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LIMIT THEOREMS FOR AGE-STRUCTURED POPULATIONS[1]

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We study populations consisting of individuals whose birth and death rates depend on both their age and the age-structure of the whole population. In particular, as the population size tends to infinity, we derive a law of large numbers and a central limit theorem for the empirical processes.

1. Introduction. We study certain random processes describing the time evolution of populations, whose members are characterized by their age. We assume that for any individual the birth and death rate depends on both *age* and the *age-structure* of the whole population. In particular, we are interested in the behaviour of these processes as the population size tends to infinity.

Let us give now the details of the model: For any $N \in \mathbf{N}$ we consider a population of initial size approximately equal to N. Suppose this population consists at some instant $t \geq 0$ of individuals with respective age a_k, $k \in R_N(t)$, where $R_N(t)$ is some enumeration of the individuals living at time t. Then we describe it, or more precisely, its age-structure, by the positive measure

$$A_N(t) = \mu = \frac{1}{N} \sum_{k \in R_N(t)} \delta_{a_k}$$

on $\mathbf{R}_+$, where δ_a is the Dirac measure at a. We assume that individual j with age a_j gives birth to a new individual with intensity $\beta(a_j, \mu)$, respectively, dies with intensity $\gamma(a_j, \mu)$. In these cases A_N jumps from μ to $\mu + (1/N)\delta_0$, respectively, to $\mu - (1/N)\delta_{a_j}$. Apart from these discontinuous transitions A_N changes continuously through aging, i.e., if in the time interval $[t, t+\Delta]$ no birth or death occurs, we obtain $A_N(t+\Delta) = (1/N)\sum_{k \in R_N(t)} \delta_{a_k+\Delta} = A_N(t) * \delta_\Delta$.

The exact regularity properties of the functions β and γ will be specified in Section 2. Let us note here only that in our framework we typically have

$$\begin{aligned} \beta(a, \mu) &= \hat{\beta}\big(a, \langle \mu, f_1(a, \cdot)\rangle, \ldots, \langle \mu, f_p(a, \cdot)\rangle\big), \\ \gamma(a, \mu) &= \hat{\gamma}\big(a, \langle \mu, g_1(a, \cdot)\rangle, \ldots, \langle \mu, g_q(a, \cdot)\rangle\big), \end{aligned} \tag{1.1}$$

where $\hat{\beta}$, $\hat{\gamma}$, $f_1, \ldots, g_1, \ldots$ are sufficiently smooth. We use here the notation

$$\langle \nu, f \rangle = \int_{\mathbf{R}} \nu(dx) f(x)$$

for any measure ν and any real-valued function f on $\mathbf{R}$.

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Usual ways to describe the asymptotics of the processes A_N as $N \to \infty$ are the *law of large numbers* (LLN) and the *central limit theorem* (CLT). The LLN states the convergence of A_N to some deterministic limit process A, whereas the CLT asserts that the limit $F = \lim_{N \to \infty} F_N$ of the fluctuations $F_N = \sqrt{N}(A_N - A)$ exists and is Gaussian.

The special form of the dynamics, in particular, a form like (1.1) of the intensities β and γ, implies that the processes A_N are examples of *weakly interacting stochastic processes*, which are studied extensively in the literature. In particular, there are enough techniques to prove the LLN, cf., e.g., [1], [9] and for related models [7], [8], [11], [16]. Nevertheless we shall give here another derivation of this theorem, both for completeness of this paper and since this proof may motivate the proof of our main result, the CLT.

As a CLT for weakly interacting stochastic processes our result has connections, e.g., to [6], [12], [15]–[19]. In particular, in the identification of the limit process, i.e., Lemma 4.2, we proceed in a similar way as in Uchiyama's work [18], [19]. This approach is based on Stroock and Varadhan's martingale characterization of diffusion processes [14], which has been adapted by Holley and Stroock [4] to many-particle systems.

Similar limit theorems for models of age-structured populations have been obtained by Wang [21], [22] and Solomon [13]. A common feature of these papers is that the lifetime is not influenced by the age distribution of the remaining population. On the other hand, the birth rates are quite general and even may depend on the age distribution in the past, cf. [13].

The results of this paper and the main ideas of their proofs are contained in [10] where the present model and a related system of weakly interacting jump processes have been investigated.

We shall state our results in the next section, whereas the proofs are deferred to Sections 3 and 4. An Appendix contains the proof of a technical lemma.

2. The results. As indicated in the Introduction, our main concern is the CLT. The LLN is stated and proved only for completeness. But first of all we need some preliminaries.

A. *Notation and technical preliminaries.* We denote by $C_b(M)$ the space of bounded continuous real-valued functions on some topological space M. For an Euclidean space E let $C_b^k(E)$ be the subset of $C_b(E)$ consisting of functions with bounded partial derivatives of order $1, \dots, k$.

As the age of any individual is always positive, the underlying space of our calculations is $\mathbf{R}_+$. However, it will be convenient to consider the processes A_N and F_N as measure-valued processes on the whole line $\mathbf{R}$. In this context we can consider any function on $\mathbf{R}_+$, e.g., the birth and death rates, as being defined on $\mathbf{R}$.

We denote by

$$\lfloor f \rfloor^L = \sup\left\{ \frac{|f(x) - f(y)|}{|x - y|} : x, y \in \mathbf{R}, x \neq y \right\}$$

the possibly infinite Lipschitz constant of a function $f \in C_b(\mathbf{R})$. Then we define on $C_b(\mathbf{R})$ a scale of norms

$$\|f\|^{(0)} = \sup\{|f(x)|: x \in \mathbf{R}\},$$

$$\|f\|^{(k)} = \sup\{\|f^{(l)}\|^{(0)} + \lfloor f^{(l)} \rfloor^{L}: l = 0, \dots, k-1\}, \qquad k = 1, 2, \dots.$$

Here $f^{(l)}$ denotes the lth derivative of f.

Associated to these norms are the Banach spaces

$$\mathscr{L}_k = \{f \in C_b(\mathbf{R}): \|f\|^{(k)} < \infty\}, \qquad k = 0, 1, \dots.$$

In order to accentuate the analogy with $\mathscr{L}_k$, $k = 1, 2, \dots$, we have dispensed with the more common notation $C_b(\mathbf{R})$, $\|\cdot\|_\infty$ for $\mathscr{L}_0$, $\|\cdot\|^{(0)}$.

The spaces $\mathscr{L}_k$, $k = 0, 1, \dots$, are not separable, in contrast to their subsets

$$\mathscr{L}_k^c = \{f \in \mathscr{L}_k: \operatorname{supp}(f) \text{ is compact}\}, \qquad k = 0, 1, \dots.$$

Next, let $\mathscr{M}_0$ be the space of signed measures μ on $\mathbf{R}$ with bounded total variation

$$\|\mu\|_0 = \sup\{\langle \mu, f \rangle: f \in \mathscr{L}_0, \|f\|^{(0)} \le 1\}.$$

Analogously to the spaces $\mathscr{L}_k$, $k = 0, 1, \dots$, we obtain a scale of spaces of generalized functions on $\mathbf{R}$ by defining $\mathscr{M}_k$, $k = 1, 2, \dots$, as the closure of the set $\mathscr{M}_0$ with respect to the norm

$$\|\mu\|_k = \sup\{\langle \mu, f \rangle: f \in \mathscr{L}_k, \|f\|^{(k)} \le 1\}, \qquad k = 1, 2, \dots.$$

By the Riesz–Markov–Kakutani theorem $\mathscr{M}_0$ is contained in the dual Banach space of $\mathscr{L}_0$, cf. [23]. Similarly, for $k = 1, 2, \dots$ the space $\mathscr{M}_k$ is a subset of the dual of $\mathscr{L}_k$. The spaces $\mathscr{M}_k$, $k = 1, 2, \dots$, are separable, since measures like $\sum_{k=1}^{M} c_m \delta_{x_m}$, $c_1, \dots, c_M, x_1, \dots, x_M \in \mathbf{Q}$, $M \in \mathbf{N}$, are dense. On the other hand, $\mathscr{M}_0$ is not separable.

We extend the use of $\langle \mu, f \rangle$ to denote the natural duality between elements $\mu \in \mathscr{M}_k$ and $f \in \mathscr{L}_k$.

$\mathscr{M}_k^+$, $k = 0, 1, \dots$, is the set of positive finite measures on $\mathbf{R}$, equipped with the metric $\|\mu - \nu\|_k$. Any $\mathscr{M}_k^+$ is closed in $\mathscr{M}_k$. Note that the restriction to the set $\mathscr{M}_0^+$ of the *weak-$*$-topology* on $\mathscr{M}_0$ is generated by the metric $\|\mu - \nu\|_1$, cf. [2], Theorem 12.

The sequences $\mathscr{L}_k$ and $\mathscr{M}_k$ are reminiscent of similar scales of Sobolev spaces, which have been employed in some of the papers mentioned in the Introduction. For example, $\mathscr{L}_k$, $k = 0, 1, \dots$, correspond to the spaces $W^{m,p,\alpha}$, $m = 0, 1, \dots$, in [9] or to the spaces H_α, $\alpha > 0$, in [6], whereas $\mathscr{M}_k$, $k = 0, 1, \dots$, are analogues of $W^{m,p,\alpha}$, $m = -1, -2, \dots$, in [9] or of H_α, $\alpha < 0$, in [6], respectively.

To illustrate the application of such scales of spaces in the context of many-particle systems we consider the sequence $\mu_N = (1/N)\sum_{k=1}^{N} \delta_{k/N}$, $N \in \mathbf{N}$. Formally this sequence converges to the uniform distribution μ on the unit interval $[0, 1]$. However, we obviously can find for any $N \in \mathbf{N}$ and $\delta > 0$ a function $f_{N,\delta} \in \{g \in \mathscr{L}_0: \|g\|^{(0)} \le 1\}$ such that $\|\langle \mu_N - \mu, f_{N,\delta} \rangle\| \ge 2 - \delta$, i.e., we obtain $\|\mu_N - \mu\|_0 = 2$. Therefore, the convergence $\mu_N \to \mu$ cannot hold in $\mathscr{M}_0$. The reason is that $\mu_N - \mu$ gets very irregular as $N \to \infty$ and that $f_{N,\delta}$ can be chosen in such a way that it emphasizes this irregularity. On the other hand, if we only use functions $f \in \{g \in \mathscr{L}_1 : \|g\|^{(1)} \le 1\}$, the irregularities are blurred in the integral $\langle \mu_N - \mu, f \rangle$. We obtain

$$|\langle \mu_N - \mu, f \rangle| = \left| \sum_{k=1}^{N} \int_{(k-1)/N}^{k/N} (f(x) - f(k/N))\, dx \right| \le 1/N,$$

uniformly for f in this set and therefore $\|\mu_N - \mu\|_1 \to 0$. In other words, to obtain the desired convergence $\mu_N \to \mu$ we have to use a sufficiently weak metric. Such a metric is conveniently defined by taking the supremum of $\langle \mu_N - \mu, f \rangle$ over a class of sufficiently smooth functions f.

These arguments hold mutatis mutandis in our particular cases of the convergence of the processes A_N and F_N. For the CLT we have to note that for absolutely continuous $A(t) = \lim_{N\to\infty} A_N(t)$ we have $\|F_N(t)\|_0 \sim \sqrt{N}$, i.e., $F(t) = \lim_{N\to\infty} F_N(t)$ cannot be measure-valued. However, as it turns out that $\|F_N(t)\|_1$ stays nondegenerate as $N \to \infty$, we can utilize a result on the compact imbedding of subsets of $\mathscr{M}_1$ into $\mathscr{M}_2$ (cf. Lemma 2.1 below) to show that $\mathscr{M}_2$ with its metric $\|\cdot\|_2$ is the right setting to formulate the convergence of the fluctuation processes F_N. This means that we have to weaken the notion of convergence once more and thereby enlarge the state space.

The background of the use of scales of different spaces of measures and generalized functions, which has been sketched in the preceding two paragraphs, is quite the same in the present paper or in [10] and in some other work employing Sobolev space- and $\mathscr{S}'$-valued processes, cf. e.g., [6], [9], [18], [19]. Our particular choice of the sequence $\mathscr{M}_k$, $k = 0, 1, \dots$, was supported both by the ease to formulate for the birth and death rates β and γ quite general, unrestrictive conditions in terms of the dual norms $\|\cdot\|^{(k)}$; cf. (2.3), (2.4), (2.10)–(2.14) and the fact that the subsequent calculations in the proofs of our results remain relatively simple and straightforward.

Another norm on $\mathscr{M}_0$, which is closely related but slightly stronger than $\|\cdot\|_1$, is defined by

$$\|\mu\|_1^{\#} = \sum_{m=-\infty}^{\infty} \left(\int_{m-1}^{m} \|\mathbf{1}_{(x,m]}\mu\|_1\, dx + \|\mathbf{1}_{(m-1,m]}\mu\|_1 \right).$$

Here we use $\mathbf{1}_A$ for the indicator function of a set A, i.e., $\mathbf{1}_A\mu$ is the restriction of μ to A.

