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RANDOM TREE-TYPE PARTITIONS AS A MODEL FOR ACYCLIC POLYMERIZATION: HOLTSMARK (3/2-STABLE) DISTRIBUTION OF THE SUPERCRITICAL GEL[1]

BY B. PITTEL,[2] W. A. WOYCZYNSKI AND J. A. MANN

Ohio State University, Case Western Reserve University and Case Western Reserve University

Random tree-type partitions for finite sets are used as a model of a chemical polymerization process when ring formation is forbidden. Technically, our series of three papers studies the asymptotic behavior (in the thermodynamic limit as $n \to \infty$) of a particular probability distribution on the set of all forests of trees on a set of n elements (monomers). The study rigorously establishes the existence of three stages of polymerization dependent upon the ratio of association and dissociation rates of monomers. The subcritical stage has been analyzed in the other two papers of this series. The present paper, second in the series, concentrates on the analysis of the near-critical and supercritical stages. In the supercritical stage we discover that the molecular weight of the largest connected component (gel) has the Holtsmark distribution.

Our study combines elements of a classical Flory–Stockmayer polymerization theory with the spirit of more recent developments in the Erdös–Rényi theory of random graphs. Although this paper has a chemical motivation, conceptually similar mathematical models have been found useful in other disciplines, such as computer science and biology, etc.

1. Introduction.

1.1. *Physicochemical motivation*: *General setting*. The idea of using the theory of random graphs to describe polymerization processes is a natural one and is not new. Polymerization occurs when larger molecules (polymers) are formed by the creation of bonds between smaller units (monomers) such as atoms or functionality groups. Due to the dynamics of the process, bonds can also break. In so-called living polymers (such as vinyl), the bond formation process can continue without termination, giving rise to "arbitrarily" large polymerized molecules limited only by the amount of monomers present.

In general, polymers are polydisperse, i.e., the probability distribution of size of polymerized molecules is nontrivial (diffuse), and theoretical predictions about such distributions are one of the main topics of this series of papers [cf. Pittel, Woyczynski and Mann (1987) for the announcement of results, Pittel, Woyczynski and Mann (1989) for the discussion of the subcritical case and

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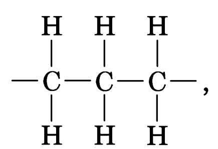
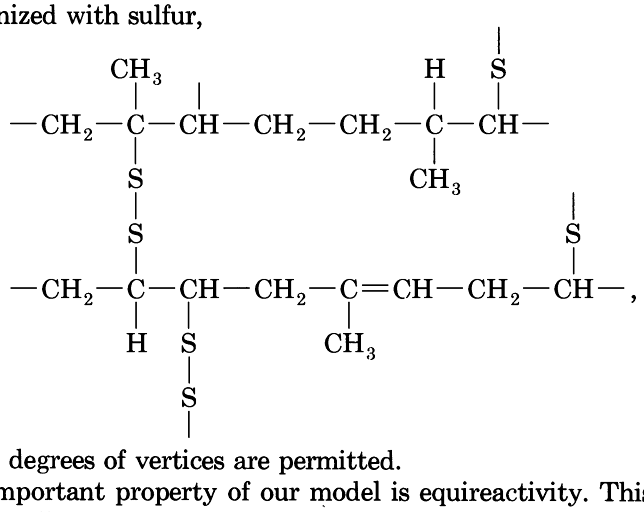
Pittel and Woyczynski (1989a) for a study of the so-called post-gelation sticking phenomenon]. A number of physical properties (such as viscosity) depend on the molecular size and shape distribution in the polymer, and for monodisperse polymers (which corresponds to conditioning on the size of molecules) quite precise empirical relations are known. Usually, when a large number of small molecules are present in the system (sol phase) the global shear viscosity is small and depends on the volume fraction, molecular weight distribution and temperature, but when a single giant molecule forms (sol–gel phase transition), the global shear viscosity goes up dramatically.

With monomer units as vertices, bonds as edges, polymerized molecules as connected components and with the random nature of bonding dictated by random collisions and the quantum-mechanical nature of bond formation, the random graph model suggests itself and has been considered by many authors, going back to Stockmayer (1943), Flory (1949, 1953), Gordon (1962), Whittle (1965 and onwards) and Spouge (1985). In the first paper of this series [cf. Pittel, Woyczynski and Mann (1989)] we traced the history of this topic in more detail.

The work presented in this series of papers concentrates on the case when the ring formation is prohibited and the resulting random graph is acyclic. There exist a large number of polymers of this kind. Examples include simple chain-like structures such as polyethylene,



where the degree of each vertex is at most 2, and branched structures such as rubber vulcanized with sulfur,



where higher degrees of vertices are permitted.

Another important property of our model is equireactivity. This means that monomers are allowed to form (or break) bonds without any regard for mutual spatial position. Sometimes this assumption is described as a graphlike state of matter, and physicists often refer to equireactive models as mean field models. It should be mentioned that, in recent years, a study of several other models of polymerization, such as reptation and self-avoiding random walk models, and models based on percolation and interacting particle systems theories, has been initiated. These models, from the very beginning, take three-dimensional spatial constraints into consideration. However, the classical equireactive model is still very much of interest, as evidenced by Whittle's (1986) book, and can account for many subtle phenomena, including phase transitions, observable in physical systems. Discovery, in this paper, of the appearance of the Holtsmark distribution in the supercritical phase, adds further interest to studying equireactive models. Besides, it is possible to include geometric constraints by embedding our random graph in a 3-space. We've done it in Pittel, Woyczynski and Mann (1987, 1989), and then computed certain geometry-dependent physical parameters (such as the friction coefficient and the intensity of scattered light) of the resulting polymer. Our computation yielded formulas very close to generally accepted and heuristic (Kirkwood's and Debye's) formulas which are well tested experimentally. Finally, it should be noted that, apart from our physicochemical motivation, similar equireactive assumptions for graph-valued processes appear quite naturally in other contexts, such as a study of set merging algorithms in computer science [cf., e.g., Yao (1976), Knuth and Schonhage (1978) and Bollobás and Simon (1985)].

1.2. *Contents and composition of the present paper.* The objective of the present paper is to study the limit distribution of sizes of connected components (trees) of the forest-of-trees-valued Markov process in the supercritical and near-critical stages. In contrast to Whittle's work, our model permits not only variable association (bond formation) rates for monomers, but also variable dissociation rates. Together with an analysis of the subcritical stage presented in the first paper of this series [cf. Pittel, Woyczynski and Mann (1989)], the results of the present paper provide a rigorous proof (the first in the literature, we believe) of the existence of three distinct stages of polymerization: subcritical, near critical and supercritical. Given the whole graph of size n, these three stages correspond to the size of the largest tree of the random forest of size n to be itself of order $\log n$, $n^{2/3}$ and n, respectively.

For the sake of completeness, let us mention that the third paper of this series [Pittel and Woyczynski (1989a)] gives a rigorous explanation, via a study of finite- and infinite-dimensional distributions arising in polymerization processes, of the phenomenon of post-gelation sticking discovered on a heuristic level by Stockmayer (1944). The term post-gelation sticking is used by physical chemists in the situation when the relative frequencies of molecular sizes of components different from the "giant" component are retained as the system evolves from near-critical phase to the supercritical phase. (In other words, the sol distribution "sticks" after gelation.) The problem of ring formation in polymers, which is excluded from the present series of papers, is addressed in a forthcoming paper by Pittel and Woyczynski (1989b).

As far as organization of the present paper is concerned, Section 2 contains the formal description of the Markov process $M(t)$ which serves as our model of polymerization. Section 3 contains formulation of the main results on the limit distributions of kth largest components in the supercritical and near-critical cases, preceded by a summary of pertinent results on the subcritical case from Pittel, Woyczynski and Mann (1989). The proofs are given in Section 4.

2. Description of the model. In the first part of this work (1989) we began a detailed asymptotic analysis of a Whittle-type polymerization process [cf. also Whittle (1965 and onwards)] which is also the object of our interest here. The process is modeled as a continuous-time Markov process $\{M(t) = M_n(t): t \geq 0\}$ whose *finite* state space $\mathscr{M}$ is a set of all forests M (no confusion will result from denoting the process and its states by the same letter) on a set V_n of n vertices labeled $1, 2, \ldots, n$, and interpreted as basic structural units (monomers). The state of the process changes in time, since bonds between the vertices may form or break. Specifically, if at a time t the process is in a state M, then the rate of bond formation between two vertices a and b (which do not belong to the same tree component of M) equals $\lambda A_{j+1}A_{k+1}/A_jA_k$, where j and k are, respectively, the degree of a and b in M [in short, $j = \deg(a, M)$, $k = \deg(b, M)$]. Furthermore, the rate of bond breaking for two vertices a and b connected by a bond in M equals $\mu D_{j-1}D_{k-1}/D_jD_k$ if $\deg(a, M) = j$, $\deg(b, M) = k$. In these formulas $\lambda > \mu > 0$ and $\{A_j: j \geq 0\}$, $\{D_j: j \geq 0\}$ are such that A_0, $D_0 \geq 0$ and A_j, $D_j > 0$ for $0 \leq j \leq j_{\max}$, where $j_{\max} \geq 3$. Note that in the case $j_{\max} \leq 2$, the trees are reduced to chains, and that in the Whittle (1980) model, $D_j = \text{const}$.

