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TIME-AVERAGE CONTROL OF MARTINGALE PROBLEMS: EXISTENCE OF A STATIONARY SOLUTION[1]

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This paper studies the average cost for controlled systems via the martingale problem for their generator. The stationary distributions for these processes on the product of the state and control spaces are characterized in terms of integration of the generator and existence of a stationary solution is established. This gives existence of a stationary control.

1. Introduction. Optimal control problems consist of two parts: the dynamics under which the system operates and the decision criterion by which the control is chosen. Formulation of the dynamics of an (uncontrolled) system as a solution of a martingale problem was initiated by Stroock and Varadhan (1969, 1979) for diffusion processes. Kushner (1978) and Fleming (1984), to name only two, formulated controlled diffusions as a martingale problem in which the diffusion and drift coefficients were parametrized by the control. The object of primary interest was the diffusion in $\mathbb{R}^d$. Interested in the control of queues, Hajek (1982) studied two interacting service stations using a martingale problem formulation. Kurtz (1987) considered an abstract controlled martingale problem which included these previous models as special cases. In this approach, the control entered the generator as an independent variable. Kurtz exploited this formulation to give a general existence theorem for optimal solutions to problems in which the decision criteria are the total cost over a finite time horizon and a discounted cost over an infinite time horizon. The present work uses the formulation of the controlled martingale problem given by Kurtz to study the long-run average cost for processes given as solutions. In this formulation, all nonanticipating controls are allowed; that is, we do not restrict attention to Markov controls.

For clarity of exposition, this work is presented in two papers. This paper is concerned with the existence of stationary solutions to controlled martingale problems. The companion paper [Stockbridge (1990)] reformulates the control problem as a linear programming problem and shows existence of an optimal solution.

Existence of stationary solutions to (uncontrolled) martingale problems has frequently assumed the generator generates a strongly continuous contraction semigroup and utilized this structure in the construction of the stationary solution. Echeverria (1982) established existence of a stationary solution under

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more relaxed conditions on the generator. The reader is referred to Ethier and Kurtz (1986), Chapter 4, Section 9 for discussion of stationary solutions to martingale problems.

This paper extends the approach of Echeverria to controlled martingale problems. Section 2 defines the control problem under consideration. Section 3 provides some motivation for the interest in stationary solutions and, in the attempt to be as self-contained as possible, gives the statements of theorems (and references) that will be used later in this paper. Section 4 extends Exheverria's theorem to establish the existence of stationary solutions to the controlled martingale problem.

2. Formulation of the model. $\hat{C}(E)$ denotes the space of continuous functions on E which vanish at infinity. $\overline{C}(E \times U)$ denotes the space of bounded, continuous functions on $E \times U$ and $\|\cdot\|$ denotes the supremum norm on the appropriate space.

Dynamics. Let the state space E and the control space U be locally compact separable metric spaces and $E^{\Delta} = E \cup \{\Delta\}$ the one-point compactification of E. Let $A: \mathscr{D}(A) \to \overline{C}(E \times U)$, $\mathscr{D}(A) \subset \hat{C}(E)$, satisfy:

(i) $\mathscr{D}(A)$ is dense in $\hat{C}(E)$.
(ii) For each $f \in \mathscr{D}(A)$ and $u \in U$, $Af(\cdot, u) \in \hat{C}(E)$.
(iii) For each $f \in \mathscr{D}(A)$ and compact $K \subset U$,

$$\lim_{x \to \Delta} \sup_{u \in K} Af(x, u) = 0.$$

(iv) For each $u \in U$, $A_u f = Af(\cdot, u)$ satisfies the positive maximum principle [i.e., if $f(x) = \sup_z f(z) > 0$, then $A_u f(x) \le 0$].

2.1 Definition. An $E \times U$-valued process $(X(\cdot), u(\cdot))$ is a solution of the controlled martingale problem for A if there exists a filtration $\{\mathscr{F}_t\}$ such that:

(a) $(X(\cdot), u(\cdot))$ is $\{\mathscr{F}_t\}$-progressive.
(b) For every $f \in \mathscr{D}(A)$, $f(X(t)) - \int_0^t Af(X(s), u(s))\, ds$ is an $\{\mathscr{F}_t\}$-martingale.

We will need to specify the initial distribution of the state. Thus for $\nu \in \mathscr{P}(E)$, $(X(\cdot), u(\cdot))$ is a solution of the controlled martingale problem for (A, ν) if, in addition to the above, $X(0)$ has distribution ν.

2.2 Remark. Conditions (i)–(iv) on A ensure the existence of solutions of the controlled martingale problem if we set $u(\cdot) \equiv u$ for some fixed $u \in U$ and allow values in E^{Δ} [Ethier and Kurtz (1986), Chapter 4, Theorem 5.4]; however, to ensure the existence of stationary solutions in this more general setting we need to allow relaxed controls.

2.3 Definition. An $E \times \mathscr{P}(U)$-valued process $(X(\cdot), \Lambda_{\cdot})$ is a relaxed solution of the controlled martingale problem for A if there exists a filtration $\{\mathscr{F}_t\}$

such that:

(a) $(X(\cdot), \Lambda_.)$ is $\{\mathscr{F}_t\}$-progressive.

(b) For every $f \in \mathscr{D}(A)$, $f(X(t)) - \int_0^t \int_U Af(X(s), u)\Lambda_s(du)\, ds$ is an $\{\mathscr{F}_t\}$-martingale.

Similarly $(X(\cdot), \Lambda_.)$ is a relaxed solution of the controlled martingale problem for (A, ν), $\nu \in \mathscr{P}(E)$, if the above holds and $X(0)$ has distribution ν.

Decision criterion. Let c: $E \times U \to \mathbb{R}$ be lower-semicontinuous, bounded below and satisfy $c(x, u) \to \infty$ as $x \to \Delta$. Define the long-run average cost to be

$$\limsup_{t \to \infty} t^{-1} E\left[\int_0^t \int_U c(X(s), u)\Lambda_s(du)\, ds\right],$$

where $(X(\cdot), \Lambda_.)$ is an $E \times \mathscr{P}(U)$-valued process or $(X(\cdot), \Lambda_.) = ((X(\cdot), \delta_{\{u(\cdot)\}}))$ if $(X(\cdot), u(\cdot))$ is an $E \times U$-valued process.

The control problem is to minimize the long-run average cost subject to the condition that $(X(\cdot), \Lambda_.)$ be a relaxed solution of the controlled martingale problem for A.

