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FUNCTIONAL LAWS OF THE ITERATED LOGARITHM FOR THE PRODUCT-LIMIT ESTIMATOR OF A DISTRIBUTION FUNCTION UNDER RANDOM CENSORSHIP OR TRUNCATION

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Functional laws of the iterated logarithm are established for a modified version of the classical product-limit estimator of a distribution function when the data are subject to random censorship or truncation. These functional laws are shown to hold for the entire interval I over which the distribution function can be consistently estimated, under basically the same assumptions that have been used in the literature to establish the weak convergence of the normalized estimator in $D(I)$. Making use of stochastic integral representations and empirical process theory, strong approximations involving i.i.d. continuous-parameter martingales are developed for the product-limit estimator, and these strong approximations are then applied to derive the functional laws of the iterated logarithm.

1. Introduction and background. Let $X_1, X_2, \dots, Y_1, Y_2, \dots, T_1, T_2, \dots$ be independent random variables such that the X_i are real valued and have a common continuous distribution function F, the Y_i are extended real valued and have a common distribution function G $[G(y) = P\{Y_i \le y\}]$ and the T_i are extended real valued and have a common distribution function H. The nonparametric maximum likelihood estimator of F based on $X_1, \dots, X_n$ is the empirical distribution function F_n defined by

$$F_n(x) = n^{-1} \sum_1^n I_{\{X_i \le x\}}. \tag{1.1}$$

In many applications, however, the X_i are not completely observable and only "censored data" of the form $(X_i \wedge Y_i, \delta_i)$, $i = 1, \dots, n$, are available, where $\delta_i = I_{\{X_i \le Y_i\}}$ and we use the notation $\wedge$ and $\vee$ to denote minimum and maximum, respectively. For example, survival data in clinical trials or failure time data in reliability studies are often subject to such censoring [cf. Kalbfleisch and Prentice (1980)]. Based on these randomly censored data, the nonparametric maximum likelihood estimator is the product-limit estimator F_n^*, first introduced by Kaplan and Meier (1958) and defined by

$$1 - F_n^*(s) = \prod_{i:\, \delta_i = 1 \text{ and } \tilde{X}_i \le s}^{n} \left\{1 - 1/N_n(\tilde{X}_i)\right\}, \tag{1.2}$$

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where

$$\tilde{X}_i = X_i \wedge Y_i, \qquad N_n(s) = \sum_{i=1}^{n} I_{\{\tilde{X}_i \geq s\}}.$$

Another model of incomplete observations is the "random truncation model," which assumes the presence of truncation variables T_i so that (X_i, T_i) can be observed only when $T_i \leq X_i$. There is no loss of generality in assuming left truncation since the case of right truncation (in which X_i can be observed only when $X_i \leq T_i$) can be transformed to a left truncated model by multiplying the observations by -1. Thus, the data consist of n observations (X_i^o, T_i^o) with $X_i^o \geq T_i^o$. Important examples of this kind of data can be found in the econometrics literature [cf. Tobin (1958) and Amemiya (1985)] and in astronomy [cf. Lynden-Bell (1971), Jackson (1974) and Nicoll and Segal (1980)]. We can regard the observed sample as being generated by a larger sample of independent random variables $X_i, T_i, i = 1, \dots, m$, where $n = n(m)$ is given by

$$\sum_{i=1}^{m} I_{\{X_i \geq T_i\}} = n. \tag{1.3}$$

The nonparametric maximum likelihood estimator in this case is the product-limit estimator of the form

$$1 - F_n^o(s) = \prod_{i:\, X_i^o \leq s}^{n} \{1 - 1/\#(X_i^o; n)\}, \tag{1.4}$$

where

$$\#(s; n) = \sum_{j=1}^{n} I_{\{X_j^o \geq s \geq T_j^o\}} \tag{1.5}$$

[cf. Lynden-Bell (1971), Jackson (1974) and Woodroofe (1985)].

Limit theorems on the empirical distribution function F_n, including its uniform strong consistency (the Glivenko–Cantelli theorem), the weak convergence of the normalized process $n^{1/2}(F_n - F)$ to Brownian bridge and associated functional laws of the iterated logarithm are classical results in probability theory. There has been considerable effort in the literature to extend these results to the product-limit estimator F_n^* in the random censorship model. Let

$$\tau = \inf\{x: F(x) = 1 \text{ or } G(x) = 1\}, \qquad \inf \varnothing = \infty, \tag{1.6}$$

and let $\tilde{X}_{(1)} \leq \cdots \leq \tilde{X}_{(n)}$ denote the order statistics of the observed sample. Then $\tilde{X}_{(n)} \leq \tau$ and $\lim_{n\to\infty} \tilde{X}_{(n)} = \tau$ a.s. Making use of martingale theory and stochastic integral representations, Wang (1987) recently showed that

$$\sup_{t \leq \tilde{X}_{(n)}} |F_n^*(t) - F(t)| \to_P 0. \tag{1.7}$$

He also indicated that if $F(\tau) = 1$ or if $G(\tau-) < 1$, then

$$\sup_{t<\tau} |F_n^*(t) - F(t)| \to 0 \quad \text{a.s.} \tag{1.8}$$

Making use of martingale functional central limit theorems and assuming that

$$(1.9)\qquad \int_{-\infty}^{\tau} dF(s)/(1-G(s-)) < \infty,$$

Gill (1983) showed that $\{n^{1/2}(F_n^*(t) - F(t \wedge \tilde{X}_{(n)})), t < \tau\}$ converges weakly in $D(-\infty, \tau)$ to a zero-mean Gaussian process $\{Z(t), t < \tau\}$ with covariance function

$$(1.10)\qquad \begin{aligned} EZ(t)Z(t') &= (1-F(t))(1-F(t')) \\ &\quad\times \int_{-\infty}^{t\wedge t'} (1-G(s-))^{-1}(1-F(s))^{-2}\,dF(s), \qquad t \vee t' < \tau. \end{aligned}$$

The limiting Gaussian process Z can therefore be written as

$$(1.11)\qquad Z(t) = (1-F(t))W(\sigma(t)), \qquad t < \tau,$$

where $\{W(s), s \geq 0\}$ is a standard Wiener process and

$$(1.12)\qquad \sigma(t) = \int_{-\infty}^{t} (1-G(s-))^{-1}(1-F(s))^{-2}\,dF(s).$$

Note that if $G(s) = 0$ for all $s < \tau$ (i.e., no censoring is present), then $\sigma(t) = F(t)/(1-F(t))$ and $\{Z(F^{-1}(s)), 0 \leq s \leq 1\}$ is a Brownian bridge, while F_n^* reduces to the empirical distribution function F_n since $N_n(\tilde{X}_{(i)}) = n - i + 1$ in this case.

Results on the laws of the iterated logarithm for F_n^*, however, are much less complete. A commonly used method in the literature to obtain such results is to regard the cumulative hazard function

$$(1.13)\qquad \Lambda_n(x) = \int_{-\infty}^{x} dL_n(s)/N_n(s), \quad \text{where} \quad L_n(s) = \sum_{i=1}^{n} I_{\{\tilde{X}_i \leq s,\, \delta_i = 1\}},$$

as a functional of the empirical subdistribution functions $n^{-1}L_n$ and $n^{-1}N_n$ and to apply empirical process theory to analyze these empirical subdistribution functions and the associated functional. A difficulty with this method is that for s near τ, $n^{-1}N_n(s)$ approaches 0, leading to a "singularity" of the functional associated with Λ_n. Assuming that G is continuous with $G(\tau) < 1 = F(\tau)$ and using instead of (1.2) the slightly modified version

$$(1.14)\qquad \begin{aligned} 1 - F_n^{\mathrm{I}}(s) &= \prod_{i:\, \delta_i = 1 \text{ and } \tilde{X}_i \leq s}^{n} \left\{1 - 1/\left(N_n(\tilde{X}_i) + 1\right)\right\}, \quad \text{if } s < \tilde{X}_{(n)}, \\ &= 0, \quad \text{if } s \geq \tilde{X}_{(n)}, \end{aligned}$$

Földes and Rejtő (1981) were able to use this approach to show that

$$(1.15)\qquad \limsup_{n\to\infty} (n/\log\log n)^{1/2} \sup_{s<\tau} |F_n^{\mathrm{I}}(s) - F(s)| < \infty \quad \text{a.s.}$$

Subsequently, Csörgő and Horváth (1983) studied an alternative left continuous modification of (1.2) defined by

$$\begin{aligned} 1 - F_n^{\mathrm{II}}(s) &= \prod_{i:\, \delta_i = 1 \text{ and } \tilde{X}_i < s}^{n} \left\{1 - 1/N_n(\tilde{X}_i)\right\}, \quad \text{if } s \le \tilde{X}_{(n)}, \\ &= 0, \quad \text{if } s > \tilde{X}_{(n)}. \end{aligned} \tag{1.16}$$

They showed that if $t < \tau$, then with probability 1, the sequence $\{(n/2 \log\log n)^{1/2}(F_n^{\mathrm{II}} - F)|_{(-\infty, t]}\}$ of functions restricted to $(-\infty, t]$ is relatively compact in the space of bounded functions on $(-\infty, t]$ with the sup-norm metric and its set of limit points is

$$\left\{(1 - F(\cdot))\sigma^{1/2}(t) g(\sigma(\cdot)/\sigma(t)) \colon g \in K\right\}, \tag{1.17}$$

where σ is defined in (1.12) and K is Strassen's (1964) unit ball of absolutely continuous functions on $[0, 1]$ defined by

$$K = \left\{g | g \colon [0,1] \to R,\ g(0) = 0,\ \int_0^1 (dg/dx)^2\, dx \le 1\right\}. \tag{1.18}$$

They also noted that if $G(\tau -) = 1 > F(\tau)$, then this result cannot be extended to $(-\infty, \tau)$ since $\tilde{X}_{(n)} < \tau$ a.s. in this case and by (1.16),

$$F_n^{\mathrm{II}}(\tilde{X}_{(n)} +) - F(\tilde{X}_{(n)} +) = 1 - F(\tilde{X}_{(n)}) \to 1 - F(\tau) \quad (> 0) \quad \text{a.s.} \tag{1.19}$$

In this paper we use a different approach to study functional laws of the iterated logarithm for the product-limit estimator. First note that since Strassen's set (1.18) is the unit ball of the reproducing kernel Hilbert space of the Wiener process W, it follows from (1.11) that (1.17) is the unit ball of the reproducing kernel Hilbert space of the limiting Gaussian process $\{Z(s), s < t\}$ in the aforementioned weak convergence result due to Gill (1983). Note, however, that if $t = \tau$ and $F(\tau) = 1$ $[\Rightarrow \sigma(\tau) = \infty]$, then the unit ball of the reproducing kernel Hilbert space of $\{Z(s), s < \tau\}$ cannot be written in the form (1.17). Since Gill's weak convergence theorem holds in $D(-\infty, \tau)$ under the assumption (1.9), it is natural to ask whether one can develop a functional law of the iterated logarithm for some suitable modification of F_n^* on the entire interval $(-\infty, \tau)$. Recalling that the uniform strong consistency property (1.8) holds for F_n^* if $F(\tau) = 1$ and that this property fails to hold in (1.19) where $G(\tau -) = 1 > F(\tau)$, we shall modify F_n^* by the following product-limit estimator $\tilde{F}_n$.

Let $c > 0$ and $\frac{1}{3} < \alpha < \frac{1}{2}$ and choose $\varepsilon_n > 0$ such that

$$\varepsilon_n \to 0 \quad \text{and} \quad \varepsilon_n n^{\gamma} \to \infty \quad \text{as } n \to \infty \text{ for some } \gamma < \alpha/2. \tag{1.20}$$

For every s, define $F_n^*(s)$ by (1.2) and let

$$\begin{aligned} \tilde{F}_n(s) &= F_n^*(s \wedge \tilde{X}_{(n - [cn^{\alpha}])}), \quad \text{if} \quad F_n^*(\tilde{X}_{(n-[cn^{\alpha}])}) < 1 - \varepsilon_n, \\ &= F_n^*(s), \quad \text{otherwise.} \end{aligned} \tag{1.21}$$

For this product-limit estimator, we are able to establish various functional laws of the iterated logarithm on the entire interval $(-\infty, \tau)$, under Gill's assumption (1.9) in the case $F(\tau) < 1$ and under a slightly stronger assumption [see (1.22)] in the case $F(\tau) = 1$, as corollaries of the following strong approximation result using i.i.d. martingales.