Intuitively speaking, the factor A_{j+1}/A_j can be interpreted as a measure of "eagerness" of a vertex to form a new bond in addition to its j present bonds. Furthermore, each of two vertices which "contemplate" forming a bond, is assumed to make up its mind independently of the other, whence the rate of bond formation is proportional to the product of these individual measures. The dissociation rates are interpreted similarly.

EXAMPLE 2.1. (i) If $A_j = 0$ for $j > m$, then each unit may form at most m bonds (which corresponds to the valency of monomers to be at most m).

(ii) If $A_j = 0$ for $j > m$ and $A_j = m!/(m-j)!$ for $j \leq m$, then $A_{j+1}A_{k+1}/(A_jA_k) = (m-j)(m-k)$, $j, k \leq m$, and the rate of bond formation between units a and b is proportional to the number of still "available" bonds of both a and b. This situation corresponds to the so-called classical Flory model, popular in the physicochemical literature.

From a general result announced in Pittel, Woyczynski and Mann (1987), and proved in the first part of this series of papers [1989, Theorem 1(ii)], it follows that the Markov process M has a stationary distribution

$$P(M) = Q^{-1}q(M), \qquad M \in \mathscr{M}, \tag{2.1}$$

where

$$q(M) = (\mu/\lambda)^{C(M)} \prod_{a \in V_n} H_{\deg(a, M)}$$

and

$$H_j := A_j D_j, \qquad j \geq 0.$$

Here, $C(M)$ is the total number of trees in forest M, and Q is a normalizing factor (partition function). Keeping in mind that $M = M_n$, $q = q_n$ and $Q = Q_n$, we seek information on the asymptotic behavior of the stationary distribution $P(M)$ in thermodynamic limit as $n \to \infty$, i.e., under assumption that $\{H_j\colon j \geq 0\}$ is fixed, but μ and λ change with n in such a way that $\mu/\lambda = n/\sigma_n$, where σ_n is bounded away from both 0 and ∞ [cf. Whittle (1980) and Pittel, Woyczynski and Mann (1987a) for a heuristic explanation].

We restrict our attention to a case when the following two technical assumptions are fulfilled.

Assumption 1. The series $H(y) = \sum_{j \geq 0} H_j y^j / j!$ has a positive radius r of convergence.

Assumption 2. The positive root $\bar{y}$ of

$$yH^{(2)}(y) - H^{(1)}(y) = 0, \qquad y \in (0, r), \tag{2.2}$$

is its only root in the *disk* $|y| \leq \bar{y}$.

Under these conditions, the equation

$$y = xH^{(1)}(y) \tag{2.3}$$

determines a function $y = R_1(x)$ which is analytic for $|x| < \bar{x}$ and continuous for $|x| \leq \bar{x}$, where $\bar{x}$ satisfies

$$\bar{y} = \bar{x}H^{(1)}(\bar{y}) \quad \text{and} \quad 1 = \bar{x}H^{(2)}(\bar{y}) \tag{2.4}$$

[the first by definition, and the latter by (2.2)]. It can be seen [cf. Pittel, Woyczynski and Mann (1989), Lemma 1] that

$$R_s(x) := xH^{(s)}(R_1(x)), \qquad s \geq 0, \tag{2.5}$$

is the exponential generating function (e.g.f.) of a nonnegative sequence $\{R_{sj}\colon j \geq 1\}$ given by the formula

$$R_{sj} = \sum_{T'} h_s(T').$$

Here, the sum extends over all *rooted* trees on V_j and, if the root of T' is a,

$$h_s(T') = \left[\prod_{b \in V_j \setminus \{a\}} H_{\deg(b, T')}\right] H_{\deg(a, T') + s}.$$

We shall also need a sequence $\{f_j\colon j \geq 1\}$ determined by

$$f_j = \sum_T h(T) \quad \text{where} \quad h(T) =: \prod_{b \in T} H_{\deg(b, T)}, \tag{2.6}$$

and where T runs through $\mathscr{T}_j$, the set of all (free) trees on V_j. By the same lemma, $F(x)$ the e.g.f. of $\{f_j\}$ satisfies, for $|x| \le \bar{x}$,

$$R_0(x) = xF^{(1)}(x), \qquad F(x) = R_0(x) - R_1^2(x)/2. \tag{2.7}$$

To better understand the meaning of identities (2.3)–(2.7) consider the case $H_j \equiv 1$. Then R_{1j} (resp. f_j) is just the total number of rooted (resp. free) trees on V_j. Furthermore, $R_1(x)$ satisfies equation $y = xe^y$, and—via Lagrange's inversion formula—we get that $R_{1j} = j^{j-1}$ (resp. $f_j = j^{j-2}$) and that $F(x) = R_1(x) - R_1^2(x)/2$ [see Moon (1970)]. Notice also that $\bar{x} = e^{-1}$ and $\bar{y} = 1$. Similar, but more complicated, formulas can be obtained for the Flory model, where $H(y) = (1+y)^m$ as well.

Importantly, the asymptotic analysis of the process M relies mostly on the behavior of $f_j = R_{0j}/j$ and R_{1j} for large j, which, for a general $H(y)$, is described by the formulas

$$R_{sj}/j! = (1 + o(1))\beta_s(\bar{x})^{-j} j^{-5/2}, \qquad j \to \infty, \tag{2.8}$$

where

$$\beta_s := \left[\bar{x}\bar{y}/2\pi H^{(3)}(\bar{y})\right]^{1/2} H^{(s+1)}(\bar{y}), \qquad s = 0,1,2,\dots \tag{2.9}$$

[see Pittel, Woyczynski and Mann (1989), (3.9) and Meir and Moon (1978)].

3. The main results. In Pittel, Woyczynski and Mann (1989), we studied the asymptotic behavior of the distribution $\{P(M)\}$ in the subcritical domain, that is, in the case when $\sigma_n \equiv \sigma < \bar{\sigma}$, where $\bar{\sigma} := R_0(\bar{x})$ $[= \bar{x}H(\bar{y})]$. Our objective in this paper is to consider the behavior of $\{P(M)\}$ in the near-critical domain ($\lim \sigma_n = \bar{\sigma}$) and in the supercritical domain ($\sigma_n \equiv \sigma > \bar{\sigma}$). The characteristics of the random forest M (or, more appropriately, M_n) which are of primary interest here are $L_n^{(1)}$, the size of the largest tree in M_n, $L_n^{(2)}$, the size of the second largest tree in M_n, etc.

To make comparisons for different domains of σ easier, let us first recall the relevant subcritical result:

THEOREM 3.1 [Pittel, Woyczynski and Mann (1989)]. *Let $\sigma_n \equiv \sigma < \bar{\sigma}$, and let ρ be the positive root of $R_0(x) = \sigma$, $x \in (0, \bar{x})$. [It exists since $\bar{\sigma} = R_0(\bar{x})$.] Then, asymptotically (in probability), the size of the largest tree in M_n,*

$$L_n^{(1)} = \eta^{-1}[\log n - (5/2)\log\log n] + O_P(1),$$

where $\eta = \log(\bar{x}/\rho)$ and $O_P(1)$ denotes random variables bounded in probability.

Since $\eta = \eta(\sigma) \to 0$ as $\sigma \to \bar{\sigma}-$, the formula for $L_n^{(1)}$ signals that in the case when $\lim \sigma_n = \bar{\sigma}$, (not to mention the case when $\lim \sigma_n > \bar{\sigma}$) we must expect a faster growth of $L_n^{(1)}$. So, it is a natural question if (and how many) such giant (i.e., of size greater than $\log n$) components exist in the random forest M_n. In the case of the Flory model [cf. Example 2.1(ii)], it has long been believed that (presumably, with high probability, for large n) such a giant tree must be unique. In fact, Donoghue (1982) [see also Donoghue and Gibbs (1979)] undertook an asymptotic (but not quite rigorous) analysis of the Flory distribution, which did indicate uniqueness of the largest tree.

Our first theorem demonstrates that the uniqueness hypothesis is indeed true, even for a general function $H(\cdot)$, and that, in the supercritical case, the size of the largest component has a Holtsmark distribution. Let $L_n^{(k)}$ stand for the size (i.e., the number of vertices) of the kth largest tree in the forest M_n, $k \geq 1$.