3. Preliminaries. It is natural in studying the long-run average cost for a system to take particular interest in the stationary processes for the system. One reason is that the cost is easily determined from knowledge of the one-dimensional distribution. Suppose $(X(\cdot), \Lambda_.)$ is a stationary solution to the controlled martingale problem for A satisfying $E[\chi_{\Gamma_1}(X(0))\Lambda_0(\Gamma_2)] = \mu(\Gamma_1 \times \Gamma_2)$ for all $\Gamma_1 \in \mathscr{B}(E)$ and $\Gamma_2 \in \mathscr{B}(U)$. [Note $\mu \in \mathscr{P}(E \times U)$.] Then

$$\limsup_{t \to \infty} t^{-1} E\left[\int_0^t \int_U c(X(s), u)\Lambda_s(du)\, ds\right] = \int_{E \times U} c(x, u)\mu(dx \times du).$$

Furthermore since for every $f \in \mathscr{D}(A)$,

$$f(X(t)) - \int_0^t \int_U Af(X(s), u)\Lambda_s(du)\, ds \tag{3.1}$$

is an $\{\mathscr{F}_t\}$-martingale, it immediately follows upon taking expectations that:

(v) For every $f \in \mathscr{D}(A)$, $\int Af(x, u)\mu(dx \times du) = 0$.

The goal of this paper is to show that, under assumptions (i)–(iv) on the generator A, condition (v) on the distribution μ is sufficient to characterize μ as a stationary distribution for the system and that there exists a stationary solution $(X(\cdot), \Lambda_.)$ corresponding to μ. The general outline of the proof is to approximate the generator A by its Yosida approximations A_n and to determine a transition function $\hat{\eta}$ for a stationary Markov chain on $E \times U$ having initial distribution μ. We then change from discrete time to continuous time by composition with a Poisson process (preserving the stationarity) and obtain a stationary solution to the controlled martingale problem for A_n (each n). The stationary solution to the (original) controlled martingale problem for A is obtained by passing to the weak limit as $n \to \infty$.

The proof of the existence of the transition function $\hat{\eta}$ heavily relies upon the Hahn–Banach theorem and the Riesz representation theorem which we now state without proof. Proofs can be found in Rudin (1974).

3.1 THEOREM (Hahn–Banach). *If M is a subspace of a normed linear space X and if f is a bounded linear functional on M, then f can be extended to a bounded linear functional F on X so that $\|F\| = \|f\|$.*

3.2 THEOREM (Riesz representation). *To each bounded linear functional Φ on $\hat{C}(X)$, where X is a locally compact Hausdorff space, there corresponds a unique regular Borel measure μ such that $\Phi(f) = \int_X f\,d\mu$, $f \in \hat{C}(X)$.*

The existence of $\hat{\eta}$ also depends upon a technical lemma and the existence of a regular conditional distribution which is given by Morando's theorem. These are adaptations to the control setting of Ethier and Kurtz (1986), Chapter 4, Lemma 9.16 and Appendix 8, respectively. First we give the statements from Ethier and Kurtz (Lemma 3.3 and Theorem 3.4) without proof and then adapt them for the present application (Lemma 3.5 and Theorem 3.6).

3.3 LEMMA. *Let $A\colon \mathscr{D}(A) \to \bar{C}(E)$, $\mathscr{D}(A) \subset \bar{C}(E)$. Suppose for each $\nu \in \mathscr{P}(E)$ that the martingale problem for (A, ν) has a solution with sample paths in $D_E[0,\infty)$. Suppose that φ is continuously differentiable and convex on $G \subset \mathbb{R}^m$, that $(f_1, g_1), \dots, (f_m, g_m) \in G(A)$, $(f_1, \dots, f_m)\colon E \to G$ and that $(\varphi(f_1, \dots, f_m), h) \in G(A)$, where $G(A)$ denotes the graph of A. Then*

$$h \geq \nabla\varphi(f_1, \dots, f_m) \cdot (g_1, \dots, g_m).$$

3.4 THEOREM (Morando). *Let $(Y, \mathscr{Y})$ be a measurable space and let Z be a complete, separable metric space. Let ν be a measure on $Y \times Z$ such that: $0 < \nu(Y \times Z) < \infty$ and define $\mu = \nu(\cdot \times Z)$. Then there exists $\eta\colon Y \times \mathscr{B}(Z) \to [0,\infty)$ such that for each $y \in Y$, $\eta(y, \cdot)$ is a measure on $\mathscr{B}(Z)$, for each $B \in \mathscr{B}(Z)$, $\eta(\cdot, B)$ is $\mathscr{Y}$-measurable and*

$$\nu(C) = \iint \chi_C(y, z)\eta(y, dz)\mu(dy).$$

3.5 LEMMA. *Let $A\colon \hat{C}(E) \to \bar{C}(E \times U)$ and for each $u \in U$ let $A_u f = Af(\cdot, u)$. Suppose for each $u \in U$ and $\nu \in \mathscr{P}(E)$, the martingale problem for (A_u, ν) has a solution with sample paths in $D_E[0,\infty)$. Suppose that φ is continuously differentiable and convex on $G \subset \mathbb{R}^m$, that $(f_1, g_1), \dots, (f_m, g_m) \in G(A)$, $(f_1, \dots, f_m)\colon E \to G$ and that $(\varphi(f_1, \dots, f_m), h) \in G(A)$. Then $h \geq \nabla\varphi(f_1, \dots, f_m) \cdot (g_1, \dots, g_m)$.*

PROOF. This is an immediate application of Lemma 3.3 to the operator A_u. □

3.6 THEOREM. *Let $(\Omega, \mathscr{F}, P)$ be a probability space. Let Λ be a random measure on $U \times [0,\infty)$ such that for all ω, $\Lambda(U \times \Gamma) = m(\Gamma)$ for all $\Gamma \in$*

$\mathscr{B}([0,\infty))$, *where* m *denotes Lebesgue measure. Then there exists* η: $\mathscr{B}(U) \times [0,\infty) \times \Omega \to [0,1]$ *such that*:

(a) $\eta(\cdot, s, \omega) \in \mathscr{P}(U)\ \forall\ s, \omega$.
(b) *For each* $B \in \mathscr{B}(U)$, $\eta(B, \cdot, \cdot)$ *is* $\mathscr{B}([0,\infty)) \times \mathscr{F}$*-measurable.*
(c) $\Lambda(C) = \int\int \chi_C(u, s)\eta(du, s)\, ds$ *a.s.* $\forall\ C \in \mathscr{B}(U) \times \mathscr{B}([0,\infty))$.

PROOF. For $\Gamma_1 \in \mathscr{B}(U)$, $\Gamma_2 \in \mathscr{B}([0,\infty))$ and $\Gamma_3 \in \mathscr{F}$, define $\tilde{\Lambda}$ by

$$\tilde{\Lambda}(\Gamma_1 \times \Gamma_2 \times \Gamma_3) = E\left[\chi_{\Gamma_3} \int \chi_{\Gamma_1}(u)\chi_{\Gamma_2}(s)e^{-s}\Lambda(du \times ds)\right]$$

and extend $\tilde{\Lambda}$ to a measure on $U \times [0,\infty) \times \Omega$. Note $\tilde{\Lambda}(U \times [0,\infty) \times \Omega) = 1$.

