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CONDITIONED LIMIT THEOREMS OF STOPPED CRITICAL BRANCHING BESSEL PROCESSES

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We consider critical branching Bessel processes initially at $r \gg 1$ and stopped at $r = 1$. Let N be the number of descendants hitting $r = 1$. We give the norming constant $k(r)$ and prove convergence, as $r \to \infty$, of $N/[k(r)]$ conditioned on $\{N > 0\}$.

The distribution of conditioned limit laws is also investigated. A feature of this study is an interplay between probabilistic insights and analytic techniques for Emden–Fowler's equation.

Introduction. Branching d-dimensional Brownian motion is used in [8] to study a geographical problem for a mutant allele. Individuals are assumed to diffuse independently of one another in R^d according to Brownian motions. After an exponential waiting time each individual is replaced by a random number of offspring which will then diffuse away and begin dying and reproducing themselves and so forth (all exponential waiting times, branchings and Brownian motions are independent of one another).

The large x asymptotic behavior of the probability that an allelic type initially at x has some descendant which diffuses within distance $a > 0$ of the origin was determined in [8] under the assumption that the branching is critical with a finite third moment. A natural question then concerns normalized limits of the number of descendants that diffuse within $a > 0$ of the origin, conditioned on some of them doing so. (The sphere of center zero and radius a serves as an absorbing set.)

Our goal is to prove some conditioned limit theorems of this kind and also demonstrate an intimate relation between critical branching processes and some semilinear elliptic differential equations.

We now introduce some notation.

Let $X(t)$ be a Bessel process of index d, $d \in R$, generated by

$$L_d \equiv \frac{1}{2}\left(D^2 + \frac{d-1}{x}D\right), \qquad x > 0,$$

and let the probability that branching does not occur in the time period (τ, s) be $\exp[-\int_\tau^s \sigma(X(t))\,dt]$ where $\sigma(x)$ is a nonnegative continuous function.

Note that the radial part of d-dimensional Brownian motion is a Bessel process. Bessel processes are considered instead of Brownian motions because the influence of d on conditioned limit laws can be seen more clearly and also because it simplifies notation.

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We shall allow the offspring distribution to depend on the location of branching, but always assume it to have mean 1 (criticality) and finite variance. Let the probability of having n offspring when a branching occurs at x be $q_n(x)$, $n = 0, 1, 2, \dots$, which is nonnegative and satisfies

$$q_n(x) \in C(0, \infty),$$

$$\sum_{n=0}^{\infty} q_n(x) = 1,$$

$$\sum_{n=1}^{\infty} n q_n(x) = 1$$

and

$$0 < \sum_{n=2}^{\infty} n(n-1) q_n(x) < \infty$$

uniformly in x. Important for us is $h(x, v)$, the probability generating function (p.g.f.)

$$h(x, v) \equiv \sum_{n=0}^{\infty} q_n(x) v^n, \qquad -1 \le v \le 1.$$

It can be shown that $h(x, v) > v$ and satisfies $\lim_{v \uparrow 1}[h(x, v) - v] / \frac{1}{2}(v-1)^2 < \infty$ by the assumptions on $\{q_n(x)\}$. We define $f(x, u)$ by $h(x, v) - v \equiv \frac{1}{2}(1-v)^2 f(x, 1-v)$ and make a further assumption on $\sigma(x) f(x, u)$,

$$\lim_{x \to \infty} \sigma(x) f(x, 0) = 1 \quad \text{and} \quad \lim_{u \to 0} \|\sigma(\cdot) f(\cdot, u) - \sigma(\cdot) f(\cdot, 0)\|_\infty = 0. \tag{0.1}$$

The second condition is satisfied when $\sum_{n=1}^{\infty} \sigma(x) n(n-1) q_n(x)$ converges uniformly in x. Note that $\lim_{x \to \infty} \sigma(x) f(x, 0) = 1$ instead of a, $a > 0$, is used only for simplicity of presentation and that $f(x, 0) = \sum n(n-1) q_n(x)$ is the variance of the offspring distribution for individuals which branch at x.

Let $x = 1$ be an absorbing point and P_x (E_x, resp.) be the probability distribution (expectation, resp.) associated with a single allele initially at x. Let N denote the number of descendants which diffuse to 1. To study the distribution of N, we focus on its p.g.f.,

$$v(x, \eta) \equiv E_x\{(1-\eta)^N\}, \quad \text{where } 1 > \eta \ge 0.$$

By looking into the events that the individual does not or does die after an infinitesimal time period, $v(x, \eta)$ is known to satisfy

$$\frac{1}{2}\left(D^2 + \frac{d-1}{x} D\right) v + \sigma(x)[h(x, v) - v] = 0,$$

$$v(1, \eta) = 1 - \eta, \qquad v(\infty, \eta) = 1 \quad \text{and} \quad 0 \le v(x, \eta) < 1,$$

where $D = D_x$, the differential w.r.t. x. For the derivation see [8] and references therein. Using f and a new variable $u(x, \eta) = 1 - v(x, \eta) \equiv 1 - E_x\{(1-\eta)^N\}$, this equation transforms into (again, $D = D_x$)

$$-\left(D^2 + \frac{d-1}{x}D\right)u + \sigma(x)f(x,u)u^2 = 0, \tag{0.2}$$

$$u(1,\eta) = \eta, \qquad u(\infty,\eta) = 0 \quad \text{and} \quad u(x,\eta) > 0.$$

If we consider a constant rate critical death and doubling process, $\sigma(x) = 1$ and $q_0(x) = q_2(x) = \frac{1}{2}$, this equation takes a simpler form (again, $D = D_x$),

$$-\left(D^2 + \frac{d-1}{x}D\right)u + u^2 = 0,$$

$$u(1,\eta) = \eta, \qquad u(\infty,\eta) = 0 \quad \text{and} \quad u(x,\eta) > 0.$$

For easy reference we name one differential equation,

$$-\left(D^2 + \frac{d-1}{x}D\right)u + u^2 = 0. \tag{A}$$

In this paper solutions of (A) are always understood to be positive and tend to zero as $x \to \infty$ unless stated otherwise. We shall specify the domain and boundary data when referring to (A). Note $P_x\{N > 0\} = \lim_{\eta \uparrow 1} 1 - E_x\{(1-\eta)^N\} \equiv u(x,1)$. Assuming $\sigma(x)f(x,u) \equiv f(u)$ with $f'(0) < \infty$, it was proved in [8] that, as $x \to \infty$,

$$P_x\{N > 0\} \sim \begin{cases} 2(4-d)x^{-2} & \text{if } d < 4, \\ 2x^{-2}(\log x)^{-1} & \text{if } d = 4, \\ \beta_d x^{2-d} & \text{if } d > 4, \end{cases} \tag{0.3}$$

where β_d is a positive constant depending on d. In Section 1 we prepare some mathematical tools and state a basic result of asymptotic probability. To find appropriate normalization for nondegenerate conditioned limits in Sections 2–6, it is useful to know that $E_x\{N\} = 1$ if $d \le 2$ and $E_x\{N\} = x^{2-d}$ (same as a nonbranching Brownian motion due to criticality). In view that $d = 2$ is the threshold of $E_x\{N\}$ and $d = 4$ is that of $P_x\{N > 0\}$, we expect the norming constants to be $E_x\{N|N > 0\} = E_x\{N\}/P_x\{N > 0\}$, behave somewhat differently in five regimes: $d < 2$, $d = 2$, $2 < d < 4$, $d = 4$ and $d > 4$. We shall prove a conditioned limit theorem for $d < 2$ in Section 2, moving to larger d in Sections 3–6 and omitting details when a previous argument works well for new situations.