THEOREM 3.2. *Suppose that* $\sigma_n \equiv \sigma > \bar{\sigma}$. *Then, with probability approaching* 1 *as* $n \to \infty$, *the forest* M_n *contains a single tree of size relatively close to* $n(1 - \bar{\sigma}/\bar{\sigma})$. *The sizes of all other trees are of order at most* $n^{2/3}$. *More precisely:*

(i) *The distribution of* $L_n^{(1)}$ *satisfies a local limit-type relation*

$$P\big(L_n^{(1)} = j\big) = (1 + o(1))p(x_j)\,\Delta x_j, \tag{3.1}$$

where

$$x_j := [n(1 - \bar{\sigma}/\sigma) - j](\bar{\sigma}n/\sigma)^{-2/3}$$

and

$$\Delta x_j := (\bar{\sigma}n/\sigma)^{-2/3}$$

and $p(\cdot)$ *is the density of a* (3/2)-*stable* (*Holtsmark*) *distribution* [*cf*., *e.g*., *Zolotarev* (1986)] *with the canonical Lévy measure concentrated on* $[0, \infty)$ *and assigning to an interval* $[0, x]$ *the measure equal to* $(\beta_0/\bar{\sigma})x^{1/2}$ [*see* (2.9) *for* β_0]. *The formula* (3.1) *holds uniformly over* j *such that* x_j *belongs to a bounded interval. Consequently, in distribution*,

$$\big[n(1 - \bar{\sigma}/\sigma) - L_n^{(1)}\big](\bar{\sigma}n/\sigma)^{-2/3} \Rightarrow X,$$

as $n \to \infty$, *where the random variable* X *has the characteristic function*

$$E[\exp(iuX)] = \exp[i\Psi(u)], \tag{3.2}$$

where

$$\Psi(u) = (4/3\bar{\sigma})\pi^{1/2}\beta_0 e^{-i3\pi/4}u^{3/2}$$

for $u \geq 0$ *and* $\Psi(u) = \overline{\Psi}(-u)$ *for* $u < 0$.

(ii) *For every fixed* $x > 0$ *and* $k \geq 2$,

$$\lim_{n\to\infty} P\big(L_n^{(k)} \leq xn^{2/3}\big) = e^{-\lambda(x)} \sum_{0 \leq j < k-2} \lambda^j(x)/j!, \tag{3.3}$$

where

$$\lambda(x) = (2\beta_0/3\sigma)x^{-3/2}.$$

REMARK 3.1. It is interesting to note that, for the Erdös–Rényi (1960) model of a random graph in a supercritical domain, a giant component of order n is also unique. However, in contrast to the situation described in Theorem 3.2, its size is asymptotically normal (i.e., 2-stable) with both mean and variance of order n [Stepanov (1970)]. Another difference is that, for the Erdös–Rényi model, $L_n^{(2)}$ is of order $\log n$ [Bollobás (1985)], while in our case $L_n^{(2)}$ is of order $n^{2/3}$. Heuristically, the nature of this deep difference can be explained as follows. First, it can be seen that, after removing the giant component from the Erdös–Rényi graph, we are left with a random subgraph (of size comparable to n) with a *subcritical* number of edges. So, by a theorem of Erdös and Rényi (1960), $L_n^{(2)}$ (the size of the largest component in this subgraph) has to be a.s. of order $\log n$. If we perform a similar removal of the giant component from the forest M_n, the remaining forest $M_{n'}$ will have n' vertices where $n' \approx n\bar{\sigma}/\sigma$. So, introducing $\sigma_{n'}$ via $\mu/\lambda = n'/\sigma_{n'}$, we notice that $\sigma_{n'} \approx \bar{\sigma}$, which means that $M_{n'}$ is in a near-critical domain, where Theorem 3.3 applied to $M_{n'}$ indicates that $L_n^{(2)}$ must be of order $(n')^{2/3}$, which is what is basically stated in (3.3) for $k = 2$.

The next result discusses the distribution of sizes of tree components of M_n in the near-critical domain.

THEOREM 3.3. *Suppose that* $\bar{\sigma}/\sigma_n = 1 - an^{-1/3}$, *where* $a \in (-\infty, \infty)$ *is fixed. Then, for every* $x > 0$ *and* $k \geq 1$,

$$\lim P\left(L_n^{(k)} \leq xn^{2/3}\right) = e^{-I(x)} \sum_{0 \leq j \leq k-1} I^j(x)/j!,$$

where

$$I(x) = \beta_0(\bar{\sigma}p(a))^{-1} \int_x^{\infty} y^{-5/2} p(a-y)\,dy.$$

In particular,

$$\lim P\left(L_n^{(1)} \leq xn^{2/3}\right) = e^{-I(x)}.$$

REMARK 3.2. Thus (see Theorem 3.1), the largest component of M_n is of order $\log n$, $n^{2/3}$ or n, depending upon whether σ_n is below and bounded away from $\bar{\sigma}$, close enough to $\bar{\sigma}$, or above $\bar{\sigma}$ and bounded away from it.

Curiously, the precise limiting behavior of $L_n^{(1)}$ for the random graph M_n in the near-critical domain itself is still unknown, but the existing results [Erdös–Rényi (1960) and Bollobás (1985)] suggest quite strongly that $L_n^{(1)}$ is also of order exactly $n^{2/3}$.

4. Proofs.

4.1. *An outline.* The structure of proofs of Theorems 3.2 and 3.3 is somewhat complex so, in this section, we include a description of the composition of our arguments.

We begin (Section 4.2) with the formulation of two auxiliary Lemmas 4.1 and 4.2, which establish, for σ_n in the supercritical and near-critical domains, the asymptotic estimates for $Q_n/n!$, where Q_n is the normalizing factor in (2.1). The long and technical Sections 4.3 and 4.4 contain proofs of auxiliary Lemmas 4.1 and 4.2.

Another result, concerning the asymptotic behavior of $Q_n/n!$ in the case when $\liminf \sigma_n > 0$ is formulated in Section 4.5 (Lemma 4.3). That section also contains a discussion of connections of the above mentioned asymptotic results with local limit theorems for stable densities.

Section 4.6 contains additional auxiliary lemmas (Lemmas 4.4–4.6), which study directly a finer structure of the random forest M_n in the supercritical case, and a result (Lemma 4.7) which gives a sharp asymptotic estimate for the expected number of trees of size relatively close to the threshold value $n[1 - \bar{\sigma}/\sigma]$. Lemmas 4.4 and 4.5 show that the forest M_n has neither a tree of size between $\omega(n)n^{2/3}$ and $n[(1 - \bar{\sigma}/\sigma) - \varepsilon]$, nor a tree of size more than or equal to $n[(1 - \bar{\sigma}/\sigma) + \varepsilon]$. Here $\varepsilon > 0$ is fixed and $\omega(n) \to \infty$ arbitrarily slowly. Lemma 4.6 shows that the forest M_n may have at most one tree of size $\geq n[(1 - \bar{\sigma}/\sigma) - \varepsilon]$.

In Section 4.7, Lemmas 4.6 and 4.7 help to establish a local limit theorem for $L_n^{(1)}$, and the combination of this result and Lemmas 4.4 and 4.6 proves part (i) of Theorem 2 which deals with the giant component of the graph M_n.

Section 4.8 gives a proof of Theorem 2(ii), and the paper concludes with Section 4.9 which contains the proof of Theorem 3.3.

4.2. *Auxiliary lemmas.* Fix μ, λ. The e.g.f. of the normalizing factors Q_n from (2.1) is given by

$$\sum_{n \geq 0} Q_n x^n / n! = \exp[(\mu/\lambda) F(x)], \tag{4.1}$$

where

$$F(x) := \sum_{j \geq 1} f_j x^j / j!$$

[cf. Pittel, Woyczynski and Mann (1989)].

LEMMA 4.1. *If* $\liminf \sigma_n \geq \bar{\sigma}$, *then*

$$Q_n / n! = (1 + o(1)) n^{-3/2} \exp[\mathscr{N}_n] \beta_0 \left[\sigma_n (1 - \bar{\sigma}/\sigma_n)^{5/2}\right]^{-1}, \tag{4.2}$$

where β_0 *is defined in* (2.9) *and*

$$\mathscr{N}_n = n\left[\sigma_n^{-1} F(\bar{x}) - \log \bar{x}\right].$$

LEMMA 4.2. *If* $a_n = n^{1/3}(1 - \bar{\sigma}/\sigma_n) = O(1)$, *then*

$$Q_n / n! = (1 + o(1)) n^{-2/3} p(a_n) \exp[\mathscr{N}_n],$$

where $p(\cdot)$ *is the density of a* 3/2-*stable distribution of the random variable* X *introduced in Theorem* 3.2.