It will be necessary to estimate the *size* of the elements of $\mathscr{M}_k$ outside compact sets in $\mathbf{R}$. Then the following notation will be convenient:

$$\|\mu\|_k^{[l]} = \sup\{\langle \mu, f \rangle : f \in \mathscr{L}_k, \|f\|^{(k)} \le 1, \operatorname{supp}(f) \subseteq \mathbf{R} \setminus (-l, l)\}, \qquad k, l = 0, 1, \dots,$$

$$\|\mu\|_1^{\#,l} = \sum_{|m| \ge l} \left(\int_{m-1}^{m} \|\mathbf{1}_{(x,m]}\mu\|_1\, dx + \|\mathbf{1}_{(m-1,m]}\mu\|_1 \right), \qquad l = 0, 1, \dots .$$

Obviously $\|\mu\|_k^{[0]} = \|\mu\|_k$ and $\|\mu\|_1^{\#,0} = \|\mu\|_1^{\#}$.

The following lemma describes some relations between these different norms and spaces. Its proof can be found in the Appendix.

LEMMA 2.1. (a)

$$\|\mu\|_{k+1}^{[l]} \le \|\mu\|_k^{[l]}, \qquad \mu \in \mathscr{M}_k,\ k, l = 0, 1, \dots, \tag{2.1}$$

$$\|\mu\|_1^{[l]} \le \|\mu\|_1^{\#, l} \le 3 \sum_{|m| \ge l} \left(\int_{m-1}^{m} |\mu((x, m])|\, dx + |\mu((m-1, m])| \right), \tag{2.2}$$

$$\mu \in \mathscr{M}_0,\ l = 0, 1, \dots .$$

(b) *For some* $k = 0, 1, \dots$ *suppose* $\mathscr{K} \subseteq \mathscr{M}_k$ *with*

$$\sup\{\|\mu\|_k : \mu \in \mathscr{K}\} < \infty$$

and

$$\lim_{n \to \infty} \sup\{\|\mu\|_k^{[n]} : \mu \in \mathscr{K}\} = 0.$$

Then $\mathscr{K}$ *is relatively compact in* $\mathscr{M}_{k+1}$.

REMARK. Since, as mentioned above, the metric $\|\mu - \nu\|_1$ generates the weak-$*$-topology in $\mathscr{M}_0^+$, Lemma 2.1(b) is a partial generalization of Prohorov's theorem.

Let $\mathscr{M}_*$ be any $\mathscr{M}_k$ or $\mathscr{M}_k^+$, $k = 0, 1, \dots$. Then for fixed $T \in (0, \infty)$ we denote by $\mathscr{D}([0, T], \mathscr{M}_*)$ the set of all right continuous functions $[0, T] \to \mathscr{M}_*$ with left limits. These spaces are equipped with the usual Skorokhod topology, cf. [3]. $\mathscr{C}([0, T], \mathscr{M}_*)$ denotes the subset of all continuous functions $[0, T] \to \mathscr{M}_*$. We use the symbol $Y = Y_t$, $0 \le t \le T$, for the canonical processes in these spaces. The $\mathscr{D}$-spaces will be used as path spaces for the processes A_N and F_N, more precisely, $\mathscr{D}([0, T], \mathscr{M}_1^+)$ for A_N and $\mathscr{D}([0, T], \mathscr{M}_2)$ for F_N. On the other hand, it will turn out that the limits of these processes are continuous, i.e., have trajectories in some $\mathscr{C}$-space.

We denote by $\mathscr{L}(X)$ the distribution of some random variable or random process X. For any topological space M let $\mathscr{P}(M)$ be the space of probability distributions on M equipped with the *weak-$*$-topology*. This means that a sequence P_n converges to some limit P in $\mathscr{P}(M)$ if and only if $\lim_{n \to \infty} \int_M f(y) P_n(dy) = \int_M f(y) P(dy)$ for any $f \in C_b(M)$.

By $\mathbf{E}_Q[X]$, respectively, $\mathbf{P}_Q[A]$, we denote the expectation of a random variable X, respectively, the probability of an event A, with respect to a probability distribution Q.

In our situation we can take as underlying probability space $\mathscr{D}([0, T], \mathscr{M}_1^+)$ in the case of the LLN, respectively, $\mathscr{D}([0, T], \mathscr{M}_2)$ for the CLT. On these spaces we have the natural filtration $\mathscr{F}_t$, $0 \le t \le T$, $\mathscr{F}_t = \sigma(Y_s : 0 \le s \le t)$. The martingale property of any random process always refers to this filtration.

Using our boundedness assumptions below one can easily show that any process occurring in this paper, which, e.g., by Itô's formula turns out to be a local martingale, even possesses the global martingale property. For this reason we do not distinguish between local and global martingales.

The following result is obtained from the general Theorem 8.6 in [3], Chapter 3.

PROPOSITION 2.2. *For some* $k = 1, 2, \ldots$ *let* X_N, $N \in \mathbf{N}$, *be a sequence of* $\mathscr{M}_k$*-valued processes. Suppose*

$$\lim_{n\to\infty} \inf_{N\in\mathbf{N}} \mathbf{P}_{\mathscr{L}(X_N)}[Y_t \in \mathscr{K}_n \text{ for all } t \in [0,T]] = 1$$

for some sequence $\mathscr{K}_n$, $n \in \mathbf{N}$, *of compact subsets of* $\mathscr{M}_k$. *Furthermore, assume that there exists a family of positive random variables* $G(\varepsilon)$, $\varepsilon > 0$, *such that for any* $0 \le s \le t \le T$,

$$\mathbf{E}_{\mathscr{L}(X_N)}[\|Y_t - Y_s\|_k | \mathscr{F}_s] \le \mathbf{E}_{\mathscr{L}(X_N)}[G(t-s)|\mathscr{F}_s]$$

and

$$\lim_{\varepsilon\to 0} \sup_{N\in\mathbf{N}} \mathbf{E}_{\mathscr{L}(X_N)}[G(\varepsilon)] = 0.$$

Then the sequence $\{\mathscr{L}(X_N): N \in \mathbf{N}\}$ *is relatively compact in* $\mathscr{P}(\mathscr{D}([0,T], \mathscr{M}_k))$.

Let $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$ for $a, b \in \mathbf{R}$. By $C, C', C'', \ldots$ we denote positive constants, which may vary from place to place and are independent of N.

B. *The law of large numbers.* The result of this section states the convergence of the processes A_N to a deterministic limit process A.

THEOREM 1. *Suppose*

$$\sup\left\{\|\beta(\cdot,\mu)\|^{(0)} + \|\gamma(\cdot,\mu)\|^{(0)}: \mu \in \mathscr{M}_1^+\right\} < \infty,$$

$$\sup\left\{\frac{1}{1+\|\mu\|_0}\left(\lfloor\beta(\cdot,\mu)\rfloor^L + \lfloor\gamma(\cdot,\mu)\rfloor^L\right): \mu \in \mathscr{M}_1^+\right\} < \infty, \tag{2.3}$$

$$\sup\left\{\frac{1}{\|\mu-\nu\|_1}\left(\|\beta(\cdot,\mu) - \beta(\cdot,\nu)\|^{(0)} + \|\gamma(\cdot,\mu) - \gamma(\cdot,\nu)\|^{(0)}\right): \mu,\nu \in \mathscr{M}_1^+\right\} < \infty, \tag{2.4}$$

$$\lim_{N\to\infty} \mathscr{L}(A_N(0)) = \delta_{A_0} \quad \text{in } \mathscr{P}(\mathscr{M}_1^+), \tag{2.5}$$

$$\sup_{N\in\mathbf{N}} \mathbf{E}_{\mathscr{L}(A_N)}[\|Y_0\|_0] < \infty, \tag{2.6}$$

$$\lim_{n\to\infty} \sup_{N\in\mathbf{N}} \mathbf{E}_{\mathscr{L}(A_N)}[\|Y_0\|_0^{[n]}] = 0. \tag{2.7}$$

Then for any $T \in (0, \infty)$ the sequence $\{\mathscr{L}(A_N): N \in \mathbf{N}\}$ of the distributions of the processes $A_N = A_N(t)$, $0 \le t \le T$, converges in $\mathscr{P}(\mathscr{D}([0, T], \mathscr{M}_1^+))$ to the Dirac measure δ_A concentrated at the unique $A = (A_t)_{0 \le t \le T} \in \mathscr{C}([0, T], \mathscr{M}_1^+)$, which solves the integral equation

$$\langle A_t, f\rangle = \langle A_0, f\rangle + \int_0^t \langle A_s, \beta(\cdot, A_s) f(0) - \gamma(\cdot, A_s) f + f'\rangle\, ds, \tag{2.8}$$
$$0 \le t \le T,\ f \in C_b^1(\mathbf{R}).$$

REMARKS. (i) The unique existence of the process A_N for any fixed N is an immediate consequence of (2.3), which implies that with probability 1 the total population size stays finite during $[0, T]$. Hence, the jump times of the process A_N are isolated. Between these jump times A_N is trivial anyway.

(ii) (2.8) is a weak version of the *Von Foerster equation*

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)\rho(t, a) = -\rho(t, a)\gamma(a, \rho(t, \cdot)), \qquad a > 0, \tag{2.9}$$
$$\rho(t, 0) = \int_0^\infty \rho(t, x)\beta(x, \rho(t, \cdot))\, dx.$$

This equation has been introduced in [20] and was studied extensively in the subsequent years. A quite general existence and uniqueness result can be found, for example, in [5].

(iii) Suppose β and γ are of the form (1.1). Then (2.3) and (2.4) are satisfied, e.g., if $\hat{\beta} \in C_b^1(\mathbf{R}^{p+1})$, $\hat{\gamma} \in C_b^1(\mathbf{R}^{q+1})$, $f_1, \dots, f_p, g_1, \dots, g_q \in C_b^1(\mathbf{R}^2)$. Another example, where (2.3) and (2.4) hold, and which is not covered by (1.1), is $\beta(x, \mu) = \bar{\beta}(x, \|\mu - \bar{\mu}\|_1)$, $\gamma(x, \mu) = \bar{\gamma}(x, \|\mu - \bar{\mu}\|_1)$ for $\bar{\beta}, \bar{\gamma} \in C_b^1(\mathbf{R}^2)$ and some fixed *reference measure* $\bar{\mu} \in \mathscr{M}_1^+$.

C. *The central limit theorem.* Now we turn to the investigation of the asymptotics of the fluctuation process $F_N(t) = \sqrt{N}(A_N(t) - A_t)$. A major problem we encounter is the choice of a suitable state space, i.e., a space where we can obtain a nice asymptotics of the processes F_N. In particular, we have to smooth out too rapid fluctuations of F_N within small distances in *age space*. As mentioned in Section 2A, this may be achieved by using $\mathscr{M}_2$ as state space.

We have to assume some additional regularity properties of the birth and death rates β and γ.

There exists a function $\beta_1: \mathbf{R}_+ \times \mathbf{R}_+ \times \mathscr{M}_1^+ \to \mathbf{R}$, such that

$$\lim_{h \to 0} (1/|h|)\,|\beta(a, \mu + h\nu) - \beta(a, \mu) - h\langle \nu, \beta_1(a, \cdot, \mu)\rangle| = 0, \tag{2.10}$$
$$a \in \mathbf{R}_+,\ \mu \in \mathscr{M}_1^+,\ \nu \in \mathscr{M}_1 \text{ with } \mu + h\nu \in \mathscr{M}_1^+ \text{ for sufficiently small } h,$$

$$\sup\left\{\|\beta_1(a, \cdot, \mu)\|^{(1)}: a \in \mathbf{R}_+, \mu \in \mathscr{M}_1^+\right\} < \infty. \tag{2.11}$$

Let $\beta_1(a,\mu;\nu) = \langle \nu, \beta_1(a,\cdot,\mu)\rangle$. Then

$$\sup\left\{\frac{1}{\|\nu\|_1(1+\|\mu\|_1)}\|\beta_1(\cdot,\mu;\nu)\|^{(1)}: \mu \in \mathscr{M}_1^+,\ \nu \in \mathscr{M}_1\right\} < \infty, \tag{2.12}$$

$$\begin{aligned}&\sup\left\{\frac{1}{\|\nu\|_1\|\mu'\|_1}\|\beta_1(\cdot,\mu+\mu';\nu) - \beta_1(\cdot,\mu;\nu)\|^{(0)}:\right.\\ &\qquad\left.\mu \in \mathscr{M}_1^+,\ \mu',\nu \in \mathscr{M}_1 \text{ with } \mu+\mu' \in \mathscr{M}_1^+\right\} < \infty.\end{aligned} \tag{2.13}$$

The same conditions have to be satisfied for the death rate:

(2.14) There exists a function $\gamma_1: \mathbf{R}_+ \times \mathbf{R}_+ \times \mathscr{M}_1^+ \to \mathbf{R}$ satisfying (2.10)–(2.13) with γ replacing β.

REMARKS. (i) (2.10) and (2.14) mean that the functions $\mu \to \beta(\cdot,\mu)$ and $\mu \to \gamma(\cdot,\mu)$ are *Gateaux-differentiable* in any direction not leaving the positive cone $\mathscr{M}_1^+$ of $\mathscr{M}_1$.