When $d \neq 2$, our result is, as expected, that $N/(E_r\{N\}/P_r\{N > 0\})$ converges in distribution to some nondegenerate law Z as $r \to \infty$ conditioned on $\{N > 0\}$. When $d = 2$, we show in Section 3 that $\log r \cdot N/(E_r\{N\}/P_r\{N > 0\})$ converges in distribution to a nondegenerate law Z as $r \to \infty$ conditioned on $\{N > 0\}$. The picture of the branching diffusion process provides this study with a good deal of insight. For instance, when tracing back the path of a descendant found on $r = 1$, a Bessel process anticipates that this descendant is the $G(r)$th generation, $G(r) \sim O(r^2)$ as $r \to \infty$, and distribution of the ancestors' birthplace is like

$H(r)\,dr$, $H(r) \sim O(r)$. In view of

$$E_r\{N\} = P_r\{N = 0\} + P_r\{N > 0\}\int_1^r E_x\{N\}H(x)\,dx$$

and $E_x\{N\} \sim 1$ if $d \leq 2$ and $\sim x^{2-d}$ if $d > 2$, we can foresee (0.3) up to the correct magnitude. This heuristic argument also anticipates that $t = r^2 \to \infty$ (instead of $t = \infty$ in this paper) is also an appropriate scaling. We shall postpone results involving scaling of t to [6]. The results in this paper and in the case when $t = r^2 \to \infty$ are closely related to that in [1] where occupation times of a critical branching Brownian motion were proved to behave very differently depending on whether $d > 4$, $d = 4$, $d = 3$, $d = 2$ or $d = 1$.

1. Basics. We first state a special form of comparison principle which will be used many times in this paper. Consider $G_a u \equiv -(D^2 + (d-1)/(x)D)u + au^2$ where $a > 0$ is a constant. We call u a supersolution for G_a if $G_a u \geq 0$ and a subsolution for G_a if $G_a u \leq 0$.

COMPARISON LEMMA. *Suppose $\bar{u}(x)$ is a positive supersolution for G_a, $\underline{u}(x)$ is a positive subsolution for G_a and $\bar{u}(Y_j) \geq \underline{u}(Y_j)$ where $j = 1, 2$ and $0 \leq Y_1 < Y_2 < \infty$. Then $\bar{u}(x) \geq \underline{u}(x)$ for all $x \in (Y_1, Y_2)$.*

$\bar{u}(x) \geq \underline{u}(x)$ for $x \in (Y_1, \infty)$ if the condition at Y_2 is replaced by $\limsup_{x\to\infty}[\bar{u}(x) - \underline{u}(x)] \geq 0$.

PROOF. Our assumption implies that

$$-\left(D^2 + \frac{d-1}{x}D\right)(\bar{u} - \underline{u}) + a(\bar{u} + \underline{u})(\bar{u} - \underline{u}) \geq 0,$$

$$(\bar{u} - \underline{u})(Y_1) \geq 0 \quad \text{and} \quad (\bar{u} - \underline{u})(Y_2) \geq 0.$$

That $(\bar{u} - \underline{u})(x) < 0$ for some $x \in (Y_1, Y_2)$ leads to a contradiction to the above differential inequality at a negative local minima. Therefore $\bar{u}(x) \geq \underline{u}(x)$. □

The scaling structure for (A) can be easily checked.

SCALING LEMMA. *If $u(x)$ satisfies* (A) *in (a, ∞), then $v_m(x) = m^2 u(mx)$, $m > 0$, satisfies* (A) *in $(a/m, \infty)$.*

If $u(x)$ satisfies (A) *in (a, ∞), then $v(x) \equiv u(kx)$, $k > 0$, satisfies*

$$-\left(D^2 + \frac{d-1}{x}D\right)v + k^2 v^2 = 0 \quad in \left(\frac{a}{k}, \infty\right).$$

The following lemma is a slight generalization of a result in [8]. We present a proof for easy reference and to introduce some arguments useful later on.

LEMMA 1.1. *Suppose $\sigma(x)$ and $f(x, u)$ satisfy* (0.1) *and $u(x, \eta)$ is as in* (0.2). *Then for all $\eta > 0$,*

$$\lim_{x\to\infty} u(x, \eta)x^2 = 2(4 - d) \quad when\ d < 4 \tag{1.1}$$

and

$$\lim_{x\to\infty} u(x,\eta)\cdot\frac{x^2\log x}{2} = 1 \quad \textit{when } d = 4. \tag{1.2}$$

PROOF. We first consider the simple case when $\sigma(x)f(x,u)\equiv 1$ and $d<4$. A key observation is that cx^{-2} is a supersolution for G_1 if $c>2(4-d)$, a solution if $c=2(4-d)$ and a subsolution for G_1 if $c<2(4-d)$. The comparison lemma implies that a solution curve of (A) can intersect $y=ax^{-2}$, $a\neq 2(4-d)$, at most at one $x\in(0,\infty)$ and that, if intersecting $y=2(4-d)x^{-2}$, it is identically $y=2(4-d)x^{-2}$. It then follows easily from the comparison lemma that $g(x)\equiv u(x,\eta)x^2$ increases as $x\to\infty$ and $g(x)\le 2(4-d)$ if $\eta<2(4-d)$ and that $g(x)$ decreases as $x\to\infty$ and $g(x)\ge 2(4-d)$ if $\eta>2(4-d)$.

It remains to prove $g(\infty)=2(4-d)$. For convenience we suppress η and use $u(x)$ for $u(x,\eta)$. The following elegant proof was pointed out to me by the referee: If $u(1)<2(4-d)$, then $u(x)x^2\uparrow A\le 2(4-d)$. If $u_m(x)=m^2u(mx)$, then $G_1u_m(x)\equiv 0$ and, as $m\to\infty$, $u_m(x)=u(mx)(mx)^2/x^2\uparrow u_\infty(x)=g(\infty)/x^2$ for all $x>0$. Since a locally bounded pointwise limit of solutions of $G_1u=0$ must be a weak solution, hence a solution by Weyl's lemma, $g(\infty)$ can only be $2(4-d)$. The case $u(1)>2(4-d)$ can be similarly treated.