4.3. *Proof of Lemma* 4.1. Using the Cauchy integral formula, we write

$$Q_n / n! = (2\pi i)^{-1} \int_{\mathscr{C}} \exp[(\mu/\lambda) F(x) - n \log x] x^{-1} \, dx,$$

where $\mathscr{C}$ is a contour surrounding the origin $x = 0$. Having arrived at this representation for $Q_n/n!$, we may and shall assume that μ/λ depend on n, namely $\mu/\lambda = n/\sigma_n$. Choose $\mathscr{C} = \{x\colon x = \bar{x}e^{i\phi},\ -\pi \le \phi < \pi\}$, where $\bar{x}$ is defined in (1.4). Then the above identity becomes

$$(4.3)\qquad Q_n/n! = (2\pi)^{-1}\int_{-\pi}^{\pi} \exp[N_n(\phi)]\,d\phi,$$

where

$$(4.4)\qquad N_n(\phi) = \mathscr{N}_n + n\sigma_n^{-1}\big[F(\bar{x}e^{i\phi}) - F(\bar{x})\big] - in\phi$$

and $\mathscr{N}_n$ is defined in (4.2). We need to find a sharp asymptotic estimate of the integral in (4.3). It turns out (not too surprisingly) that the dominant part of the integral corresponds to the small values of ϕ. To see it, we have to find an expansion for $N_n(\phi)$ in powers of ϕ. A key relation is, for $m \ge 4$ and $\phi \ge 0$,

$$(4.5)\qquad F(\bar{x}e^{i\phi}) - F(\bar{x}) = i\bar{\sigma}\phi + \alpha_3\phi^{3/2} + \sum_{4\le l\le m} \alpha_l\phi^{l/2} + O\big(n\phi^{(m+1)/2}\big),$$

where

$$\alpha_3 = -(4/3)\pi^{1/2}\beta_0 e^{i\pi/4}.$$

Let us outline the derivation of (4.5). According to (2.5) and (2.7),

$$F(x) = R_0(x) - R_1^2(x)/2 = H(R_1)f(R_1) - R_1^2/2,$$

where $R_1 = R_1(x)$ and $f(y) = y/H^{(1)}(y)$. By (1.2), $f^{(1)}(\bar{y}) = 0$, so expanding F in powers of $R_1 - R_1(\bar{x}) = R_1 - \bar{y}$, we have

$$(4.6)\qquad \begin{aligned} F(x) &= F(\bar{x}) + 2^{-1}(R_1 - \bar{y})^2 Hf^{(2)} \\ &\quad + 6^{-1}(R_1 - \bar{y})^3\big(2H^{(1)}f^{(2)} + Hf^{(3)}\big) + \cdots, \end{aligned}$$

where H, f and their derivatives are evaluated at $y = \bar{y}$. In particular,

$$(4.7)\qquad f^{(2)}(\bar{y}) = -\bar{y}H^{(3)}(\bar{y})/\big[H^{(1)}(\bar{y})\big]^2.$$

According to (2.5), $R_1(x)$ satisfies $f(R_1) = x$. Since $f^{(1)}(\bar{y}) = 0$ and $f^{(2)}(\bar{y}) \ne 0$ [see (4.7)], by the implicit function theorem we have, for x close enough to $\bar{x}$ and $|x| \le \bar{x}$, that

$$(4.8)\qquad R_1(x) - \bar{y} = \sum_{j\ge 1} r_j(x - \bar{x})^{j/2},$$

where we select the main branch of the square root function. Plugging (4.8) into $f(R_1) = x$, we get after some work that

$$(4.9)\qquad 2^{-1}f^{(2)}r_1^2 = 1 \quad\text{and}\quad f^{(2)}r_1r_2 + 6^{-1}f^{(3)}r_1^3 = 0.$$

Consequently [see (4.7)],

$$(4.10)\qquad r_1 = \pm i\big[2(H^{(1)})^2/\bar{y}H^{(3)}\big]^{1/2},$$

and we must choose $+$, since $R_1(x) < \bar{y}$ for $x < \bar{x}$. A combination of (4.6)–(4.10) leads to

$$F(x) = F(\bar{x}) + \sum_{j\geq 2} d_j(x-\bar{x})^{j/2},$$

where

$$\begin{aligned} d_2 &= \left(2^{-1}f^{(2)}r_1^2\right)H = H,\\ d_3 &= \left(f^{(2)}r_1r_2 + 6^{-1}f^{(3)}r_1^3\right)H + 3^{-1}f^{(2)}r_1^3H^{(1)}\\ &= 3^{-1}f^{(2)}r_1^3H^{(1)} = i3^{-1}2^{3/2}\left[\left(H^{(1)}\right)^2/\bar{y}H^{(3)}\right]^{1/2}H^{(1)}. \end{aligned}$$

These formulas yield (4.5) [see (1.9) for β_0], since $\bar{x}e^{i\phi} - \bar{x} = \bar{x}i\phi + O(\phi^2)$.

Using (4.4) and (4.5), we obtain, for $m \geq 2$ and $\phi \geq 0$,

$$N_n(\phi) = \mathscr{N}_n + n\sum_{2\leq j\leq m}\gamma_j\phi^{j/2} + O\left(n\phi^{(m+1/2)}\right), \tag{4.11}$$

where

$$\begin{aligned} \gamma_2 &= i\left[\bar{\sigma}/\sigma_n - 1\right],\\ \gamma_3 &= (4/3\sigma_n)\pi^{1/2}\beta_0\exp(-3\pi i/4). \end{aligned} \tag{4.12}$$

As we'll see later, explicit formulas for γ_j, $j\geq 3$, are not needed.

With the expansion (4.11) at hand, we can begin estimating the integral in (4.3). Choose s from $(0, 2/3)$, set $\phi_0 = n^{-s}$ and rewrite (4.3) as

$$\begin{gathered} Q_n/n! = (2\pi)^{-1}\int_1 + (2\pi)^{-1}\int_2,\\ \int_1 = 2\operatorname{Re}\left\{\int_0^{\phi_0}\exp[N_n(\phi)]\,d\phi\right\},\qquad \int_2 = 2\operatorname{Re}\left\{\int_{\phi_0}^{\pi}\exp[N_n(\phi)]\,d\phi\right\}. \end{gathered} \tag{4.13}$$

[Re(u) is the real part of a complex number u.]

First, let us estimate $\int_2$. By (4.4), for $\phi \in (-\pi, \pi]$,

$$\operatorname{Re}[N_n(\phi)] = \mathscr{N}_n + n\sigma_n^{-1}\sum_{j\geq 1} f_j(\bar{x}^j/j!)(\cos j\phi - 1) \leq \mathscr{N}_n - cn\phi^2.$$

[Here and below we use the letter c—with (or without) a subscript (superscript) —to denote various positive numbers *independent* of n.] In addition [see (4.11) and (4.12)], for *small* $\phi \geq 0$,

$$\operatorname{Re}[N_n(\phi)] \leq \mathscr{N}_n - c_1n\phi^{3/2}.$$

Since $\phi_0 = n^{-s}$, $s \in (0, 2/3)$, from these estimates it follows that

$$\int_2 = O\left(\exp[\mathscr{N}_n - c_1n^{\nu}]\right),\qquad \nu = 1 - 3s/2 > 0. \tag{4.14}$$

Second, let us asymptotically evaluate $\int_1$. By (4.12), for a fixed $m \geq 2$ and $\phi \in [0, \phi_0]$,

$$\operatorname{Re}\left[n\sum_{2\leq j\leq m}\gamma_j\phi^{j/2}\right] = -cn\phi^{3/2}\left[1 + O(n^{-s/2})\right] \leq 0. \tag{4.15}$$

Select m so large that $m > (2/s) - 1$; then

$$n\phi_0^{(m+1)/2} = n^{-s(m+1)/2+1} = o(1). \tag{4.16}$$

Introducing

$$\tilde{N}_n(\phi) = \mathcal{N}_n + A_n(\phi), \qquad A_n(\phi) = n \sum_{2 \le j \le m} \gamma_j \phi^{j/2} \tag{4.17}$$

and using (2.15) and (2.16), we obtain

$$\begin{aligned} &|\exp[N_n(\phi)] - \exp[\tilde{N}_n(\phi)]| \\ &\quad = \exp\left(\mathrm{Re}[\tilde{N}_n(\phi)]|\exp[O(n\phi_0^{(m+1)/2})] - 1|\right) \\ &\quad = O\left(\exp[\mathcal{N}_n] n^{-s(m+1)/2+1}\right). \end{aligned} \tag{4.18}$$

Therefore

$$\int_0^{\phi_0} \exp[N_n(\phi)]\, d\phi = \int_0^{\phi_0} \exp[\tilde{N}_n(\phi)]\, d\phi + o\left(n^{-3/2}\exp[\mathcal{N}_n]\right), \tag{4.19}$$

provided that

$$s + s(m+1)/2 - 1 > 3/2 \quad \text{or} \quad m > (5/s) - 3, \tag{4.20}$$

which we may, and shall, assume. Thus, it remains to show that

$$\mathrm{Re}(J) = (1 + o(1)) \cdot \text{const.} \cdot n^{-3/2} \quad \text{where} \quad J := \int_0^{\phi_0} \exp[A_n(\phi)]\, d\phi.$$

For this purpose, set first $u = \phi^{1/2}$ and write

$$J = 2\int_0^{u_0} u \exp[\mathscr{A}_n(u)]\, du \quad \text{where} \quad \mathscr{A}_n(u) := A_n(u^2) \quad \text{and} \quad u_0 = \phi_0^{1/2}.$$

Second, introduce in the complex plane u a closed (clockwise-oriented) contour $\mathscr{D} = \mathscr{D}_1 \cup \mathscr{D}_2 \cup \mathscr{D}_3$. Here $\mathscr{D}_1$ is the interval $[0, u_0]$ of the real line, $\mathscr{D}_2$ is the arc $u = u_0 e^{i\psi}$, $0 \ge \psi \ge -\pi/4$ and $\mathscr{D}_3$ is the line segment connecting the points $u_0 e^{-i\pi/4}$ and the origin 0. Since the integrand $2u\exp[\mathscr{A}_n(u)]$ is an analytic function of u, denoting it by $w(u)$ we have

$$J = -\int_{\mathscr{D}_2} w(u)\, du - \int_{\mathscr{D}_3} w(u)\, du.$$

Consider the integral along $\mathscr{D}_2$. On $\mathscr{D}_2$, according to (4.12),

$$n\,\mathrm{Re}[\gamma_2 u^2] = -n\phi_0[(1 - \bar{\sigma}/\sigma_n)]\sin 2|\psi| \le 0$$

and

$$n\,\mathrm{Re}(\gamma_3 u^3) \le -cn\phi_0^{3/2}\cos[(\pi/4) + 3\psi].$$

Hence,

$$\begin{aligned} \mathrm{Re}[\mathscr{A}_n(u)] &= n\,\mathrm{Re}[\gamma_2 u^2] + n\,\mathrm{Re}(\gamma_3 u^3) + O(n\phi_0^2) \\ &\le \begin{cases} -c_1 n\phi_0^{3/2} & \text{if } -\pi/8 \le \psi \le 0, \\ -c_2 n\phi_0 & \text{if } -\pi/4 \le \psi \le -\pi/8. \end{cases} \end{aligned}$$