(ii) Suppose β and γ are of the form (1.1). Then (2.10)–(2.14) are satisfied, e.g., if $\hat\beta \in C_b^2(\mathbf{R}^{p+1})$, $\hat\gamma \in C_b^2(\mathbf{R}^{q+1})$, $f_1,\dots,f_p, g_1,\dots,g_q \in C_b^2(\mathbf{R}^2)$. In this situation we have

$$\beta_1(a,x,\mu) = \sum_{k=1}^{p} f_k(a,x)\frac{\partial}{\partial z_k}\hat\beta(a,z_1,\dots,z_p)\bigg|_{z_1=\langle\mu, f_1(a,\cdot)\rangle,\dots,z_p=\langle\mu, f_p(a,\cdot)\rangle}$$

and

$$\gamma_1(a,x,\mu) = \sum_{k=1}^{q} g_k(a,x)\frac{\partial}{\partial z_k}\hat\gamma(a,z_1,\dots,z_q)\bigg|_{z_1=\langle\mu, g_1(a,\cdot)\rangle,\dots,z_q=\langle\mu, g_q(a,\cdot)\rangle}$$

To formulate the CLT, in particular, to describe the *drift* and the *diffusion* of the limit Gaussian process, we need a certain *time evolution* $S = S_s^t$, $0 \le s \le t \le T$, on $\mathscr{L}_1$, i.e., a family of bounded operators S_s^t on $\mathscr{L}_1$ satisfying the semigroup property $S_s^t \circ S_u^s = S_u^t$, $0 \le u \le s \le t \le T$. These operators are defined by

$$(S_s^t f)(x) = (\bar S_s^t f(\cdot + t))(x-s), \qquad 0 \le s \le t \le T,\ f \in \mathscr{L}_1,\ x \in \mathbf{R}, \tag{2.15}$$

where the auxiliary operators $\bar S_s^t$ solve the system of integral equations

$$\begin{aligned}(\bar S_s^t f)(x) = f(x) + \int_s^t &\Big((\bar S_u^t f)(-u)\big(\beta(x+u,A_u) + \langle A_u, \beta_1(\cdot, x+u, A_u)\rangle\big)\\ &- (\bar S_u^t f)(x)\gamma(x+u,A_u)\\ &- \big\langle A_u, \gamma_1(\cdot, x+u, A_u)(\bar S_u^t f)(\cdot - u)\big\rangle\Big)\,du,\\ &\qquad 0 \le s \le t \le T,\ f \in \mathscr{L}_1,\ x \in \mathbf{R}.\end{aligned} \tag{2.16}$$

The regularity assumptions (2.3) (2.4), (2.10), (2.11), (2.14) imply that (2.16) determines a unique family $\bar S = \bar S_s^t$, $0 \le s \le t \le T$, of uniformly bounded operators in $\mathscr{L}_1$, i.e.,

$$\sup_{0\le s\le t\le T,\ f\in\mathscr{L}_1}\frac{\|\bar{S}_s^t f\|^{(1)}}{\|f\|^{(1)}}=\sup_{0\le s\le t\le T,\ f\in\mathscr{L}_1}\frac{\|S_s^t f\|^{(1)}}{\|f\|^{(1)}}<\infty. \tag{2.17}$$

REMARK. The transformation (2.15) hints at a trick in the proof of our results, namely the application of a time-dependent shift in age space. This procedure allows getting rid of the aging of the individuals and therefore provides a simplified description of the dynamics of A_N and F_N.

Finally, we need a quadratic functional determining the diffusion of the limit process:

$$\Sigma_{s,t}^2(f)=\int_s^t\left\langle A_u,\beta(\cdot,A_u)(S_u^t f)(0)^2+\gamma(\cdot,A_u)(S_u^t f)(\cdot)^2\right\rangle du, \qquad 0\le s\le t\le T,\ f\in\mathscr{L}_1. \tag{2.18}$$

Now we can formulate the main result of this paper.

THEOREM 2. *Assume* (2.3)–(2.7), (2.10)–(2.14) *and*

$$\sup_{N\in\mathbf{N}}\mathbf{E}_{\mathscr{L}(A_N)}[\langle Y_0,\varphi_2\rangle]<\infty,\quad \textit{where } \varphi_2(x)=1+x^2, \tag{2.19}$$

$$\sup_{N\in\mathbf{N}}\mathbf{E}_{\mathscr{L}(F_N)}\left[\|Y_0\|_1^{\#}\right]<\infty, \tag{2.20}$$

$$\lim_{n\to\infty}\sup_{N\in\mathbf{N}}\mathbf{E}_{\mathscr{L}(F_N)}\left[\|Y_0\|_1^{\#,n}\right]=0, \tag{2.21}$$

$$\lim_{N\to\infty}\mathscr{L}(F_N(0))=\mathscr{L}_0^{\#}\quad \textit{in } \mathscr{P}(\mathscr{M}_2). \tag{2.22}$$

Then for any $T\in(0,\infty)$ the sequence $\{\mathscr{L}(F_N)\colon N\in\mathbf{N}\}$ of the distributions of the processes $F_N=F_N(t)$, $0\le t\le T$, converges in $\mathscr{P}(\mathscr{D}([0,T],\mathscr{M}_2))$ to the distribution $\mathscr{L}^{\#}=\mathscr{L}(F)$ of a continuous Markov process $F=F_t$, $0\le t\le T$, on $\mathscr{M}_1$, which is uniquely determined by the initial distribution $\mathscr{L}(F_0)=\mathscr{L}_0^{\#}$ and the Gaussian transition probabilities

$$\mathbf{P}_{\mathscr{L}(F)}[\langle Y_t,f\rangle\in A|\mathscr{F}_s]=\frac{1}{\sqrt{2\pi\Sigma_{s,t}^2(f)}}\int_A\exp\left(-\frac{(x-\langle Y_s,S_s^t f\rangle)^2}{2\Sigma_{s,t}^2(f)}\right)dx, \tag{2.23}$$

$$0\le s\le t\le T,\ f\in\mathscr{L}_1,\ A\subseteq\mathbf{R}\ \textit{some Borel set}.$$

REMARKS. (i) The time evolution S on $\mathscr{L}_1$ has the *dual* $S^*=S_s^{t,*}$, $0\le s\le t\le T$, on $\mathscr{M}_1$, which is defined by

$$\langle S_s^{t,*}\mu,f\rangle=\langle\mu,S_s^t f\rangle.$$

By (2.23) S^* determines the drift of the limit process F.

(ii) Assumptions (2.19)–(2.22) are satisfied, e.g., if $A_N(0) = (1/N)\Sigma_{k=1}^N \delta_{a_k}$, where a_k, $k \in \mathbf{N}$, are i.i.d. random variables with a sufficiently nice density $\alpha(\cdot)$, i.e.,

$$\alpha \in C_b(\mathbf{R}_+), \qquad \int_0^\infty \alpha(x)x^2\,dx < \infty.$$

In this case A_0 has density α and F_0 is a Gaussian random field on $\mathbf{R}$ with

$$\mathbf{E}_{\mathscr{L}^*}[\langle Y_0, f\rangle] = 0, \qquad \mathbf{E}_{\mathscr{L}^*}\left[\langle Y_0, f\rangle^2\right] = \langle \alpha, f^2\rangle - \langle \alpha, f\rangle^2, \qquad f \in \mathscr{L}_1.$$

(iii) Our results and methods may easily be generalized to multitype age-structured populations, at least, if the birth and death rates of the mth subpopulation are of a form like (1.1), but additionally depending on $\langle \mu_i, f_{i,k,m}(a,\cdot)\rangle$, $i, k = 1, 2, \ldots,$ where μ_i is the age distribution of the ith subpopulation.

3. Proof of Theorem 1. The time evolution of the processes A_N is determined by two ingredients. The birth and death of individuals contribute jump components, i.e., A_N jumps from $\mu = (1/N)\Sigma_{k=1}^K \delta_{a_k}$ to $\mu + (1/N)\,\delta_0$ with intensity $\Sigma_{k=1}^K \beta(a_k, \mu) = N\langle \mu, \beta(\cdot, \mu)\rangle$ and to $\mu - (1/N)\,\delta_{a_m}$ with intensity $\gamma(a_m, \mu)$. Furthermore, the aging of the individuals adds a continuous component, i.e., in the absence of any birth or death in $[t, t + \Delta]$ the state μ of the population changes to $\mu * \delta_\Delta$ during that time interval.

An essential point of our proof consists in the transformation of the processes A_N into processes $\bar{A}_N = \bar{A}_N(t)$, $0 \le t \le T$, $\bar{A}_N(t) = A_N(t) * \delta_{-t}$. Since the convolution with δ_{-t} cancels the aging, $\bar{A}_N$ is a pure jump process, however, with time-inhomogeneous transition mechanisms. $\bar{A}_N$ jumps at time t from $\bar{A}_N(t) = \mu = (1/N)\Sigma_{k=1}^K \delta_{a_k}$ to $\mu + (1/N)\,\delta_{-t}$ with intensity $N\langle \mu, \bar{\beta}(t, \cdot, \mu)\rangle$, where $\bar{\beta}(t, a, \mu) = \beta(a + t, \mu * \delta_t)$, and to $\mu - (1/N)\,\delta_{a_m}$ with intensity $\bar{\gamma}(t, a_m, \mu) = \gamma(a_m + t, \mu * \delta_t)$.

First, we have to show the existence of a limit of the processes $\bar{A}_N$, i.e., the relative compactness of the sequence $\{\mathscr{L}(\bar{A}_N)\colon N \in \mathbf{N}\}$ in $\mathscr{P}(\mathscr{D}([0, T], \mathscr{M}_1^+))$. Second, we identify the dynamics of this limit and thereby obtain its uniqueness.

Note that we try to employ in this proof quite the same arguments as in the proof of the CLT below. This method should help understanding the proof of Theorem 2.

A. *The relative compactness of the sequence* $\{\mathscr{L}(\bar{A}_N)\colon N \in \mathbf{N}\}$ *in* $\mathscr{P}(\mathscr{D}([0, T], \mathscr{M}_1^+))$. We can define for $\mathscr{L}(\bar{A}_N)$-almost all $Y \in \mathscr{D}([0, T], \mathscr{M}_1^+)$ and any $s \in [0, T]$ the functions

$$u \to B_N(s, u) = B_N(s, u, Y)$$
$$= \text{number of individuals that are born during the time interval } (s, u], \qquad s \le u \le T$$

and

$$u \to D_N(s, u) = D_N(s, u, Y)$$
$$= \text{number of individuals that died during } (s, u], \qquad s \le u \le T.$$

With $B_N(s, u)$ and $D_N(s, u)$ we obtain an upper bound for $\|\bar{A}_N(u) - \bar{A}_N(s)\|_0$,

$$\|Y_u - Y_s\|_0 \le \frac{1}{N}(B_N(s, u, Y) + D_N(s, u, Y)), \tag{3.1}$$
$$0 \le s \le u \le T,\ \mathscr{L}(\bar{A}_N)\text{-a.s.}$$

The definition of the dynamics of the random process $\bar{A}_N$ implies that B_N and D_N can be described by time-changed Poisson processes. More precisely, $B_N(s, \cdot\,)$ and $D_N(s, \cdot\,)$ are distributed like

$$u \to P_\beta\left(N\int_s^u \langle Y_v, \bar{\beta}(v, \cdot\,, Y_v)\rangle\, dv\right)$$

and

$$u \to P_\gamma\left(N\int_s^u \langle Y_v, \bar{\gamma}(v, \cdot\,, Y_v)\rangle\, dv\right),$$

where P_β and P_γ are independent Poisson processes with intensity 1.