An alternate proof is to rewrite (A) as $(x^{d-1}u')'=x^{d-1}u^2$ and integrate it: $x^{d-1}u'(x)-M^{d-1}u'(M)=-\int_x^M z^{d-1}u^2(z)\,dz$. We assert existence of $M_n\to\infty$ such that $\lim_{r\to\infty}M_n^{d-1}u'(M_n)=0$. This implies that $z^{d-1}u^2(z)$ is integrable and also

$$x^{d-1}u'(x) = -\int_x^\infty z^{d-1}u^2(z)\,dz. \tag{1.3}$$

(1.3) is useful later on. We prove our assertion by contradiction: Note that $u'(x)\le 0$ for all $x>1$ because the structure of (A) obstructs existence of positive local maxima of $u(x)$. If no such M_n as asserted exists, there must be $\delta>0$ such that $-\delta x^{1-d}\ge u'(x)$ for all large x, say, $x\ge Y$. Integration then yields that $u(x)\ge[\delta/(d-2)]x^{2-d}$ for $x\ge Y$ for $d>2$ and that $u'(x)\notin L^1(Y,\infty)$ for $d\le 2$. Both conclusions contradict $u(x)\sim O(x^{-2})$. Multiplying (1.3) by x^{1-d} and integrating yields

$$u(x)=\int_x^\infty y^{1-d}\int_y^\infty z^{d-1}u^2(z)\,dz\,dy,$$

i.e.,

$$g(x)=x^2\int_x^\infty y^{1-d}\int_y^\infty z^{d-5}g^2(z)\,dz. \tag{1.4}$$

Note that $g(x)\equiv 2(4-d)$ is a solution.

As $x\to\infty$, (1.4) implies that

$$g(\infty)=\frac{g(\infty)^2}{[2(4-d)]},\quad \text{i.e.,}\quad g(\infty)=2(4-d).$$

(1.1) is proved for (A).

When $d = 4$, we have that $cx^{-2}(\log x)^{-1}$ is a supersolution for large x if $c > 2$ and is a subsolution if $c < 2$. The above argument with simple modification yields (1.2) for (A).

For general $\sigma(x)$ and $f(x, u)$, we first show $\limsup_{x\to\infty} u(x)x^2 \le 2(4 - d)$ when $d < 4$. Let $\bar{u}(x)$ be the solution of

$$-\left(D^2 + \frac{d-1}{x}D\right)\bar{u} + \left[\inf_{\substack{u\le u(Y)\\ x\ge Y}} \sigma(x)f(x,u)\right]\bar{u}^2 = 0$$

with $\bar{u}(\infty) = 0$ and $\bar{u}(Y) = u(Y)$.

Consider $\bar{u}(x)$ as a solution and u as a subsolution for G_a, $a = \inf_{u\le u(Y), x\ge Y} \sigma(x)f(x,u)$. The comparison lemma implies that $u(x) \le \bar{u}(x)$ for all $x \ge Y$.

Results for (A) and the scaling lemma imply that

$$\limsup_{x\to\infty} x^2u(x) \le \lim_{x\to\infty} x^2\bar{u}(x) \le \left[\inf_{\substack{u\le u(Y)\\ x\ge Y}} \sigma(x)f(x,u)\right]^{-1} 2(4-d).$$

Letting Y tend to ∞, it follows that $\limsup_{x\to\infty} u(x)x^2 \le 2(4 - d)$. Since the same kind of argument works nicely to prove $\liminf_{x\to\infty} u(x)x^2 \ge 2(4 - d)$, the proof of (1.1) is complete. The above-mentioned procedure also yields (1.2). □

2. The case when $d < 2$. When $d < 2$, $P_r\{N > 0\} \sim 2(4 - d)r^{-2}$ (Lemma 1.1) and $E_r\{N\} = 1$ suggests existence of $\lim_{r\to\infty} E_r\{e^{-\eta(N/r^2)}|N > 0\}$. In view of

$$\begin{aligned} E_r\{e^{-\eta(N/r^2)}; N > 0\} &= E_r\{e^{-\eta(N/r^2)}\} - P_r\{N = 0\} \\ &= E_r\{e^{-\eta(N/r^2)}\} - 1 + P_r\{N > 0\} \\ &= P_r\{N > 0\} - u(r, 1 - e^{-\eta/r_2}), \end{aligned} \tag{2.1}$$

where $u(x, \cdot\,)$ is as in (0.1)–(0.2), we first establish

THEOREM 2.1. *For all $\eta > 0$ we have*

$$\lim_{r\to\infty} r^2u(r, r^{-2}\eta) = \eta U(\eta^{1/2}) \tag{2.2}$$

for $u(x, \cdot\,)$ as in (0.2) *where $U(x)$ is the unique solution of (A) for $x > 0$ with* $U(0) = 1$.

Small x asymptotics of solutions of (A) in $(0, \infty)$ are pivotal. We first show

LEMMA 2.1. *There exists a unique $U(x)$ which satisfies (A) in $(0, \infty)$ with* $U(0+) = 1$.

PROOF. First note that solutions of (A) in $(0, \infty)$ are monotonically decreasing since the structure of (A) implies the absence of positive local maxima. It suffices to find a constant upper bound at $x = 0$ for some solution $u(x)$ of (A) in

$(0, \infty)$ because the monotonicity of $u(x)$ will then ensure the existence of $u(0)$ and an appropriate scaling, namely, $U(x) \equiv u(0)^{-1} \cdot u(u(0)^{-1/2}x)$ will yield U. Uniqueness follows from the comparison lemma. Our upper bound is obtained using a subsolution. For $d > 0$ choosing $0 < \varepsilon < 2 - d$ and $c \equiv [\varepsilon(2 - d - \varepsilon)]^{\varepsilon/(\varepsilon+2)}$ and letting $\underline{u}(x) = c - x^\varepsilon$, we see that

$$-\left(D^2 + \frac{d-1}{x}D\right)\underline{u} + \underline{u}^2 \le 0 \quad \text{on} \quad (0, c^{1/\varepsilon}) \equiv \{x\colon \underline{u}(x) \ge 0\}.$$

Note that $m^2u(mx, \eta)$ is, by the scaling lemma, a solution of (A) in $(1/m, \infty)$ with boundary value $m^2\eta$ at $x = 1/m$. Let m be so large that $1/m < c^{1/\varepsilon}$ and then let η be so small that $m^2\eta < \underline{u}(1/m)$. The comparison lemma now ensures that the extension of $m^2u(mx, \eta)$ to $x \in (0, 1/m)$ gives a solution of (A) which is bounded by c. For $d \le 0$, it was pointed out to me by the referee that the only modification needed is to have $0 < \varepsilon < 2$ instead of $0 < \varepsilon < 2 - d$. This can be easily checked. □

Proof of Theorem 2.1. We first consider a simple case where $\sigma(x)f(x, u) \equiv 1$. Fix $\eta > 0$. For sufficiently large r, there exists $m = m(r)$ such that $m^2U(m) = r^{-2}\eta$ with $\lim_{r\to\infty} m(r) = 0$. More precisely, $rm(r) \to \eta^{1/2}$ and by the scaling lemma $u(x, r^{-2}\eta) = m^2U(mx)$ and hence $\lim_{r\to\infty} r^2u(r, r^{-2}\eta) = \lim_{r\to\infty} r^2m^2U(mr) = \eta U(\eta^{1/2})$.

