}}, \mathbf{1}_{\{z_1>L\}}, \dots, N_{z_k}\mathbf{1}_{\{z_k>L\}}, \mathbf{1}_{\{z_k>L\}}\right)/\mathscr{F}_L\right],$$

which completes the proof of (3.1). □

A companion to the notion of stopping lines is the concept of optional increasing path which is defined to be a random continuous increasing path $L = L(t)$ such that for any $z \in \mathbb{R}_+^2$ and $t \geq 0$, $\{L(t) \leq z\} \in \mathscr{F}_z$. An optional increasing path L splits the positive quadrant into two regions: the right-bottom side denoted by $\vec{L}$ and the left-top side denoted by $\overleftarrow{L}$, as was done in [12]. L can be approximated in $\vec{L}$ or in $\overleftarrow{L}$ by stepped optional increasing paths.

Suppose that N is strictly simple and that N has no discontinuity on any optional increasing path L: $N(L) = 0$ a.s. For instance, this property is true when N is the Poisson sheet (cf. [11]). For any optional increasing path, let $Z_0 = (S_0, T_0)$ be its end point and consider the random sets

$$L^+ = LU\{(S_0, t), 0 \leq t \leq T_0\} \quad \text{and} \quad D^+(L) = \left\{(s,t): 0 \leq s \leq S_0,\ t \in \vec{L}\right\}.$$

Then, by the same arguments as before, the strong Markov property holds for L^+. That means, $\mathscr{F}_{D^+(L)} \perp \mathscr{F}_{D\overline{(L^+)^c}}/\mathscr{F}_{L^+}$.

4. Transformations of point processes. Using the same ideas as in [22], we can consider absolutely continuous transformations of measures that leave the Markov property (or the SMP) invariant. Let $\mathscr{B}$ denote the family of Borel subsets A of $\mathbb{R}_+^2$ such that $|\partial A| = 0$ and A or A^c is bounded. Consider a family of random variables $\{\alpha(D),\ D \in \mathscr{B}\}$, satisfying the following properties:

1. $\alpha(D_1 \cup D_2) = \alpha(D_1) + \alpha(D_2)$ if $D_1 \cap D_2 = \phi$.
2. $\alpha(D)$ is $\mathscr{F}_D$-measurable.
3. $E(\alpha(D)) < \infty$ for any D.

Let $L = [\exp \alpha(\mathbb{R}_+^2)]/(E(\exp \alpha(\mathbb{R}_+^2)))$. Then we have (cf. Proposition 3 of [2])

PROPOSITION 4.1. *If the point process N has the SMP relative to sets $A \in \mathscr{B}$, then it has also the SMP relative to sets $A \in \mathscr{B}$, under the new probability measure $dQ = L\,dP$.*

Consider the following example of application. Suppose that N is the Poisson process on $\mathbb{R} \times \mathbb{R}_+$ with intensity $\mu(ds, dt) = \mu_1(ds)\,dt$, where μ_1 is some σ-finite measure on $\mathbb{R}$. Let $\{h(z),\ z \in \mathbb{R} \times \mathbb{R}_+\}$ be a 2-predictable process such that $|h(z)| \le 1/2$ and h is uniformly bounded by a square integrable deterministic function. That means $|h(s,t)| \le \phi(s)$ and $\int_{\mathbb{R}} \phi^2(s)\mu_1(ds) < \infty$.

Define

$$L_t = \int_{\mathbb{R} \times [0,t]} h(u)(N(du) - \mu(du)), \qquad t \ge 0,$$

and

$$M_t = \exp L_t \prod_{\nu < t} \left[(1 + \Delta^2 L_\nu) \exp(-\Delta^2 L_\nu) \right].$$

We introduce the random measure

$$N^1(s,t) = \int_{\mathbb{R} \times [0,t]} \mathbf{1}_{[0,s]}\left(x + \int_0^x h(\sigma, y)\, d\sigma \right) N(dx, dy), \qquad s \in \mathbb{R},\ t \ge 0.$$

Let Q be a new probability measure given by

$$\left. \frac{dQ}{dP} \right|_{\mathscr{F}_t} = M_t,$$

where $\mathscr{F}_t = \sigma\{N(A \times [0,\tau]),\ 0 \le \tau \le t,\ A \in \mathscr{B}(\mathbb{R})\}$ completed with the null sets of $\mathscr{F}$.

We remark that M is a strictly positive martingale, because $|\Delta L_t| \le 1/2$. Then, following the arguments used in [2], we can show the next result.

PROPOSITION 4.2. *Under Q, N^1 is a Poisson process with intensity μ.*

PROOF. First note that N^1 is a point process since it is obtained as an integral of an indicator function with respect to N. The one parameter process $\{N^1(s,t),\ t \ge 0\}$ is $\mathscr{F}_t$-adapted. Then it suffices to show that $\{N(A \times [0,t]) - t\mu_1(A),\ t \ge 0\}$ is an $\mathscr{F}_t$-local martingale under Q, for any $A \in \mathscr{B}(\mathbb{R})$. For the proof of this fact we refer to [2]. □

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