THEOREM 1. *Let $\tilde{F}_n$ be the product-limit estimator* (1.21) *based on the censored data* $(\tilde{X}_i, \delta_i)$, $i = 1, \dots, n$, *where* $\tilde{X}_i = X_i \wedge Y_i$ *and* $\delta_i = I_{\{X_i \le Y_i\}}$. *Suppose that the distribution function F of the X_i is continuous. Assume that the distribution function G of the Y_i satisfies* (1.9) *in the case* $F(\tau) < 1$, *where τ is defined in* (1.6). *In the case* $F(\tau) = 1$, *assume the stronger condition*

$$(1.22)\qquad (1 - F(s))^{\beta} = O(1 - G(s-)) \quad \text{as } s \uparrow \tau, \text{ for some } 0 \le \beta < 1.$$

For $s < \tau$, *let* $\Lambda(s) = -\log(1 - F(s))$ *and define*

$$(1.23)\qquad \xi_i(s) = I_{\{X_i \le Y_i \wedge s\}} - \int_{-\infty}^{s} I_{\{X_i \wedge Y_i \ge u\}}\, d\Lambda(u),$$

$$(1.24)\qquad X_i(s) = (1 - F(s)) \int_{-\infty}^{s} (1 - F(u))^{-1} (1 - G(u-))^{-1}\, d\xi_i(u).$$

(i) *Suppose that* $F(\tau) = 1$. *Then for every* $q \in (\beta/(\beta + 1), \frac{1}{2})$,

$$(1.25)\qquad \sup_{s<\tau} \left| n\big(\tilde{F}_n(s) - F(s)\big) - \sum_{1}^{n} X_i\big(s \wedge F^{-1}(1 - n^{-(1-q)})\big) \right| = O(n^q) \quad a.s.$$

(ii) *Suppose that* $F(\tau) < 1$. *Let* $s_n = \tilde{X}_{(n-[cn^{\alpha}])}$, *where* $c > 0$ *and* $\frac{1}{3} < \alpha < \frac{1}{2}$ *are the same as in the definition* (1.21) *of* $\tilde{F}_n$. *Then for every* $q \in (1 - 3\alpha/2, \frac{1}{2})$,

$$(1.26)\qquad \sup_{s<\tau} \left| n\big(\tilde{F}_n(s) - F(s)\big) - \sum_{1}^{n} X_i(s \wedge s_n) \right| = o(n^q) + O(n^{\alpha}) \quad a.s.$$

COROLLARY 1. *Under the same assumptions as in Theorem* 1, *with probability* 1, *the sequence* $\{(n/2\log\log n)^{1/2}(F_n - F)\}$ *is relatively compact in* $D(-\infty, \tau)$ *and its set of limit points is the unit ball K_1 of the reproducing kernel Hilbert space of the covariance kernel* (1.10). *Consequently,*

$$(1.27)\qquad \begin{aligned} &\limsup_{n\to\infty} (n/2\log\log n)^{1/2} \sup_{s<\tau} |\tilde{F}_n(s) - F(s)| \\ &\quad = \sup_{t<\tau} (1 - F(t))\sigma(t) \quad a.s., \end{aligned}$$

where σ is defined in (1.12).

COROLLARY 2. *Define*

$$U_n(r, s) = r\big(\tilde{F}_{[nr]}(s) - F(s)\big), \qquad 0 \le r \le 1,\ s < \tau.$$

Under the same assumptions as in Theorem 1, *with probability* 1, *the sequence* $\{(n/2\log\log n)^{1/2}U_n\}$ *is relatively compact in* $D([0,1] \times (-\infty, \tau))$ *and its set of limit points is the unit ball K_2 of the reproducing kernel Hilbert space of the covariance kernel*

$$(1.28)\qquad \begin{aligned} &\Gamma((r, t), (r', t')) = (r \wedge r') EZ(t)Z(t'), \\ &\qquad\qquad 0 \le r,\ r' \le 1,\ t \vee t' < \tau, \end{aligned}$$

where $EZ(t)Z(t')$ *is given by* (1.10).

As will be shown in Section 2, Theorem 1 and Corollaries 1 and 2 on the modified product-limit estimator $\tilde{F}_n$ also imply laws of the iterated logarithm and strong approximations for the classical product-limit estimator (1.2) and its variants (1.14) and (1.16). This is the content of

COROLLARY 3. *Under the same assumptions as in Theorem* 1, *consider the classical product-limit estimator* F_n^* *defined in* (1.2) *and its variants* F_n^{I} *and* F_n^{II} *defined in* (1.14) *and* (1.16), *respectively. Then* $F_n^{\mathrm{II}}(s) = F_n^*(s-)$ *for all* s.

(i) *Suppose that* $F(\tau) = 1$. *Then for every* $q \in (\beta/(\beta+1), \frac{1}{2})$, (1.25) *still holds with* $\tilde{F}_n$ *replaced by* F_n^* *or* F_n^{I}. *Consequently, the laws of the iterated logarithm in Corollaries* 1 *and* 2 *still hold if* $\tilde{F}_n$ *is replaced by any of* F_n^*, F_n^{I}, F_n^{II}.

(ii) *Suppose that* $F(\tau) < 1$. *Let* $\frac{1}{3} < \alpha < \frac{1}{2}$ *and let* $\tau_n = \sup\{t: 1 - G(t-) \geq n^{-(1-\alpha)}\}$. *Then for every* $q \in (1 - 3\alpha/2, \frac{1}{2})$,

$$\sup_{t \leq \tau_n} \left| n(F_n^*(s) - F(s)) - \sum_1^n X_i(s) \right| = o(n^q) \quad a.s. \tag{1.29a}$$

Consequently, the law of the iterated logarithm (1.27) *still holds with* τ *replaced by* τ_n *if* $\tilde{F}_n$ *is replaced by* F_n^* *or* F_n^{II}. *Moreover, letting* $\frac{1}{4} < q < \frac{1}{2}$ *and* $\tau_n(q) = \sup\{t: 1 - G(t-) \geq n^{-q}\}$, *we have*

$$\sup_{t \leq \tau_n(q)} \left| n\big(F_n^{\mathrm{I}}(s) - F(s)\big) - \sum_1^n X_i(s) \right| = O(n^q) \quad a.s. \tag{1.29b}$$

and the law of the iterated logarithm (1.27) *still holds with* τ *replaced by* $\tau_n(q)$ *if* $\tilde{F}_n$ *is replaced by* F_n^{I}.

REMARK. The step functions F_n^*, $\tilde{F}_n$ and F_n^{I} are right continuous and therefore are elements of $D(-\infty, \tau)$. However, F_n^{II} is left continuous and therefore for this case, $D(-\infty, \tau)$ in Corollary 3(i) refers to the class of left continuous functions with right-hand limits.

The proofs of Theorem 1 and Corollaries 1–3 will be given in Section 2. If the X_i are uniform random variables on (0,1) and $Y_i \geq 1$ a.s., then $\tau = 1$ and $EZ(t)Z(t')$ is the covariance kernel of Brownian bridge. Thus, (1.28) reduces to the covariance kernel of the Kiefer process and Corollary 2 reduces to the Kiefer–Wichura functional law of the iterated logarithm for sequential empirical processes [cf. Kiefer (1972), Wichura (1973) and Lai (1974)]. Note also that in the case $F(\tau) = 1$, Corollary 3(i) provides a definitive form of the Földes–Rejtő result (1.15) under (1.22), which is much weaker than their assumption $G(\tau) < 1$. As will be shown in Section 2, the limit set K_1 in Corollary 1 has the explicit representation

$$\begin{aligned} K_1 = \Big\{\psi: \psi(t) = (1 - F(t)) \int_{-\infty}^t (1 - G(s-))^{-1/2} (1 - F(s))^{-1} \\ \times h(F(s))\, dF(s) \text{ for } t < \tau, \\ \text{for some } h \in L_2[0, F(\tau)] \text{ with } \int_0^{F(\tau)} h^2(u)\, du \leq 1\Big\}. \end{aligned} \tag{1.30}$$

Likewise the limit set K_2 in Corollary 2 can be represented in the form

$$(1.31)\quad \begin{aligned} K_2 = \bigg\{\psi\colon \psi(r,t) &= (1-F(t))\iint_{0\le\rho\le r,\, s\le t}(1-G(s-))^{-1/2} \\ &\qquad \times(1-F(s))^{-1}h(\rho,F(s))\,d\rho\,dF(s) \\ &\text{for } t<\tau \text{ and } 0\le r\le 1, \\ &\text{for some } h\in L_2([0,1]\times[0,F(\tau)]) \\ &\text{with } \iint_{[0,1]\times[0,F(\tau)]} h^2(\rho,u)\,d\rho\,du\le 1\bigg\}. \end{aligned}$$

Theorem 1 is very different from previous strong approximations of the Kaplan–Meier estimator due to Burke, Csörgő and Horváth (1981) and to Lo and Singh (1986). Assuming that G is continuous, Burke, Csörgő and Horváth (1981) used strong approximations of empirical subdistribution functions and expressed the Kaplan–Meier estimator as a functional of these subdistribution functions to show that if $\lambda_n \ge \{2(1+\delta)n^{-1}\log n\}^{1/2}$ for some $\delta > 0$, then by redefining the random variables on a new probability space if necessary, there exist Gaussian processes Z_n with limiting covariance (1.10) such that

$$(1.32)\quad \begin{aligned} &\sup_{(1-F(s))(1-G(s))\ge\lambda_n} |n^{1/2}(F_n^*(s)-F(s))-Z_n(s)| \\ &\quad = O\Big(\max\big\{n^{-1/3}\lambda_n^{-2}(\log n)^{3/2},\, n^{-1/2}\lambda_n^{-4}\log n,\, n^{-3/2}\lambda_n^{-6}(\log n)^2\big\}\Big) \quad \text{a.s.} \end{aligned}$$

Taking $t_0 < \tau$ and partitioning the interval $[0, t_0]$ into sufficiently small subintervals (assuming the random variables to be nonnegative), Lo and Singh (1986) made use of uniform probability bounds for sums of i.i.d. bounded random variables to prove a strong approximation of the form

$$(1.33)\quad \sup_{0\le s\le t_0}\left|F_n^*(s)-F(s)-n^{-1}\sum_1^n f(\tilde{X}_i,\delta_i;s)\right| = O\Big(\big(n^{-1}\log n\big)^{3/4}\Big) \quad \text{a.s.}$$

where

$$f(x,\delta;s) = (1-F(s))\bigg\{-\int_0^{x\wedge s}(\Psi(u))^{-2}\,d\Psi^*(u) + (\Psi(x))^{-1}I_{\{x\le s,\,\delta=1\}}\bigg\}$$

for $\delta = 0$ or 1 and $\Psi^*(u) = P\{X_1 \le u \wedge Y_1\}$, $\Psi(u) = P\{X_1 \wedge Y_1 \ge u\}$. It is interesting to note that (1.32) provides a sharper rate of approximation than (1.33) since (1.32) implies that

$$\sup_{s\le t_0}|F_n^*(s)-F(s)-n^{-1/2}Z_n(s)| = O\big(n^{-5/6}(\log n)^{3/2}\big) \quad \text{a.s.}$$

However, the approximations (1.32) and (1.33) are clearly not strong enough to yield functional laws of the iterated logarithm in $D(-\infty,\tau)$ or $D([0,1]\times(-\infty,\tau))$, as given in Corollaries 1 and 2.

In Section 3 we develop analogues of these results for a modified version of the product-limit estimator (1.4) in the random truncation model and also study the case in which the data are subject to both censoring and truncation. In Section 4 we extend these results to time-sequential product-limit estimators which are of particular interest in the analysis of sequentially monitored survival data during the course of a long-term clinical trial [cf. Tsiatis (1982), Jennison and Turnbull (1985) and Gu (1987)].