We now extend the result to general $\sigma(x)$ and $f(x, u)$. Let $a(Y, \varepsilon) \equiv \inf_{u\le\varepsilon, x\ge Y} \sigma(x)f(x, u)$, $\delta \equiv a(1, \varepsilon)$ and $A(Y, \varepsilon) \equiv \sup_{u\le\varepsilon, x\ge Y} \sigma(x)f(x, u)$ and $M = A(1, \varepsilon)$. It follows from our assumptions that

$$\lim_{\substack{Y\to\infty \\ \varepsilon\to 0}} a(Y, \varepsilon) = 1 = \lim_{\substack{Y\to\infty \\ \varepsilon\to 0}} A(Y, \varepsilon).$$

First, we prove

$$u(x, r^{-2}\eta) \sim r^{-2}\eta \quad \text{as } r \to \infty, \text{all } x \in (1, \infty). \tag{2.3}$$

By the scaling lemma, $u(x) = m^2U(mkx)$ is a solution of

$$-\left(D^2 + \frac{d-1}{x}D\right)u + k^2u^2 = 0, \qquad u(\infty) = 0.$$

Thus if $m_i = m_i(r)$ are chosen such that

$$m_1^2U(m_1M^{1/2}) = r^{-2}\eta = m_2^2U(m_2\delta^{1/2}), \tag{2.4}$$

where $\lim_{r\to\infty} m_i(r) = 0$, then by the comparison lemma

$$m_1^2U(m_1M^{1/2}x) \le u(x, r^{-2}\eta) \le m_2^2U(m_2\delta^{1/2}x) \tag{2.5}$$

for $x \in (1, \infty)$. By (2.4), $m_i(r)^2 \sim r^{-2}\eta$ as $r \to \infty$ and (2.3) follows from $U(0+) = 1$ and (2.5) for fixed x. Since $u(x, \eta)$ is an increasing function of η by the comparison lemma, one can choose Y sufficiently large so that $u(x, \eta) < \varepsilon$ for $x \ge Y$. Similarly, if $m_i(r)$ are now defined by

$$m_1^2U(m_1A(Y, \varepsilon)^{1/2}Y) = u(Y, r^{-2}\eta) = m_2^2U(m_2a(Y, \varepsilon)^{1/2}Y), \tag{2.6}$$

where $m_i(r) \to 0$ as $r \to \infty$, then

$$m_1^2 U\big(m_1 A(Y,\varepsilon)^{1/2} x\big) \le u(x, r^{-2}\eta) \le m_2^2 U\big(m_2 a(Y,\varepsilon)^{1/2} x\big) \tag{2.7}$$

for $x \in (Y, \infty)$. Again $m_i r \to \eta^{\frac{1}{2}}$ as $r \to \infty$ by (2.3) and (2.6) for fixed Y. (2.7) with $x = r$ implies that

$$\eta U\big(\eta^{1/2} A(Y,\varepsilon)^{1/2}\big) \le \liminf_{r\to\infty} r^2 u(r, r^{-2}\eta) \le \limsup_{r\to\infty} r^2 u(r, r^{-2}\eta)$$
$$\le \eta U\big(\eta^{1/2} a(Y,\varepsilon)^{1/2}\big).$$

Letting Y tend to ∞ and ε to 0, the proof is completed by the continuity of U. □

REMARK. From the viewpoint of differential equations, note that the variation equation of (0.2) about identical zero is $-(D^2 + (d-1)/(x)D)w = 0$, i.e., $w =$ const. and that the linear approximation of $u(x, r^{-2}\eta)$ by $w = r^{-2}\eta$ can at best be valid for intervals like $[1, O(r)]$ due to x^{-2} decay of $u(x, r^{-2}\eta)$. The scaling in Theorem 2.1 makes this clear by showing that $u(r, r^{-2}\eta) \sim \eta U(\eta^{1/2}) r^{-2}$ as $r \to \infty$, indeed not $\sim w(r) = \eta r^{-2}$. For the other four regimes of d, similar structures persist which we shall not dwell on. Our treatment on scaled limits of solutions of differential equations (Theorems 2.1, 3.1, 4.1 and 5.1) applies to a large class of semilinear equations, e.g., equations with u^2 replaced by u^p, $p > 1$ (five regions: $d < 2$, $d = 2$, $2 < d < 2p/(p-1)$, $d = 2p/(p-1)$ and $2p/(p-1) < d$ show very different behavior).

THEOREM 2.2. *Suppose $X(t)$ is a critical branching Bessel process initially at r generated by $L_d = \frac{1}{2}(D^2 + (d-1)/(x)D)$, $-\infty < d < 2$, associated with branching rate $\sigma(x)$ and offspring distribution $\{q_n(x);\, n = 0, 1, \dots\}$ that satisfy* (0.1). *Then:*

$$P_r\{N > 0\} \sim 2(4-d) r^{-2} \quad \text{as } r \to \infty. \tag{2.8}$$

$$\text{Conditional on } \{N > 0\},\ \frac{N}{r^2} \text{ converges in distribution to } Z \tag{2.9}$$

which is characterized by

$$E\{e^{-\eta Z}\} = 1 - \frac{\eta U(\eta^{1/2})}{2(4-d)}.$$

Let $k(d) = \min(2 - d/2, 2)$. *We also have have* $\operatorname{Prob}\{Z = 0\} = 0$ *and*

$$E\{Z^{k(d)-\varepsilon}\} < \infty \quad \text{and} \quad E\{Z^{k(d)+\varepsilon}\} = \infty \tag{2.10}$$
$$\text{for every small } \varepsilon < 0.$$

PROOF. Recalling from $P_r\{N > 0\} = u(r, 1)$, Lemma 1.1 implies (2.8). As in (2.1),

$$\lim_{r\to\infty} \frac{E_r\{e^{-\eta(N/r^2)};\, N > 0\}}{P_r\{N > 0\}} = 1 - \lim_{r\to\infty} \frac{u(r, 1 - e^{-r^{-2}\eta})}{2(4-d) r^{-2}},$$

and

$$1 - e^{-r^{-2}\eta} = r^{-2}\eta + O(r^{-4}) \quad \text{as } r \to \infty.$$

Since $u(x, \eta)$ is monotonic in η by the comparison lemma, (2.9) follows from Theorem 2.1 and the continuity of $U(x)$. It follows from $\lim_{\eta\to\infty} \eta U(\eta^{1/2}) = 2(4-d)$ that $\text{Prob}\{Z = 0\} = \lim_{\eta\to\infty} E\{e^{-\eta Z}\} = 0$. Recall from (1.3) that $U'(x) = -x^{1-d}\int_x^\infty z^{d-1}U^2(z)\,dz$. Integration yields

$$U(x) - 1 = -\int_0^x y^{1-d}\int_y^\infty z^{d-1}U^2(z)\,dz.$$

When $0 < d < 2$, we have $\int_0^\infty z^{d-1}U^2(z)\,dz = b < \infty$ and

$$U(x) - 1 \sim -\frac{b}{(2-d)}x^{2-d} \quad \text{as } x \to 0. \tag{2.11}$$

When $d \le 0$, a simple asymptotic evaluation shows

$$U(x) - 1 \sim \begin{cases} -\dfrac{x^2}{2}\log\dfrac{1}{x} & \text{for } d = 0, \\ -\dfrac{1}{(-2d)}x^2 & \text{for } d < 0. \end{cases} \tag{2.12}$$

Substituting x by $\eta^{1/2}$ in (2.11) and (2.12), (2.9) implies (2.10). The proof is complete. □

3. The case when $d = 2$. When $d = 2$, we naturally believe in the same heuristics as stated before Theorem 2.1. The suggested scaling N/r^2 nevertheless converges in distribution to $Z = 0$. We prove the appropriate normalization to be $(\log r)\cdot N/r^2$ by first establishing

THEOREM 3.1. *For all $\eta > 0$ we have*

$$\lim_{r\to\infty} r^2 u(r, r^{-2}\eta) = 0, \tag{3.1}$$

$$\lim_{r\to\infty} r^2 u(r, (r^{-2}\log r)\eta) = \eta U(\eta^{1/2}) \tag{3.2}$$

for $u(x,\cdot)$ in (0.1)–(0.2) *where U is the unique solution of* (A) *for $x > 0$ with* $\lim_{x\downarrow 0}[(U(x))/-\log x] = 1$.