Using (2.3), (3.1) and this representation we obtain for $0 \le s \le t \le T$,

$$\begin{aligned}
&\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\sup_{s \le u \le t} \|Y_u - Y_s\|_0 \,\middle|\, \mathscr{F}_s\right] \\
&\le \frac{1}{N}\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\sup_{s\le u\le t} B_N(s, u)\,\middle|\,\mathscr{F}_s\right] + \frac{1}{N}\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\sup_{s\le u\le t} D_N(s, u)\,\middle|\,\mathscr{F}_s\right] \\
&= \frac{1}{N}\mathbf{E}_{\mathscr{L}(\bar{A}_N)}[B_N(s, t)|\mathscr{F}_s] + \frac{1}{N}\mathbf{E}_{\mathscr{L}(\bar{A}_N)}[D_N(s, t)|\mathscr{F}_s] \\
&\le C\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\int_s^t \|Y_v\|_0\, dv\,\middle|\,\mathscr{F}_s\right].
\end{aligned} \tag{3.2}$$

Setting $s = 0$ and integrating with respect to $\mathscr{L}(A_N(0))$ we obtain

$$\mathbf{E}_{\mathscr{L}(\bar{A}_N)}[\|Y_t\|_0] \le \mathbf{E}_{\mathscr{L}(\bar{A}_N)}[\|Y_0\|_0] + C\int_0^t \mathbf{E}_{\mathscr{L}(\bar{A}_N)}[\|Y_v\|_0]\, dv,$$

i.e., by Gronwall's inequality and (2.6),

$$\sup_{N\in\mathbf{N},\, t\le T} \mathbf{E}_{\mathscr{L}(\bar{A}_N)}[\|Y_t\|_0] < \infty. \tag{3.3}$$

Inserting (3.3) into (3.2) yields

$$\sup_{N\in\mathbf{N}} \mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\sup_{t\le T}\|Y_t\|_0\right] < \infty. \tag{3.4}$$

Since those individuals that are born during $[0, T]$ are represented by some δ_a, $-T \le a \le 0$, we have

$$\|Y_t\|_0^{[n]} \le \|Y_0\|_0^{[n]}, \qquad 0 \le t \le T,\ n > T,\ \mathscr{L}(\bar{A}_N)\text{-a.s.}, \tag{3.5}$$

i.e., (2.7) implies

$$\lim_{n\to\infty} \sup_{N\in\mathbf{N}} \mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\sup_{t\le T}\|Y_t\|_0^{[n]}\right] = 0.$$

In particular, there exists for any $\varepsilon > 0$, $r \in \mathbf{N}$, some $n_r > 0$, such that

$$\sup_{N \in \mathbf{N}} \mathbf{P}_{\mathscr{L}(\bar{A}_N)}\left[\sup_{t \le T} \|Y_t\|_0^{[n_r]} \ge 2^{-r}\right] \le \varepsilon 2^{-r}.$$

Hence,

$$\sup_{N \in \mathbf{N}} \mathbf{P}_{\mathscr{L}(\bar{A}_N)}\left[\sup_{t \le T} \|Y_t\|_0^{[n_r]} \ge 2^{-r} \text{ for some } r = 1, 2, \dots\right] \le \varepsilon. \tag{3.6}$$

From this estimate, (3.4) and Lemma 2.1(b) we derive:

(3.7) For any $\varepsilon > 0$ there exists a compact set $\mathscr{K}_\varepsilon \subseteq \mathscr{M}_1^+$ such that

$$\inf_{N \in \mathbf{N}} \mathbf{P}_{\mathscr{L}(\bar{A}_N)}\left[Y_t \in \mathscr{K}_\varepsilon \text{ for all } t \in [0, T]\right] \ge 1 - \varepsilon.$$

Next, by (2.1) and (3.2) for any $0 \le s \le t \le T$,

$$\begin{aligned} \mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\|Y_t - Y_s\|_1 | \mathscr{F}_s\right] &\le \mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\sup_{s \le u \le t} \|Y_u - Y_s\|_0 \Big| \mathscr{F}_s\right] \\ &\le C(t-s)\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\sup_{u \le T} \|Y_u\|_0 \Big| \mathscr{F}_s\right]. \end{aligned} \tag{3.8}$$

By (3.4), (3.7) and (3.8) the conditions of Proposition 2.2 are satisfied. Consequently, the sequence $\{\mathscr{L}(\bar{A}_N)\colon N \in \mathbf{N}\}$ is relatively compact in $\mathscr{P}(\mathscr{D}([0, T], \mathscr{M}_1^+))$.

Since the jump times of the processes $\bar{A}_N$ are isolated, we have

$$\sup_{t \le T} \|Y_t - Y_{t-}\|_1 \le \frac{1}{N}, \qquad \mathscr{L}(\bar{A}_N)\text{-a.s.},$$

where $Y_{t-} = \lim_{s \nearrow t} Y_s$. Hence by Theorem 10.2 in [3], Chapter 3,

$$\mathbf{P}_{\bar{\mathscr{L}}^*}\left[\mathscr{C}([0, T], \mathscr{M}_1^+)\right] = 1 \tag{3.9}$$

for any limit $\bar{\mathscr{L}}^*$ of the sequence $\{\mathscr{L}(\bar{A}_N)\colon N \in \mathbf{N}\}$.

B. *Description of the limit and its uniqueness.* Let us now fix some subsequence $\{\mathscr{L}(\bar{A}_{N_k})\colon k \in \mathbf{N}\}$ with

$$\lim_{k \to \infty} \mathscr{L}\left(\bar{A}_{N_k}\right) = \bar{\mathscr{L}}^* \quad \text{in } \mathscr{P}\left(\mathscr{D}([0, T], \mathscr{M}_1^+)\right). \tag{3.10}$$

We define for $\theta \in \mathbf{R}$, $f \in \mathscr{L}_1$,

$$\begin{aligned} e_{\theta, f}^*(t) &= e_{\theta, f}^*(t)(Y) \\ &= \exp\left(-i\theta \int_0^t \left\langle Y_s, \bar{\beta}(s, \cdot\,, Y_s) f(-s) - \bar{\gamma}(s, \cdot\,, Y_s) f \right\rangle ds\right) \end{aligned}$$

and

$$H_{\theta, f}^*(t) = H_{\theta, f}^*(t)(Y) = \exp(i\theta\langle Y_t, f\rangle) e_{\theta, f}^*(t)(Y).$$

LEMMA 3.1. *$H_{\theta, f}^*$ is a martingale with respect to $\bar{\mathscr{L}}^*$ for any $\theta \in \mathbf{R}$ and $f \in \mathscr{L}_1$.*

Proof. Let us first define

$$
\begin{aligned}
H^*_{N,\theta,f}(t) &= H^*_{N,\theta,f}(t)(Y)\\
&= \exp(i\theta\langle Y_t, f\rangle)\\
&\quad - \int_0^t \exp(i\theta\langle Y_s, f\rangle)\left(N\left(\exp\left(i\theta\frac{f(-s)}{N}\right) - 1\right)\langle Y_s, \bar\beta(s,\cdot\,,Y_s)\rangle\right.\\
&\qquad\qquad \left. + \left\langle Y_s, \bar\gamma(s,\cdot\,,Y_s) N\left(\exp\left(-i\theta\frac{f(\cdot)}{N}\right) - 1\right)\right\rangle\right) ds.
\end{aligned}
$$

Itô's formula implies

(3.11) $\qquad H^*_{N,\theta,f}$ is a martingale with respect to $\mathscr{L}(\bar A_N)$.

Next, let

$$
\begin{aligned}
H^{**}_{N,\theta,f}(t) &= H^{**}_{N,\theta,f}(t)(Y)\\
&= H^*_{N,\theta,f}(t)(Y) e^*_{\theta,f}(t)(Y) - \int_0^t H^*_{N,\theta,f}(s)(Y)\frac{d}{ds} e^*_{\theta,f}(s)(Y)\,ds.
\end{aligned}
$$

Theorem 1.2.8 in [14] and (3.11) yield

(3.12) $\qquad H^{**}_{N,\theta,f}$ is a martingale with respect to $\mathscr{L}(\bar A_N)$.

Using integration by parts we easily check

$$
\begin{aligned}
&H^{**}_{N,\theta,f}(t)(Y)\\
&\quad = H^*_{\theta,f}(t)(Y)\\
&\qquad - \int_0^t H^*_{\theta,f}(s)(Y)\left(N\left(\exp\left(i\theta\frac{f(-s)}{N}\right) - 1 - i\theta\frac{f(-s)}{N}\right)\langle Y_s, \bar\beta(s,\cdot\,,Y_s)\rangle\right.\\
&\qquad\qquad \left. + \left\langle Y_s, \bar\gamma(s,\cdot\,,Y_s) N\left(\exp\left(-i\theta\frac{f(\cdot)}{N}\right) - 1 + i\theta\frac{f(\cdot)}{N}\right)\right\rangle\right) ds
\end{aligned}
$$

and therefore we obtain by (2.3) and (3.4),

$$
\lim_{N\to\infty} \mathbf{E}_{\mathscr{L}(\bar A_N)}\left[\sup_{t\le T}\left|H^{**}_{N,\theta,f}(t) - H^*_{\theta,f}(t)\right|\right] = 0. \tag{3.13}
$$

By (3.9) the function $Y \to H^*_{\theta,f}(t)(Y)$, t fixed, is continuous on $\mathscr{D}([0,T], \mathscr{M}_1^+)$, $\mathscr{L}^*$-a.s., and we obtain by (3.10) and (3.13) for fixed $t \in [0,T]$ and $\Xi \in C_b(\mathscr{D}([0,T],\mathscr{M}_1^+))$,

$$
\lim_{k\to\infty} \mathbf{E}_{\mathscr{L}(\bar A_{N_k})}\left[H^{**}_{N_k,\theta,f}(t)\Xi\right] = \mathbf{E}_{\mathscr{L}^*}\left[H^*_{\theta,f}(t)\Xi\right].
$$

In particular, if Ξ is $\mathscr{F}_s$-measurable, $s \le t$,

$$\begin{aligned}\mathbf{E}_{\bar{\mathscr{L}}^*}\big[H^*_{\theta,f}(t)\Xi\big] &= \lim_{k\to\infty} \mathbf{E}_{\mathscr{L}(\bar{A}_{N_k})}\big[H^{**}_{N_k,\theta,f}(t)\Xi\big]\\ &= \lim_{k\to\infty} \mathbf{E}_{\mathscr{L}(\bar{A}_{N_k})}\big[H^{**}_{N_k,\theta,f}(s)\Xi\big] \quad \text{[by (3.12)]}\\ &= \mathbf{E}_{\bar{\mathscr{L}}^*}\big[H^*_{\theta,f}(s)\Xi\big].\end{aligned}$$

This is the desired martingale property of $H^*_{\theta,f}$. □

By Lemma 3.1 and (2.5) we obtain for fixed $t \in [0,T]$, $f \in \mathscr{L}_1$,

$$\begin{aligned}&\mathbf{E}_{\bar{\mathscr{L}}^*}\left[\exp\left(i\theta\left(\langle Y_t, f\rangle - \int_0^t \big\langle Y_s, \bar{\beta}(s,\cdot,Y_s)f(-s) - \bar{\gamma}(s,\cdot,Y_s)f(\cdot)\big\rangle\, ds\right)\right)\right]\\ &\quad= \mathbf{E}_{\bar{\mathscr{L}}^*}[\exp(i\theta\langle Y_0, f\rangle)]\\ &\quad= \exp(i\theta\langle A_0, f\rangle), \qquad \theta \in \mathbf{R}.\end{aligned}$$

Therefore

$$\begin{aligned}&\langle Y_t, f\rangle - \int_0^t \big\langle Y_s, \bar{\beta}(s,\cdot,Y_s)f(-s) - \bar{\gamma}(s,\cdot,Y_s)f(\cdot)\big\rangle\, ds = \langle A_0, f\rangle,\\ &\qquad t \in \mathscr{T},\ f \in \mathscr{Q},\ \bar{\mathscr{L}}^*\text{-a.s. for countable sets } \mathscr{T} \subseteq [0,T],\ \mathscr{Q} \subseteq \mathscr{L}_1.\end{aligned} \tag{3.14}$$

By (3.9) we can replace $\mathscr{T}$ and $\mathscr{Q}$ in (3.14) by $[0,T]$ and the separable space $\mathscr{L}_1^c$. The functions $Y \to \sup_{t\le T}\|Y_t\|_1^{[n]}$, $n \ge 0$, are continuous in $\mathscr{D}([0,T], \mathscr{M}_1^+)$. Therefore (3.6) and (3.10) imply for any $\varepsilon > 0$ the existence of a sequence n'_r, $r \in \mathbf{N}$, in $\mathbf{N}$ such that

$$\mathbf{P}_{\bar{\mathscr{L}}^*}\left[\sup_{t\le T}\|Y_t\|_1^{[n'_r]} \ge 2^{-r} \text{ for some } r = 1,2,\dots\right] < \varepsilon.$$

For this reason $\bar{\mathscr{L}}^*$ is uniquely determined by the distribution of the processes $t \to \langle Y_t, f\rangle$, $f \in \mathscr{L}_1^c$, and therefore we even may replace in (3.14) $\mathscr{Q}$ by $\mathscr{L}_1$.

For the unique characterization of $\bar{\mathscr{L}}^*$ we now need

LEMMA 3.2. *The integral equation*

$$\begin{aligned}&\langle \bar{A}_t, f\rangle = \langle A_0, f\rangle + \int_0^t \big\langle \bar{A}_s, \bar{\beta}(s,\cdot,\bar{A}_s)f(-s) - \bar{\gamma}(s,\cdot,\bar{A}_s)f(\cdot)\big\rangle\, ds,\\ &\qquad f \in \mathscr{L}_1,\ t \in [0,T]\end{aligned} \tag{3.15}$$

has a unique solution $\bar{A} = (\bar{A}_t)_{0\le t\le T} \in \mathscr{C}([0,T], \mathscr{M}_1^+)$.