2. Proof of Theorem 1 and its corollaries. Throughout this section we shall use the same notations and assumptions as in Theorem 1. Let $\mathscr{F}(s)$ denote the complete σ-field generated by $I_{\{\tilde{X}_i \le s\}}$, $\delta_i I_{\{\tilde{X}_i \le s\}}$, $\tilde{X}_i I_{\{\tilde{X}_i \le s\}}$, $i = 1, 2, \dots$, and let

$$(2.1) \qquad M_n(t) = \sum_1^n \xi_i(t) = \sum_1^n I_{\{\tilde{X}_i \le t, \delta_i = 1\}} - \int_{-\infty}^t N_n(s)\, d\Lambda(s),$$

where N_n is defined in (1.2). To prove Theorem 1, we make use of exponential inequalities for martingales (in Lemmas 3 and 4) and for weighted empirical processes (in Lemma 2) together with the following well-known martingale property of M_n [cf. Gill (1980) and Métivier (1982)]. We shall use the notation $\Delta M_n(s)$ to denote $M_n(s) - M_n(s-)$.

LEMMA 1. *For every* $n \ge 1$, $\{M_n(s), \mathscr{F}(s), -\infty < s < \infty\}$ *is a square integrable martingale whose predictable variation process is* $\langle M_n \rangle(t) = \int_{-\infty}^t N_n(s)\, d\Lambda(s)$.

LEMMA 2. *For every* $0 < \delta < 1$ *and* $\varepsilon > 0$,

$$(2.2) \qquad \sup_{s:\, EN_n(s) \ge n^\delta} (EN_n(s))^{1/2-\varepsilon} |EN_n(s)/N_n(s) - 1| \to 0 \quad a.s.$$

PROOF. Recall that $n^{-1}N_n(s) = n^{-1}\sum_1^n I_{\{\tilde{X}_i \ge s\}}$ and that the $\tilde{X}_i$ are i.i.d. with $P\{\tilde{X}_i \ge s\} = (1 - F(s))(1 - G(s-))$, where F is continuous. If G is also continuous, then we can apply Lemma 3 of Wellner (1978) to conclude that for every $1 \ge \lambda > 0$ and $0 < \theta < \lambda/2$,

$$P\left\{ \sup_{s:\, n^{-1}EN_n(s) \ge n^{-(1-\lambda)}} |EN_n(s)/N_n(s) - 1| \ge n^{-\lambda/2+\theta} \right\} = O\left(\exp\left\{-\tfrac{1}{3}n^{2\theta}\right\}\right).$$

Hence by the Borel–Cantelli lemma, for every $1 \ge \lambda > 0$ and $\theta > 0$,

$$(2.3) \qquad \sup_{s:\, EN_n(s) \ge n^\lambda} |EN_n(s)/N_n(s) - 1| = o(n^{-\lambda/2+\theta}) \quad \text{a.s.}$$

Without assuming the distribution function of the $\tilde{X}_i$ to be continuous, we can apply Theorem 1.3.1 of van Zuijlen (1977) together with the result (2.3) under the continuity assumption to show that (2.3) still holds in the general case.

Given $0 < \varepsilon < \frac{1}{2}$ and $1 > \delta > 0$, take $\gamma > 0$ such that $\gamma/2 < \delta\varepsilon$. Let J be the smallest nonnegative integer such that $\delta + (J+1)\gamma \geq 1$. For $j = 0, 1, \dots, J$,

$$\begin{aligned}
&\sup_{n^{\delta+j\gamma} \leq EN_n(s) \leq n^{\delta+(j+1)\gamma}} (EN_n(s))^{1/2-\varepsilon} |EN_n(s)/N_n(s) - 1| \\
&\quad \leq \sup_{EN_n(s) \geq n^{\delta+j\gamma}} n^{(\delta+j\gamma+\gamma)(1/2-\varepsilon)} |EN_n(s)/N_n(s) - 1| \\
&\quad \leq \Big\{ \sup_{EN_n(s) \geq n^{\delta+j\gamma}} |EN_n(s)/N_n(s) - 1| \Big\} n^{(\delta+j\gamma)/2-(\delta\varepsilon-\gamma/2)} \to 0 \quad \text{a.s.}
\end{aligned}$$

by (2.3). Hence (2.2) follows. □

LEMMA 3. *For every* $0 < \rho < 1$ *and* $\varepsilon > 0$,

$$\sup_{t:\, EN_n(t) \geq n^\rho} |(1 - F_n^*(t))/(1 - F(t)) - 1| = o(n^{-\rho/2+\varepsilon}) \quad a.s.$$

Moreover,

$$\sup_{t:\, EN_n(t) \geq n^\rho \text{ or } N_n(t) \geq n^\rho} |N_n(t)/EN_n(t) - 1| = O(1) \quad a.s. \tag{2.4}$$

PROOF. (2.4) follows from Lemma 2 (with $\delta < \rho$). Let

$$Z_n(t) = \int_{-\infty}^t \big[I_{\{N_n(s) \geq n^\rho/2\}} / N_n(s) \big] \, dM_n(s).$$

Then by Lemma 1, $\{Z_n(t), \mathscr{F}(t), -\infty < t < \infty\}$ is a martingale whose predictable variation process is

$$\langle Z_n \rangle(t) = \int_{-\infty}^t \big[I_{\{N_n(s) \geq n^\rho/2\}} / N_n(s) \big] \, d\Lambda(s).$$

Note that

$$\sup_{EN_n(t) \geq n^\rho} \langle Z_n \rangle(t) \leq 2n^{-\rho} \sup_{1-F(t) \geq n^{-(1-\rho)}} \int_{-\infty}^t d\Lambda(t) \leq 2n^{-\rho} \log n. \tag{2.5}$$

Since $\sup_s |\Delta M_n(s)| \leq 1$ a.s. by (2.1) and the continuity of F, $\sup_s |\Delta Z_n(s)| \leq 2n^{-\rho}$ a.s. Hence by the exponential inequality for martingales [cf. Shorack and Wellner (1986), page 899], for every $0 < \varepsilon < \rho/2$,

$$P\Big\{ \sup_{t \leq b_n} |Z_n(t)| \geq n^{-\rho/2+\varepsilon} \Big\} = \exp\big\{ -\big(\tfrac{1}{2} + o(1)\big) n^{2\varepsilon} (2 \log n)^{-1} \big\},$$

where

$$\begin{aligned}
b_n &= \sup\{t: EN_n(t) \geq n^\rho\} \\
&= \sup\{t: (1 - F(t))(1 - G(t-)) \geq n^{-(1-\rho)}\}.
\end{aligned} \tag{2.6}$$

Therefore by the Borel–Cantelli lemma,

$$\sup_{t \le b_n} |Z_n(t)| = o(n^{-\rho/2+\varepsilon}) \quad \text{a.s.} \tag{2.7}$$

Defining L_n and Λ_n as in (1.13) and using the convention $0/0 = 0$, it follows from (1.2) that

$$\log(1 - F_n^*(t)) = \sum_{s \le t} \log\{1 - \Delta L_n(s)/N_n(s)\}.$$

Therefore, by (2.4), with probability 1, we have uniformly in $t \le b_n$,

$$\begin{aligned} \log(1 - F_n^*(t)) &= -\int_{-\infty}^{t} \{(N_n(s))^{-1} + O((N_n(s))^{-2})\}\, dL_n(s) \\ &= -\Lambda_n(t)\{1 + O(n^{-\rho})\}. \end{aligned} \tag{2.8}$$

Moreover, by (2.4), with probability 1, for all large n,

$$\inf_{s \le b_n} I_{\{N_n(s) \ge n^\rho/2\}} = I_{\{N_n(b_n) \ge n^\rho/2\}} = 1$$

and therefore

$$Z_n(t) = \int_{-\infty}^{t} dM_n(s)/N_n(s) = \Lambda_n(t) - \Lambda(t) \quad \text{for all } t \le b_n. \tag{2.9}$$

Combining (2.7)–(2.9), we obtain that with probability 1, for all large n,

$$\begin{aligned} \log\{(1 - F_n^*(t))/(1 - F(t))\} &= -\{\Lambda_n(t) - \Lambda(t)\} + O(n^{-\rho}\Lambda_n(t)) \\ &= -Z_n(t)\{1 + O(n^{-\rho})\} + O(n^{-\rho}\Lambda(t)) \\ &= o(n^{-\rho/2+\varepsilon}) \quad \text{uniformly in } t \le b_n, \end{aligned}$$

since $\Lambda(b_n) \le \int_{1-F(t) \ge n^{-(1-\rho)}} d\Lambda(t) = O(\log n)$. □

LEMMA 4. *Define*

$$U_n(t) = n\int_{-\infty}^{t} (1 - F(s))\left\{\left(\frac{1 - F_n^*(s-)}{1 - F(s)}\right)\frac{1}{N_n(s)} - \frac{1}{EN_n(s)}\right\} dM_n(s).$$

(i) *Suppose that* $F(\tau) = 1$ *and* (1.22) *holds. Then for every* $q < 1$ *such that* $q > \beta/(\beta + 1)$, *where* β *is given in* (1.22), $\sup_{1-F(t) \ge n^{-(1-q)}} |U_n(t)| = o(n^q)$ *a.s.*

(ii) *Suppose that* $F(\tau) < 1$ *and* (1.9) *holds. Let* $c > 0$ *and* $0 < \alpha < \frac{1}{2}$ *and define* $b_n = \sup\{t\colon (1 - F(t))(1 - G(t-)) \ge \frac{1}{2}cn^{-(1-\alpha)}\}$. *Then for every* $q < 1$ *such that* $q > 1 - 3\alpha/2$, $\sup_{t \le b_n} |U_n(t)| = o(n^q)$ *a.s.*

PROOF. Using Lemma 3 and an argument similar to the proof of Lemma 2, it can be shown that for every $1 > \delta > 0$ and $\varepsilon > 0$,

$$\sup_{s\colon EN_n(s) \ge n^\delta} (EN_n(s))^{1/2-\varepsilon} |(1 - F_n^*(s))/(1 - F(s)) - 1| \to 0 \quad \text{a.s.} \tag{2.10}$$

First consider the case $F(\tau) = 1$. Let $1 > q > \beta/(\beta + 1)$ and let $a_{n,q} = F^{-1}(1 - n^{-(1-q)})$. By (1.22), there exists $\lambda > 0$ such that for all large n,

$$EN_n(a_{n,q}) \ge n(1 - F(a_{n,q}))(1 - G(a_{n,q}-)) \ge \lambda n^{1-(1+\beta)(1-q)}. \tag{2.11}$$

Since $1 - (1 + \beta)(1 - q) > 0$, it follows from (2.11) together with (2.10) and (2.2) that with probability 1, uniformly in $s \le a_{n,q}$,

$$\left(\frac{1-F_n^*(s-)}{1-F(s)}\right)\frac{1}{N_n(s)}-\frac{1}{EN_n(s)}$$
$$(2.12)\qquad =\frac{1}{N_n(s)}-\frac{1}{EN_n(s)}+O\left(\frac{(EN_n(s))^{-1/2+\varepsilon}}{N_n(s)}\right)$$
$$=O\left(\frac{(EN_n(s))^{-1/2+\varepsilon}}{N_n(s)}\right)$$
$$=O\left(\frac{\{n(1-F(s))(1-G(s-))\}^{-1+\varepsilon}}{N_n^{1/2}(s)}\right).$$