We first prove

LEMMA 3.1. *If $d = 2$, then there exists a unique $U(x)$ which satisfies* (A) *and* $\lim_{x\downarrow 0}[U(x)]/(-\log x) = 1$.

PROOF. We first show that there exists $V(x)$ satisfying (A) and $\int_0^\infty xV^2(x)\,dx < \infty$.

In view that $\underline{u} \equiv \varepsilon^2(x^{-\varepsilon} - 1)$ satisfies $-(D^2 + (1/x)D)\underline{u}^2 + \underline{u}^2 \le 0$ on $(0, 1]$ for $0 < \varepsilon < 2$ with $\underline{u}(1) = 0$, the same argument as in the proof of Lemma 2.1 yields the existence of $V(x)$, the solution to (A), such that $V(x) \le \underline{u}(x)$ for small

x. When $\varepsilon < 1$, we see that $\int_0^\infty xV^2(x)\,dx < \infty$. It then follows from (1.3) for $d = 2$ that

$$(3.3)\qquad -xV'(x) = \int_x^\infty (zV')'\,dz = \int_x^\infty zV^2(z)\,dz.$$

(3.3) implies that $-V'(x) \sim O(x^{-3})$ as $x \to \infty$ and $\sim [\int_0^\infty zV^2(z)\,dz]x^{-1}$ as $x \to 0$, hence that $V(x) \sim [\int_0^\infty zV^2(z)\,dz](-\log x)$. Again, an appropriate m makes $m^2V(mx) \sim (-\log x)$ which we choose as U. Lemma 3.1 is proved. □

PROOF OF THEOREM 3.1. The case when $\sigma(x)f(x,u) \equiv 1$ is again simple. Since $\phi(m) = m^2U(m) \sim m^2\log(1/m) \to 0$ as $m \to 0$, $m = m(r)$ can be defined for large r by $\phi(m) = m^2U(m) = r^{-2}\eta$. Then $rm(r) \to 0$ and $r^2u(r, r^{-2}\eta) \to 0$ since $u(x, r^{-2}\eta) = m^2U(mx)$. For $u(r,(r^{-2}\log r)\eta)$, the above computation needs to be modified slightly. We now have $u(x,(r^{-2}\log r)\eta) = m^2U(mx)$ where m satisfies $r^{-2}(\log r)\eta = m^2U(m)$. This leads to $\lim_{r\to\infty} mr = \eta^{1/2}$ and therefore $\lim_{r\to\infty} r^2u(r,(r^{-2}\log r)\eta) = \eta U(\eta^{1/2})$.

For the general $\sigma(x)$ and $f(x,u)$ the same kind of argument as used for $d < 2$ goes through. We omit the details. □

Theorem 3.1 has the following probabilistic interpretation.

THEOREM 3.2. *Suppose $X(t)$ is a critical branching Bessel process initially at r generated by $\frac{1}{2}(D^2 + (1/x)D)$ with branching rate $\sigma(x)$ and offspring distribution $\{q_n(x); n \geq 0\}$ that satisfy (0.1). Then*

$$P_r\{N > 0\} \sim 4r^{-2} \quad \text{as } r \to \infty.$$

Conditioned on $\{N > 0\}$, $(\log r)N/r^2$ converges, as $r \to \infty$, in distribution to Z which is characterized by

$$E\{e^{-\eta Z}\} = 1 - \tfrac{1}{4}\eta U(\eta^{1/2}).$$

In particular, $E\{Z\} = \infty$ and $E\{Z^{1-\varepsilon}\} < \infty$ for any small $\varepsilon > 0$.

4. The case when $2 < d < 4$. When $2 < d < 4$, $P_r\{N > 0\} \sim 2(4-d)r^{-2}$ and $E_r\{N\} = r^{2-d}$ suggests existence of $\lim_{r\to\infty} E_r\{e^{-\eta(N/r^{4-d})}|N > 0\}$. Since

$$E_r\{e^{-\eta(N/r^{4-d})}|N > 0\} = 1 - \frac{1 - E_r\{e^{-\eta(N/r^{4-d})}\}}{P_r\{N > 0\}} = 1 - \frac{r^2u(r, 1 - e^{-\eta/r^{4-d}})}{2(4-d)},$$

we first prove

THEOREM 4.1. *For all $\eta > 0$ we have*

$$\lim_{r\to\infty} r^2u(r, r^{d-4}\eta) = \eta^{2/(4-d)}U(\eta^{1/(4-d)})$$

for $u(x,\cdot)$ in (0.1)–(0.2) where U is the unique solution of (A) such that $U(x) \sim x^{2-d}$ as $x \downarrow 0$.

First we show the existence of U.

LEMMA 4.1. *When* $2 < d < 4$, *there exists a unique* $U(x)$ *which satisfies* (A) *and* $\lim_{x\downarrow 0} x^{d-2}U(x) = 1$.

PROOF. Since cx^{2-d}, $c > 0$, is a supersolution, the comparison lemma implies that $x^{d-2}u(x)$ increases as $x \downarrow 0$ if u is a solution of (A) on $(0,\infty)$. We prove that $\lim_{x\downarrow 0} x^{d-2}u(x) < \infty$ for some solution of (A) on $(0,\infty)$ again by constructing a subsolution as in the case when $d < 2$.

Choosing $0 < \varepsilon < \min(4-d, d-2)$ and $v(x) = cx^{2-d}(1-x^{\varepsilon})$, which is positive for $0 \le x < 1$, computation shows that

$$-\left(D^2 + \frac{d-1}{x}D\right)v + v^2 = -c\varepsilon(d-2-\varepsilon)x^{\varepsilon-d} + c^2(1-x^{\varepsilon})^2x^{4-2d}$$
$$\le c\left[c - \varepsilon(d-2-\varepsilon)\right]x^{\varepsilon-d} \le 0$$
$$\text{for } 0 \le x \le 1 \text{ if } c \le \varepsilon(d-2-\varepsilon).$$

The same argument as used in the proof of Lemma 2.1 ensures existence of u such that $\lim_{x\to 0} x^{d-2}u(x) = \alpha$ is a positive constant. $U(x) \equiv m^2u(mx)$ with $m = \alpha^{-1/(4-d)}$ then has $U(x) \sim x^{2-d}$ as $x \to 0$. The proof is complete. □

PROOF OF THEOREM 4.1. When $\sigma(x)f(x,u) \equiv 1$, $u(x, r^{d-4}\eta) = m^2U(mx)$ where m is determined by $m^2U(m) = r^{d-4}\eta$. This implies $\lim_{r\to\infty} mr = \eta^{1/(4-d)}$ and therefore

$$\lim_{r\to\infty} r^2u(r, r^{d-4}\eta) = \lim_{r\to\infty} m^2r^2U(mr) = \eta^{2/(4-d)}U(\eta^{1/(4-d)}).$$