PROOF. The existence of a solution is settled by (3.9) and (3.14) and the subsequent considerations. Any possible limit $\bar{\mathscr{L}}^*$ is concentrated in the set of solutions of (3.15) in $\mathscr{C}([0,T], \mathscr{M}_1^+)$. Since the functions $\bar{\beta}$ and $\bar{\gamma}$ are uniformly bounded, we obtain at once $\sup_{t\le T}\|\bar{A}_t\|_0 < \infty$ for any solution $\bar{A}$. Next, suppose that both $\bar{A}^1$ and $\bar{A}^2$ solve (3.15). Then by (2.3) and (2.4),

$$\begin{aligned}\langle \bar{A}^1_t - \bar{A}^2_t, f \rangle = \int_0^t \Big(f(-s)\Big(\big\langle \bar{A}^1_s - \bar{A}^2_s, \bar{\beta}(s,\cdot,\bar{A}^1_s)\big\rangle \\ + \big\langle \bar{A}^2_s, \bar{\beta}(s,\cdot,\bar{A}^1_s) - \bar{\beta}(s,\cdot,\bar{A}^2_s)\big\rangle\Big) \\ - \big\langle \bar{A}^1_s - \bar{A}^2_s, \bar{\gamma}(s,\cdot,\bar{A}^1_s) f\big\rangle \\ - \big\langle \bar{A}^2_s, \big(\bar{\gamma}(s,\cdot,\bar{A}^1_s) - \bar{\gamma}(s,\cdot,\bar{A}^2_s)\big) f\big\rangle\Big)\, ds \\ \leq C \int_0^t \|\bar{A}^1_s - \bar{A}^2_s\|_1\, ds, \quad \text{uniformly in } f \in \mathscr{L}_1 \text{ with } \|f\|^{(1)} \leq 1.\end{aligned}$$

Taking the supremum over such f 's and applying Gronwall's lemma finishes the proof of Lemma 3.2. □

To complete the proof of Theorem 1 we only have to note that in $\mathscr{C}([0, T], \mathscr{M}_1^+)$ solutions $A = A_t$, $t \leq T$, of (2.8) and solutions $\bar{A}$ of (3.15) are in a one-to-one correspondence through the relation $A_t = \bar{A}_t * \delta_t$. As a consequence of the occurrence of the term f' in (2.8) the space $C_b^1(\mathbf{R}^d)$ of allowed test functions in (2.8) is slightly smaller than that in (3.15), which is $\mathscr{L}_1$.

4. Proof of Theorem 2. We shall proceed in this section along quite similar lines as in the proof of Theorem 1, although the lines are now sometimes slightly more lengthy. In particular, it will be convenient to study the *time-shifted* fluctuation process $\bar{F}_N = \bar{F}_N(t)$, $0 \leq t \leq T$, where $\bar{F}_N(t) = \sqrt{N}(\bar{A}_N(t) - \bar{A}_t)$ for the unique solution $\bar{A}_t = A_t * \delta_{-t}$ of (3.15). Furthermore, the use of martingales will be decisive for this proof. First, in the derivation of (4.5), which is the core of the proof of the relative compactness of the sequence $\{\mathscr{L}(\bar{F}_N)\colon N \in \mathbf{N}\}$, we apply the martingale property of $P_\gamma^{\#}(w) - w$ and $(P_\gamma^{\#}(w) - w)^2 - w$ to control the fluctuations of $\bar{F}_N$ within finite intervals. Next, having established in Lemma 4.2 the martingale property of $H^{\#}_{\theta, f}$ the identification of the limit process is almost automatic, cf. (4.26).

By (3.4) we are allowed to replace (2.3) by

$$\text{(2.3}') \qquad \sup\left\{ \|\beta(\cdot,\mu)\|^{(1)} + \|\gamma(\cdot,\mu)\|^{(1)}\colon \mu \in \mathscr{M}_1^+ \right\} < \infty$$

to gain some notational and computational simplicity.

A. *The relative compactness of the sequence* $\{\mathscr{L}(\bar{F}_N)\colon N \in \mathbf{N}\}$ *in* $\mathscr{P}(\mathscr{D}([0, T], \mathscr{M}_2))$. Our aim is the derivation of suitable estimates involving the norms $\|\cdot\|_1$ and $\|\cdot\|_1^{\#}$ such that we can apply our knowledge on compact subsets of $\mathscr{M}_2$ (Lemma 2.1) and the criterion for relative compactness of the distributions of $\mathscr{M}_2$-valued processes (Proposition 2.2).

Fundamental for our calculations is the following analogue of (3.2).

LEMMA 4.1. *For any* $N \in \mathbf{N}$, $0 \le s \le t \le T$, $-\infty < k-1 \le x \le k < \infty$,

$$\sqrt{N}\,\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\Big[\sup_{s \le u \le t} \big|\big(Y_u - \bar{A}_u - (Y_s - \bar{A}_s)\big)((x,k])\big| \,\Big|\, \mathscr{F}_s\Big]$$

$$\le C\Bigg(\bigg(\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\Big[\int_s^t Y_v((k-1,k])\,dv \,\Big|\, \mathscr{F}_s\Big]\bigg)^{1/2}$$

$$(4.1)\qquad + \sqrt{N}\,\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\Big[\int_s^t \Big(\big\|\mathbf{1}_{(x,k]}\big(Y_v - \bar{A}_v\big)\big\|_1 + \|Y_v - \bar{A}_v\|_1 \bar{A}_v((k-1,k])\Big)\,dv \,\Big|\, \mathscr{F}_s\Big]$$

$$+ \chi_k\bigg(\Big(\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\Big[\int_s^t \|Y_v\|_0\,dv \,\Big|\, \mathscr{F}_s\Big]\Big)^{1/2} + \sqrt{N}\,\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\Big[\int_s^t \|Y_v - \bar{A}_v\|_1\,dv \,\Big|\, \mathscr{F}_s\Big]\bigg)\Bigg),$$

where $\chi_k = 1$ *if* $(k-1,k] \cap [-T,0] \ne \varnothing$ *and* $\chi_k = 0$ *elsewhere.*

PROOF. Let us take for simplicity $-1 \le x \le k = 0$. We can define for $\mathscr{L}(\bar{A}_N)$-almost all Y the functions

$$u \to B_N(x,s,u) = B_N(x,s,u,Y)$$
$$= \text{number of individuals that are born during the time interval } (s,u] \cap [0,-x), \qquad s \le u \le T$$

and

$$u \to D_N(x,s,u) = D_N(x,s,u,Y)$$
$$= \text{number of individuals that are born during } [0,u] \cap [0,-x) \text{ and die in } (s,u], \qquad s \le u \le T.$$

$B_N(x,s,u)$, respectively, $D_N(x,s,u)$, describe the gain, respectively, the loss, of $\bar{A}_N((x,0])$ during $(s,u]$, i.e.,

$$(4.2)\qquad Y_u((x,0]) = \frac{1}{N}\big(B_N(x,s,u,Y) - D_N(x,s,u,Y)\big) + Y_s((x,0]),$$
$$s \le u \le T,\ \mathscr{L}(\bar{A}_N)\text{-a.s.}$$

By the definition of $\bar{A}_N$ the processes $B_N(x,s,\cdot\,)$, respectively, $D_N(x,s,\cdot\,)$, are distributed like

$$u \to P_\beta^{\#}\bigg(N\int_s^{s \vee ((-x)\wedge u)} \big\langle Y_v, \bar{\beta}(v,\cdot\,,Y_v)\big\rangle\,dv\bigg),$$

respectively,

$$u \to P_\gamma^{\#}\bigg(N\int_s^u \big\langle Y_v, \bar{\gamma}(v,\cdot\,,Y_v)\mathbf{1}_{(x,0]}\big\rangle\,dv\bigg),$$

where $P_\beta^{\#}$ and $P_\gamma^{\#}$ are Poisson processes with intensity 1. Similarly, we have by (3.15) for the deterministic limit

$$(4.3)\qquad \bar{A}_u((x,0]) = B(x,s,u) - D(x,s,u) + \bar{A}_s((x,0]),$$

with

$$B(x,s,u) = \int_s^{s \vee ((-x)\wedge u)} \big\langle \bar{A}_v, \bar{\beta}(v,\cdot\,,\bar{A}_v)\big\rangle\,dv,$$

respectively,

$$D(x,s,u)=\int_s^u\left\langle \bar{A}_v,\bar{\gamma}(v,\cdot,\bar{A}_v)\mathbf{1}_{(x,0]}\right\rangle dv.$$

(4.2) and (4.3) imply

$$\begin{aligned}
&\sqrt{N}\,\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\sup_{s\le u\le t}\left|\left(Y_u-\bar{A}_u-(Y_s-\bar{A}_s)\right)((x,0])\right|\Big|\mathscr{F}_s\right]\\
(4.4)\qquad &\le\frac{1}{\sqrt{N}}\Bigg(\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\sup_{s\le u\le t}\left|B_N(x,s,u)-NB(x,s,u)\right|\Big|\mathscr{F}_s\right]\\
&\qquad+\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\sup_{s\le u\le t}\left|D_N(x,s,u)-ND(x,s,u)\right|\Big|\mathscr{F}_s\right]\Bigg).
\end{aligned}$$

With the above representation of D_N by the Poisson process $P_\gamma^{\#}$ we obtain for the second term on the right side of (4.4),

$$\begin{aligned}
&\frac{1}{\sqrt{N}}\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\sup_{s\le u\le t}\left|D_N(x,s,u)-ND(x,s,u)\right|\Big|\mathscr{F}_s\right]\\
&\le\frac{1}{\sqrt{N}}\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\Bigg[\sup_{s\le u\le t}\Bigg|P_\gamma^{\#}\left(N\int_s^u\left\langle Y_v,\bar{\gamma}(v,\cdot,Y_v)\mathbf{1}_{(x,0]}\right\rangle dv\right)\\
&\qquad\qquad-N\int_s^u\left\langle Y_v,\bar{\gamma}(v,\cdot,Y_v)\mathbf{1}_{(x,0]}\right\rangle dv\Bigg|\Bigg|\mathscr{F}_s\Bigg]\\
&\quad+\sqrt{N}\,\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\int_s^t\left|\left\langle Y_v,\bar{\gamma}(v,\cdot,Y_v)\mathbf{1}_{(x,0]}\right\rangle-\left\langle \bar{A}_v,\bar{\gamma}(v,\cdot,\bar{A}_v)\mathbf{1}_{(x,0]}\right\rangle\right|dv\Big|\mathscr{F}_s\right]\\
&\le\frac{2}{\sqrt{N}}\left(\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[N\int_s^t\left\langle Y_v,\bar{\gamma}(v,\cdot,Y_v)\mathbf{1}_{(x,0]}\right\rangle dv\Big|\mathscr{F}_s\right]\right)^{1/2}\\
(4.5)\qquad&\quad+\sqrt{N}\,\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\Bigg[\int_s^t\Big(\left|\left\langle Y_v-\bar{A}_v,\bar{\gamma}(v,\cdot,Y_v)\mathbf{1}_{(x,0]}\right\rangle\right|\\
&\quad+\left\langle \bar{A}_v,\left|\bar{\gamma}(v,\cdot,Y_v)-\bar{\gamma}(v,\cdot,\bar{A}_v)\right|\mathbf{1}_{(x,0]}\right\rangle\Big)dv\Big|\mathscr{F}_s\Bigg]
\end{aligned}$$

[since $w\to P_\gamma^{\#}(w)-w$ and $w\to\left(P_\gamma^{\#}(w)-w\right)^2-w$ are martingales and by Doob's inequality]

$$\begin{aligned}
&\le C\Bigg(\left(\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\int_s^t Y_v((-1,0])\,dv\Big|\mathscr{F}_s\right]\right)^{1/2}\\
&\quad+\sqrt{N}\,\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\int_s^t\left(\left\|\mathbf{1}_{(\ x,0]}(Yv-\bar{A}_v)\right\|_1+\|Y_v-\bar{A}_v\|_1\bar{A}_v((-1,08])\right)dv\Big|\mathscr{F}_s\right]\Bigg)
\end{aligned}$$

[by (2.4), (2.3′)].