By Lemma 1, $\{U_n(t), \mathscr{F}(t), -\infty < t \le a_{n,q}\}$ is a martingale and by (2.12),

$$\langle U_n\rangle(a_{n,q}) = n^2\int_{-\infty}^{a_{n,q}}(1-F(s))^2\left\{\left(\frac{1-F_n^*(s-)}{1-F(s)}\right)\frac{1}{N_n(s)}-\frac{1}{EN_n(s)}\right\}^2$$
$$(2.13)\qquad \times N_n(s)\,d\Lambda(s)$$
$$=O\left(n^{2\varepsilon}\int_{-\infty}^{a_{n,q}}(1-F(s))^{2\varepsilon-1}(1-G(s-))^{-2+2\varepsilon}\,dF(s)\right)$$
$$=o(n^{2\varepsilon+2\beta(1-q)})\quad \text{a.s., by (1.22)},$$

for every $\varepsilon > 0$. Moreover, letting

$$v_n(s) = n(1-F(s))\{(1-F_n^*(s-))/[(1-F(s))N_n(s)] - 1/EN_n(s)\},$$

we obtain from (2.12), (2.10) and (1.22) that for every $\varepsilon > 0$,

$$(2.14)\qquad \sup_{s\le a_{n,q}}|v_n(s)| = O\left(\sup_{s\le a_{n,q}}\left\{n(1-F(s))(EN_n(s))^{-3/2+\varepsilon}\right\}\right)$$
$$= O(n^{-1/2+\varepsilon+(3\beta+1)(1-q)/2})\quad \text{a.s.}$$

Since $q > \beta/(\beta+1)$, it then follows from (2.13) and (2.14) that for some $\rho > 0$,

$$(2.15)\qquad P\left\{\langle U_n\rangle(a_{n,q}) \le n^{2q-\rho} \text{ and } \sup_{s\le a_{n,q}}|v_n(s)| \le n^{q-2\rho} \text{ for all large } n\right\} = 1.$$

Let $\tilde{v}_n(s) = (-n^{q-2\rho}) \vee (v_n(s) \wedge n^{q-2\rho})$, $\tilde{U}_n(t) = \int_{-\infty}^t \tilde{v}_n(s)\,dM_n(s)$. By (2.15), $P\{\tilde{v}_n = v_n$ and $\tilde{U}_n = U_n$ for all large $n\} = 1$. Since $\sup_s|\Delta M_n(s)| \le 1$ a.s. and since $\sup_s|\tilde{v}_n(s)| \le n^{q-2\rho}$, application of the exponential inequality [cf. Shorack and Wellner (1986), page 899] to the martingale $\{\tilde{U}_n(t), \mathscr{F}(t), t \le a_{n,q}\}$ gives that for every $\varepsilon > 0$,

$$(2.16)\qquad P\left\{\sup_{t\le a_{n,q}}|\tilde{U}_n(t)| \ge \varepsilon n^q, \langle\tilde{U}_n\rangle(a_{n,q}) \le n^{2q-\rho}\right\} = \exp\left\{-\left(\tfrac{1}{2}+o(1)\right)\varepsilon^2 n^\rho\right\}.$$

Applying the Borel–Cantelli lemma to (2.16) and combining the result with (2.15), we then obtain the desired conclusion that $\sup_{t \le a_{n,q}} |U_n(t)| = o(n^q)$ a.s.

Now consider the case $F(\tau) < 1$. Since $b_n = \sup\{t\colon EN_n(t) \ge \frac{1}{2}cn^\alpha\}$ and since $EN_n(t)$ is left continuous, it then follows from (2.2) and (2.10) that (2.12) still holds uniformly in $s \le b_n$, with probability 1. In place of (2.13), we now have for every $\varepsilon > 0$,

$$(2.17)\qquad \begin{aligned}\langle U_n\rangle(b_n) &= O\Big(n^{2\varepsilon}\int_{-\infty}^{b_n}(1-G(s-))^{-2+2\varepsilon}\,dF(s)\Big)\\ &= O\big(n^{2\varepsilon}\{1-G(b_n-)\}^{-1}\big) = O(n^{2\varepsilon+(1-\alpha)})\quad \text{a.s.},\end{aligned}$$

recalling that $\int_{-\infty}^{\tau} dF(s)/(1-G(s-)) < \infty$ by (1.9) and that $1 - G(b_n -) \ge (\frac{1}{2}c + o(1))(1-F(\tau))^{-1}n^{-(1-\alpha)}$ by the definition of b_n. Moreover, in place of (2.14), we now have

$$(2.18)\qquad \sup_{s \le b_n} |v_n(s)| = O(n^{1-(3/2-\varepsilon)\alpha})\quad \text{a.s.},$$

for every $\varepsilon > 0$. Hence using the exponential inequality for màrtingales as in (2.16), it can be shown that $\sup_{t \le b_n}|U_n(t)| = o(n^q)$ a.s. for every $1 > q > 1 - 3\alpha/2$ [$> \frac{1}{2}(1-\alpha)$, since $\alpha < \frac{1}{2}$]. □

Proof of Theorem 1. First note that $s_n (= \tilde{X}_{(n-[cn^\alpha])}) \to \tau$ a.s. Since $N_n(s_n) = [cn^\alpha] + 1$, it follows from (2.4) that with probability 1,

$$(2.19)\qquad N_n(s_n) \sim cn^\alpha \sim EN_n(s_n) = n(1-F(s_n))(1-G(s_n-)).$$

Hence by Lemma 3,

$$(2.20)\qquad F_n^*(s_n) - F(s_n) = o(n^{-\alpha/2+\theta})\quad \text{a.s., for every } \theta > 0.$$

We shall apply the martingale integral representation [cf. Gill (1980), page 37]

$$(2.21)\qquad \frac{F_n^*(t) - F(t)}{1-F(t)} = \int_{-\infty}^{t} \frac{1-F_n^*(s-)}{1-F(s)}\,\frac{dM_n(s)}{N_n(s)},\qquad t \le \tilde{X}_{(n)}.$$

First consider the case $F(\tau) = 1$. By (1.22) and (2.19),

$$(2.22)\qquad 1 - F(s_n) = O(n^{-(1-\alpha)/(1+\beta)})\quad \text{a.s.}$$

Combining (2.20) and (2.22), we obtain that for every $\theta > 0$, $F_n^*(s_n) \ge 1 + o(n^{-\alpha/2+\theta})$ a.s. In view of (1.20), this implies that $F_n^*(s_n) > 1 - \varepsilon_n$ and therefore by (1.21),

$$(2.23)\qquad \tilde{F}_n = F_n^*\quad \text{for all large } n \text{ with probability 1}.$$

Let $\frac{1}{2} > q > \beta/(\beta+1)$ and let $a_{n,q} = F^{-1}(1 - n^{-(1-q)})$. By (2.21) and (1.24), for $t \le a_{n,q}$,

$$\begin{aligned} n(F_n^*(t) - F(t)) &- \sum_1^n X_i(t)\\ &= n(1-F(s))\int_{-\infty}^{t}\left\{\left(\frac{1-F^*(s-)}{1-F(s)}\right)\frac{1}{N_n(s)} - \frac{1}{EN_n(s)}\right\} dM_n(s),\end{aligned}$$

recalling that $n/EN_n(s) = (1 - F(s))^{-1}(1 - G(s-))^{-1}$ and that $M_n = \sum_1^n \xi_i$. Hence by Lemma 2.9 of Gill (1983),

$$\sup_{t \le a_{n,q}} \left| n(F_n^*(t) - F(t)) - \sum_1^n X_i(t) \right| \le 2 \sup_{t \le a_{n,q}} |U_n(t)|, \tag{2.24}$$

where $U_n(t)$ is defined in Lemma 4. From (2.24) and Lemma 4,

$$\sup_{t \le a_{n,q}} \left| n(F_n^*(t) - F(t)) - \sum_1^n X_i(t) \right| = o(n^q) \quad \text{a.s.} \tag{2.25}$$

For $t \ge a_{n,q}$,

$$\begin{aligned} &\left| n(F_n^*(t) - F(t)) - \sum_1^n X_i(t \wedge a_{n,q}) \right| \\ &\quad \le \left| n\{F_n^*(a_{n,q}) - F(a_{n,q})\} - \sum_1^n X_i(a_{n,q}) \right| \\ &\qquad + n\{(1 - F(a_{n,q})) \vee (1 - F_n^*(a_{n,q}))\}. \end{aligned} \tag{2.26}$$

By Lemma 3 and (2.11), with probability 1, $1 - F_n^*(a_{n,q}) \sim 1 - F(a_{n,q}) = n^{-(1-q)}$. From (2.25) and (2.26), the desired conclusion (1.25) follows.

Now consider the case $F(\tau) < 1$. By (2.20), $F_n^*(s_n) \to F(\tau)$ a.s., since F is continuous and $s_n \to \tau$ a.s. Since $F(\tau) < 1$, it then follows that $F_n^*(s_n) < 1 - \varepsilon_n$ and therefore $\tilde{F}_n(s) = F_n^*(s \wedge s_n)$ by (1.21), for all large n, with probability 1. Defining $U_n(t)$ and b_n as in Lemma 4, it follows from (2.4) that with probability 1, $s_n \le b_n$ for all large n. Moreover, by Lemma 2.9 of Gill (1983) and Lemma 4,

$$\sup_{t \le b_n} \left| n(F_n^*(t) - F(t)) - \sum_1^n X_i(t) \right| \le 2 \sup_{t \le b_n} |U_n(t)| = o(n^q) \quad \text{a.s.,} \tag{2.27}$$

analogous to (2.24) and (2.25). With probability 1, for all large n and all $s_n \le t < \tau$, since $\tilde{F}_n(t) = F_n^*(s_n)$,

$$\begin{aligned} &\left| n(\tilde{F}_n(t) - F(t)) - \sum_1^n X_i(t \wedge s_n) \right| \\ &\quad \le \left| n(F_n^*(s_n) - F(s_n)) - \sum_1^n X_i(s_n) \right| + n(F(\tau) - F(s_n)). \end{aligned} \tag{2.28}$$

By (1.9), $\infty > \int_{-\infty}^{\tau} dF(s)/(1 - G(s-)) \ge (1 - G(s_n -))^{-1}(F(\tau) - F(s_n))$ and therefore

$$F(\tau) - F(s_n) = O(1 - G(s_n -)) \quad \text{a.s.} \tag{2.29}$$

Since $1 - F(s_n) \ge 1 - F(\tau) > 0$, we obtain from (2.19) and (2.29) that $n(F(\tau) - F(s_n)) = O(n^\alpha)$ a.s. and therefore the desired conclusion (1.26) follows from (2.27) and (2.28). □

Corollaries 1 and 2 are easy consequences of Theorem 1 and the following.

LEMMA 5. *Let* $-\infty \le a < \tau$. *For* $a < s < \tau$, *define* $\xi_i(s)$ *by* (1.23) *and*

$$X_{i,a}(s) = (1 - F(s)) \int_{(a,s]} \{(1 - G(u-))(1 - F(u))\}^{-1} d\xi_i(u).$$

Suppose that (1.9) *holds. Then with probability* 1, *the sequence* $\{(2n \log\log n)^{-1/2} \sum_1^n X_{i,a}\}$ *is relatively compact in* $D(a, \tau)$ *and its set of limit points is the unit ball of the reproducing kernel Hilbert space of the covariance kernel*

$$\Gamma_a(t, t') = (1 - F(t))(1 - F(t')) \int_a^{t \wedge t'} (1 - G(s-))^{-1} (1 - F(s))^{-2} dF(s),$$
$$a < t,\ t' < \tau.$$

PROOF. Let $\|X_{i,a}\| = \sup_{a<s<\tau} |X_{i,a}(s)|$. Then by Lemma 2.9 of Gill (1983),

$$\|X_{i,a}\| \le 2 \sup_{a<s<\tau} \left| \int_{(a,s]} (1 - F(u)) \{(1 - G(u-))(1 - F(u))\}^{-1} d\xi_i(u) \right|.$$

Hence by Lemma 1 and martingale inequalities [cf. Métivier (1982), page 128],

$$E\|X_{i,a}\|^2 \le \limsup_{s \uparrow \tau} 16 E\left\{ \int_{(a,s]} (1 - G(u-))^{-2} I_{\{X_i \wedge Y_i \ge u\}}\, d\Lambda(u) \right\}$$
$$= 16 \int_a^{\tau} (1 - G(u-))^{-1} dF(u) < \infty \quad \text{by (1.9)}.$$

Likewise

$$\left(E \left\| \sum_1^n X_{i,a} \right\| \right)^2 \le E \left\| \sum_1^n X_{i,a} \right\|^2$$
$$\le \limsup_{s \uparrow \tau} 16 E\left\{ \int_{(a,s]} (1 - G(u-))^{-2} \sum_1^n I_{\{X_i \wedge Y_i \ge u\}}\, d\Lambda(u) \right\}$$
$$= 16 n \int_a^{\tau} (1 - G(u-))^{-1} dF(u) = O(n).$$