We apply Theorem 3.4 to $\tilde{\Lambda}$, where $Y = [0,\infty) \times \Omega$ and the Y-marginal μ satisfies $d\mu = e^{-s}\, ds \times dP$, to get the existence of η. This η fulfills the requirements. □

To pass to the weak limit, we require the following two theorems [Ethier and Kurtz (1986), Chapter 3, Theorems 9.1 and 9.4] in order to establish the relative compactness of the stationary solutions of the controlled martingale problem for the Yosida approximations.

3.7 THEOREM. *Let* (E, r) *be complete and separable and let* $\{X_\alpha\}$ *be a family of processes with sample paths in* $D_E[0,\infty)$. *Suppose that for every* $\eta > 0$ *and* $T > 0$ *there exists a compact set* $\Gamma_{\eta,T} \subset E$ *for which* $\inf_\alpha P\{X_\alpha(t) \in \Gamma_{\eta,T}$ *for* $0 \le t \le T\} \ge 1 - \eta$. *Let* H *be a dense subset of* $\bar{C}(E)$ *in the topology of uniform convergence on compact sets. Then* $\{X_\alpha\}$ *is relatively compact if and only if* $\{f \circ X_\alpha\}$ *is relatively compact (as a family of processes with sample paths in* $D_{\mathbb{R}}[0,\infty)$) *for each* $f \in H$.

The next theorem addresses the problem of verifying the relative compactness of $\{f \circ X_\alpha\}$ for fixed $f \in \bar{C}(E)$. In order to state the theorem, a little notation is necessary.

For each α, let X_α be a process with sample paths in $D_E[0,\infty)$ defined on a probability space $(\Omega_\alpha, \mathscr{F}^\alpha, P_\alpha)$ and adapted to a filtration $\{\mathscr{F}_t^\alpha\}$. Let $\mathscr{L}_\alpha$ be the Banach space of real-valued $\{\mathscr{F}_t^\alpha\}$-progressive processes with norm $\|Y\| = \sup_{t\ge0} E[|Y(t)|] < \infty$. Let

$$\mathbb{A}_\alpha = \left\{(Y, Z) \in \mathscr{L}_\alpha \times \mathscr{L}_\alpha \colon Y(t) - \int_0^t Z(s)\, ds \text{ is an } \{\mathscr{F}_t^\alpha\}\text{-martingale}\right\}.$$

3.8 THEOREM. *Let* (E, r) *be arbitrary and let* $\{X_\alpha\}$ *be a family of processes defined as above. Let* C_α *be a subalgebra of* $\bar{C}(E)$ *and let* D *be the collection of* $f \in \bar{C}(E)$ *such that for every* $\varepsilon > 0$ *and* $T > 0$ *there exists* $(Y_\alpha, Z_\alpha) \in \mathbb{A}_\alpha$ *with*

$$\sup_\alpha E\left[\sup_{t\in[0,T]\cap\mathbb{Q}} |Y_\alpha(t) - f(X_\alpha(t))|\right] < \varepsilon \tag{3.2}$$

and

$$\sup_\alpha E\left[\|Z_\alpha\|_{p,T}\right] < \infty \quad \textit{for some } p \in (1, \infty]. \tag{3.3}$$

($\|h\|_{p,T} = [\int_0^T |h(t)|^p\,dt]^{1/p}$ *if* $p < \infty$; $\|h\|_{\infty,T} = \operatorname{ess\,sup}_{0 \le t \le T}|h(t)|$.) *If* C_a *is contained in the closure of* D *(in the* sup *norm), then* $\{f \circ X_\alpha\}$ *is relatively compact for each* $f \in C_a$; *more generally*, $\{(f_1, \dots, f_k) \circ X_\alpha\}$ *is relatively compact in* $D_{\mathbb{R}^k}[0, \infty)$ *for all* $f_1, \dots, f_k \in C_a$, $1 \le k \le \infty$.

4. Existence of stationary solutions. We now give the statement and proof of the extension of Echeverria's theorem to the control setting (Theorem 4.1). Recall it is assumed Af is bounded for each $f \in \mathscr{D}(A)$, but this condition can be relaxed. We conclude this section by relaxing the boundedness condition (Theorem 4.7).

4.1 THEOREM. *Let* E *and* U *be locally compact separable metric spaces. Let* $A\colon \mathscr{D}(A) \to \bar{C}(E \times U)$, $\mathscr{D}(A) \subset \hat{C}(E)$ *satisfy conditions* (i)–(iv) *and* $\mathscr{D}(A)$ *is an algebra. Let* $\mu \in \mathscr{P}(E \times U)$ *satisfy condition* (v). *Then there exists a stationary relaxed solution* $(X(\cdot), \Lambda_.)$ *to the controlled martingale problem for* A *with*

$$E\left[\chi_{\Gamma_1}(X(0))\Lambda_0(\Gamma_2)\right] = \mu(\Gamma_1 \times \Gamma_2) \quad \forall\, \Gamma_1 \in \mathscr{B}(E),\ \Gamma_2 \in \mathscr{B}(U).$$

PROOF. Without loss of generality we may assume E is compact and $(1, 0) \in G(A)$. If not, let E^Δ be the one-point compactification of E. Define $A^\Delta\colon C(E^\Delta) \to \bar{C}(E \times U)$ by $(A^\Delta f)|_E = A(f - f(\Delta)|_E)$ and $A^\Delta f(\Delta, u) = 0$ for those $f \in C(E^\Delta)$ such that $f - f(\Delta) \in \mathscr{D}(A)$. Extend μ to $E^\Delta \times U$ by setting $\mu(\{\Delta\} \times U) = 0$. Then (A^Δ, μ) satisfies the hypotheses of the theorem.

For $n = 1, 2, 3, \dots,$ define $A_n g = n[(I - n^{-1}A)^{-1} - I]g$ for $g \in \mathscr{R}(I - n^{-1}A)$. Here $\mathscr{R}(\cdot)$ denotes the range of the operator. Note that for all $f \in \mathscr{D}(A)$ and $g = (I - n^{-1}A)f$ we have $\int A_n g\,d\mu = \int Af\,d\mu = 0$. The existence of a stationary solution to the martingale problem for the Yosida approximation A_n is given by the next lemma.