For general $\sigma(x)$ and $f(x,u)$ let δ, $a(Y,\varepsilon)$, $A(Y,\varepsilon)$ and M be as in the proof of Theorem 2.1. Again, our first step is to prove

$$u(x, r^{d-4}\eta) \sim \eta r^{d-4}x^{2-d} \quad \text{as } r \to \infty \tag{4.1}$$

for all $x \in (0,\infty)$. First, since $m^2U(mC) \sim m^{4-d}C^{2-d}$ as $m \to 0$,

$$m^2U(mCx) \sim m^2U(mC)x^{2-d} \tag{4.2}$$

as $m \to 0$ for all constants C and x. Define $m_i = m_i(r)$ by

$$m_1^2U(m_1M^{1/2}) = r^{d-4}\eta = m_2^2U(m_2\delta^{1/2}). \tag{4.3}$$

Then by the comparison lemma

$$m_1^2U(m_1M^{1/2}x) \le u(x, r^{d-4}\eta) \le m_2^2U(m_2\delta^{1/2}x) \tag{4.4}$$

for all $x \in (1,\infty)$. The relations (4.2)–(4.4) imply (4.1). Similarly if $m_i(r)$ are defined by

$$m_1^2U\left(m_1A(Y,\varepsilon)^{1/2}Y\right) = u(Y, r^{d-4}\eta) = m_2^2U\left(m_2a(Y,\varepsilon)^{1/2}Y\right), \tag{4.5}$$

then $m_1^2U(m_1A(Y,\varepsilon)^{1/2}x) \le u(x, r^{d-4}\eta) \le m_2^2U(m_2a(Y,\varepsilon)^{1/2}x)$ for all $x \ge Y$. The relations (4.5), (4.2) and (4.1) imply that $m_1r \to A(Y,\varepsilon)^{1/(4-d)}\eta^{1/(4-d)}$.

$m_2 r \to a(Y,\varepsilon)^{1/(4-d)}\eta^{1/(4-d)}$ as $r \to \infty$, and

$$\begin{aligned} A(Y,\varepsilon)^{(d-2)/(4-d)}\eta^{2/(4-d)}U\big(A(Y,\varepsilon)^{1/(4-d)}\eta^{1/(4-d)}\big) \\ \le \liminf_{r\to\infty} r^2 u(r, r^{d-4}\eta) \\ \le \limsup_{r\to\infty} r^2 u(r, r^{d-4}\eta) \\ \le a(Y,\varepsilon)^{(d-2)/(4-d)}\eta^{2/(4-d)}U\big(a(Y,\varepsilon)^{1/(4-d)}\eta^{1/(4-d)}\big). \end{aligned}$$

Letting Y tend to ∞ and ε to 0, Theorem 4.1 is completely proved. □

A probabilistic counterpart of Theorem 4.1 is

THEOREM 4.2. *Suppose $X(t)$ is a critical branching Bessel process initially at r, generated by $L_d \equiv \frac{1}{2}(D^2 + (d-1)/(x)D)$, $2 < d < 4$, and with branching rate $\sigma(x)$ and offspring distribution $\{q_n(x); n \ge 0\}$ that satisfy* (0.1). *Then*:

(4.6) $$P_r\{N > 0\} \sim 2(4-d)r^{-2} \quad \text{as } r \to \infty.$$

(4.7) *Conditional on* $\{N > 0\}$, $\dfrac{N}{r^{4-d}}$ *converges, as* $r \to \infty$,

in distribution to Z which is characterized by

$$E\{e^{-\eta Z}\} = 1 - \frac{\eta^{2/(4-d)}U(\eta^{1/(4-d)})}{2(4-d)},$$

(4.8) *We have that* $\text{Prob}\{Z = 0\} = 0$,
$E\{Z^{2/(4-d)-\varepsilon}\} < \infty$ *and* $E\{Z^{2/(4-d)+\varepsilon}\} = \infty$ *for any small* $\varepsilon > 0$.

PROOF. Again, (4.6) is a probabilistic interpretation of (1.1) and (4.7) is an interpretation of Theorem 4.1 because

$$E_r\{e^{-\eta(N/r^{4-d})}|N > 0\} = 1 - \frac{u(r, 1 - e^{-r^{d-4}\eta})}{P_r\{N > 0\}}.$$

We now turn to prove (4.8).

$$\text{Prob}\{Z = 0\} = \lim_{\eta\to\infty} E\{e^{-\eta Z}\} = 1 - \frac{1}{2(4-d)} \lim_{\eta\to\infty} \eta^{2/(4-d)}U(\eta^{1/(4-d)}) = 0.$$

To study $\eta^{2/(4-d)}U(\eta^{1/(4-d)})$ for a small η, first rewrite the differential equation of U as $[x^{3-d}(x^{d-2}U)']' = xU^2$ which can be integrated to yield

(4.9) $$x^{d-2}U(x) - 1 = -\int_0^x y^{d-3}\int_y^\infty zU^2\,dz\,dy.$$

The proof of (4.8) will be carried out for (i) $2/(4-d) < 2$, (ii) $2/(4-d) = 2$, (iii) $2 < 2/(4-d) < 3$ and (iv) $2/(4-d) = 3$. The rest of d can then be treated using the same procedures over and over again and we shall omit the details.

(i) $1 < 2/(4-d) < 2$, i.e., $2 < d < 3$. In view of

$$zU^2(z) \sim \begin{cases} O(z^{-3}) & \text{as } z \to \infty, \\ z^{5-2d} & \text{as } z \to 0, \end{cases} \tag{4.10}$$

we have

$$y^{d-3} \int_y^\infty zU^2(z)\,dz \sim \int_0^\infty zU^2(z)\,dz \cdot y^{d-3} \quad \text{as } y \to 0. \tag{4.11}$$

(4.9) therefore implies that

$$U(x) \sim x^{2-d}\left[1 - \frac{a}{d-2}x^{d-2} + o(x^{d-2})\right] \quad \text{as } x \to 0,$$

i.e.,

$$\eta^{2/(4-d)}U\left(\eta^{1/(4-d)}\right) \sim \eta - \frac{a}{d-2}\eta^{2/(4-d)} + o\left(\eta^{2/(4-d)}\right) \quad \text{as } \eta \to 0, \tag{4.12}$$

where $a \equiv \int_0^\infty zU^2\,dz$.

(ii) $2/(4-d) = 2$, i.e., $d = 3$. Noting $5 - 2d = -1$, (4.11) therefore is replaced by

$$\int_y^\infty zU^2(z)\,dz \sim (-\log y) \quad \text{as } y \to 0$$

and (4.12) is replaced by

$$\eta^{2/(4-d)}U\left(\eta^{1/(4-d)}\right) = \eta + \eta^2 \log \eta + o\left(\eta^2 \log \eta\right) \quad \text{as } \eta \to 0.$$

This implies (4.8).