Hence the desired conclusion follows from Theorem 4.1 of Kuelbs (1977) (see also his remarks on pages 786, 787, 789 and 790). □

PROOF OF COROLLARY 1. First consider the case $F(\tau) < 1$. Then in view of Theorem 1(ii) and Lemma 5, it suffices to show that for every $\varepsilon > 0$,

$$\limsup_{n \to \infty} \left\{ \sup_{s_n < s < \tau} \left| \sum_1^n X_i(s) - \sum_1^n X_i(s_n) \right| / (2n \log\log n)^{1/2} \right\} \le 2\varepsilon \quad \text{a.s.} \tag{2.30}$$

Since $s_n \to \tau$ a.s., (2.30) will follow if it can be shown that given $\varepsilon > 0$, there exists $a < \tau$ for which

$$(2.31)\quad \limsup_{n\to\infty}\left\{\sup_{a<s<\tau}\left|\sum_i^n X_i(s) - \sum_1^n X_i(a)\right|/(2n\log\log n)^{1/2}\right\} \le \varepsilon \quad \text{a.s.}$$

Defining $X_{i,a}$ as in Lemma 5, note that $X_i = X_{i,-\infty}$ and that $X_i(s) - X_i(a) = X_{i,a}(s) + (F(s) - F(a))(1 - F(a))^{-1}X_i(a)$. Moreover,

$$\sup_{a<t<\tau} \operatorname{Var}(X_{i,a}(t))\Big(= \sup_{a<t<\tau}\Gamma_a(t,t)\Big)$$
$$\le \int_a^\tau (1 - G(s-))^{-1}\,dF(s) \to 0 \quad \text{as } a \to \tau, \text{by (1.9)}.$$

Hence by Lemma 5, we can choose a sufficiently near τ such that (2.31) holds. The case $F(\tau) = 1$ can be proved similarly, using Theorem 1(i) instead of Theorem 1(ii). □

An obvious extension of Lemma 5 and Kuelbs' (1977) related results to the sequence $\{(2n\log\log n)^{-1/2}\sum_1^{[nr]} X_i(s)\}$ of functions on $\{0 \le r \le 1, s < \tau\}$ also leads to Corollary 2, in view of (2.30). Note that the reproducing kernel Hilbert space of Corollary 2 is the direct product $H \otimes H_1$, where H is the reproducing kernel Hilbert space of the covariance kernel $r \wedge r'$ of Brownian motion and H_1 is the reproducing kernel Hilbert space of the covariance kernel $EZ(t)Z(t')$ [cf. Aronszain (1950), Section 8].

We now give a probabilistic proof of the representations (1.30) and (1.31) for the unit balls K_1 and K_2 of H_1 and $H \otimes H_1$, respectively. Using here the convention $F^{-1}(u) = \sup\{t: F(t) \le u\}$ and setting $u = F(t)$ in (1.11), we can express $Z(F^{-1}(u))$ as

$$(2.32)\quad \begin{aligned}\tilde{Z}(u) &\triangleq Z(F^{-1}(u))\\ &= (1-u)\int_0^u \{1 - G(F^{-1}(v)-)\}^{-1/2}(1-v)^{-1}\,dW(v),\\ &\qquad\qquad u < F(\tau),\end{aligned}$$

where $\{W(v), v \ge 0\}$ is the standard Wiener process. Integration by parts then gives

$$(2.33)\quad \begin{aligned}\tilde{Z}(u) &= \{1 - G(F^{-1}(u)-)\}^{-1/2}W(u)\\ &\quad -(1-u)\int_0^u W(v)\,d\Big[\{1 - G(F^{-1}(v)-)\}^{-1/2}(1-v)^{-1}\Big].\end{aligned}$$

Take any $0 < \varepsilon < F(\tau)$ and let $\tilde{Z}_1, \tilde{Z}_2, \dots$ be i.i.d. Gaussian processes on $[0, F(\tau) - \varepsilon]$ having the same distribution as $\tilde{Z}$ and let $W_1, W_2, \dots$ be i.i.d. Wiener processes on $[0,1]$. Then by (2.33), the set $\tilde{K}_1$ of limit points of $\{\tilde{Z}_n/(2\log n)^{1/2}\}$ in $C[0, F(\tau) - \varepsilon]$ is given by

$$(2.34)\quad \begin{aligned}\tilde{K}_1 = \Big\{\tilde{\psi}: \tilde{\psi}(u) &= \{1 - G(F^{-1}(u)-)\}^{-1/2}g(u)\\ &\quad -(1-u)\int_0^u g(v)\,d\Big[\{1 - G(F^{-1}(v)-)\}^{-1/2}(1-v)^{-1}\Big]\\ &\quad \text{for } 0 \le u \le F(\tau) - \varepsilon \text{ for some } g \in \tilde{K}\Big\},\end{aligned}$$

where $\tilde{K} = \{g \in C[0, F(\tau) - \varepsilon]: g(0) = 0 \text{ and } \int_0^{F(\tau)-\varepsilon}(dg/dx)^2 \le 1\}$ is the set of limit points of $\{W_n/(2\log n)^{1/2}\}$ in $C[0, F(\tau) - \varepsilon]$ [cf. Lai (1974), Theorem 1]. Noting that $dg(v) = g'(v)\,dv$ for $g \in \tilde{K}$, we can use integration by parts to express $\tilde{\psi}$ in (2.34) as

$$\tilde{\psi}(u) = (1-u)\int_0^u \{1 - G(F^{-1}(v)-)\}^{-1/2}(1-v)^{-1}g'(v)\,dv.$$

Moreover, by Theorem 1 of Lai (1974), $\tilde{K}_1$ is the unit ball of the reproducing kernel Hilbert space of $\tilde{Z}$. A change of variables $u = F(t)$ then gives the desired representation (1.30) of K_1 upon letting $\varepsilon \to 0$ and applying (1.9); see also (2.31). The representation (1.31) of K_2 can be proved by a similar argument using the two-parameter process

$$\tilde{Z}(u;r) = (1-u)\int_0^u \{1 - G(F^{-1}(v)-)\}^{-1/2}(1-v)^{-1}\,dW(v;r),$$

where $W(v;r)$ is the Brownian sheet with covariance kernel

$$E\{W(v;r)W(v';r')\} = (v \wedge v')(r \wedge r').$$

PROOF OF COROLLARY 3. In the case $F(\tau) = 1$, the desired conclusions for F_n^* follow from Theorem 1 and Corollaries 1 and 2, in view of (2.23). In the case $F(\tau) < 1$, the strong approximation (1.29a) follows from (2.27) and the law of the iterated logarithm (1.27) follows from (1.29a), (2.31) and Lemma 5. Since $F_n^{\mathrm{II}}(s) = F_n^*(s-)$ and since F is continuous, laws of the iterated logarithm for F_n^{II} follow from those for F_n^*.

To prove strong approximations for F_n^{I}, we make use of the inequality $|F_n^{\mathrm{I}}(s) - F_n^*(s)| < 2(1 - F_n^{\mathrm{I}}(s))/N_n(s)$, due to Cuzick [(1985), Equation (4)], which in turn implies that

$$|F_n^{\mathrm{I}}(s) - F_n^*(s)| < 4(1 - F_n^*(s))/N_n(s) \quad \text{if } N_n(s) \ge 4. \tag{2.35}$$

In the case $F(\tau) = 1$, let $\frac{1}{2} > q > \beta/(\beta+1)$ and let $a_{n,q} = F^{-1}(1 - n^{-(1-q)})$, as in the proof of Theorem 1. Then from (2.35) and Lemma 3, it follows that with probability 1, for all large n,

$$\sup_{s \le a_{n,q}} n|F_n^{\mathrm{I}}(s) - F_n^*(s)| < 5 \sup_{s \le a_{n,q}} (1 - G(s-))^{-1}, \tag{2.36}$$

noting that $EN_n(s) = n(1 - F(s))(1 - G(s-))$. By (1.22),

$$\{1 - G(a_{n,q}-)\}^{-1} = O\left(\{1 - F(a_{n,q})\}^{-\beta}\right) = O(n^{\beta(1-q)}) = o(n^q),$$

since $q > \beta/(\beta+1)$. Hence by (2.36),

$$\sup_{s \le a_{n,q}} n|F_n^{\mathrm{I}}(s) - F_n^*(s)| = o(n^q). \tag{2.37}$$

Combining (2.37) with (2.25) and (2.26), we obtain the desired conclusion (1.25) with F_n^{I} in place of F_n^*.

We next consider the case $F(\tau) < 1$ and assume that $\frac{1}{4} < q < \frac{1}{2}$. First note that $\tau_n(q) < \tau_n$ for every $\alpha < \frac{1}{2}$. Analogous to (2.36) and (2.37), we now have

$$\sup_{s \le \tau_n(q)} n|F_n^{\mathrm{I}}(s) - F_n^*(s)| = O\left(\{1 - G(\tau_n(q)-)\}^{-1}\right) = O(n^q). \tag{2.38}$$

From (2.38) and (1.29a), the desired conclusion (1.29b) follows. □

3. Extensions to random truncation models and mixed censorship-truncation models. For the product-limit estimator (1.4) based on randomly truncated data (X_i^o, T_i^o) with $X_i^o \ge T_i^o$, $i = 1, \dots, n$, Woodroofe (1985) showed that

$$\sup_{t > \tau_H} |F_n^o(t) - F_H(t)| \to_P 0, \tag{3.1}$$

where

$$\tau_H = \inf\{s\colon H(s) > 0\}, \qquad F_H(t) = P(X_1 \le t | X_1 > \tau_H), \tag{3.2}$$

under the assumption that H is continuous with $F(\tau_H) < 1$. Moreover, assuming that H is continuous, $F(\tau_H) < 1$ and

$$\int_{\tau_H}^{\infty} dF/H < \infty, \tag{3.3}$$

he also showed that $n^{1/2}(F_n^o - F_H)$ converges weakly to a Gaussian process. Recently Lai and Ying (1989) studied consistency and weak convergence questions of the product-limit estimator when the T_i are independent but need not be identically distributed. They showed that (3.1) may fail to hold when the T_i are not identically distributed and proposed minor modifications of F_n^o which are uniformly strongly consistent [i.e., the convergence in probability in (3.1) can be replaced by a.s. convergence] and, under a condition similar to (3.3), also asymptotically Gaussian, without assuming F to be continuous and without assuming the T_i to be identically distributed or continuous. Instead of (1.4), they proposed to define F_n^o by

$$1 - F_n^o(s) = \prod_{i\colon X_i^o \le s}^{n} \left(1 - I_{\{\#(X_i^o;\, n) \ge cn^\alpha\}} / \#(X_i^o; n)\right), \tag{3.4}$$

where $c > 0$, $0 < \alpha < \frac{1}{2}$ and $\#(s; n)$ is defined in (1.5). Their analysis also shows that the following alternative modification of the product-limit estimator, which we shall use here, is also uniformly strongly consistent and asymptotically Gaussian:

$$\begin{aligned} 1 - F_n^o(s) &= \prod_{i\colon X_i^o \le s}^{n} \left(1 - I_{\{\nu(X_i^o;\, n) \ge cn^\alpha\}} / \#(X_i^o; n)\right) \\ &\text{where } \nu(s; n) = \sum_{i=1}^{n} I_{\{T_i^o \le s\}}. \end{aligned} \tag{3.5}$$

In this section we develop functional laws of the iterated logarithm for this modified version of the product-limit estimator. More generally, we shall consider the case where censoring variables Y_i may also be present. As described in Section 1, $X_1, X_1, \dots, Y_1, Y_2, \dots, T_1, T_2, \dots$ are assumed to be independent such that the X_i have a common continuous distribution function F and the Y_i and T_i have possibly defective distributions G and H, respectively. The case $G \equiv 0$ (corresponding to $Y_i = \infty$) reduces to the random truncation model considered by Woodroofe (1985), while the case $H \equiv 1$ (corresponding to $T_i = -\infty$) reduces to the random censorship model studied in the previous section.