4.2 LEMMA. *There exists a stationary solution* $(X(\cdot), u(\cdot))$ *of the controlled martingale problem for* A_n.

PROOF. Let $M \subset \bar{C}(E \times E \times U)$ be the linear subspace of functions of the form

$$F(x, y, u) = \sum_{i=1}^m h_i(x) g_i(y, u) + f(y, u),$$

where $h_1, \dots, h_m \in \bar{C}(E)$, $f \in \bar{C}(E \times U)$ and $g_1, \dots, g_m \in \mathscr{R}(I - n^{-1}A)$. Define the linear functional Ψ on M by

$$\Psi F = \int \sum_{i=1}^m h_i(x)(I - n^{-1}A)^{-1} g_i(x)\mu(dx \times du) + \int f(y, u)\mu(dy \times du).$$

We will show that $|\Psi F| \le \|F\|$ so that we can apply Theorem 3.1 to extend Ψ to $\bar{C}(E \times E \times U)$ with $\|\Psi\| \le 1$. In addition once $|\Psi F| \le \|F\|$ is shown to be true, since $\Psi 1 = 1$, we have for $F \geq 0$,

$$\|F\| - \Psi F = \Psi(\|F\| - F) \leq \| \|F\| - F\| \leq \|F\|$$

so that $\Psi F \geq 0$. Thus we will be able to apply Theorem 3.2.

Let $f_1, f_2, \ldots, f_m \in \mathscr{D}(A)$ and let $\alpha_k = \|(I - n^{-1}A)f_k\|$, $k = 1, \ldots, m$. Let φ be a polynomial on $\mathbb{R}^m$ which is convex on $\prod_{i=1}^m [-\alpha_i, \alpha_i]$. Since $\mathscr{D}(A)$ is an algebra, $\varphi(f_1, \ldots, f_m) \in \mathscr{D}(A)$. By Lemma 3.5, we have

$$A\varphi(f_1, \ldots, f_m) \geq \nabla\varphi(f_1, \ldots, f_m) \cdot (Af_1, \ldots, Af_m).$$

Thus

$$\begin{aligned}\varphi\big((I - n^{-1}A)f_1, \ldots, (I - n^{-1}A)f_m\big) &\geq \varphi(f_1, \ldots, f_m) - n^{-1}\nabla\varphi(f_1, \ldots, f_m) \cdot (Af_1, \ldots, Af_m) \\ &\geq \varphi(f_1, \ldots, f_m) - n^{-1}A\varphi(f_1, \ldots, f_m).\end{aligned}$$

Integrating this inequality and changing notation yields

$$\int \varphi(g_1, \ldots, g_m)\, d\mu \geq \int \varphi\Big((I - n^{-1}A)^{-1} g_1, \ldots, (I - n^{-1}A)^{-1} g_m\Big)\, d\mu \tag{4.1}$$

for $g_1, \ldots, g_m \in \mathscr{R}(I - n^{-1}A)$. Since all convex functions on $\mathbb{R}^m$ can be approximated uniformly on any compact set $K \subset \mathbb{R}^m$ by a polynomial that is convex on K, (4.1) holds for all φ convex on $\mathbb{R}^m$.

Fix $F \in M$, as $F(x, y, u) = \sum_{i=1}^m h_i(x) g_i(y, u) + f(y, u)$. Define $\varphi: \mathbb{R}^m \to \mathbb{R}$ by $\varphi(r_1, \ldots, r_m) = \sup_x \sum_{i=1}^m h_i(x) r_i$ and note that φ is convex. We then have

$$\begin{aligned}\Psi F &= \int \sum_{i=1}^m h_i(y)(I - n^{-1}A)^{-1} g_i(y)\mu(dy \times du) + \int f(y, u)\mu(dy \times du) \\ &\leq \int \varphi\Big((I - n^{-1}A)^{-1} g_1, \ldots, (I - n^{-1}A)^{-1} g_m\Big)(y)\mu(dy \times du) \\ &\quad + \int f(y, u)\mu(dy \times du) \\ &\leq \int \{\varphi(g_1, \ldots, g_m)(y, u) + f(y, u)\}\mu(dy \times du) \\ &= \int \left\{ \sup_x \sum_{i=1}^m h_i(x) g_i(y, u) + f(y, u) \right\} \mu(dy \times du) \\ &\leq \|F\|.\end{aligned}$$

Also, $-\Psi F = \Psi(-F) \leq \| -F\| = \|F\|$, so $|\Psi F| \leq \|F\|$.

By Theorem 3.1 there exists an extension of Ψ to a bounded linear functional on $\bar{C}(E \times E \times U)$ and by Theorem 3.2 there exists a $\nu \in \mathscr{P}(E \times E \times U)$ such that:

$$\Psi F = \int F\, d\nu \quad \text{for all } F \in \bar{C}(E \times E \times U).$$

Considering the case in which $F(x, y, u) = f(y, u)$ with $f \in \bar{C}(E \times U)$ we have

$$\int f(y, u)\nu(dx \times dy \times du) = \int f(y, u)\mu(dy \times du),$$

so $\nu(E \times \cdot \times \cdot) = \mu(\cdot \times \cdot)$.

Considering $F(x, y, u) = h(x)g(y, u)$ where $h \in \bar{C}(E)$ and $g \in \mathscr{R}(I - n^{-1}A)$, we have

$$\int h(x)g(y, u)\nu(dx \times dy \times du) = \int h(x)\left(I - \frac{1}{n}A\right)^{-1} g(x)\mu(dx \times du).$$

Let $\mu_x(\cdot) = \mu(\cdot \times U)$. By Theorem 3.4, there exists $\hat{\eta}$: $E \times \mathscr{B}(E \times U) \to [0,1]$ such that:

(a) For each $x \in E$, $\hat{\eta}(x, \cdot) \in \mathscr{P}(E \times U)$.
(b) For each $\Gamma \in \mathscr{B}(E \times U)$, $\hat{\eta}(\cdot, \Gamma)$ is $\mathscr{B}(E)$-measurable.
(c) $\nu(G) = \int\int \chi_G(x, y, u)\hat{\eta}(x, dy \times du)\mu_x(dx)$ $\forall$ $G \in \mathscr{B}(E \times E \times U)$.

Thus

$$\int h(x)\left(\int g(y, u)\hat{\eta}(x, dy \times du)\right)\mu_x(dx) = \int h(x)(I - n^{-1}A)^{-1}g(x)\mu_x(dx)$$

for all $h \in \bar{C}(E)$ and so for each $g \in \mathscr{R}(I - n^{-1}A)$,

$$\int g(y,u)\hat{\eta}(x, dy \times du) = (I - n^{-1}A)^{-1}g(x) \quad \text{a.e. } \mu_x.$$

Let $(X_0, u_0), (X_1, u_1), (X_2, u_2), \dots$ be a Markov chain on $E \times U$ having initial distribution μ and transition function $\hat{\eta}$. Then $\{(X_k, u_k)\}$ is stationary. Also, it is easy to verify that for each $g \in \mathscr{R}(I - n^{-1}A)$,

$$g(X_k, u_k) - \sum_{i=0}^{k-1} n^{-1}A_n g(X_i, u_i)$$

is an $\{\mathscr{F}_k\}$-martingale where $\mathscr{F}_k = \sigma((X_0, \mu_0), \dots, (X_k, u_k))$.