(iii) $2 < 2/(4-d) < 3$, i.e., $3 < d < \frac{10}{3}$. In view of $5 - 2d < -1$, (4.11) is replaced by

$$y^{d-3} \int_y^\infty zU^2(z)\,dz \sim \frac{1}{(6-2d)}y^{3-d} \quad \text{as } y \to 0,$$

which leads to

$$U(x) = x^{2-d}\left[1 - \frac{1}{2(3-d)(4-d)}x^{4-d} + o(x^{4-d})\right] \quad \text{as } x \to 0.$$

Using this finer expansion in (4.9) and the same argument, we obtain

$$\begin{aligned} zU^2(z) &= z^{5-2d} - \frac{1}{(3-d)(4-d)}z^{9-3d} + o(z^{9-3d}) \quad \text{as } z \to 0, \\ y^{d-3} \int_y^\infty zU^2(z)\,dz &= \frac{1}{2(d-3)}y^{3-d} - by^{d-3} + o\left(y^{d-3}\right) \quad \text{as } y \to 0, \end{aligned} \tag{4.13}$$

where b is a positive constant. This leads to

$$U(x) = x^{2-d}\left[1 - \frac{1}{2(d-3)(4-d)}x^{4-d} + \frac{b}{(d-2)}x^{d-2} + o(x^{d-2})\right] \quad \text{as } x \to 0$$

and hence,

$$\eta^{2/(4-d)}U\big(\eta^{1/(4-d)}\big) = \eta - \frac{1}{2(d-3)(4-d)}\eta^2 + \frac{b}{(d-2)}\eta^{2/(4-d)} + o\big(\eta^{2/(4-d)}\big) \quad \text{as } \eta \to 0.$$

This implies (4.8).

(iv) $2/(4-d) = 3$, i.e., $d = \frac{10}{3}$. We follow the computation in (iii) with (4.13) replaced by

$$y^{d-3}\int_y^\infty zU^2(z)\,dz = \frac{1}{2(d-3)}y^{3-d} + \frac{y^{d-3}\log y}{(d-3)(4-d)} + o\big(y^{d-3}\log y\big) \quad \text{as } y \to 0.$$

This leads to

$$\eta^{2/(4-d)}U\big(\eta^{1/(4-d)}\big) = \eta - \frac{1}{2(d-3)(4-d)}\eta^2 - \frac{\eta^3\log\eta}{(d-3)(4-d)(d-2)} + o\big(\eta^3\log\eta\big) \quad \text{as } \eta \to 0,$$

which implies (4.8) □

5. The case when $d = 4$. When $d = 4$, $P_r\{N > 0\} \sim 2(r^2\log r)^{-1}$ as $r \to \infty$ and $E_r\{N\} = r^{-2}$ suggests the convergence of $N/\log r$ conditioned on $\{N > 0\}$ which amounts to

THEOREM 5.1. *For all $\eta > 0$ we have, for $u(x,\cdot)$ in (0.1)–(0.2), that*

$$\lim_{r\to\infty}\frac{1}{2}r^2(\log r)u\big(r, 2(\log r)^{-1}\eta\big) = \frac{\eta}{1+\eta}.$$

PROOF. First, assume $\sigma(x)f(x,u) \equiv 1$ and let $U(x) = u(x,1)$ in (0.2). Since $m^2U(m)\downarrow 0$ as $m\downarrow 0$ for $d = 4$, there exists $m = m(r) \to \infty$ such that $u(x, 2(\log r)^{-1}\eta) = m^2U(mx)$ where $m^2U(m) = 2(\log r)^{-1}\eta$. It then follows that $\lim_{r\to\infty}(\log r/\log m) = \eta$ and therefore

$$\lim_{r\to\infty}\frac{1}{2}r^2(\log r)u\big(r,2(\log r)^{-1}\eta\big) = \lim_{r\to\infty}\frac{1}{2}m^2r^2(\log r)U(mr) = \lim_{r\to\infty}\frac{\log r}{\log mr} = \frac{\eta}{1+\eta}.$$

For general $\sigma(x)$ and $f(x,u)$ the proof is similar to that in Theorem 2.1. We only sketch it. If $m \to \infty$, then $m^2U(mC) \sim 2(C^2\log m)^{-1}$ for constant C and thus

$$m^2U(mCx) \sim m^2U(mC)x^{-2} \quad \text{as } m \to \infty \tag{5.1}$$

for constants C and x. Hence,

$$u\big(x, 2(\log r)^{-1}\eta\big) \sim 2(\log r)^{-1}\eta x^{-2} \quad \text{as } m \to \infty \tag{5.2}$$

for all fixed $x \in (1, \infty)$. If $m_i(r)$ are defined by

$$m_1^2 U\big(m_1 A(Y,\varepsilon)^{1/2} Y\big) = u\big(Y, 2(\log r)^{-1}\eta\big) = m_2^2 U\big(m_2 a(Y,\varepsilon)^{1/2} Y\big),$$

then

$$\lim_{r\to\infty} \frac{\log r}{\log m_1} = \eta A(Y,\varepsilon) \quad \text{and} \quad \lim_{r\to\infty} \frac{\log r}{\log m_2} = \eta a(Y,\varepsilon) \tag{5.3}$$

and, by the comparison lemma,

$$m_1^2 U\big(m_1 A(Y,\varepsilon)^{1/2} x\big) \le u\big(x, 2(\log r)^{-1}\big) \le m_2^2 U\big(m_2 a(Y,\varepsilon)^{1/2} x\big)$$

for $x \ge Y$. Substituting x by r, (5.3) and (1.2) imply

$$\begin{aligned}\frac{\eta}{1 + A(Y,\varepsilon)\eta} \cdot \frac{1}{A(Y,\varepsilon)} &\le \liminf_{r\to\infty} \frac{1}{2}(r^2 \log r) u\big(r, 2(\log r)^{-1}\big) \\ &\le \limsup_{r\to\infty} \frac{1}{2}(r^2 \log r) u\big(r, 2(\log r)^{-1}\big) \\ &\le \frac{\eta}{1 + a(Y,\varepsilon)} \cdot \frac{1}{a(Y,\varepsilon)}.\end{aligned}$$

Letting Y tend to ∞ and ε tend to 0 completes the proof. □

A probabilistic interpretation of Theorem 5.1 is

THEOREM 5.2. *Suppose $X(t)$ is a critical branching process initially at r, generated by $\frac{1}{2}(D^2 + (3/x)D)$ and with branching rate $\sigma(x)$ and offspring distribution $\{q_n(x); n \ge 0\}$ that satisfy* (0.1). *Then*

$$P_r\{N > 0\} \sim 2(r^2 \log r)^{-1}$$

and, conditioned on $\{N > 0\}$, $N/\log r$ converges, as $r \to \infty$, in distribution to an exponential variable of mean $\frac{1}{2}$.

6. The case when $d > 4$.

We first establish a crucial analytic result.