Letting $\tilde{X}_i = X_i \wedge Y_i$ and $\delta_i = I_{\{X_i \le Y_i\}}$, this mixed censorship–truncation model assumes that $(\tilde{X}_i, \delta_i)$ is observable only when $\tilde{X}_i \ge T_i$. We can regard an observed sample $(\tilde{X}_i^o, \delta_i^o, T_i^o)$ with $\tilde{X}_i^o \ge T_i^o$, $i = 1, \dots, n$, as being generated by a larger sample of independent random variables X_i, Y_i, T_i, $i = 1, \dots, m$, where $n = n(m)$ is given by

$$\sum_{i=1}^{m} I_{\{X_i \wedge Y_i \ge T_i\}} = n. \tag{3.6}$$

In place of (1.5), we now define

$$\#_m(s) = \sum_{j=1}^{n} I_{\{\tilde{X}_j^o \ge s \ge T_j^o\}} = \sum_{i=1}^{m} I_{\{\tilde{X}_i \ge s \ge T_i\}}, \tag{3.7}$$

$$\nu_m(s) = \sum_{j=1}^{n} I_{\{s \ge T_j^o\}} = \sum_{i=1}^{m} I_{\{T_i \le \tilde{X}_i \wedge s\}}. \tag{3.8}$$

Moreover, combining the ideas behind (1.2) and (3.5), we now define

$$1 - F_m^*(t) = \prod_{j:\, \delta_j^o = 1 \text{ and } \tilde{X}_j^o \le t} \left(1 - I_{\{\nu_m(\tilde{X}_j^o) \ge cn^\alpha\}} / \#_m\left(\tilde{X}_j^o\right)\right), \tag{3.9}$$

where $c > 0$ and $\frac{1}{3} < \alpha < \frac{1}{2}$. Choosing $\varepsilon_n > 0$ such that (1.20) holds, define in analogy with (1.21),

$$\begin{aligned} \hat{F}_m(s) &= F_m^*\left(s \wedge \tilde{X}^o_{(n-[cn^\alpha])}\right), \quad \text{if } F_m^*\left(\tilde{X}^o_{(n-[cn^\alpha])}\right) < 1 - \varepsilon_n, \\ &= F_m^*(s), \quad \text{otherwise.} \end{aligned} \tag{3.10}$$

Noting that $\hat{F}_m$ reduces to $\tilde{F}_n$ when $T_i = -\infty$, we now generalize Theorem 1 and its corollaries to $\hat{F}_m$. Recall that F is assumed to be continuous and that H and G need not be continuous and may be defective.

Theorem 2. *Defining τ_H and F_H by* (3.2), *suppose that $F(\tau_H) < 1$ and $G(\tau_H) < 1$. Defining τ as in* (1.6), *assume that* (1.9) *holds in the case $F(\tau) < 1$ and that the stronger condition* (1.22) *holds in the case $F(\tau) = 1$. Moreover, assume that* (3.3) *holds. For $s < \tau$, let $\Lambda(s) = -\log(1 - F(s))$ and define*

$$\zeta_i(s) = I_{\{T_i \le X_i \le Y_i \wedge s\}} - \int_{-\infty}^{s} I_{\{X_i \wedge Y_i \ge u \ge T_i\}}\, d\Lambda(u). \tag{3.11}$$

For $\tau_H \le \sigma \le s < \tau$, *define*

$$(3.12)\quad X_i(\sigma, s) = (1 - F_H(s)) \int_{(\sigma, s]} \{H(u)(1 - F(u))(1 - G(u-))\}^{-1} d\zeta_i(u).$$

Let $\sigma_m = \inf\{s: \nu_m(s) \ge cn^{\alpha}\}$, $s_m = \tilde{X}^o_{(n-[cn^{\alpha}])}$, *where* $n = n(m)$ *is given by* (3.6) *and* $c > 0$, $\frac{1}{3} < \alpha < \frac{1}{2}$ *are the same as in the definition* (3.10) *of* $\hat{F}_m$.

(i) *Suppose that* $F(\tau) = 1$. *Then for every* $q < \frac{1}{2}$ *such that* $q > \max\{1 - 3\alpha/2, \beta/(\beta + 1)\}$, *where* β *is given in* (1.22),

$$(3.13)\quad \sup_{\tau_H < s < \tau} \left| m(\hat{F}_m(s) - F_H(s)) - \sum_1^m X_i(\sigma_m, s \wedge F^{-1}(1 - m^{-(1-q)})) \right| = O(m^{q \vee \alpha}) \quad a.s.$$

(ii) *Suppose that* $F(\tau) < 1$. *Then for every* $q \in (1 - 3\alpha/2, \frac{1}{2})$,

$$(3.14)\quad \sup_{\tau_H < s < \tau} \left| m(\hat{F}_m(s) - F_H(s)) - \sum_1^m X_i(\sigma_m, s \wedge s_m) \right| = o(m^q) + O(m^{\alpha}) \quad a.s.$$

PROOF. Let $\mathscr{G}(s)$ be the complete σ-field generated by $T_i, Y_i, I_{\{T_i \le X_i \wedge Y_i\}}$, $I_{\{T_i \le u \le X_i\}}$, $I_{\{T_i \le X_i \le u\}}$ $(u \le s,\ i = 1, 2, \dots)$. Then

$$I_{\{T_i \le X_i \le Y_i \wedge s\}} - \int_{[T_i, Y_i \wedge s]} I_{\{X_i \ge u\}}\, d\Lambda(u) = \zeta_i(s)$$

is a martingale with respect to $\mathscr{G}(s)$. Letting $M_m(s) = \sum_1^m \zeta_i(s)$, it then follows that $\{M_m(s), \mathscr{G}(s), -\infty < s < \infty\}$ is a square integrable martingale whose predictable variation process is

$$(3.15)\quad \langle M_m \rangle(t) = \int_{-\infty}^{t} \#_m(s)\, d\Lambda(s).$$

Note also that T_i, Y_i and $n = n(m)$ [defined by (3.6)] are measurable with respect to $\mathscr{G}_{-\infty} \triangleq \bigcap_{s=-\infty}^{\infty} \mathscr{G}(s)$. Moreover, the random variables $\nu_m(s)$ defined in (3.8) are measurable with respect to $\mathscr{G}_{-\infty}$.

Following the arguments of the proof of Lemma 2 but replacing Wellner's (1978) and van Zuijlen's (1977) results used in the proof by Alexander [(1985), Theorem 2.1] (which we apply to the class of sets of the form $\{X \ge s_1, T \le s_2$ (or $T < s_2$), $Y \le s_3$ (or $Y < s_3$)$\}$; see also the proof of his Theorem 1.1), it can be shown that analogous to Lemma 2, for every $1 > \delta > 0$ and $\varepsilon > 0$,

$$(3.16)\quad \sup_{s:\, E\#_m(s) \ge m^{\delta}} (E\#_m(s))^{1/2-\varepsilon} |E\#_m(s)/\#_m(s) - 1| \to 0 \quad \text{a.s.}$$

Likewise, since $\nu_m(s) = \sum_1^m (I_{\{T_i \le \tilde{X}_i \le s\}} + I_{\{T_i \le s < \tilde{X}_i\}})$, for every $1 > \delta > 0$ and $\varepsilon > 0$,

$$(3.17)\quad \sup_{s:\, E\nu_m(s) \ge m^{\delta}} (E\nu_m(s))^{1/2-\varepsilon} |E\nu_m(s)/\nu_m(s) - 1| \to 0 \quad \text{a.s.}$$

By (3.6) and the law of large numbers,

$$(3.18)\qquad \lim_{m\to\infty} n/m = \int_{-\infty}^{\infty} (1 - F(s))(1 - G(s-))\,dH(s) \quad (>0) \text{ a.s.}$$

Since $\nu_m(s)$ is right continuous and nondecreasing in s, $\nu_m(s) \geq cn^\alpha$ for all $s \geq \sigma_m$ and $\nu_m(s) < cn^\alpha$ if $s < \sigma_m$. Moreover, σ_m is measurable with respect to $\mathscr{G}_{-\infty}$. Defining $L_m(s) = \sum_{i=1}^m I_{\{T_i \leq \tilde{X}_i \leq s, \delta_i = 1\}} = \sum_{i=1}^m I_{\{T_i \leq X_i \leq Y_i \wedge s\}}$, it follows from (3.9) that

$$(3.19)\qquad \log(1 - F_m^*(t)) = \sum_{\sigma_m \leq s \leq t} \log\{1 - \Delta L_m(s)/\#_m(s)\}.$$

Hence an argument similar to the proof of Lemma 3 can be used to show that for every $0 < \rho < 1$ and $\varepsilon > 0$,

$$(3.20)\qquad \sup_{t:\, E\#_m(t) \geq m^\rho} |(1 - F_m^*(t))/(1 - F_m(t)) - 1| = o(m^{-\rho/2+\varepsilon}) \quad \text{a.s.},$$

where

$$(3.21)\quad 1 - F_m(t) = \exp\left\{-\int_{[\sigma_m, t]} d\Lambda(s)\right\} = (1 - F(t \vee \sigma_m))/(1 - F(\sigma_m)).$$

Define

$$U_m(t) = m\int_{[\sigma_m, t]} (1 - F_m(s))\left\{\left(\frac{1 - F_m^*(s-)}{1 - F_m(s)}\right)\frac{1}{\#_m(s)} - \frac{1}{E\#_m(s)}\right\} dM_m(s)$$
$$\text{for } t \geq \sigma_m.$$

Since $E\nu_m(s) \leq mH(s)$ and since $\nu_m(\sigma_m) \geq cn^\alpha$, it follows from (3.17) and (3.18) that with probability 1,

$$(3.22)\qquad H(\sigma_m) \geq \tfrac{1}{2}\theta m^{-(1-\alpha)} \quad \text{for all large } m,$$

$$\text{where } \theta = c\left\{\int_{-\infty}^{\tau} (1 - F(s))(1 - G(s-))\,dH(s)\right\}^\alpha.$$

Note also that

$$(3.23)\qquad E\#_m(s) = mH(s)(1 - F(s))(1 - G(s-)).$$

Take $a \in (\tau_H, \tau)$. Then $H(a) > 0$ and $(1 - F(a))(1 - G(a)) > 0$. An argument similar to the proof of Lemma 4 can be used to show that $\sup_{\sigma_m \leq t \leq a} |U_m(t)| = o(n^q)$ a.s. for every $q \in (1 - 3\alpha/2, 1)$. Moreover, as in Lemma 4, we still have $\sup_{a \leq t \leq b_m} |U_m(t)| = (m^q)$ a.s. for every $q \in (1 - 3\alpha/2, 1)$ in the case $F(\tau) < 1$, where b_m is defined in Lemma 4(ii) and $\sup_{a \leq t \leq F^{-1}(1 - m^{-(1-q)})} |U_m(t)| = o(m^q)$ a.s. for every $q \in (\beta/(\beta + 1), 1)$ in the case $F(\tau) = 1$.