Consequently,

$$\int_{\mathscr{D}_2} w(u)\,du = O\left[\phi_0^{1/2}\exp\left(-c_1 n\phi_0^{3/2}\right)\right] = O\left[\exp\left(-c_1 n^{1-(3/2)s}\right)\right] = o\left(n^{-3/2}\right). \tag{4.21}$$

Turn to the integral along $\mathscr{D}_3$. On $\mathscr{D}_3$, $u = ve^{-\pi/4}$, where $v \geq 0$ runs from u_0 to 0. Hence [see again (4.12)]

$$\int_{\mathscr{D}_3} w(u)\,du = -2i\int_{u_0}^{0} v\exp\left[\mathscr{A}_n\left(ve^{-i\pi/4}\right)\right]dv = \int_0^{u_0} 2iv\exp\left[-a_2nv^2 + ia_3nv^3 + O\left(nv^4\right)\right]dv = \int_0^{u_0},$$

where

$$a_2 = 1 - \bar{\sigma}/\sigma_n, \qquad a_3 = \left(4/3\sigma_n\right)\pi^{1/2}\beta_0. \tag{4.22}$$

Break $[0, u_0]$ into $[0, u_1]$ and $[u_1, u_0]$ where $u_1 = n^{-t}$, $t \in (s/2, 1/2)$. Observe that

$$\left|\int_{u_1}^{u_0}\right| \leq u_0^2\exp\left(-cn^{1-2t}\right) = o\left(n^{-3/2}\right). \tag{4.23}$$

It remains to evaluate asymptotically the integral $\int_0^{u_1}$, or rather its real part. Let us confine t to a subinterval of $(s/2, 1/2)$; namely, let $t \in (t_1, 1/2)$, where $t_1 \overset{\text{def}}{=} \max(s/2, 1/3) = 1/3$. [Recall that $s \in (0, 2/3)$.] For this t, $nu_1^3 = n^{1-3t} = o(1)$; therefore

$$\int_0^{u_1} = \int_0^{u_1} 2iv\exp\left(-a_2nv^2\right)\left[1 + ia_3nv^3 + O\left(nv^4 + n^2v^6\right)\right]dv = 2i\int_0^{u_1} v\exp\left(-a_2nv^2\right)dv - 2a_3n\int_0^{u_1} v^4\exp\left(-a_2nv^2\right)dv + \mathscr{R}_n. \tag{4.24}$$

Here

$$\mathscr{R}_n = O\left[n\int_0^{u_1} v^5\exp\left(-a_2nv^2\right)dv + n^2\int_0^{u_1} v^7\exp\left(-a_2nv^2\right)dv\right] = O\left[n/\left(n^{1/2}\right)^6 + n^2/\left(n^{1/2}\right)^8\right] = O\left(n^{-2}\right) = o\left(n^{-3/2}\right). \tag{4.25}$$

Putting together (4.21)–(4.25), we arrive at

$$\begin{aligned}\operatorname{Re}(J) &= o\left(n^{-3/2}\right) - \operatorname{Re}\left(\int_{\mathscr{D}_3}\right) = o\left(n^{-3/2}\right) - \operatorname{Re}\left(\int_0^{u_1}\right)\\ &= o\left(n^{-3/2}\right) + 2a_3n\int_0^{u_1} v^4\exp\left(-a_2nv^2\right)dv\\ &= o\left(n^{-3/2}\right) + (1 + o(1))2a_3a_2^{-5/2}n^{-3/2}\int_0^{\infty} v^4\exp\left(-v^2\right)dv\\ &= (1 + o(1))\pi\beta_0\left[\sigma_n\left(1 - \bar{\sigma}/\sigma_n\right)^{5/2}\right]^{-1}n^{-3/2}.\end{aligned} \tag{4.26}$$

A combination of (4.13), (4.14), (4.19) and (4.26) leads to

$$Q_n/n! = (1 + o(1))\exp[\mathcal{N}_n]\beta_0\left[\sigma_n(1 - \bar{\sigma}/\sigma_n)^{5/2}\right]^{-1} n^{-3/2}. \qquad \square$$

4.4. *Proof of Lemma* 4.2. Consider again the relation (4.13). No changes are needed to obtain the estimate (4.14) and an analogue of (4.19), that is,

$$(4.27) \qquad \int_0^{\phi_0} \exp[N_n(\phi)]\, d\phi = \int_0^{\phi_0} \exp[\tilde{N}_n(\phi)]\, d\phi + o\left(n^{-2/3}\exp[\mathcal{N}_n]\right),$$

provided that

$$s + s(m+1)/2 - 1 > 2/3 \quad \text{or} \quad m > (10/3s) - 3.$$

Restricting s to $(5/9, 2/3)$, we can use therefore (2.27) with $m = 3$. It remains to evaluate [see (4.17)]

$$(4.28) \qquad \int_0^{\phi_0} \exp[\tilde{N}_n(\phi)]\, d\phi = \exp[\mathcal{N}_n] \int_0^{\phi_0} \exp\left[n\gamma_2\phi + n\gamma_3\phi^{3/2}\right] d\phi = \exp[\mathcal{N}_n] I_n.$$

Substitute $u = (n\bar{\sigma}/\sigma_n)^{2/3}\Phi$ and write [see (4.12)]

$$(4.29) \qquad I_n = (n\bar{\sigma}/\sigma_n)^{-2/3} \int_0^{u_0} \exp\left[-ia_n u + \gamma u^{3/2}\right] du,$$

where $u_0 = n(\bar{\sigma}/\sigma_n)^{-2/3}\phi_0$, and since $\sigma_n/\bar{\sigma} = 1 + O(n^{-1/3})$, we have

$$a_n = -n^{1/3}[(\bar{\sigma}/\sigma_n - 1)]$$

and

$$\gamma = (4/3\bar{\sigma})\pi^{1/2}\exp(-3\pi i/4).$$

Introduce a (3/2)-stably distributed random variable X (cf. Theorem 3.2) such that $E[\exp(iuX)] = \exp(\gamma u^{3/2})$, $u \geq 0$, and denote its density by $p(\cdot)$. According to the inversion formula, we have

$$p(a_n) = (2\pi)^{-1}\int_{-\infty}^{\infty} \exp[ia_n u] E[\exp(iuX)]\, du$$

$$(4.30) \qquad = \pi^{-1}\operatorname{Re}\left\{\int_0^{\infty} \exp\left[-ia_n u + \gamma u^{3/2}\right] du\right\}$$

$$= \pi^{-1}\operatorname{Re}\left\{\int_0^{u_0} \exp\left[-ia_n u + \gamma u^{3/2}\right] du\right\} + O\left[\exp\left(-\delta u_0^{3/2}\right)\right],$$

$\delta = -\operatorname{Re}(\gamma) > 0$. In view of (4.27)–(4.30), we conclude

$$Q_n/n! = (1 + o(1))n^{-2/3}p(a_n)\exp[\mathcal{N}_n]. \qquad \square$$

4.5. *Another auxiliary lemma and local limit theorems.* The Lemmas 4.1 and 4.2 contain the asymptotic formulas for $Q_n/n!$ in the near critical and supercritical cases. Below, we shall also need an estimate for $Q_n/n!$ in the case when $\liminf \sigma_n > 0$. Such an estimate is contained in Lemma 4.3. Its proof is similar to, but much simpler than, the proof of Lemma 4.2 and we omit it here.