In quite the same way we obtain for the first summand in (4.4),

$$
\begin{aligned}
&\frac{1}{\sqrt{N}}\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\sup_{s\le u\le t}\left|B_N(x,s,u)-NB(x,s,u)\right|\,\middle|\,\mathscr{F}_s\right]\\
&\quad\le C\left(\left(\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\int_s^t\|Y_v\|_0\,dv\,\middle|\,\mathscr{F}_s\right]\right)^{1/2}\right.\\
&\qquad\left.+\sqrt{N}\,\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\int_s^t\|Y_v-\bar{A}_v\|_1\left(1+\|\bar{A}_v\|_0\right)dv\,\middle|\,\mathscr{F}_s\right]\right).
\end{aligned}
\tag{4.6}
$$

Estimates like (4.5) and (4.6) can be proved for any interval $(x,k]$, $-\infty < k-1 \le x \le k < \infty$. The proof of (4.1) is finished now, since contributions like (4.6) coming from the birth of individuals can occur only if $(x,k]\cap[-T,0]\neq\varnothing$. □

Since outside $[-T,0]$ individuals only can die during $[0,T]$, cf. (3.5), we immediately obtain by (2.19), (3.4) and Theorem 1,

$$
\sup_{N\in\mathbf{N}}\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\sup_{t\le T}\langle Y_t,\varphi_2\rangle\right]+\sup_{t\le T}\langle\bar{A}_t,\varphi_2\rangle<\infty,\quad\text{where } \varphi_2(x)=1+x^2.
\tag{4.7}
$$

Next, we have for any $\mu\in\mathscr{M}_0^+$ and $m=0,1,\dots,$

$$
\begin{aligned}
\sum_{|k|\ge m}\left\langle\mu,\mathbf{1}_{(k-1,k]}\right\rangle^{1/2}&\le C\sum_{|k|\ge m}\frac{1}{|k|+1}\left\langle\mu,\mathbf{1}_{(k-1,k]}\varphi_2\right\rangle^{1/2}\\
&\le C\frac{1}{\sqrt{m+1}}\langle\mu,\varphi_2\rangle^{1/2}.
\end{aligned}
\tag{4.8}
$$

(2.2), (4.1), (4.7) and (4.8) yield

$$
\begin{aligned}
&\sqrt{N}\,\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\sup_{s\le u\le t}\left\|Y_u-\bar{A}_u-\left(Y_s-\bar{A}_s\right)\right\|_1^{\#}\,\middle|\,\mathscr{F}_s\right]\\
&\le C\left(\sum_{k=-\infty}^{\infty}\left(\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\int_s^t Y_v((k-1,k])\,dv\,\middle|\,\mathscr{F}_s\right]\right)^{1/2}\right.\\
&\quad+\sum_{k=-\infty}^{\infty}\int_{k-1}^{k}\sqrt{N}\,\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\int_s^t\left\|\mathbf{1}_{(x,k]}\left(Y_v-\bar{A}_v\right)\right\|_1dv\,\middle|\,\mathscr{F}_s\right]dx\\
&\quad+\sum_{k=-\infty}^{\infty}\sqrt{N}\,\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\int_s^t\left\|\mathbf{1}_{(k-1,k]}\left(Y_v-\bar{A}_v\right)\right\|_1dv\,\middle|\,\mathscr{F}_s\right]\\
&\quad\left.+\sqrt{N}\,\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\int_s^t\|Y_v-\bar{A}_v\|_1\,dv\,\middle|\,\mathscr{F}_s\right]+\left(\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\int_s^t\|Y_v\|_0\,dv\,\middle|\,\mathscr{F}_s\right]\right)^{1/2}\right)\\
&\le C\left(\sqrt{N}\,\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\int_s^t\|Y_v-\bar{A}_v\|_1^{\#}\,dv\,\middle|\,\mathscr{F}_s\right]+\left(\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\left[\int_s^t\langle Y_v,\varphi_2\rangle\,dv\,\middle|\,\mathscr{F}_s\right]\right)^{1/2}\right).
\end{aligned}
\tag{4.9}
$$

In particular, we obtain from (4.7) and (4.9) for $0 \le t \le T$,

$$\mathbf{E}_{\mathscr{L}(\bar{F}_N)}\big[\|Y_t\|_1^{\#}\big] \le C + C'\int_0^t \mathbf{E}_{\mathscr{L}(\bar{F}_N)}\big[\|Y_v\|_1^{\#}\big]\,dv + \mathbf{E}_{\mathscr{L}(\bar{F}_N)}\big[\|Y_0\|_1^{\#}\big],$$

i.e., by (2.20) and Gronwall's inequality

$$\sup_{N\in\mathbf{N},\, t\le T} \mathbf{E}_{\mathscr{L}(\bar{F}_N)}\big[\|Y_t\|_1^{\#}\big] < \infty.$$

Now (4.7) and (4.9) yield

$$\sup_{N\in\mathbf{N}} \mathbf{E}_{\mathscr{L}(\bar{F}_N)}\Big[\sup_{t\le T}\|Y_t\|_1^{\#}\Big] < \infty. \tag{4.10}$$

Next, we obtain from (4.1) by summation over $|k| \ge m \ge T + 2$ in the same way as (4.9),

$$\begin{aligned}\sqrt{N}\,&\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\Big[\sup_{u\le t}\|Y_u - \bar{A}_u\|_1^{\#,m}\Big]\\ &\le \sqrt{N}\,\mathbf{E}_{\mathscr{L}(\bar{A}_N)}\big[\|Y_0 - \bar{A}_0\|_1^{\#,m}\big]\\ &\quad + C\Bigg(\bigg(\int_0^t \mathbf{E}_{\mathscr{L}(\bar{A}_N)}[\langle Y_v, \varphi_2\rangle]\,dv\bigg)^{1/2}\frac{1}{\sqrt{m}} + \sqrt{N}\int_0^t \mathbf{E}_{\mathscr{L}(\bar{A}_N)}\big[\|Y_v - \bar{A}_v\|_1^{\#,m}\big]\,dv\\ &\qquad + \sqrt{N}\int_0^t \mathbf{E}_{\mathscr{L}(\bar{A}_N)}\big[\|Y_v - \bar{A}_v\|_1^{\#}\big]\frac{1}{m^2}\langle \bar{A}_v, \varphi_2\rangle\,dv\Bigg),\end{aligned}$$

i.e., by (2.21), (4.7) and (4.10) and again by Gronwall's inequality

$$\lim_{m\to\infty}\sup_{N\in\mathbf{N}} \mathbf{E}_{\mathscr{L}(\bar{F}_N)}\Big[\sup_{u\le T}\|Y_u\|_1^{\#,m}\Big] = 0. \tag{4.11}$$

In the same way as in the derivation of (3.7) we can conclude from (4.10) and (4.11) and Lemma 2.1:

(4.12) For any $\varepsilon > 0$ there exists a compact set $\mathscr{K}_\varepsilon' \subseteq \mathscr{M}_2$ such that

$$\inf_{N\in\mathbf{N}} \mathbf{P}_{\mathscr{L}(\bar{F}_N)}\big[Y_t \in \mathscr{K}_\varepsilon' \text{ for all } t\in[0,T]\big] \ge 1-\varepsilon.$$

Another application of (2.1), (2.2) and (4.9) yields for any $0 \le s \le t \le T$,

$$\begin{aligned}\mathbf{E}_{\mathscr{L}(\bar{F}_N)}\big[\|Y_t - Y_s\|_2|\mathscr{F}_s\big] &\le \mathbf{E}_{\mathscr{L}(\bar{F}_N)}\big[\|Y_t - Y_s\|_1^{\#}|\mathscr{F}_s\big]\\ &\le C\sqrt{t-s}\Bigg(\bigg(\mathbf{E}_{\mathscr{L}(\bar{F}_N)}\bigg[\sup_{u\le T}\bigg\langle \frac{Y_u}{\sqrt{N}} + \bar{A}_u, \varphi_2\bigg\rangle\bigg|\mathscr{F}_s\bigg]\bigg)^{1/2}\\ &\qquad + \mathbf{E}_{\mathscr{L}(\bar{F}_N)}\bigg[\sup_{u\le T}\|Y_u\|_1^{\#}\bigg|\mathscr{F}_s\bigg]\Bigg)\\ &\le \sqrt{t-s}\,\mathbf{E}_{\mathscr{L}(\bar{F}_N)}\big[G^{\#}|\mathscr{F}_s\big],\end{aligned} \tag{4.13}$$

where

$$G^{\#} = G^{\#}(Y) = C \sup_{u \le T} \left(1 + \|Y_u\|_1^{\#} + \left\langle \frac{Y_u}{\sqrt{N}} + \bar{A}_u, \varphi_2 \right\rangle \right).$$

By (4.7), (4.10), (4.12) and (4.13) the assumptions of Proposition 2.2 are satisfied and we obtain the relative compactness of the sequence $\{\mathscr{L}(\bar{F}_N): N \in \mathbf{N}\}$ in $\mathscr{P}(\mathscr{D}([0, T], \mathscr{M}_2))$.

Moreover, by (4.10) and (4.13), any limit $\bar{\mathscr{L}}^{\#}$ of this sequence satisfies

$$\mathbf{P}_{\bar{\mathscr{L}}^{\#}}[\mathscr{D}([0, T], \mathscr{M}_1)] = 1.$$

Since additionally

$$\sup_{t \le T} \|Y_t - Y_{t-}\|_1 \le \frac{1}{\sqrt{N}}, \qquad \mathscr{L}(\bar{F}_N)\text{-a.s.},$$

we even have by Theorem 10.2 in [3], Chapter 3,

$$\mathbf{P}_{\bar{\mathscr{L}}^{\#}}[\mathscr{C}([0, T], \mathscr{M}_1)] = 1. \tag{4.14}$$

B. *Description of the limit and its uniqueness.* Let us introduce the additional notation

$$\bar{\beta}_1(s, a, x, \mu) = \beta_1(a + s, x + s, \mu * \delta_s), \qquad \bar{\beta}_1(s, a, \mu; \nu) = \langle \nu, \bar{\beta}_1(s, a, \cdot\,, \mu) \rangle,$$

$$\bar{\gamma}_1(s, a, x, \mu) = \gamma_1(a + s, x + s, \mu * \delta_s), \qquad \bar{\gamma}_1(s, a, \mu; \nu) = \langle \nu, \bar{\gamma}_1(s, a, \cdot\,, \mu) \rangle.$$

We now fix some subsequence $\{\mathscr{L}(\bar{F}_{N_k}): k \in \mathbf{N}\}$ satisfying

$$\lim_{k \to \infty} \mathscr{L}(\bar{F}_{N_k}) = \bar{\mathscr{L}}^{\#} \quad \text{in } \mathscr{P}(\mathscr{D}([0, T], \mathscr{M}_2)). \tag{4.15}$$

Our aim is the derivation of an analogue of Lemma 3.1. Therefore we define for any $\theta \in \mathbf{R}$ and

$$f \in \mathscr{L}_{1,T} = \left\{ f \in C_b([0, T] \times \mathbf{R}): \sup_{s \le T} \left\{ \| f(s, \cdot) \|^{(1)} + \left\| \frac{\partial}{\partial s} f(s, \cdot) \right\|^{(1)} \right\} < \infty \right\},$$

the random processes

$$\begin{aligned} e^{\#}_{\theta, f}(t) &= e^{\#}_{\theta, f}(t)(Y) \\ &= \exp\Bigg(-i\theta \int_0^t \bigg(f(s, -s)\big(\langle Y_s, \bar{\beta}(s, \cdot\,, \bar{A}_s)\rangle + \langle \bar{A}_s, \bar{\beta}_1(s, \cdot\,, \bar{A}_s; Y_s)\rangle\big) \\ &\qquad - \langle Y_s, \bar{\gamma}(s, \cdot\,, \bar{A}_s) f(s, \cdot)\rangle - \langle \bar{A}_s, \bar{\gamma}_1(s, \cdot\,, \bar{A}_s; Y_s) f(s, \cdot)\rangle \\ &\qquad + \left\langle Y_s, \frac{\partial}{\partial s} f(s, \cdot) \right\rangle \bigg)\, ds \\ &\qquad + \frac{\theta^2}{2} \int_0^t \langle \bar{A}_s, \bar{\beta}(s, \cdot\,, \bar{A}_s) f(s, -s)^2 + \bar{\gamma}(s, \cdot\,, \bar{A}_s) f(s, \cdot)^2 \rangle\, ds \Bigg), \end{aligned}$$

and

$$H^{\#}_{\theta,f}(t) = H^{\#}_{\theta,f}(t)(Y) = \exp\bigl(i\theta\langle Y_t, f(t,\cdot)\rangle\bigr)e^{\#}_{\theta,f}(t)(Y).$$

LEMMA 4.2. *$H^{\#}_{\theta,f}$ is a martingale with respect to $\bar{\mathscr{L}}^{\#}$ for any $\theta \in \mathbf{R}$ and $f \in \mathscr{L}_{1,T}$.*

PROOF. For $K > 0$ let $T_K = T_K(Y) = \inf\{t \geq 0: \|Y_t\|_1 \geq K\}$ and $H^{\#,K}_{\theta,f}(t) = H^{\#}_{\theta,f}(t \wedge T_K)$, $t \geq 0$. (4.14) implies $\lim_{K\to\infty} \mathbf{P}_{\mathscr{L}^{\#}}[T_K \leq T] = 0$ and therefore, since $H^{\#}_{\theta,f}(t)$, $t \leq T$, is uniformly bounded,

$$\lim_{K\to\infty} \mathbf{E}_{\mathscr{L}^{\#}}\left[\sup_{t\leq T}\left|H^{\#,K}_{\theta,f}(t) - H^{\#}_{\theta,f}(t)\right|\right] = 0.$$

Hence, we only have to prove for any fixed $K > 0$ the martingale property of the process $H^{\#,K}_{\theta,f}$. Let

$$\begin{aligned}
H^{\#}_{N,\theta,f}(t) &= H^{\#}_{N,\theta,f}(t)(Y)\\
&= \exp\bigl(i\theta\langle Y_t, f(t,\cdot)\rangle\bigr) - \int_0^t \exp\bigl(i\theta\langle Y_s, f(s,\cdot)\rangle\bigr)\\
&\quad\times\left(\left(\exp\left(i\theta\frac{f(s,-s)}{\sqrt{N}}\right) - 1\right)N\left\langle\frac{Y_s}{\sqrt{N}} + \bar{A}_s, \bar{\beta}\left(s,\cdot,\frac{Y_s}{\sqrt{N}} + \bar{A}_s\right)\right\rangle\right.\\
&\quad + N\left\langle\frac{Y_s}{\sqrt{N}} + \bar{A}_s, \bar{\gamma}\left(s,\cdot,\frac{Y_s}{\sqrt{N}} + \bar{A}_s\right)\left(\exp\left(-i\theta\frac{f(s,\cdot)}{\sqrt{N}}\right) - 1\right)\right\rangle\\
&\quad - i\theta\sqrt{N}\,\langle\bar{A}_s, \bar{\beta}(s,\cdot,\bar{A}_s)f(s,-s) - \bar{\gamma}(s,\cdot,\bar{A}_s)f(s,\cdot)\rangle\\
&\quad\left. + i\theta\left\langle Y_s, \frac{\partial}{\partial s}f(s,\cdot)\right\rangle\right)ds.
\end{aligned}$$

Itô's lemma and (3.15) imply

(4.16) $\qquad H^{\#}_{N,\theta,f}$ is a martingale with respect to $\mathscr{L}(\bar{F}_N)$.