Since

$$E\nu_m(s) \geq mH(s)(1 - F(a))(1 - G(a -))$$

for $s \leq a$ and since $\nu_m(\sigma_m -) \leq cn^{\alpha}$ with $\sigma_m \to \tau_H$ a.s., it follows from (3.17) and (3.18) that with probability 1,

$$(3.24)\quad (1 - F(a))(1 - G(a -))H(\sigma_m -) \leq 2\theta m^{-(1-\alpha)} \quad \text{for all large } m,$$

where θ is given in (3.22). By (3.3), $\infty > \int_{\tau_H}^{\infty} dF/H \geq \int_{(\tau_H, \sigma_m)} dF/H \geq (F(\sigma_m) - F(\tau_H))/H(\sigma_m -)$ and therefore

$$(3.25)\qquad m(F(\sigma_m) - F(\tau_H)) = O(mH(\sigma_m -)) = O(m^{\alpha}) \quad \text{a.s.},$$

by (3.24). By (3.21), for $t \geq \sigma_m$,

$$\begin{aligned} |m(F_m(t) - F_H(t))| &\leq m\left|\{1 - F(\sigma_m)\}^{-1} - \{1 - F(\tau_H)\}^{-1}\right| \\ &= O(m^{\alpha}) \quad \text{a.s. by (3.25)}, \end{aligned}$$

while for $t < \sigma_m$,

$$\begin{aligned} |m(F_m^*(t) - F_H(t))| &= mF_H(t) \leq m(F(\sigma_m) - F(\tau_H))/(1 - F(\tau_H)) \\ &= O(m^{\alpha}) \quad \text{a.s. by (3.25)}. \end{aligned}$$

Analogous to (2.21), we now have

$$\frac{F_m^*(t) - F_m(t)}{1 - F_m(t)} = \int_{[\sigma_m, t]} \frac{1 - F_m^*(s -)}{1 - F_m(s)} \frac{dM_m(s)}{\#_m(s)}, \qquad t \geq \sigma_m$$

[cf. Gill (1980), Proposition A.4.1, pages 153–155]. The rest of the proof is similar to that of Theorem 1. □

By an argument similar to the proof of Lemma 5 and Corollary 1, together with representations of unit balls of reproducing kernel Hilbert spaces analogous to (1.30) and (1.31), we obtain from Theorem 2 the following functional laws of the iterated logarithm for the modified product-limit estimator in the mixed censorship–truncation model.

COROLLARY 4. *Under the same assumptions as in Theorem* 2, *with probability* 1, *the sequence* $\{(m/2\log\log m)^{1/2}(\hat{F}_m - F_H)\}$ *is relatively compact in* $D(\tau_H, \tau)$ *and its set of limit points is*

$$\begin{aligned} K_1^* = \Big\{\psi\colon \psi(u) = (1 - F_H(u))\int_{\tau_H}^{u} &\{H(s)(1 - G(s -))^{-1/2}(1 - F(s))\}^{-1} \\ (3.26)\qquad &\phi(F(s))\, dF(s) \text{ for } \tau_H < u < \tau, \text{ for some } \phi \in L_2[F(\tau_H), F(\tau)] \\ &\text{with } \int_{F(\tau_H)}^{F(\tau)} \phi^2(\theta)\, d\theta \leq 1\Big\}. \end{aligned}$$

Define

$$(3.27)\qquad U_m(r, s) = r\big(\hat{F}_{[mr]}(s) - F_H(s)\big), \qquad 0 \leq r \leq 1,\ s < \tau.$$

Then with probability 1, *the sequence* $\{(m/2\log\log m)^{1/2}U_m\}$ *is relatively compact in* $D([0,1]\times(\tau_H,\tau))$ *and its set of limit points is*

$$K_2^* = \Big\{\psi\colon \psi(r,u) = (1-F_H(u))\iint_{0\le\rho\le r,\, s\le u}\{H(s)(1-G(s-))\}^{-1/2} \times(1-F(s))^{-1}\phi(\rho,F(s))\,d\rho\,dF(s) \tag{3.28}$$
$$\text{for } \tau_H<u<\tau \text{ and } 0\le r\le 1, \text{ for some } \phi\in L_2([0,1]\times[F(\tau_H),F(\tau)])$$
$$\text{such that } \iint_{[0,1]\times[F(\tau_H),F(\tau)]}\phi^2(\rho,\theta)\,d\rho\,d\theta\le 1\Big\}.$$

4. Functional laws of the iterated logarithm for time-sequential product-limit estimators. In this section we shall assume that X_i, Y_i and T_i are nonnegative random variables and study time-sequential product-limit estimators $\hat F_{n,t}$, at (times) $t\ge 0$, of the data $(\tilde X_i(t),\delta_i(t))$, $i=1,\dots,n$, where

$$\tilde X_i(t) = X_i\wedge Y_i\wedge(t-T_i)^+,\qquad \delta_i(t) = I_{\{X_i\le Y_i\wedge(t-T_i)^+\}}. \tag{4.1}$$

A prototypical example of this kind of censored data is a long-term clinical trial in which n patients enter the study serially and are followed until they fail or withdraw from the study. The data are monitored sequentially and analyzed at various calendar times t. Let T_i denote the entry time and X_i the time to failure (measured with the entry time as the origin) of the ith subject and let Y_i denote the censoring time (also measured with the entry time as the origin) that the ith subject is "lost" in follow-up. On the basis of these available data at calendar time t, one can compute the Kaplan–Meier estimator

$$F_{n,t}^*(s) = 1-\prod_{i:\,\delta_i(t)=1 \text{ and } \tilde X_i(t)\le s}^{n}\{1-1/N_{n,t}(\tilde X_i(t))\}. \tag{4.2}$$

Here and in the sequel we define

$$N_{n,t}(s) = \sum_{i=1}^n I_{\{\tilde X_i(t)\ge s\}},$$
$$L_{n,t}(s) = \sum_{i=1}^n I_{\{\tilde X_i(t)\le s,\,\delta_i(t)=1\}} = \sum_1^n I_{\{X_i\le s\wedge Y_i\wedge(t-T_i)^+\}}. \tag{4.3}$$

For the time-sequential Kaplan–Meier estimator $F_{n,t}^*(s)$ with $s\le s_0$, Jennison and Turnbull (1985) showed that for fixed $t(1)<\cdots<t(k)$, $n^{1/2}(F_{n,t(1)}^*-F,\dots,F_{n,t(k)}^*-F)$ converges weakly to a k-dimensional Gaussian process on $[0,s_0]$, for every s_0 such that

$$(1-F(s_0))(1-G(s_0-))H(t(1)-s_0)>0.$$

Gu (1987) strengthened this result into weak convergence of the sequence of two-parameter processes $n^{1/2}(F_{n,t}^*(s)-F(s))$ to a two-parameter Gaussian process on $\{(t,s)\colon 0\le s\le t, (1-F(s))(1-G(s-))H(t-s)\ge\varepsilon\}$, for every $\varepsilon>0$. We now modify $F_{n,t}^*$ by using the same ideas as in (1.21) together with a suitable discretization of t and show that for the modified estimator, the weak convergence theory can be further developed into a functional law of the iterated logarithm in which the limit set is the unit ball of the reproducing kernel Hilbert space associated with the limiting Gaussian process in the weak convergence theory.

Let $\tau_H = \inf\{u: H(u) > 0\}$, as in (3.2). At calendar time t, order the observations $\tilde{X}_1(t), \dots, \tilde{X}_n(t)$ as $\tilde{X}_{(1)}(t) \leq \cdots \leq \tilde{X}_{(n)}(t)$ and note that there is no information to estimate $F(s)$ if $s > t - \tau_H$ ($\geq t - T_i$ for all i). Letting $a > \tau_H$ so that $H(a-) > 0$, we shall estimate $F(s)$ at calendar time t ($\geq a$) only for $s \leq t - a$. Throughout the sequel we shall also assume that H is continuous on $[a, \infty)$ and that

$$E\{(X_1 \wedge Y_1)^\delta\} < \infty \quad \text{for some } \delta > 0. \tag{4.4}$$

To define the modified product-limit estimator $\tilde{F}_{n,t}$, first partition the interval $[a, \infty)$ by the points $a = \tau_{n,0} < \tau_{n,1} < \cdots < \tau_{n,k_n} < \infty = \tau_{n,k_n+1}$ such that as $n \to \infty$,

$$\max_{1 \leq j \leq k_n} (\tau_{n,j} - \tau_{n,j-1}) \to 0, \tag{4.5a}$$

$$\tau_{n,k_n} \to \infty \quad \text{such that} \quad n^\rho = O(\tau_{n,k_n}) \quad \text{for some } \rho > 1/\delta, \tag{4.5b}$$

$$k_n = O(n^r) \quad \text{for some } r > \rho. \tag{4.5c}$$

Given $t \in [a, \infty)$, define

$$t_n = \tau_{n,j} \quad \text{if } \tau_{n,j} \leq t < \tau_{n,j+1} \ (j = 0, \dots, k_n). \tag{4.6}$$

Let $c > 0$ and $\frac{1}{3} < \alpha < \frac{1}{2}$ and choose $\varepsilon_n > 0$ such that (1.20) holds. For $t \geq a$ and $0 \leq s \leq t - a$ define, in analogy with (1.21),

$$\begin{aligned} \tilde{F}_{n,t}(s) &= F^*_{n,t_n}\big(s \wedge \tilde{X}_{(n-[cn^\alpha])}(t_n)\big), \\ &\qquad \text{if } F^*_{n,t_n}\big(\tilde{X}_{(n-[cn^\alpha])}(t_n)\big) < 1 - \varepsilon_n, \\ &= F^*_{n,t_n}(s), \quad \text{otherwise.} \end{aligned} \tag{4.7}$$

We now generalize Theorem 1 and its corollaries to the time-sequential product-limit estimator $\tilde{F}_{n,t}$.

THEOREM 3. *Suppose that X_i, Y_i and T_i are nonnegative random variables with respective distribution functions F (continuous), G (possibly defective) and H such that $H(a-) > 0$ and H is continuous on $[a, \infty)$. Assume that* (4.4) *holds; moreover, assume that* (1.9) *holds in the case $F(\tau) < 1$ and that the stronger condition* (1.22) *is satisfied in the case $F(\tau) = 1$, where $\tau(\leq \infty)$ is defined in* (1.6). *For $t \geq a$ and $0 \leq s \leq t - a$, define the product-limit estimator*

$\tilde{F}_{n,t}(s)$ *by* (4.7). *Letting* $\Lambda(u) = -\log(1 - F(u))$, *define*

$$\zeta_{i,t}(s) = I_{\{X_i \le Y_i \wedge (t-T_i)^+ \wedge s\}} - \int_0^s I_{\{X_i \wedge Y_i \wedge (t-T_i)^+ \ge u\}}\, d\Lambda(u), \tag{4.8}$$

$$Z_i(t,s) = (1 - F(s)) \int_{[0,s]} \{H(t-u)(1 - G(u-))(1 - F(u))\}^{-1} d\zeta_{i,t}(u), \tag{4.9}$$

$$\Delta_a = \{(t,s): a \le t < a + \tau, 0 \le s \le t - a (< \tau)\} \cup \{(t,s): t \ge a + \tau, 0 \le s < \tau\}. \tag{4.10}$$

(i) *Suppose that* $F(\tau) = 1$. *Then for every* $q < \frac{1}{2}$ *such that* $q > \max\{\beta/(\beta + 1), 1 - \rho\delta/(1 + \beta)\}$, *where* $0 \le \beta < 1$ *is given in* (1.22) *and* $\delta > 0$, $\rho > 1/\delta$ *are given in* (4.4) *and* (4.5b),

$$\sup_{(t,s) \in \Delta_a} \left| n(\tilde{F}_{n,t}(s) - F(s)) - \sum_1^n Z_i(t_n, s \wedge F^{-1}(1 - n^{-(1-q)})) \right| = O(n^q) \quad a.s. \tag{4.11}$$

(ii) *Suppose that* $F(\tau) < 1$. *Then for every* $q \in (1 - 3\alpha/2, \frac{1}{2})$,

$$\sup_{(t,s) \in \Delta_a} \left| n(\tilde{F}_{n,t}(s) - F(s)) - \sum_1^n Z_i(t_n, s \wedge \tilde{X}_{(n-[cn^\alpha])}(t_n)) \right| = o(n^q) + O(n^\alpha) \quad a.s. \tag{4.12}$$

PROOF. For $(t,s) \in \Delta_a$, $t_n = \tau_{n,j}$ for some $j \le k_n$ and $s \le t - a < \tau_{n,j+1} - a$. Therefore

$$\{(t_n, s): (t,s) \in \Delta_a, s \le \tau_{n,k_n} - a\} \subset \bigcup_{j=0}^{k_n} \{(\tau_{n,j}, s): s \le (\tau_{n,j+1} \wedge \tau_{n,k_n}) - a\}. \tag{4.13}$$