Let $V_n(t)$ be a Poisson process with parameter n independent of $\{(X_k, u_k)\}$ and define $(X_n(\cdot), u_n(\cdot)) = (X_{V_n(\cdot)}, u_{V_n(\cdot)})$ and $\mathscr{F}_t^n = \sigma((X_n(s), u_n(s))\colon 0 \le s \le t)$. Then for each $g \in \mathscr{R}(I - n^{-1}A)$ we have $g(X_n(t), u_n(t)) - \int_0^t A_n g(X_n(s), u_n(s))\,ds$ is an $\{\mathscr{F}_t^n\}$-martingale. In addition, $\{(X_n(\cdot), u_n(\cdot))\}$ is stationary and is thus a stationary solution of the martingale problem for (A_n, μ). □

Proof of Theorem 4.1 (continued). By Lemma 4.2, there exists a stationary solution $(X_n(\cdot), u_n(\cdot))$ of the controlled martingale problem for A_n (each n). By Theorems 3.7 and 3.8, the state processes $\{X_n(\cdot)\}$ are relatively compact and therefore for each subsequence of $\{X_n(\cdot)\}$, there exists a weak limit. The particular subsequence we use is determined in passing to a weak limit for the control processes. The next lemma establishes the existence of a relaxed control on $U \times [0, \infty)$ which is a weak limit, for every $t \ge 0$, of the occupation measures for the control on $U \times [0, t]$.

4.3 LEMMA. *Let $\nu \in \mathscr{P}(U)$ and $\{u_n(\cdot)\}$ be a sequence of stationary control processes satisfying $E[\chi_B(u_n(t))] = \nu(B)$ for all $B \in \mathscr{B}(U)$. Let*

$$\Lambda_n^*(B \times C) = \int_0^\infty \chi_{B\times C}(u_n(s), s)\, ds \qquad \forall\, B \in \mathscr{B}(U), C \in \mathscr{B}([0,\infty))$$

and define the sequence of (random) occupation measures on $U \times [0,\infty)$ by extending Λ_n^* to all of $\mathscr{B}(U \times [0,\infty))$. Then there exists a relaxed control process $\Lambda\,\cdot$ such that $\Lambda_n^* \Rightarrow \Lambda^*$ on bounded time intervals, where

$$\Lambda^*(C) = \iint \chi_C(u,s)\Lambda_s(du)\, ds \qquad \forall\, C \in \mathscr{B}(U) \times \mathscr{B}([0,\infty)).$$

PROOF. We begin by normalizing in t. For $n = 1, 2, 3, \ldots$ define

$$\Lambda_n^*(B \times C) = t^{-1}\int_0^t \chi_{B\times C}(u_n(s), s)\, ds \qquad \forall\, B \in \mathscr{B}(U), C \in \mathscr{B}([0,t])$$

and extend Λ_n^* to a measure on $U \times [0, t]$. For each n,

$$E[\Lambda_n^*(B \times C)] = \nu(B)m(C)/t$$

by stationarity. Given $\delta > 0$, select a compact set $K \subset U$ with $\nu(K) \geq 1 - \delta$. Then for each n, $P\{\Lambda_n^*(K^c \times [0,t]) \geq \alpha\} \leq \delta/\alpha$.

This estimate is enough to show tightness of the distributions of $\{\Lambda_n^*\}$ on the space of probability measures on $U \times [0, t]$. Therefore the distributions of $\{\Lambda_n^*\}$ are relatively compact and there is some subsequence which converges weakly to a limit Λ^*. It follows that

$$E[\Lambda^*(B \times C)] = \nu(B)m(C)/t \qquad \forall\, B \in \mathscr{B}(U), C \in \mathscr{B}([0,t]).$$

This result remains valid for $\Lambda_n^*(\cdot) = \int_0^t \chi_{(\cdot)}(U_n(s), s)\, ds$ without normalizing in t and so the existence of a random measure Λ^* on $U \times [0,\infty)$, which is a weak limit of the occupation measures on every bounded time interval, follows by a diagonal argument. In addition,

$$E[\Lambda^*(B \times C)] = \nu(B)m(C) \qquad \forall\, B \in \mathscr{B}(U), C \in \mathscr{B}([0,\infty])).$$

By Theorem 3.6, Λ^* can be written as

$$\Lambda^*(C) = \iint \mathrm{X}_C(u,s)\Lambda_s(du)\, ds \qquad \forall\, C \in \mathscr{B}(U) \times \mathscr{B}([0,\infty])).$$

The process $\Lambda.$ is the candidate for the relaxed control. □

PROOF OF THEOREM 4.1 (continued). By Lemma 4.3, there exists a relaxed control which is the weak limit of the occupation measures of the control processes $\{u_n(\cdot)\}$. By considering a further subsequence to that obtained from the diagonal argument (if necessary), let $X(\cdot)$ be the corresponding weak limit of the state processes. Note that $X(\cdot)$ is stationary since each $X_n(\cdot)$ is stationary.

We now proceed to show that $(X(\cdot), \Lambda_.)$ is a solution of the controlled martingale problem for A. This will be true provided that for all $f \in \mathscr{D}(A)$,

$$f(X(t)) - \int_{U\times[0,t]} Af(X(s),u)\Lambda^*(du \times ds) \tag{4.2}$$

is a martingale with respect to some filtration; in fact, the filtration is

$$\mathscr{F}_t^{X,\Lambda} = \sigma\left(X(s), \int_{s-\delta}^{s} \Lambda_r \, dr \colon 0 \le \delta \le s \le t \right).$$

By a monotone class argument, (4.2) is a martingale if and only if for all $f \in \mathscr{D}(A)$,

$$E\left[\left(f(X(b)) - f(X(a)) - \int_{U\times[a,b]} Af(X(s),u)\Lambda^*(du \times ds)\right) \times \prod_{i=1}^{m} \int_{U\times[t_i-\delta_i, t_i]} h_i(X(t_i),u)\Lambda^*(du \times ds)\right] = 0$$

whenever $0 \le a < b$, $t_1,\dots,t_m \in [0,a]$, $0 \le \delta_i < t_i$, $i = 1,\dots,m$, and $h_1,\dots, h_m \in \hat{C}(E \times U)$.