LEMMA 4.3. *If* $\liminf \sigma_n > 0$, *then*

$$Q_n/n! = O\left\{n^{-2/3}\left[\exp\left(\sigma_n^{-1}F(\bar{x})\right)/\bar{x}\right]^n\right\}.$$

REMARK 4.1. The assertions of Lemma 4.1 and 4.2 strongly suggest that there ought to be a local limit theorem-type connection lurking beneath the surface. This is actually true. To see it, confine ourselves to the case when $\sigma_n \equiv \sigma$. Observe that in that case, according to (2.1),

$$Q_n/n! = \operatorname{coeff}_{x^n} \exp\left[n\sigma^{-1}F(x)\right].$$

Introduce a random variable $Y = \sum_{j \geq 1} j y_j$, where $\{y_j\colon\ j \geq 1\}$ are independent, Poisson random variables with parameters $\{\sigma^{-1}f_j\bar{x}^j/j!\colon\ j \geq 1\}$. Then

$$E(z^Y) = \exp\left[\sigma^{-1}\left(F(\bar{x}z) - F(\bar{x})\right)\right],$$

so that the formula for $Q_n/n!$ becomes

$$\text{(a)}\qquad Q_n/n! = \left[\exp\left(\sigma^{-1}F(\bar{x})\right)/\bar{x}\right]^n P(Z_n = n), \qquad Z_n = \sum_{1 \leq m \leq n} Y_m,$$

where $Y_1, Y_2, \ldots$ are independent copies of Y. Define $Z_n^* = (Z_n - n\bar{\sigma}/\sigma)n^{-2/3}$. A straightforward computation based on (4.5) and the formula for $E(z^Y)$ shows that

$$\lim_{n\to\infty} E\left[\exp(iuZ_n^*)\right] = \exp\left[-(4/3\sigma)\pi^{1/2}\beta_0 e^{i\pi/4}u^{3/2}\right], \qquad u \geq 0,$$

i.e., $Z_n^* \Rightarrow Z^*$, where Z^* is (3/2)-stably distributed. Hence, the distribution of Y is attracted to the distribution of stable Z^* and, by a local limit theorem for the stable distributions [Ibragimov and Linnik (1965)], we have

$$\text{(b)}\qquad \begin{aligned} P(Z_n = n) &= P\left(Z_n^* = n^{1/3}(1 - \bar{\sigma}/\sigma)\right) \\ &= n^{-2/3}\left[p_\sigma\left(n^{1/3}(1 - \bar{\sigma}/\sigma)\right) + o(1)\right], \end{aligned}$$

where $p_\sigma(\cdot)$ is the density of Z^*. Now, if $\sigma = \bar{\sigma}$, then we get the statement of Lemma 4.2 for $\sigma_n \equiv \bar{\sigma}$, since $p_{\bar{\sigma}}(0) = p(0) > 0$. If $\sigma > \bar{\sigma}$, then $n^{1/3}(1 - \bar{\sigma}/\sigma) \to \infty$ and, using an asymptotic formula for the stable density [cf. Ibragimov and Linnik (1965)], we get

$$\text{(c)}\qquad p_\sigma\left(n^{1/3}(1 - \bar{\sigma}/\sigma)\right) = (1 + o(1))\beta_0\left[\sigma(1 - \bar{\sigma}/\sigma)^{5/2}\right]^{-1} n^{-5/6}.$$

The relations (a)–(c) would imply the statement of Lemma 4.1 if we could claim that $o(1)$ in (b) is, in fact, $o(n^{-5/6})$. However, this does not follow from the general theory. Thus, there does exist a connection with the local limit theorem for a stable distribution, but only in case $\sigma_n \equiv \bar{\sigma}$ can it be used as a basis for an alternative argument.

Note added in the proof. We have found that a more general result on large deviations for stable distributions was proven by Tkachuk (1973).

4.6. *The fine structure of forest* M_n. Let us begin with a supercritical case $\sigma_n \equiv \sigma > \bar{\sigma}$. For convenience, we break the argument into several steps.

LEMMA 4.4. *Let $\varepsilon \in (0, 1 - \bar{\sigma}/\sigma)$ be fixed and $\omega(n) \to \infty$ (however slowly). Then, with probability approaching* 1 *as $n \to \infty$, the random forest M_n does not contain a tree of size j between $j_1 = n^{2/3}\omega(n)$ and $j_2 = n[(1 - \bar{\sigma}/\sigma) - \varepsilon]$.*

PROOF. For a given j let c_{nj} denote the total number of trees of size j in M_n. By (2.1) [see also (2.6)]

$$E(c_{nj}) = \left[\binom{n}{j} Q_n^{-1}(\mu/\lambda) \sum_{M_{n-j}} (\mu/\lambda)^{C(M_{n-j})} \prod_{T \in M_{n-j}} h(T)\right] \sum_{T_j} h(T_j). \tag{4.31}$$

Here M_{n-j} is a forest on the vertices $(j+1), \dots, n$, $C(M_{n-j})$ is the number of trees in it, the product $\prod$ is taken over all the trees of M_{n-j} and the summation inside the square brackets is over all such forests, while the summation outside is over all trees on the vertices $1, 2, \dots, j$. By the definition of f and Q, we can simplify (4.31) to

$$E(c_{nj}) = (\mu/\lambda) f_j \binom{n}{j} (Q_{n'}/Q_n), \qquad n' = n - j.$$

If $j \le j_2$, then $n' \ge n(\bar{\sigma}/\sigma + \varepsilon)$, whence $n/\sigma = \mu/\lambda 0 := n'/\sigma_{n'}$, where $\sigma_{n'} = (n'/n)\sigma \ge \bar{\sigma} + \varepsilon\sigma > \bar{\sigma}$. Thus, we can use Lemma 4.1 to evaluate both $Q_{n'}$ and Q_n. For $j \to \infty$, there is also available an asymptotic formula (2.8) for $f_j/j!$ ($f_j = R_{0j}/j$), which in conjunction with Lemma 4.3 gives

$$\begin{aligned} E(c_{nj}) &= (1 + o(1)) n\sigma^{-1}\beta_0(\bar{x})^{-j} j^{-5/2} j! [n!/(n')! j!] \\ &\quad \times (n')! (n')^{-3/2} \exp[(\mu/\lambda) F(\bar{x})] (\bar{x})^{-n'} q(\sigma_n') \\ &\quad \times \{n! n^{-3/2} \exp[(\mu/\lambda) F(\bar{x})] (\bar{x})^{-n} q(\sigma)\}^{-1} \\ &= (1 + o(1)) [\beta_0 q(\sigma_{n'})/\sigma q(\sigma)] n^{-1/2} z_j^{-5/2} (1 - z_j)^{-3/2} \Delta z_j, \end{aligned}$$

where $z_j = j/n$ and $\Delta z_j = n^{-1}$. Since $z_j \le j_2/n < 1 - \bar{\sigma}/\sigma$, we have then that

$$\begin{aligned} E\left(\sum_{j_1 \le j \le j_2} c_{nj}\right) &= O\left(n^{-1/2} \sum_{j_1 \le j \le j_2} z_j^{-5/2} \Delta z_j\right) \\ &= O\left(n^{-1/2} \int_{(j_1 - 1)/n}^{1} z^{-5/2}\, dz\right) = O(\omega^{-3/2}(n)) = o(1). \end{aligned}$$

As $\sum_{j_1 \le j \le j_2} c_{nj}$ is integer-valued, the last estimate implies that $c_{nj} \equiv 0$, $j_1 \le j \le j_2$, with probability approaching 1. □

LEMMA 4.5. *Let $\varepsilon \in (0, \bar{\sigma}/\sigma)$ be fixed. Then, with probability approaching* 1 *as $n \to \infty$, the forest M_n does not contain a tree of size $j \ge j_1 = n[(1 - \bar{\sigma}/\sigma) + \varepsilon]$.*

PROOF. (i) According to (4.1), for fixed μ and λ and all $\nu \ge 0$,

$$Q_\nu/\nu! \le \exp[(\mu/\lambda) F(x) - \nu \log x], \qquad \forall x > 0.$$

Therefore, for $\nu/(\mu/\lambda) < \bar{\sigma}$,

$$Q_\nu/\nu! \le \exp[(\mu/\lambda)F(\rho) - \nu \log \rho],$$

where $\rho \in (0, \bar{x})$ satisfies

$$R_0(\rho) = \nu/(\mu/\lambda). \tag{4.32}$$

(ii) Recall that in our case $\mu/\lambda = n/\sigma$, $\sigma > \bar{\sigma}$. If $j \ge j_1$, then $n' = n - j \le n(\bar{\sigma}/\sigma - \varepsilon)$ so that

$$\sigma_{n'} = n'/(\mu/\lambda) \le \bar{\sigma} - \varepsilon\sigma < \bar{\sigma}. \tag{4.33}$$

As a consequence, by (a),

$$Q_{n'}/(n')! \le \exp[(\mu/\lambda)F(\rho) - n' \log \rho],$$

where ρ satisfies (4.32) with $\nu = n'$, so that [see (4.33)] $\rho \le \rho(\varepsilon) := R_0^{-1}(\bar{\sigma} - \varepsilon\sigma)$. So, as in the proof of Lemma 4.4,

$$\begin{aligned} E(c_{nj}) &= (\mu/\lambda) f_j \binom{n}{j} (Q_{n'}/Q_n) \\ &\le cn(\bar{x})^{-j} j^{-5/2} j! [n!/(n')!j!](n')! \exp[(\mu/\lambda)F(\rho) - n' \log \rho] \\ &\quad \times \{n! n^{-3/2} \exp[(\mu/\lambda)F(\bar{x}) - n \log \bar{x}]\}^{-1} \end{aligned} \tag{4.34}$$

or

$$E(c_{nj}) \le c' n^{5/2} \exp[(\mu/\lambda)(F(\rho) - F(\bar{x})) - n'(\log \rho - \log \bar{x})].$$