LEMMA 6.1. *Let $u(x,\eta)$ be as in* (0.2) *for the case $\sigma(x)f(x,u) \equiv 1$. Then $u(r,\eta) \sim \beta(\eta) r^{2-d}$ as $r \to \infty$ where $\beta(\eta)$ is a positive increasing function of η.*

PROOF. We prove this lemma by the method of supersolution and subsolution. Since bx^{2-d} is a supersolution for any b, the comparison lemma implies that $x^{d-2}u(x,\eta)$ decreases. We now prove $\lim_{x\to\infty} x^{d-2}u(x,\eta) > 0$ by constructing a subsolution $\underline{u}(x)$, $\underline{u}(1) < \eta$, on $(1,\infty)$. Use a fixed positive $\varepsilon < d - 4$

and consider $\underline{u}(x,\eta) \equiv cx^{2-d}(1 + x^{-\varepsilon})$. Computation shows that

$$-\left(D^2 + \frac{d-1}{x}D\right)\underline{u} + \underline{u}^2 = -cx^{-d-\varepsilon}\varepsilon(\varepsilon + d - 2) + c^2(1 + x^{-\varepsilon})x^{4-2d},$$

which is negative if c is sufficiently small and $x \geq 1$. This implies $\lim_{x\to\infty} x^{d-2}u(x,\eta) \geq c > 0$ and we denote it by $\beta(\eta)$. The proof of the lemma is complete. □

Lemma 6.1 says that $P_r\{N > 0\} \sim \beta(1)r^{2-d}$, the same order as $E_r\{N\} = r^{2-d}$. This anticipates convergence of N as $r \to \infty$ conditioned on $\{N > 0\}$. This is very different from the other four regimes of d where $N/[k(r)]$ with $\lim_{r\to\infty} k(r) = \infty$ has to be considered.

If we consider general $\sigma(x)$ and $f(x,u)$ in Lemma 6.1, $\beta(\eta) \equiv \lim_{x\to\infty} x^{d-2}u(x,\eta)$ depends on $\sigma(x)f(x,u)$. We therefore expect the conditioned limit of N to depend on $\sigma(x)f(x,u)$. This is indeed so and we now illustrate the result for $d > 4$ only for the simple situation $\sigma(x)f(x,u) \equiv 1$.

Let $U(x)$ be $u(x,1)$. We have

THEOREM 6.1. *Suppose $X(t)$ is a critical branching Bessel process initially at r, generated by $L_d \equiv \frac{1}{2}(D^2 + (d-1)/(x)D)$, $d > 4$, and with branching rate $\sigma(x)$ and offspring distribution $\{q_n(x); n \geq 0\}$ that satisfy $\sigma(x)f(x,u) \equiv 1$. Then*

(6.1) $P_r\{N > 0\} \sim \beta(1)r^{2-d}$ *as $r \to \infty$ where $0 < \beta(1) < 1$ is a constant*

and

(6.2) *conditional on $\{N > 0\}$, N converges in distribution to Z,*

characterized by $E\{(1-\eta)^Z\} = 1 - m^{4-d}(\eta)$, $1 > \eta > 0$, where $m(\eta)$ is such that $m^2U(m) = \eta$ and

(6.3) *there exists $c > 0$ such that $E\{(1+c)^Z\} < \infty$,*

i.e., Prob$\{Z \geq n\}$ *is less than some geometric decay as $n \to \infty$.*

PROOF. (6.1) is a result of Lemma 6.1. In view of

$$E\{(1-\eta)^z\} = \lim_{r\to\infty} 1 - \frac{1 - E_r\{(1-\eta)^N\}}{P_r\{N > 0\}} = 1 - \frac{\beta(\eta)}{\beta(1)},$$

we need only relate $\beta(\eta)$ to the special function U. The scaling lemma implies that $u(x,\eta) = m^2U(mx)$ where $m = m(\eta)$ is determined by $m^2U(m) = \eta$ and therefore $\beta(\eta) = \lim_{x\to\infty} u(x,\eta)x^{d-2} = \lim_{x\to\infty} m^2x^{d-2}U(mx) = \beta(1)m^{4-d}(\eta)$. (6.2) is proved. To prove (6.3) note that there exist $c > 0$ and $w(x) > 0$ for $x \geq 1$ such that

(6.4) $$\left(D^2 + \frac{d-1}{x}D\right)w + w^2 = 0, \quad w(1) = c \text{ and } w(x) \sim O(x^{2-d}) \quad \text{as } x \to \infty.$$

This was proved in [2], Table 2, Cases 2 to 4, and can also be proved by a monotonic iterative method (see, e.g., [5]) which we now sketch. Consider $0 < \varepsilon < d - 4$ and $\bar{u}(x) = (c/2)(1 + x^{-\varepsilon})x^{2-d}$, which satisfies $(D^2 + (d-1)/(x)D)\bar{u} + \bar{u}^2 < 0$, $\bar{u}(1) = c > 0$ and $\bar{u}(x) \sim O(x^{2-d})$ as $x \to \infty$ when c is small. Consider also $\underline{u}(x) = (c/2)x^{2-d}$ which satisfies $(D^2 + (d-1)/(x)D)\underline{u} + \underline{u}^2 = \underline{u}^2 > 0$ for all c and $\bar{u}(x) > \underline{u}(x)$ for $x \ge 1$. Let $u_0(x) = \bar{u}(x)$ and $u_{k+1}(x)$ be the solution of $(2c - \Delta)u_{k+1} = 2cu_k - u_k^2$ with $u_{k+1}(1) = c$ and $\lim_{x\to\infty} u_{k+1}(x) = 0$ for $k \ge 0$. It follows easily from the maximum principle that $u \ge u_1 \ge u_2 \ge \cdots \ge \bar{u}$. $\lim_{k\to\infty} u_k(x)$ then serves as the desired $w(x)$.

We next show that

$$E_x\{(1+c)^N\} \le 1 + w(x) \quad \text{for } x \ge 1. \tag{6.5}$$

This is ensured by the minimality of the probabilistic solution as follows. Let $\mathscr{F}_t$ be the σ-algebra generated by our stopped branching process up to time t and $N(t)$ the total number of particles (including those at $x = 1$) at time t. Due to the Markovian property $(\prod_{j=1}^{N(t)}[1 + w(x_j(t))], \mathscr{F}_t)$ is a positive martingale (see, e.g., [3, 7]). Now letting t tend to ∞, we see that $N(t)$ tends to N and, from Fatou's lemma, that

$$\begin{aligned}[1 + w(x)] &= E_x\left\{\prod_{j=1}^{N(t)}\left[1 + w\left(x_j(t)\right)\right]\right\} = \lim_{t\to\infty} E_x\left\{\prod_{j=1}^{N(t)}\left[1 + w\left(x_j(t)\right)\right]\right\}\\ &\ge E_x\left\{\prod_{j=1}^{N}(1+c)\right\} = E_x\left\{(1+c)^N\right\}.\end{aligned}$$

From (6.4) and (6.5) we now have

$$E\left\{(1+c)^Z\right\} = \lim_{r\to\infty}\frac{E_r\{(1+c)^N; N > 0\}}{P_r\{N > 0\}} = 1 + \lim_{r\to\infty}\frac{E_r\{(1+c)^N\} - 1}{\beta(1)r^{2-d}} < \infty.$$

This proves (6.3). □

This paper encounters monotonic solutions of some Emden–Fowler equations (see [2]). We were unable to find in the literature the asymptotics as $x \to 0$ of solutions with known asymptotics at $x \to \infty$. These small x asymptotic behaviors are important and are proved in the present paper using a comparison principle which can be extended to more general Emden–Fowler equations. Also, the scaling limit in Theorems 2.1, 3.1, 4.1 and 5.1 seems an aspect worth studying for a general class of semilinear differential equations (see a brief remark following Theorem 2.1). The result in [8] has been extended in [4] to the case of measure-valued critical branching process and the case of an infinite variance branching mechanism. It can be seen that the result in this paper is extendable to the same direction.

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