Let $D = \{t\colon P\{X(t) = X(t-)\} = 1\}$ and first fix $t_1,\dots,t_m, a, b \in D$ and fix $f \in \mathscr{D}(A)$ and $h_1,\dots,h_m \in \overline{C}(E)$. To simplify notation, let

$$H(X,\Lambda^*) = \left(f(X(b)) - f(X(a)) - \int_{U\times[a,b]} Af(X(s),u)\Lambda^*(du \times ds)\right) \times \prod_{i=1}^{m} \int_{U\times[t_i-\delta_i, t_i]} h_i(X(t_i),u)\Lambda^*(du \times ds)$$

and $H_n(X_n, \Lambda_n^*)$ be the corresponding expression with X_n, Λ_n^*, A_n and f_n replacing X, Λ^*, A and f, respectively [recall $f_n = (I - n^{-1}AW)f$]. We will show

$$E[H(X,\Lambda^*)] = \lim_{n\to\infty} E[H_n(X_n, \Lambda_n^*)] = 0.$$

First note that for each n,

$$H_n(X_n,\Lambda_n^*) = \left\{ f_n(X_n(t_{m+1}), u_n(t_{m+1})) - f_n(X_n(t_m), u(t_m)) - \int_{t_m}^{t_{m+1}} A_n f_n(X_n(s), u_n(s))\, ds \right\} \times \prod_{i=1}^{m} \int_{t_i-\delta_i}^{t_i} h_i(X_n(t_i), u_n(s))\, ds$$

by the definition of Λ_n^* and so $E[H_n(X_n, \Lambda_n^*)] = 0$ since $(X_n(\cdot), u_n(\cdot))$ is a solution of the controlled martingale problem for A_n. Since Af is bounded, $\Lambda_n^* \Rightarrow \Lambda^*$ on $U \times [0,a]$ and $X_n \Rightarrow X$ implies $(X_n(t_1),\dots, X_n(t_m), X_n(a),$

$X_n(b)) \Rightarrow (X(t_1), \ldots, X(t_m), X(a), X(b))$, we have for $t = a$ or b,

$$E\left[f_n(X_n(t), U_n(t)) \prod_{i=1}^{m} \int_{U\times[t_i-\delta_i, t_i]} h_i(X_n(t_i), u)\Lambda_n^*(du \times ds)\right]$$

$$\to E\left[f(X(t)) \prod_{i=1}^{m} \int_{U\times[t_i-\delta_i, t_i]} h_i(x(t_i), u)\Lambda^*(du \times ds)\right]$$

as $n \to \infty$. Therefore, we only need to consider the integral term.

Observe that $A_n f_n = n[(I - n^{-1}A)^{-1} - I](I - n^{-1}A)f = Af$ and so we need to establish

$$\lim_{n\to\infty} E\left[\left(\int_{U\times[a,b]} Af(X_n(s), u)\Lambda_n^*(du \times ds)\right) \times \prod_{i=1}^{m} \int_{U\times[t_i-\delta_i, t_i]} h_i(X_n(t_i), u)\Lambda_n^*(du \times ds)\right]$$

$$= E\left[\left(\int_{U\times[a,b]} Af(X(s), u)\Lambda^*(du \times ds)\right) \times \prod_{i=1}^{m} \int_{U\times[t_i-\delta_i, t_i]} h_i(X(t_i), u)\Lambda^*(du \times ds)\right].$$

This follows from the definition of weak convergence and the fact that F defined by

$$F(x, \lambda) = \int_{U\times[a,b]} Af(x(s), u)\lambda(du \times ds)$$

is a bounded, continuous functional on $D_E[0, \infty) \times \mathscr{M}$. $\mathscr{M}$ denotes the space of measures on $U \times [0, \infty)$ having Lebesgue measure as its time marginal.

The only part which still needs to be checked is the stationarity of $\Lambda_\cdot$. We have that $u_n(\cdot)$ is stationary for each n and so defining $\Lambda^n_\cdot \in \mathscr{P}(U)$ by $\Lambda^n_\cdot = \delta_{\{u_n(\cdot)\}}$ we have $\Lambda^n_\cdot$ is stationary. Furthermore for $\delta > 0$, $\int_{t-\delta}^{t} \Lambda^n_s\, ds$ will be stationary in t for $t \geq \delta$. But $\int_{t-\delta}^{t} \Lambda^n_s\, ds = \Lambda_n^*$ restricted to $U \times [t - \delta, t]$ and thus converges weakly as $n \to \infty$ to Λ^* restricted to $U \times [t - \delta, t]$. Thus for $\delta > 0$, Λ^* restricted to $U \times [t - \delta, t]$ is stationary in t (for $t > \delta$) being a limit of stationary processes. The next lemma shows that the relaxed control $\Lambda_\cdot$ can be taken to be stationary, which concludes the proof of Theorem 4.1. □

4.4 Lemma. *Let Λ^* be a random measure on $U \times [0, \infty)$ such that its restriction to $U \times [t - \delta, t]$ is stationary in t for $t > \delta$. Then there exists a stationary relaxed control process $\Lambda_\cdot$ which satisfies*

$$\Lambda^*(C) = \iint \chi_C(u, s)\Lambda_s(du)\, ds \qquad \forall\, C \in \mathscr{B}(U) \times \mathscr{B}([0, \infty)). \tag{4.3}$$

PROOF. By Theorem 3.6, there exists a relaxed control satisfying (4.3). We show that there is a modification of $\Lambda_.$ which is stationary.

Let $\{u_i\}$ be a countable dense subset of U and let $\{B_k\}$ be an enumeration of the balls $\{B(u_i, 1/j)\colon i, j = 1,2,3,\dots\}$. For each B_k, define the stochastic process $\tilde{\Lambda}(B_k)$ by

$$\tilde{\Lambda}_t(B_k) = \limsup_{m\to\infty} 2^m \int_{t-2^{-m}}^{t} \Lambda_s(B_k)\, ds.$$

Since for each k and ω, $\tilde{\Lambda}_t(B_k) = \Lambda_t(B_k)$ a.e. (ds), we have

$$\begin{aligned} 0 &= E\left[\int_0^\infty \chi\{\tilde{\Lambda}_t(B_k) \neq \Lambda_t(B_k) \text{ for some } k\}\, dt\right] \\ &= \int_0^\infty P\{\tilde{\Lambda}_t(B_k) \neq \Lambda_t(B_k) \text{ for some } k\}\, dt, \end{aligned}$$

which implies that for a.e. t, $P\{\tilde{\Lambda}_t(B_k) = \Lambda_t(B_k)\ \forall\ k\} = 1$. Therefore, for a.e. t, $\tilde{\Lambda}_t$ extends to a measure a.s. and, in fact, $\tilde{\Lambda}_t(\cdot) = \Lambda_t(\cdot)$. By the definition, $\{\tilde{\Lambda}_t(B_k)\colon k = 1,2,3,\dots\}$ is stationary which implies that $\tilde{\Lambda}_t$ extends to a measure a.s. for every t. Finally observe that

$$\Lambda^*(C) = \iint \chi_C(u,s)\tilde{\Lambda}_s(du)\, ds \qquad \forall\, C \in \mathscr{B}(u) \times \mathscr{B}([0,\infty)). \qquad \square$$

As mentioned at the beginning of this section, the requirement that Af be bounded can be relaxed. In order to relax this assumption, we require an extension of Theorem 3.8 to demonstrate the compactness of the state processes. The notation is the same as for Theorem 3.8 and a little new notation is needed.