Here [see (4.7)]

$$\begin{aligned} &(\mu/\lambda)(F(\rho) - F(\bar{x})) - n'(\log \rho - \log \bar{x}) \\ &\quad = (\mu/\lambda)\left[-\int_\rho^{\bar{x}} R_0(y) y^{-1}\,dy + R_0(\rho)\int_\rho^{\bar{x}} y^{-1}\,dy\right] \\ &\quad = -n\sigma^{-1}\int_\rho^{\bar{x}} [R_0(y) - R_0(\rho)]\, y^{-1}\,dy \\ &\quad \le -n\sigma^{-1}\int_{\rho(\varepsilon)}^{\bar{x}} [R_0(y) - R_0(\rho(\varepsilon))]\, y^{-1}\,dy = -nc'', \\ &\qquad\qquad c'' = c''(\varepsilon) > 0. \end{aligned} \tag{4.35}$$

It follows that, uniformly over $j \ge j_1$,

$$E\left(\sum_{j \ge j_1} c_{nj}\right) = O(n^{7/2} \exp(-nc'')) = o(1), \tag{4.36}$$

so again $c_{nj} \equiv 0$ for $j \ge j_1$, with probability approaching 1. □

LEMMA 4.6. *For every $\varepsilon \in (0, 1 - \bar{\sigma}/\sigma)$, with probability approaching* 1 *as $n \to \infty$, the forest M_n may contain at most one tree of size $j \ge j_1 = n[(1 - \bar{\sigma}/\sigma) - \varepsilon]$.*

PROOF. This statement follows immediately from Lemma 4 if $\bar{\sigma}/\sigma < 1/2$, since in this case

$$n - 2j_1 = n(2\bar{\sigma}/\sigma - 1 + \varepsilon) < 0$$

for all small enough $\varepsilon < 0$. Suppose then that $\bar{\sigma}/\sigma \geq 1/2$. By the same Lemma 4.4 we may, and shall, consider sufficiently small $\varepsilon > 0$. If $j', j'' \geq j_1$, and also $j' + j'' \leq n$, then

$$0 \leq n' = n - (j' + j'') \leq n - 2j_1 = n(2\bar{\sigma}/\sigma - 1 + 2\varepsilon)$$

and

$$\sigma_{n'} \leq \sigma(2\bar{\sigma}/\sigma - 1 + 2\varepsilon) < \bar{\sigma},$$

provided that $\varepsilon < (1 - \bar{\sigma}/\sigma)/2$, which we assume from now on. Introduce $C_n(\varepsilon) = \sum_{j \geq j_1} c_{nj}$; we want to show that with high probability (w.h.p.) $C_n(\varepsilon) \leq 1$ when $n \to \infty$. Since

$$P[C_n(\varepsilon) \geq 2] \leq E(n, \varepsilon) := E[(C_n(\varepsilon))_2], \qquad ((C_n(\varepsilon))_2 = C_n(\varepsilon)(C_n(\varepsilon) - 1)),$$

it suffices to show that $E(n, \varepsilon) \to 0$ when $n \to \infty$. Notice that $(C_n(\varepsilon))_2$ is the total number of ordered pairs of the different trees each of size $\geq j_1$. Therefore (cf. the proof of Lemmas 4.4 and 4.5),

$$E(n, \varepsilon) = \sum_{j', j''} E_{j'j''}(n, \varepsilon), \tag{4.37}$$

where

$$E_{j'j''}(n, \varepsilon) = (\mu/\lambda)^2 f_{j'} f_{j''} \binom{n}{j'j''n'} (Q_{n'}/Q_n), \tag{4.38}$$
$$j', j'' \geq j_1 \text{ and } n' = n - j' - j'' \geq 0.$$

Consequently [cf. (4.34)–(4.36)],

$$E_{j'j''}(n, \varepsilon) \leq cn^{7/2} \exp\left\{-n\sigma^{-1} \int_{\rho_1(\varepsilon)}^{\bar{x}} [R_0(y) - R_0(\rho_1(\varepsilon))] y^{-1} dy\right\}$$
$$= O(n^{7/2} \exp(-nc')), \qquad c' = c'(\varepsilon) > 0,$$

where $\rho_1(\varepsilon) = R_0^{-1}[\sigma(2\bar{\sigma}/\sigma - 1 + 2\varepsilon)]$ and

$$E(n, \varepsilon) = O(n^{11/2} \exp(-nc')) = o(1)$$

as $n \to \infty$. □

LEMMA 4.7. *Set*

$$j = n(1 - \bar{\sigma}/\sigma) - (\bar{\sigma}n/\sigma)^{2/3} a. \tag{4.39}$$

Then, uniformly over the parameter a from a bounded interval,

$$E(c_{nj}) = (1 + o(1)) p(a) \Delta a,$$

where $\Delta a := (\bar{\sigma}n/\sigma)^{-2/3}$ *and* $p(\cdot)$ *is the density of the* (3/2)*-stable distribution defined in Theorem* 3.2.

PROOF. This time, $n' = n - j = n\bar{\sigma}/\sigma + (\bar{\sigma}n/\sigma)^{2/3}a$, so that $\sigma_{n'} = n'\sigma/n = \bar{\sigma} + (\bar{\sigma}n/\sigma)^{2/3}(\sigma/n)a$ and $a_{n'} = (n')^{1/3}(1 - \bar{\sigma}/\sigma_{n'}) = a + o(1)$. By Lemmas 4.2 and 4.3, we have

$$\begin{aligned} E(c_{nj}) &= (\mu/\lambda)f_j\binom{n}{j}(Q_{n'}/Q_n) \\ &= (1 + o(1))(n/\sigma)\beta_0(\bar{x})^{-j}j^{-5/2}(n')^{-2/3}\exp[(\mu/\lambda)F(\bar{x})](\bar{x})^{-n'}p(a_{n'}) \\ &\quad \times\left\{n^{-3/2}\exp[(\mu/\lambda)F(\bar{x})](\bar{x})^{-n}\beta_0\left[\sigma(1 - \bar{\sigma}/\sigma)^{5/2}\right]^{-1}\right\}^{-1} \\ &= (1 + o(1))p(a)(n\bar{\sigma}/\sigma)^{-2/3}. \end{aligned}$$

□

4.7. *Proof of Theorem* 3.2(i). To finish the proof of Theorem 2(i), we need only to prove the local limit relation for $P(L_n^{(1)} = j)$.

Let j be defined by (4.39). We want to show that, uniformly over the parameter a from a bounded interval,

$$P\left(L_n^{(1)} = j\right) = (1 + o(1))p(a)\,\Delta a,$$

or that this probability is equivalent to $E(c_{nj})$ (cf. Stepanov [1970]) given in Lemma 4.7. To this end, write first

$$\begin{aligned} E(c_{nj}) &= P(c_{nj} = 1) + \sum_{k\geq 2} kP(c_{nj} = k) \\ &= P(c_{nj} = 1) + O\left[nP(c_{nj} \geq 2)\right] \\ &= P(c_{nj} = 1) + O\left[nP(c_{nj}^{+} \geq 2)\right], \end{aligned} \tag{4.40}$$

where $c_{nj}^{+} = \sum_{k\geq j} c_{nk}$. Furthermore,

$$\begin{aligned} P(c_{nj} = 1) &= P\left(c_{nj} = 1,\ L_n^{(1)} = j\right) + P\left(c_{nj} = 1,\ L_n^{(1)} > j\right) \\ &= P\left(L_n^{(1)} = j\right) - P\left(L_n^{(1)} = j,\ c_{nj} \geq 2\right) + P\left(c_{nj} = 1,\ L_n^{(1)} > j\right) \\ &= P\left(L_n^{(1)} = j\right) + O\left[P\left(c_{nj}^{+} \geq 2\right)\right]. \end{aligned} \tag{4.41}$$

Combining (4.40) and (4.41), we get

$$P\left(L_n^{(1)} = j\right) = E(c_{nj}) + O\left[nP\left(c_{nj}^{+} \geq 2\right)\right]. \tag{4.42}$$

Furthermore, for $k', k'' \geq j$,

$$n' = n - (k' + k'') = n(2\bar{\sigma}/\sigma - 1) + O(n^{2/3})$$

so, assuming that $n' \geq 0$,

$$\sigma_{n'} \leq 2\bar{\sigma} - \sigma + o(1) \leq \bar{\sigma} - (\sigma - \bar{\sigma})/2.$$

Therefore, proceeding as in the proof of Lemma 4.6, we obtain that

$$P\left(c_{nj}^{+} \geq 2\right) = O\left[n^4\exp(-cn)\right].$$

Hence, by (4.42) and Lemma 4.7,

$$P\big(L_n^{(1)}=j\big) = (1+o(1))p(a)\,\Delta a + O\big[n^5\exp(-cn)\big] = (1+o(1))p(a)\,\Delta a. \qquad \square$$