Next, we define

$$\begin{aligned}
H^{\#\#}_{N,\theta,f}(t) &= H^{\#\#}_{N,\theta,f}(t)(Y)\\
&= H^{\#}_{N,\theta,f}(t)(Y)e^{\#}_{\theta,f}(t)(Y) - \int_0^t H^{\#}_{N,\theta,f}(s)(Y)\frac{d}{ds}e^{\#}_{\theta,f}(s)(Y)\,ds.
\end{aligned}$$

Theorem 1.2.8 in [14] and (4.16) yield

(4.17) $\qquad H^{\#\#}_{N,\theta,f}$ is a martingale with respect to $\mathscr{L}(\bar{F}_N)$.

Using integration by parts we easily check

$$
\begin{aligned}
&H^{\#\#}_{N,\theta,f}(t)(Y)\\
&\quad = H^{\#}_{\theta,f}(t)(Y) - \int_0^t H^{\#}_{\theta,f}(s)(Y)\\
&\quad\times\Bigg(N\Bigg(\left(\exp\left(i\theta\frac{f(s,-s)}{\sqrt{N}}\right)-1\right)\left\langle\frac{Y_s}{\sqrt{N}}+\bar{A}_s,\bar{\beta}\left(s,\cdot,\frac{Y_s}{\sqrt{N}}+\bar{A}_s\right)\right\rangle\\
&\qquad+\left\langle\frac{Y_s}{\sqrt{N}}+\bar{A}_s,\bar{\gamma}\left(s,\cdot,\frac{Y_s}{\sqrt{N}}+\bar{A}_s\right)\left(\exp\left(-i\theta\frac{f(s,\cdot)}{\sqrt{N}}\right)-1\right)\right\rangle\Bigg)\\
&\qquad-i\theta\sqrt{N}\left\langle\bar{A}_s,\bar{\beta}(s,\cdot,\bar{A}_s)f(s,-s)-\bar{\gamma}(s,\cdot,\bar{A}_s)f(s,\cdot)\right\rangle\\
&\qquad+i\theta\left\langle Y_s,\frac{\partial}{\partial s}f(s,\cdot)\right\rangle\\
&\qquad-i\theta\Big(f(s,-s)\big(\langle Y_s,\bar{\beta}(s,\cdot,\bar{A}_s)\rangle+\langle\bar{A}_s,\bar{\beta}_1(s,\cdot,\bar{A}_s;Y_s)\rangle\big)\\
&\qquad\qquad-\langle Y_s,\bar{\gamma}(s,\cdot,\bar{A}_s)f(s,\cdot)\rangle-\langle\bar{A}_s,\bar{\gamma}_1(s,\cdot,\bar{A}_s;Y_s)f(s,\cdot)\rangle\\
&\qquad\qquad+\left\langle Y_s,\frac{\partial}{\partial s}f(s,\cdot)\right\rangle\Big)\\
&\qquad+\frac{\theta^2}{2}\left\langle\bar{A}_s,\bar{\beta}(s,\cdot,\bar{A}_s)f(s,-s)^2+\bar{\gamma}(s,\cdot,\bar{A}_s)f(s,\cdot)^2\right\rangle\Bigg)\,ds.
\end{aligned}
\tag{4.18}
$$

Next we try to find an analogue of (3.13). For that we have to investigate on the right side of (4.18) the integrand with respect to s-integration. This integrand equals

$$
\begin{aligned}
&H^{\#}_{\theta,f}(s)(Y)\Bigg(\Bigg(\sqrt{N}i\theta f(s,-s)-\frac{\theta^2}{2}f(s,-s)^2+N\sum_{k=3}^{\infty}\left(\frac{i\theta}{\sqrt{N}}f(s,-s)\right)^k\frac{1}{k!}\Bigg)\\
&\qquad\times\left\langle\frac{Y_s}{\sqrt{N}}+\bar{A}_s,\bar{\beta}\left(s,\cdot,\frac{Y_s}{\sqrt{N}}+\bar{A}_s\right)\right\rangle\\
&\qquad+\Bigg\langle\frac{Y_s}{\sqrt{N}}+\bar{A}_s,\bar{\gamma}\left(s,\cdot,\frac{Y_s}{\sqrt{N}}+\bar{A}_s\right)\\
&\qquad\qquad\times\left(-\sqrt{N}i\theta f(s,\cdot)-\frac{\theta^2}{2}f(s,\cdot)^2+N\sum_{k=3}^{\infty}\left(\frac{-i\theta}{\sqrt{N}}f(s,\cdot)\right)^k\frac{1}{k!}\right)\Bigg\rangle\\
&\qquad-i\theta\sqrt{N}\left\langle\bar{A}_s,\bar{\beta}(s,\cdot,\bar{A}_s)f(s,-s)-\bar{\gamma}(s,\cdot,\bar{A}_s)f(s,\cdot)\right\rangle\\
&\qquad-i\theta\Big(f(s,-s)\big(\langle Y_s,\bar{\beta}(s,\cdot,\bar{A}_s)\rangle+\langle\bar{A}_s,\bar{\beta}_1(s,\cdot,\bar{A}_s;Y_s)\rangle\big)\\
&\qquad\qquad-\langle Y_s,\bar{\gamma}(s,\cdot,\bar{A}_s)f(s,\cdot)\rangle-\langle\bar{A}_s,\bar{\gamma}_1(s,\cdot,\bar{A}_s;Y_s)f(s,\cdot)\rangle\Big)\\
&\qquad+\frac{\theta^2}{2}\left\langle\bar{A}_s,\bar{\beta}(s,\cdot,\bar{A}_s)f(s,-s)^2+\bar{\gamma}(s,\cdot,\bar{A}_s)f(s,\cdot)^2\right\rangle\Bigg).
\end{aligned}
$$

Using the linearity $\bar\beta_1(\cdots;a\mu)=a\bar\beta_1(\cdots;\mu)$, $\bar\gamma_1(\cdots;a\mu)=a\bar\gamma_1(\cdots;\mu)$, $a\in\mathbf{R}$, $\mu\in\mathscr{M}_1$, we observe that some terms in this sum cancel, whereas the sum of the absolute values of the remaining terms is less than

$$
\begin{aligned}
C\Bigg(&\left|\theta f(s,-s)\left\langle Y_s,\bar\beta\left(s,\cdot,\frac{Y_s}{\sqrt N}+\bar A_s\right)-\bar\beta(s,\cdot,\bar A_s)\right\rangle\right|\\
&+\sqrt N\left|\theta f(s,-s)\left\langle \bar A_s,\bar\beta\left(s,\cdot,\frac{Y_s}{\sqrt N}+\bar A_s\right)-\bar\beta(s,\cdot,\bar A_s)\right.\right.\\
&\qquad\left.\left.-\bar\beta_1\left(s,\cdot,\bar A_s;\frac{Y_s}{\sqrt N}\right)\right\rangle\right|\\
&+\frac{\theta^2}{2}f(s,-s)^2\left|\left\langle\frac{Y_s}{\sqrt N},\bar\beta\left(s,\cdot,\frac{Y_s}{\sqrt N}+\bar A_s\right)\right\rangle\right|\\
&+\frac{\theta^2}{2}f(s,-s)^2\left|\left\langle\bar A_s,\bar\beta\left(s,\cdot,\frac{Y_s}{\sqrt N}+\bar A_s\right)-\bar\beta(s,\cdot,\bar A_s)\right\rangle\right|\\
&+N\left|\sum_{k=3}^{\infty}\left(\frac{i\theta}{\sqrt N}f(s,-s)\right)^k\frac{1}{k!}\left\langle\frac{Y_s}{\sqrt N}+\bar A_s,\bar\beta\left(s,\cdot,\frac{Y_s}{\sqrt N}+\bar A_s\right)\right\rangle\right|\\
&+\left|\theta\left\langle Y_s,\left(\bar\gamma\left(s,\cdot,\frac{Y_s}{\sqrt N}+\bar A_s\right)-\bar\gamma(s,\cdot,\bar A_s)\right)f(s,\cdot)\right\rangle\right|\\
&+\sqrt N\left|\theta\left\langle\bar A_s,\left(\bar\gamma\left(s,\cdot,\frac{Y_s}{\sqrt N}+\bar A_s\right)-\bar\gamma(s,\cdot,\bar A_s)\right.\right.\right.\\
&\qquad\left.\left.\left.-\bar\gamma_1\left(s,\cdot,\bar A_s;\frac{Y_s}{\sqrt N}\right)\right)f(s,\cdot)\right\rangle\right|\\
&+\frac{\theta^2}{2}\left|\left\langle\frac{Y_s}{\sqrt N},\bar\gamma\left(s,\cdot,\frac{Y_s}{\sqrt N}+\bar A_s\right)f(s,\cdot)^2\right\rangle\right|\\
&+\frac{\theta^2}{2}\left|\left\langle\bar A_s,\left(\bar\gamma\left(s,\cdot,\frac{Y_s}{\sqrt N}+\bar A_s\right)-\bar\gamma(s,\cdot,\bar A_s)\right)f(s,\cdot)^2\right\rangle\right|\\
&+N\left|\left\langle\frac{Y_s}{\sqrt N}+\bar A_s,\bar\gamma\left(s,\cdot,\frac{Y_s}{\sqrt N}+\bar A_s\right)\left(\sum_{k=3}^{\infty}\left(\frac{-i\theta}{\sqrt N}f(s,\cdot)\right)^k\frac{1}{k!}\right)\right\rangle\right|\Bigg)\\
=\ &\sum_{l=1}^{10}Q_{N,l}(s,Y).
\end{aligned}
$$

To obtain the desired analogue of (3.13) it suffices by (4.18) to show

$$\lim_{N\to\infty} \mathbf{E}_{\mathscr{L}(\bar{F}_N)}\left[\sup_{s\le T_K} Q_{N,l}(s,Y)\right] = 0, \qquad l = 1,\dots,10,\ K > 0.$$

First we have by (2.10) and (2.12),

$$Q_{N,1}(s,Y) = C\frac{1}{\sqrt{N}}\left|\theta f(s,-s)\int_0^1 \left\langle Y_s, \bar{\beta}_1\left(s,\cdot\,,\alpha\frac{Y_s}{\sqrt{N}} + \bar{A}_s; Y_s\right)\right\rangle d\alpha\right|$$

$$\le C\frac{\|Y_s\|_1^2}{\sqrt{N}}\left(1 + \left\|\frac{Y_s}{\sqrt{N}}\right\|_1 + \|\bar{A}_s\|_1\right),$$

and therefore

$$\lim_{N\to\infty} \mathbf{E}_{\mathscr{L}(\bar{F}_N)}\left[\sup_{s\le T_K} Q_{N,1}(s,Y)\right] = 0. \tag{4.19}$$

Next we conclude from (2.13),

$$Q_{N,2}(s,Y) = C\left|\theta f(s,-s)\int_0^1 \left\langle \bar{A}_s, \bar{\beta}_1\left(s,\cdot\,,\alpha\frac{Y_s}{\sqrt{N}} + \bar{A}_s; Y_s\right) - \bar{\beta}_1(s,\cdot\,,\bar{A}_s; Y_s)\right\rangle d\alpha\right|$$

$$\le C\|\bar{A}_s\|_0\frac{\|Y_s\|_1^2}{\sqrt{N}}$$

$$\le C\frac{\|Y_s\|_1^2}{\sqrt{N}}.$$

Therefore, as in (4.19),

$$\lim_{N\to\infty} \mathbf{E}_{\mathscr{L}(\bar{F}_N)}\left[\sup_{s\le T_K} Q_{N,2}(s,Y)\right] = 0. \tag{4.20}$$

Finally, (2.4) and (2.3′) imply

$$Q_{N,3}(s,Y) \le C\frac{\|Y_s\|_1}{\sqrt{N}},$$

$$Q_{N,4}(s,Y) \le C\|\bar{A}_s\|_0\frac{\|Y_s\|_1}{\sqrt{N}}$$

and

$$Q_{N,5}(s,Y) \le C\left(\frac{1}{N}\|Y_s\|_1 + \frac{\|\bar{A}_s\|_0}{\sqrt{N}}\right).$$

Hence,

$$\lim_{N\to\infty} \mathbf{E}_{\mathscr{L}(\bar{F}_N)}\left[\sup_{s\le T_K} \left(Q_{N,3}(s,Y) + Q_{N,4}(s,Y) + Q_{N,5}(s,Y)\right)\right] = 0. \tag{4.21}$$