Let $\mathfrak{S}$ denote the set of all real-valued convex functions Φ defined on $\mathbb{R}^+$ satisfying $\Phi(0) = 0$, Φ is nondecreasing and $\Phi(t)/t \to \infty$ as $t \to \infty$. Refer to the Appendix for the definition of, and general results pertaining to, $\|\cdot\|_{\Phi,T}$.

4.5 THEOREM. *Let (E, r) be arbitrary and let $\{X_\alpha\}$ be a family of processes. Let C_a be a subalgebra of $\bar{C}(E)$ and let D be the collection of $f \in \bar{C}(E)$ such that for every $\varepsilon > 0$ and $T > 0$ there exist $(Y_\alpha, Z_\alpha) \in \mathbb{A}_\alpha$ with*

$$\sup_\alpha E\left[\sup_{t\in[0,T]\cap\mathbb{Q}} |Y_\alpha(t) - f(X_\alpha(t))|\right] < \varepsilon \tag{4.4}$$

and

$$\sup_\alpha E[\|Z_\alpha\|_{\Phi,T}] < \infty \quad \textit{for some } \Phi \in \mathfrak{S}. \tag{4.5}$$

If C_a is contained in the closure of D (in the sup *norm), then $\{f \circ X_\alpha\}$ is relatively compact for each $f \in C_a$; more generally, $\{(f_1,\dots,f_k)\circ X_\alpha\}$ is relatively compact in $D_{\mathbb{R}^k}[0,\infty)$ for all $f_1,\dots,f_k \in C_a$, $1 \le k \le \infty$.*

4.6 REMARK. Condition (4.5) will be satisfied if and only if $\{Z_\alpha(t)\colon 0 \le t \le T$, all $\alpha\}$ is uniformly integrable [Ethier and Kurtz (1986), Appendix, Proposition 2.2].

PROOF OF THEOREM 4.5. We must modify the proof of Theorem 3.8 only slightly. Using the generalization of Holder's inequality, for $0 \leq r \leq T$,

$$\int_r^{r+\delta} |h(s)|\, ds \leq 2\|h\|_{\Phi, T+1}\|\chi_{(r, r+\delta)}\|_{\Psi, T+1}$$

and observe that

$$\begin{aligned} \|\chi_{(r, r+\delta)}\|_{\Psi, T+1} &= \inf\left\{t > 0: \int_0^{T+1} \Psi\left(\chi_{(r, r+\delta)}(s)/t\right) ds \leq 1\right\} \\ &= \inf\{t > 0: \delta\Psi(1/t) \leq 1\} \\ &= 1/\Psi^{-1}(1/\delta). \end{aligned}$$

Thus as $\delta \to 0$, $\|\chi_{(r, r+\delta)}\|_{\Psi, T+1} \to 0$. Replacing $\|\cdot\|_{p, T+1}$ in the proof by $\|\cdot\|_{\Psi, T+1}$ yields the result. □

We now extend Theorem 4.1.

4.7 THEOREM . *Let E and U be locally compact, separable metric spaces. Let A: $\mathscr{D}(A) \to C(E \times U)$, $\mathscr{D}(A) \subset \hat{C}(E)$ satisfy conditions* (i) *and* (iv) *and $\mathscr{D}(A)$ is an algebra and let μ satisfy condition (v). Suppose $\psi \in C(U)$ is strictly positive and satisfies*:

(a) $\int \psi\, d\mu < \infty$.
(b) *For each $f \in \mathscr{D}(A)$, $\exists$ constants a_f, $b_f > 0$ such that*

$$|Af(x, u)| \leq a_f + b_f \cdot \psi(u) \qquad \forall\, x, u.$$

Then there exists a stationary relaxed solution $(X(\cdot), \Lambda_.)$ to the controlled martingale problem for A with

$$E\left[\chi_{\Gamma_1}(X(0))\Lambda_0(\Gamma_2)\right] = \mu(\Gamma_1 \times \Gamma_2) \qquad \forall\, \Gamma_1 \in \mathscr{B}(E),\ \Gamma_2 \in \mathscr{B}(U).$$

PROOF. Without loss of generality, we can assume ψ is bounded away from 0. Otherwise, replace ψ by $\psi + 1$.

For $m = 1, 2, 3, \ldots$, let $\psi_m = 2^{-m}(\psi \vee 2^m)$ and $k_m = \int \psi_m\, d\mu$. The fact that $k_m < \infty$ follows from (a). Define A_m on $\mathscr{D}(A)$ with images in $\bar{C}(E \times U)$ by $A_m f(x, u) = Af(x, u)/\psi_m(u)$ and define $\mu_m \in \mathscr{P}(E \times U)$ by $\mu_m(\Gamma) = k_m^{-1}\int_\Gamma \psi_m\, d\mu$. The boundedness of $A_m f$ is due to (b). For each m, (A_m, μ, m) satisfies the conditions of Theorem 4.1 and therefore there is a stationary relaxed solution $(X^m(\cdot), \Lambda^m_.)$ of the controlled martingale problem for (A_m, μ_m). We will obtain a stationary relaxed solution to the controlled martingale problem for (A, μ) as a weak limit of $\{(X^m(\cdot), \Lambda^m_.)\}$.

Compactness of $\{\Lambda^m_.\}$ is established in a similar manner to that in the proof of Theorem 4.1. This requires two observations. First, $\mu_m \Rightarrow \mu$ since $\psi_m/k_m \leq \psi + 1$ for each m and $\psi_m/k_m \to 1$ pointwise as $m \to \infty$. Second, $E[\int_0^t \Lambda^m_s(B)\, ds] = t\mu_m(E \times B)$. Therefore there exists a relaxed control $\Lambda_.$ such that $\int_0^t \Lambda^m_s(\cdot)\, ds \Rightarrow \int_0^t \Lambda(\cdot)\, ds$ for all $t \geq 0$. The stationarity of $\Lambda_.$ follows as in Theorem 4.1.

To establish compactness of $\{X^m\}$ we apply Theorems 3.7 and 4.5. We check the conditions of Theorem 4.5. For fixed $f \in \mathscr{D}(A)$, set $Y^m = f \circ X^m$ and $Z^m(\cdot) = \int_U A_m f(X^m(\cdot), u)\Lambda^m_\cdot(du)$. We only need to verify (3.3) since (3.2) is trivially satisfied.