4.8. *Proof of Theorem* 3.2(ii). (i) For fixed $x_1 > x > 0$, introduce $c_n(x, x_1) =$ the total number of trees in M_n with size $j \in [xn^{2/3}, x_1 n^{2/3}]$. Let us show that $c_n(x, x_1)$ is, in the limit, Poisson with parameter $\lambda(x, x_1) = (2\beta_0/3\sigma)(x^{-3/2} - x_1^{-3/2})$. It suffices to show that, for every fixed $k \geq 1$,

$$E\big([c_n(x, x_1)]_k\big) \to \lambda^k(x, x_1), \qquad n \to \infty.$$

Here $[c_n(x, x_1)]_k$ is the total number of the ordered k-tuples of different trees of size $j \in [xn^{2/3}, x_1 n^{2/3}]$. Analogously to (4.37) and (4.38), we write

$$E\big([c_n(x, x_1)]_k\big) = \sum_{j_1,\dots,j_k} E_n(j_1,\dots,j_k),$$

where $j_s \in [xn^{2/3}, x_1 n^{2/3}]$, $1 \leq s \leq k$, and

$$E_n(j_1,\dots,j_k) = (\mu/\lambda)^k\Big(\prod_{1\leq s\leq k} f_{j_s}\Big)\binom{n}{j_1,\dots,j_k,n'}(Q_{n'}/Q_n),$$

$n' = n - \sum_{1\leq s\leq k} j_s$. Notice that

$$\sigma_{n'} = \sigma + O(n^{-1/3}) \geq \bar\sigma + (\sigma - \bar\sigma)/2.$$

Applying Lemma 4.3 to both $Q_{n'}$ and Q_n, we see that, uniformly over $(j_1, j_2, \dots, j_k)$, with $z_s = j_s/n$ and $\Delta z_s = n^{-1}$,

$$\begin{aligned} E_n(j_1,\dots,j_k) &= (1+o(1))\big[(\beta_0/\sigma)n^{-1/2}\big]^k \prod_{1\leq s\leq k} z_s^{-5/2}\,\Delta z_s \\ &= (1+o(1))(\beta_0/\sigma)^k \prod_{1\leq s\leq k} y_s^{-5/2}\,\Delta y_s, \end{aligned}$$

where $y_s = j_s/n^{2/3}$, $\Delta y_s = n^{-2/3}$. Therefore, since $x \leq y_s \leq x_1$ $(1 \leq s \leq k)$,

$$\begin{aligned} E\big([c_n(x, x_1)]_k\big) &= (1+o(1))\Big[(\beta_0/\sigma)\int_x^{x_1} y^{-5/2}\,dy\Big]^k \\ &\to \big[(2\beta_0/3\sigma)\big(x^{-3/2} - x_1^{-3/2}\big)\big]^k, \qquad n \to \infty. \end{aligned}$$

(ii) We have proved that w.h.p. the forest M_n has exactly one tree [of size $\sim n(1 - \bar\sigma/\sigma)$] in the range from $\omega(n)n^{2/3}$ to n, where $\omega(n) \to \infty$ however slowly. Therefore, for each $\varepsilon > 0$, there exists $x_1 = x_1(\varepsilon) > x$ so large that

$$\limsup P\big(L_n^{(2)} > x_1 n^{2/3}\big) \leq \varepsilon. \tag{4.43}$$

On the other hand, for every $k \geq 2$,

$$\begin{aligned} P\big(L_n^{(k)} \geq xn^{2/3}\big) &= P\big(xn^{2/3} \leq L_n^{(k)} \leq L_n^{(2)} \leq x_1 n^{2/3}\big) \\ &\quad + O\big[P\big(L_n^{(2)} > x_1 n^{2/3}\big)\big] \qquad (4.44) \\ &= P\big(c_n(x, x_1) > k-1\big) + O\big[P\big(L_n^{(2)} > x_1 n^{2/3}\big)\big]. \end{aligned}$$

According to part (i) of the present proof, $c_n(x, x_1)$ is, in the limit, Poisson with parameter $\lambda = \lambda(x, x_1)$. So, by (4.43) and (4.44),

$$\limsup \left| P\left(L_n^{(k)} > xn^{2/3}\right) - e^{-\lambda} \sum_{j \geq k-1} \lambda^j/j! \right| \leq \varepsilon$$

and, letting $\varepsilon \to 0+$, we get that

$$\lim P\left(L_n^{(k)} \geq xn^{2/3}\right) = e^{-\lambda(x)} \sum_{j \geq k-1} \lambda^j(x)/j!,$$

where

$$\lambda(x) = (2\beta_0/3\sigma)x^{-3/2}.$$

In particular,

$$\lim P\left(L_n^{(2)} \leq xn^{2/3}\right) = e^{-\lambda(x)}. \qquad \square$$

4.9. *Proof of Theorem* 3.3. Now $\bar{\sigma}/\sigma_n = 1 - an^{1/3}$.

(i) We show first that the forest M_n w.h.p. does not contain a tree of size $j \geq \varepsilon n$, $\forall\, \varepsilon \in (0,1)$. Indeed, for every such j, $n' = n - j \leq n - \varepsilon n$ and $\sigma_{n'} \leq \bar{\sigma}(1 - \varepsilon/2)$. So, estimating $Q_{n'}/(n')!$ as in (4.34)–(4.36) and applying the Lemma 4.2 to $Q_n/n!$, we obtain

$$E(c_{nj}) = (\mu/\lambda) f_j \binom{n}{j} (Q_{n'}/Q_n) = O\left[n^{5/3} \exp\left(-n\sigma_n^{-1}c\right)\right],$$

where

$$c = c(\varepsilon) = \int_{\rho(\varepsilon)}^{\bar{x}} \left[R_0(y) - R_0(\rho(\varepsilon))\right] y^{-1}\,dy, \qquad \rho(\varepsilon) = R_0^{-1}\left[\bar{\sigma}(1 - \varepsilon/2)\right].$$

Hence,

$$E\left(\sum_{j \geq \varepsilon n} c_{nj}\right) = O\left(n^{8/3} \exp(-nc_1)\right) = o(1), \qquad n \to \infty.$$

(ii) Next, we prove that, even stronger, w.h.p. there is no tree of size $j \geq \omega(n)n^{2/3}$, where $\omega(n) \to \infty$ however slowly. According to part (i), it suffices to consider $j \in [j_1, j_2] = [\omega(n)n^{2/3}, n/2]$. Since, uniformly over such j, $\liminf \sigma_{n'} \geq \bar{\sigma}/2$, we can apply now Lemma 4.3 to estimate $Q_{n'}/(n')!$. It follows then that $E(c_{nj}) = O(nj^{-5/2})$, whence

$$E\left(\sum_{j_1 \leq j \leq j_2} c_{nj}\right) = O\left(n \sum_{j \geq j_1} j^{-5/2}\right) = O\left(nj_1^{-3/2}\right) = O\left(\omega^{-3/2}(n)\right) = o(1).$$

(iii) Now that we have demonstrated that $L_n^{(1)}/n^{2/3}$ is bounded in probability, the rest of the argument is patterned after step (ii) of the proof of Theorem 2(ii). Fix $x_1 \geq x$, set $j_1 = xn^{2/3}$, $j_2 = x_1 n^{2/3}$ and introduce $c_n(x, x_1) =$

$\sum_{j_1 \le j \le j_2} c_{nj}$. By Lemma 4.2,

$$E(c_n(x, x_1)) = \sum_{j_1 \le j \le j_2} E(c_{nj}) = \sum_{j_1 \le j \le j_2} (\mu/\lambda) f_j \binom{n}{j} Q_{n'}/Q_n$$
$$= (1 + o(1)) n\beta_0/(\bar{\sigma} p(a)) \sum_{j_1 \le j \le j_2} j^{-5/2} p(a_{n'}),$$

where

$$a_{n'} = (n')^{1/3}(1 - \bar{\sigma}/\sigma_{n'}) = (n/n')^{2/3}(a - j/n^{2/3}) = a - j/n^{2/3} + O(n^{-1/3}).$$

So, introducing $x_j = j/n^{2/3}$, $\Delta x_j = n^{-2/3}$, we conclude that

$$E(c_n(x, x_1)) = (1 + o(1))\beta_0/(\bar{\sigma} p(a)) \sum_{j_1 \le j \le j_2} x_j^{-5/2} p\left[a - x_j + O(n^{-1/3})\right] \Delta x_j$$
$$\to \beta_0/(\bar{\sigma} p(a)) \int_x^{x_1} y^{-5/2} p(a - y)\, dy := I(x, x_1), \qquad n \to \infty.$$

More generally, for every $k \ge 1$,

$$E([c_n(x, x_1)]_k) \to I^k(x, x_1), \qquad n \to \infty,$$

whence $c_n(x, x_1)$ is, in the limit, Poisson with parameter $I(x, x_1)$. Since $L_n^{(1)}/n^{2/3} = O_p(1)$, it follows that, for each $k \ge 1$,

$$P(L_n^{(k)} \ge x n^{2/3}) \to \exp(-I(x)) \sum_{j \ge k} I^j(x)/j!,$$

where

$$I(x) = \beta_0/(\bar{\sigma} p(a)) \int_x^{\infty} y^{-5/2} p(a - y)\, dy.$$

The proof of Theorem 3 is now complete. □

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B. PITTEL
DEPARTMENT OF MATHEMATICS
OHIO STATE UNIVERSITY
COLUMBUS, OHIO 43210

W. A. WOYCZYNSKI
DEPARTMENT OF MATHEMATICS AND STATISTICS
CASE WESTERN RESERVE UNIVERSITY
CLEVELAND, OHIO 44106

J. A. MANN
DEPARTMENT OF CHEMICAL ENGINEERING
CASE WESTERN RESERVE UNIVERSITY
CLEVELAND, OHIO 44106