The same arguments that lead to (4.19)–(4.21) yield

$$\lim_{N\to\infty} \mathbf{E}_{\mathscr{L}(\bar{F}_N)}\left[\sup_{s\le T_K}\sum_{l=6}^{10} Q_{N,l}(s,Y)\right] = 0. \tag{4.22}$$

We conclude from (4.18)–(4.22),

$$\lim_{N\to\infty} \mathbf{E}_{\mathscr{L}(\bar{F}_N)}\left[\sup_{t\le T_K}\left|H^{\#\#}_{N,\theta,f}(t\wedge T_K) - H^{\#;K}_{\theta,f}(t)\right|\right] = 0.$$

By (4.14) the arguments following (3.13) in the proof of Lemma 3.1 can be employed once more and imply the martingale property of $H^{\#;K}_{\theta,f}$ and therefore complete the proof of Lemma 4.2. □

We now observe that the time evolution $\bar{S} = \bar{S}^t_s$, $0 \le s \le t \le T$, introduced in (2.16) satisfies

$$\begin{aligned}\frac{\partial}{\partial s}(\bar{S}^t_s f)(x) &= -(\bar{S}^t_s f)(-s)\left(\bar{\beta}(s,x,\bar{A}_s) + \langle \bar{A}_s, \bar{\beta}_1(s,\cdot,x,\bar{A}_s)\rangle\right)\\ &\quad + (\bar{S}^t_s f)(x)\bar{\gamma}(s,x,\bar{A}_s) + \langle \bar{A}_s, \bar{\gamma}_1(s,\cdot,x,\bar{A}_s)(\bar{S}^t_s f)(\cdot)\rangle,\\ &\qquad f\in\mathscr{L}_1,\ 0\le s\le t\le T,\ x\in\mathbf{R}.\end{aligned} \tag{4.23}$$

Hence, by (2.3), (2.11), (2.14) and (2.17),

$$\sup_{0\le s\le t\le T,\ f\in\mathscr{L}_1} \frac{\|(\partial/\partial s)\bar{S}^t_s f\|^{(1)}}{\|f\|^{(1)}} < \infty. \tag{4.24}$$

(2.17) and (4.24) imply

$$(\bar{S}^{\bar{t}}_{\cdot} f)(\cdot)\colon (s,x) \to (\bar{S}^{\bar{t}}_s f)(x) \in \mathscr{L}_{1,\bar{t}}, \qquad 0\le \bar{t}\le T,\ f\in\mathscr{L}_1.$$

Hence, we obtain by (4.23) and Lemma 4.2:

The process

$$\begin{aligned} t \to H^{\#;\bar{t}}_{\theta,f}(t) &= H^{\#;\bar{t}}_{\theta,f}(t)(Y)\\ &= \exp\Bigg(i\theta\langle Y_{t\wedge\bar{t}}, \bar{S}^{\bar{t}}_{t\wedge\bar{t}} f\rangle + \frac{\theta^2}{2}\int_0^{t\wedge\bar{t}}\langle \bar{A}_u, \bar{\beta}(u,\cdot,\bar{A}_u)(\bar{S}^{\bar{t}}_u f)(-u)^2\\ &\qquad + \bar{\gamma}(u,\cdot,\bar{A}_u)(\bar{S}^{\bar{t}}_u f)(\cdot)^2\rangle\,du\Bigg)\end{aligned} \tag{4.25}$$

is a martingale with respect to $\bar{\mathscr{P}}^{\#}$.

In particular, for any $0 \le s \le t \le T$,

$$\begin{aligned}&\mathbf{E}_{\bar{\mathscr{P}}^{\#}}\left[\exp(i\theta\langle Y_t, f\rangle)\,|\,\mathscr{F}_s\right]\\ &\quad = \exp\Bigg(i\theta\langle Y_s, \bar{S}^t_s f\rangle - \frac{\theta^2}{2}\int_s^t\langle \bar{A}_u, \bar{\beta}(u,\cdot,\bar{A}_u)(\bar{S}^t_u f)(-u)^2\\ &\qquad + \bar{\gamma}(u,\cdot,\bar{A}_u)(\bar{S}^t_u f)(\cdot)^2\rangle\,du\Bigg).\end{aligned} \tag{4.26}$$

(2.22) and (4.26) show the uniqueness of $\bar{\mathscr{L}}^{\#}$. Let us now denote by $\mathscr{L}^{\#}$ the probability measure on $\mathscr{D}([0,T],\mathscr{M}_1)$ obtained from $\bar{\mathscr{L}}^{\#}$ by the transformation

$$Z\colon \bar{Y} = (\bar{Y}_t)_{0\le t\le T} \to Y = (\bar{Y}_t * \delta_t)_{0\le t\le T},$$

i.e., we invert the time shift leading from the process F_N to $\bar{F}_N$. Then we obtain for any $f \in \mathscr{L}_1$, $0 \le s \le t \le T$, and any $\mathscr{F}_s$-measurable bounded Ξ: $\mathscr{D}([0,T],\mathscr{M}_1) \to \mathbf{R}$,

$$\begin{aligned}
&\mathbf{E}_{\mathscr{L}^{\#}}[\exp(i\theta\langle Y_t, f\rangle)\Xi(Y)] \\
&\quad = \mathbf{E}_{\bar{\mathscr{L}}^{\#}}\big[\exp\big(i\theta\langle Y_t, f(\cdot+t)\rangle\big)(\Xi\circ Z)(Y)\big] \\
&\quad = \mathbf{E}_{\bar{\mathscr{L}}^{\#}}\Bigg[\exp\Bigg(i\theta\big\langle Y_s, (\bar{S}_s^t f(\cdot+t))(\cdot)\big\rangle \\
&\qquad\qquad - \frac{\theta^2}{2}\int_s^t \Big\langle \bar{A}_u, \bar{\beta}(u,\cdot,\bar{A}_u)\big(\bar{S}_u^t f(\cdot+t)\big)(-u)^2 \\
&\qquad\qquad\qquad + \bar{\gamma}(u,\cdot,\bar{A}_u)\big(\bar{S}_u^t f(\cdot+t)\big)(\cdot)^2\Big\rangle\,du\Bigg)(\Xi\circ Z)(Y)\Bigg] \\
&\qquad\qquad [\text{by (4.26) and since } \Xi\circ Z \text{ is } \mathscr{F}_s\text{-measurable}] \\
&\quad = \mathbf{E}_{\mathscr{L}^{\#}}\Bigg[\exp\Bigg(i\theta\big\langle Y_s, (S_s^t f)(\cdot)\big\rangle \\
&\qquad\qquad - \frac{\theta^2}{2}\int_s^t \Big\langle A_u, \beta(\cdot, A_u)(S_u^t f)(0)^2 \\
&\qquad\qquad\qquad + \gamma(\cdot, A_u)(S_u^t f)(\cdot)^2\Big\rangle\,du\Bigg)\Xi(Y)\Bigg] \\
&\qquad\qquad [\text{by (2.15)}].
\end{aligned}$$

Now, by (2.18), we have characterized the conditional distribution of $\langle Y_t, f\rangle$ given $\mathscr{F}_s$ as Gaussian with mean $\langle Y_s, S_s^t f\rangle$ and variance $\Sigma_{s,t}^2(f)$.

Hence, the proof of Theorem 2 is complete now.

APPENDIX

PROOF OF LEMMA 2.1. (a) (2.1) is obvious. Next, by integration by parts we obtain for any interval $(a,b]$ and $f \in C_b^1(\mathbf{R})$,

$$\langle \mathbf{1}_{(a,b]}\mu, f\rangle = \int_{(a,b]} f(y)\mu(dy) = \int_a^b f'(y)\mu((y,b])\,dy + f(a)\mu((a,b]).$$

Obviously, for $\nu \in \mathscr{M}_0$,

$$\|\nu\|_1 = \sup\{\langle \nu, f\rangle \colon f \in C_b^1(\mathbf{R}), \|f\|^{(1)} \le 1\}$$

and therefore

$$\|\mathbf{1}_{(a,b]}\mu\|_1 \le \int_a^b |\mu((y,b])|\,dy + |\mu((a,b])|.$$

In particular,

$$\begin{aligned}\|\mu\|_1^{\#,l} &\le \sum_{|m|\ge l}\biggl(\int_{m-1}^{m}\int_x^m |\mu((y,m])|\,dy\,dx \\ &\qquad + 2\int_{m-1}^{m}|\mu((x,m])|\,dx + |\mu((m-1,m])|\biggr) \\ &\le 3\sum_{|m|\ge l}\biggl(\int_{m-1}^{m}|\mu((x,m])|\,dx + |\mu((m-1,m])|\biggr).\end{aligned}$$

For the first inequality in (2.2) we observe

$$\begin{aligned}\|\mu\|_1^{[l]} &= \sup\{\langle\mu, f\rangle : f\in\mathscr{L}_1,\ \|f\|^{(1)}\le 1,\ \operatorname{supp}(f)\subseteq \mathbf{R}\setminus(-l,l]\} \\ &\qquad\qquad [\text{by the continuity of } f\in\mathscr{L}_1] \\ &\le \sup\{\langle \mathbf{1}_{\mathbf{R}\setminus(-l,l]}\mu, f\rangle : f\in\mathscr{L}_1,\ \|f\|^{(1)}\le 1\} \\ &\le \sum_{|m|\ge l}\|\mathbf{1}_{(m-1,m]}\mu\|_1 \\ &\le \|\mu\|_1^{\#,l}.\end{aligned}$$

(b) Let $\{f_l: l\in\mathbf{N}\}$ be dense in $\mathscr{L}_k^c$. The first condition implies that any sequence in $\mathscr{K}$ contains some subsequence μ_m, $m\in\mathbf{N}$, such that $\hat{I}(f_l) = \lim_{m\to\infty}\langle\mu_m, f_l\rangle$ exists for any $l\in\mathbf{N}$. By our assumptions $\hat{I}$ can be extended to a continuous linear functional I on $\mathscr{L}_k$. In particular, $\langle\mu_m,\cdot\,\rangle$ converges pointwise to $I(\cdot)$ on $\mathscr{L}_k$, i.e.,

$$\lim_{m\to\infty}\langle\mu_m, f\rangle = I(f),\qquad f\in\mathscr{L}_k. \tag{A.1}$$

Additionally, we obtain

$$\sup\{I(f): f\in\mathscr{L}_k,\ \|f\|^{(k)}\le 1\} < \infty \tag{A.2}$$

and

$$\lim_{n\to\infty}\sup\{I(f): f\in\mathscr{L}_k,\ \|f\|^{(k)}\le 1,\ \operatorname{supp}(f)\subseteq\mathbf{R}\setminus(-n,n)\} = 0. \tag{A.3}$$

To finish the proof of Lemma 2.1, we have to show that the convergence in (A.1) is uniform in $\{f\in\mathscr{L}_{k+1}: \|f\|^{(k+1)}\le 1\}$. We obtain

$$\begin{aligned}&\sup\{\langle\mu_m, f\rangle - I(f): f\in\mathscr{L}_{k+1},\ \|f\|^{(k+1)}\le 1\} \\ &\quad\le \sup\{\langle\mu_m, f\rangle - I(f): f\in\mathscr{L}_{k+1},\ \|f\|^{(k+1)}\le 1, \\ &\qquad\qquad \operatorname{supp}(f)\subseteq\mathbf{R}\setminus(-l+1, l-1)\} \\ &\qquad + \sup\{\langle\mu_m, f\rangle - I(f): f\in\mathscr{L}_{k+1},\ \|f\|^{(k+1)}\le 1, \\ &\qquad\qquad \operatorname{supp}(f)\subseteq[-l,l]\}.\end{aligned} \tag{A.4}$$

Due to our second assumption and (A.3) the first term can be made arbitrarily small uniformly in m by choosing l large enough. Next, by the theorem of Arzela and Ascoli we may find for fixed l and $\varepsilon > 0$ a finite set $g_1, \dots, g_r \in \mathscr{L}_{k+1}$ such that for any $f \in \mathscr{L}_{k+1}$ with $\|f\|^{(k+1)} \le 1$ and $\operatorname{supp}(f) \subseteq [-l, l]$ there exists a g_i such that $\|g_i - f\|^{(k)} \le \varepsilon$. Hence, by (A.1) and (A.2) and the first condition

$$\begin{aligned}
&\lim_{m \to \infty} |\langle \mu_m, f \rangle - I(f)| \\
&\quad \le \lim_{m \to \infty} |\langle \mu_m, f \rangle - \langle \mu_m, g_i \rangle| \\
&\qquad + \lim_{m \to \infty} |\langle \mu_m, g_i \rangle - I(g_i)| + \lim_{m \to \infty} |I(g_i) - I(f)| \\
&\quad \le C\varepsilon, \quad \text{uniformly in } f \in \mathscr{L}_{k+1}, \|f\|^{(k+1)} \le 1, \operatorname{supp}(f) \subseteq [-l, l].
\end{aligned}$$

Therefore for fixed l the second term on the right side of (A.4) tends to 0 as $m \to \infty$. □

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