By (a), there exists $\Phi \in \mathfrak{S}$ such that $\int \Phi \circ \psi \, d\mu < \infty$ [Ethier and Kurtz (1986), Appendix, Proposition 2.2]. Then for each m,

$$
\begin{aligned}
E&\left[\int_0^T \Phi\left(\left|\int_U A_m f(X^m(t), u)\Lambda^m_t(du)\right|\right) dt\right] \\
&\le E\left[\int_0^T \int_U \Phi(|A_m f(X^m(t), u)|)\Lambda^m_t(du)\, dt\right] \\
&= T\int \Phi(|A_m f|)\, d\mu_m \\
&= T\int \Phi(|Af|/\psi_m)\psi_m \, d\mu \\
&\le T\int \Phi(|Af|)/\psi_m \,\psi_m \, d\mu \le T\int \Phi(a_f + b_f\psi)\, d\mu < \infty.
\end{aligned}
$$

The first inequality involves Jensen's inequality and the second follows from the convexity of Φ. Since $\|Z_\alpha\|_{\Phi, T} \le \int_0^T \Phi(|Z_\alpha(t)|)\, dt + 1$, (3.3) is satisfied and $\{X^m\}$ is relatively compact.

Let $(X(\cdot), \Lambda_\cdot)$ be a weak limit of $(X^m(\cdot), \Lambda^m_\cdot)$ and without loss of generality assume the entire sequence converges. We now show that $(X(\cdot), \Lambda_\cdot)$ is a solution of the controlled martingale problem for A. The stationarity follows directly from the stationarity of $(X^m(\cdot), \Lambda^m_\cdot)$.

Again, by a monotone class argument, for each $f \in \mathscr{D}(A)$, $f(X(t)) - \int_0^t \int_U Af(X(s), u)\, \Lambda_s(du) ds$ is an $\{\mathscr{F}_t^{X, \Lambda}\}$-martingale if and only if

$$
\begin{aligned}
E&\Bigg[\left(f(X(b)) - f(X(a)) - \int_a^b \int_U Af(X(s), u)\Lambda_s(du)\, ds\right) \\
&\qquad \times \prod_{i=1}^n \int_{t_i - \delta_i}^{t_i} \int_U h_i(X(t_i), u)\Lambda_s(du)\, ds\Bigg] = 0
\end{aligned} \tag{4.6}
$$

whenever $0 \le a < b$, $t_1, \dots, t_n \in [0, a]$, $0 \le \delta_i < t_i$, $i = 1, \dots, n$, and $h_1, \dots, h_n \in \bar{C}(E \times U)$.

Fix $f \in \mathscr{D}(A)$. Let $D = \{t \ge 0\colon P\{X(t) = X(t-)\} = 1\}$ and fix $a, b, t_1, \dots, t_n, \delta_1, \dots, \delta_n \in D$. Let $H(X, \Lambda)$ denote the LHS of (4.6) and let $H_m(X^m, \Lambda^m)$ denote the same expression with A, X and Λ replaced by A_m, X^m and Λ^m, respectively. We will show

$$
H(X, \Lambda) = \lim_{m \to \infty} H_m(X^m, \Lambda^m)
$$

and hence equals 0 since $(X^m(\cdot), \Lambda^m_\cdot)$ is a solution of the controlled martingale problem for A_m. As in Theorem 4.1, we only need to consider the integral parts of $H(X, \Lambda)$ and $H_m(X^m, \Lambda^m)$.

To simplify notation, for any function F, define

$$I_m(F) = E\left[\int_a^b \int_U F(X^m(s), u)/\psi_m(u)\Lambda_s^m(du)\, ds \times \prod_{i=1}^n \int_{t_i - \delta_i}^{t_i} \int_U h_i(X^m(t_i), u)\Lambda_s^m(du)\, ds\right]$$

and

$$I(F) = E\left[\int_a^b \int_U F(X(s), u)\Lambda_s(du)\, ds \prod_{i=1}^n \int_{t_i - \delta_i}^{t_i} \int_U h_i(X(t_i), u)\Lambda_s(du)\, ds\right].$$

First note that conditions (a) and (b) imply that there exists $g_j \in \overline{C}(E \times U)$, $j = 1, 2, 3 \ldots$ with $g_j \to Af$ in $L^1(d\mu)$. It follows that $I_m(g_j) \to I_m(Af)$ uniformly in m as $j \to \infty$. Similarly, as $j \to \infty$, $I(g_j) \to I(Af)$. Finally, for each g_j, $I_m(g_j) \to I(g_j)$ as $m \to \infty$. Therefore (4.6) holds for $a, b, t_1, \ldots, t_n, \delta_1, \ldots, \delta_n \in D$ and hence (4.6) holds for all $0 \le a < b$, $t_1, \ldots, t_n \in [0, a]$, $0 \le \delta_i < t_i$ by the right continuity of X. Therefore $(X(\cdot), \Lambda_\cdot)$ is a solution of the controlled martingale problem for A. □

APPENDIX

Let Φ be a nonnegative convex function on $[0, \infty)$ with $\Phi(0) = 0$. Define $L_\Phi(\mathbb{R})$ to be the space of equivalence classes of functions f on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \Phi(|f(t)|/s)\, dt < \infty$$

for some $s > 0$. Define the Luxemburg norm $\|\cdot\|_\Phi$ on $L_\Phi(\mathbb{R})$ by

$$\|f\|_\Phi = \inf\left\{s > 0: \int_{\mathbb{R}} \Phi(|f(t)|/s)\, dt \le 1\right\}$$

and note that $L_\Phi(\mathbb{R})$ is a Banach space under $\|\cdot\|_\Phi$. $L_\Phi(\mathbb{R})$ is an Orlicz space.

We now introduce the conjugate of Φ in the sense of Young. Let $\varphi = \Phi'$ and define the inverse ψ of φ by $\psi(t) = \inf\{s \ge 0: \varphi(s) \ge t\}$. Then the conjugate of Φ is defined to be

$$\Psi(t) = \int_0^t \psi(s)\, ds.$$

Note that Ψ is a nonnegative convex function on $[0, \infty)$ with $\Psi(0) = 0$ and so the norm $\|\cdot\|_\Psi$ can be defined as above.

We state the following inequalities without proof [see Weiss (1956)].

1. Young's inequality: $ab \le \Phi(a) + \Psi(b)$.
2. Holder's inequality $\int |f(t)g(t)| dt \le 2\|f\|_\Phi \|g\|_\Psi$.
3. $\|f\|_\Phi \le \int \Phi(|f(t)|)\, dt + 1$.

For $T > 0$, we can define the norm $\|\cdot\|_{\Phi,T}$ by

$$\|f\|_{\Phi,T} = \inf\left\{s > 0: \int_0^T \Phi(|f(t)|/s)\,dt \le 1\right\}$$

and note that each of these inequalities remains valid using the norm $\|\cdot\|_{\Phi,T}$.

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