By (4.3), $N_{n,t}(s) = \sum_1^n I_{\{X_i \wedge Y_i \wedge (t-T_i)^+ \ge s\}} = \sum_1^n I_{\{X_i \wedge Y_i \ge s\}} I_{\{T_i \le t-s\}}$. Hence for $s \le (\tau_{n,j+1} \wedge \tau_{n,k_n}) - a$,

$$\begin{aligned} EN_{n,\tau_{n,j}}(s) &= nH(\tau_{n,j} - s)(1 - G(s-))(1 - F(s)) \\ &\ge n(1 - G(s-))(1 - F(s)) \inf_{0 \le i < k_n} H(a - (\tau_{n,i+1} - \tau_{n,i})) \\ &\to n(1 - G(s-))(1 - F(s))H(a-), \end{aligned} \tag{4.14}$$

as $n \to \infty$, by (4.5a). Since $H(a-) > 0$ and $k_n = O(n^r)$ by (4.5c), we obtain by a straightforward modification of the proof of Lemma 2 that for every $1 > \lambda > 0$ and $\varepsilon > 0$,

$$\sup_{0 \le j \le k_n} \sup_{s \in I_{n,j}(\lambda)} (EN_{n,\tau_{n,j}}(s))^{1/2-\varepsilon} |EN_{n,\tau_{n,j}}(s)/N_{n,\tau_{n,j}}(s) - 1| \to 0 \quad \text{a.s.}, \tag{4.15}$$

where

$$(4.16)\qquad \begin{aligned} I_{n,j}(\lambda) = \{s\colon s \le (\tau_{n,j+1} \wedge \tau_{n,k_n}) - a,\\ (1 - G(s-))(1 - F(s))H(a-) \ge \tfrac{1}{2}cn^{-(1-\lambda)}\}. \end{aligned}$$

Let $M_{n,j}(s) = L_{n,\tau_{n,j}}(s) - \int_0^s N_{n,\tau_{n,j}}(u)\, d\Lambda(u)$ and define

$$(4.17)\qquad \begin{aligned} U_{n,j}(s) &= n\int_0^s (1 - F(u))\\ &\quad\times \left\{\left(\frac{1 - F^*_{n,\tau_{n,j}}(u-)}{1 - F(u)}\right)\frac{1}{N_{n,\tau_{n,j}}(u)} - \frac{1}{EN_{n,\tau_{n,j}}(u)}\right\} dM_{n,j}(u). \end{aligned}$$

Summing over $j \le k_n$ [$= O(n^r)$] the exponential inequalities for the martingale integrals similar to those used in the proofs of Lemmas 3 and 4, we obtain by an argument similar to those proofs that for every $0 < \lambda < 1$ and $\varepsilon > 0$,

$$(4.18)\qquad \begin{aligned} \sup_{0 \le j \le k_n}\ \sup_{s \in I_{n,j}(\lambda)} \left|(1 - F_{n,\tau_{n,j}}(s))/(1 - F(s)) - 1\right|\\ = o(n^{-\lambda/2+\varepsilon}) \quad \text{a.s.}; \end{aligned}$$

moreover, in the case $F(\tau) < 1$, for every $q \in (1 - 3\alpha/2, 1)$,

$$(4.19\text{a})\qquad \sup_{0 \le j \le k_n}\ \sup_{s \in I_{n,j}(\alpha)} |U_{n,j}(s)| = o(n^q) \quad \text{a.s.},$$

while in the case $F(\tau) = 1$, for every $q \in (\beta/(\beta + 1), 1)$,

$$(4.19\text{b})\qquad \sup_{0 \le j \le k_n} \left\{ \sup_{s \le (\tau_{n,j+1} \wedge \tau_{n,k_n}) - a,\, 1 - F(s) \ge n^{-(1-q)}} |U_{n,j}(s)| \right\} = o(n^q) \quad \text{a.s.}$$

With probability 1, $\tilde{X}_{(n-[cn^\alpha])}(t_n) \to (t - \tau_H) \wedge \tau$, uniformly in $t \ge a$ ($> \tau_H$). If $\tau < \infty$, then for all large n, $\tau < \tau_{n,k_n} - a$ and therefore $(t, s) \in \Delta_a \Rightarrow s < \tau_{n,k_n} - a$. Suppose that $\tau = \infty$. Then $F(\tau) = 1$, in which case (1.22) is assumed. We now make use of the assumption (4.4) to show that as $n \to \infty$,

$$(4.20)\qquad \tau_{n,k_n} - a > F^{-1}(1 - n^{-(1-q)}) \quad \text{if } q > 1 - \rho\delta/(1 + \beta).$$

To prove (4.20), note that by (1.22), as $x \to \infty$,

$$(4.21)\qquad \begin{aligned} 1 - F(x) &= O(\{(1 - F(x))(1 - G(x-))\}^{1/(1+\beta)})\\ &= O(P^{1/(1+\beta)}\{X_1 \wedge Y_1 \ge x\}) = O(x^{-\delta/(1+\beta)}), \end{aligned}$$

by (4.4). Since $1 - q < \rho\delta/(1 + \beta)$, it follows from (4.5b) and (4.21) that $1 - F(\tau_{n,k_n} - a) = o(n^{-(1-q)})$ and therefore (4.20) holds. Hence, in the case $F(\tau) = 1$, $1 - F(s) \ge n^{-(1-q)} \Rightarrow s < \tau_{n,k_n}^{-a}$, for all large n. The rest of the proof is similar to that of Theorem 1. □

COROLLARY 5. *With the same notation and assumptions as in Theorem* 3, *let* $W_n(t, s) = \tilde{F}_{n,t}(s) - F(s)$, $V_n(r, t, s) = r(\tilde{F}_{[nr],t}(s) - F(s))$, *for* $(t, s) \in \Delta_a$ *and* $0 \le r \le 1$.

(i) *With probability* 1, *the sequence* $\{(n/2 \log\log n)^{1/2} W_n\}$ *is relatively compact in* $D(\Delta_a)$ *and its set of limit points is the unit ball* K_1^a *of the reproducing kernel Hilbert space of the covariance kernel*

$$\Gamma_1((t, s), (t', s')) = (1 - F(s))(1 - F(s'))g(t \vee t', s \wedge s'), \tag{4.22}$$

where

$$g(t, s) = \int_0^s \{H(t - u)(1 - G(u -))(1 - F(u))^2\}^{-1} dF(u).$$

(ii) *With probability* 1, *the sequence* $\{(n/2 \log\log n)^{1/2} V_n\}$ *is relatively compact in* $D([0, 1] \times \Delta_a)$ *and its set of limits points is the unit ball* K_2^a *of the reproducing kernel Hilbert space of the covariance kernel*

$$\Gamma_2((r, t, s), (r', t', s')) = (r \wedge r')\Gamma_1((t, s), (t', s')),$$

where Γ_1 *is defined in* (4.22).

PROOF. We first establish functional laws of the iterated logarithm for $\sum_1^n Z_i$. By Lemma 2.9 of Gill (1983),

$$\sup_{(t, s) \in \Delta_a} |Z_i(t, s)|$$

$$\le 2 \sup_{t \ge a} \left\{ \sup_{0 \le s \le t - a,\, s < \tau} \left| \int_{[0, s]} \{H(t - u)(1 - G(u -))\}^{-1} d\zeta_{i,t}(u) \right| \right\}. \tag{4.23}$$

Define $z_{i,t}(s) = \int_{[0,s]} (1 - G(u -))^{-1} d\zeta_{i,t}(u)$ and let $\xi_i(s) = I_{\{X_i \le Y_i \wedge s\}} - \int_0^s I_{\{X_i \wedge Y_i \ge u\}} \, d\Lambda(u)$, as in (1.23). Then by (4.8),

$$z_{i,t}(s) = \int_{[0, s \wedge (t - T_i)^+]} (1 - G(u -))^{-1} d\xi_i(u). \tag{4.24}$$

In view of the representation (4.24), we can use martingale inequalities as in the proof of Lemma 5 to show that

$$\begin{gathered} E\left[\sup_{t \ge 0, 0 \le s < \tau} z_{i,t}^2(s) \middle| T_i \right] \le 4 \int_0^\tau (1 - G(u -))^{-1} dF(u), \\ E\left[\sup_{t \ge 0, 0 \le s < \tau} \left(\sum_1^n z_{i,t}(s) \right)^2 \middle| T_1, \dots, T_n \right] \le 4n \int_0^\tau (1 - G(u -))^{-1} dF(u). \end{gathered} \tag{4.25}$$

Integration by parts gives that for $s \le t - a$,

$$\begin{aligned}&\left|\int_{[0,s]}\{H(t-u)(1-G(u-))\}^{-1}\,d\zeta_{i,t}(u)\right|\\
(4.26)\qquad &=\left|\int_{[0,s]}\{H(t-u)\}^{-1}\,dz_{i,t}(u)\right|\\
&=\left|z_{i,t}(s)/H(t-s)-\int_{[0,s]}z_{i,t}(u)\,d(1/H(t-u))\right|\\
&\le 2\sup_{0\le u\le s}|z_{i,t}(u)|/H(t-s),\end{aligned}$$

recalling that F is continuous. Combining (4.23), (4.25) and (4.26) gives

$$E\Big\{\sup_{(t,s)\in\Delta_a} Z_i^2(t,s)\Big\} \le 64(H(a))^{-2}\int_0^{\tau}(1-G(u-))^{-1}\,dF(u).$$

A similar argument also shows that

$$\left(E\left\{\sup_{(t,s)\in\Delta_a}\left|\sum_1^n Z_i(t,s)\right|\right\}\right)^2 \le E\left\{\sup_{(t,s)\in\Delta_a}\left|\sum_1^n Z_i(t,s)\right|^2\right\} \le 64n(H(a))^{-2}\int_0^{\tau}(1-G(u-))^{-1}\,dF(u).$$

Hence, as in the proof of Lemma 5, we can apply Theorem 4.1 of Kuelbs (1977) to conclude that with probability 1, $\{(2n\log\log n)^{-1/2}\sum_1^n Z_i\}$ is relatively compact in $D(\Delta_a)$ and its set of limit points is the unit ball K_1^a of the covariance kernel $EZ_1(t,s)Z_1(t',s') = \Gamma_1((t,s),(t',s'))$, noting in this connection that

$$\langle\zeta_{i,t},\zeta_{i,t'}\rangle(s) = \int_0^s I_{\{X_i\wedge Y_i\wedge[(t\wedge t')-T_i]^+\ge u\}}\,d\Lambda(u)$$

[cf. Gu (1987)]. Moreover, with probability 1, the sequence

$$\left\{(2n\log\log n)^{-1/2}\sum_1^{[nr]} Z_i(t,s)\right\}$$

of functions on $\{0\le r\le 1,\ (t,s)\in\Delta_a\}$ is relatively compact in $D([0,1]\times\Delta_a)$ and its set of limit points is K_2^a.

From the continuity of H (on $[a, \infty)$) and of F, it then follows (as in the proof of Corollary 1) that given $\varepsilon > 0$, there exists $\delta > 0$ for which

$$\sup_{(t,s)\in\Delta_a,\, t\geq t'\geq(t-\delta)\vee(s+a)} \left|\sum_1^n Z_i(t,s) - \sum_1^n Z_i(t',s)\right| / (2n\log\log n)^{1/2} \leq \varepsilon \quad \text{a.s.},$$

$$\sup_{(t,s)\in\Delta_a,\, \tau-\delta\leq s<\tau} \left|\sum_1^n Z_i(t,s) - \sum_1^n Z_i(t,\tau-\delta)\right| / (2n\log\log n)^{1/2} \leq \varepsilon$$

$$\text{a.s. if } \tau < \infty,$$

$$\sup_{s\geq 1/\delta,\, t\geq s+a} \left|\sum_1^n Z_i(t,s) - \sum_1^n Z_i(t,1/\delta)\right| / (2n\log\log n)^{1/2} \leq \varepsilon$$

$$\text{a.s. if } \tau = \infty.$$

Moreover, with probability 1, $\tilde{X}_{(n-[cn^a])}(t_n) \to (t - \tau_H) \wedge \tau$ uniformly in $t \geq a$ $(> \tau_H)$. Hence the desired conclusion follows from Theorem 1 and the functional laws of the iterated logarithm for $\Sigma_1^n Z_i$. □

Suppose that H is continuous everywhere [so $H(0) = 0$]. Let

$$\{W(x, y), (x, y) \in [0,1]^2\}$$

be a Brownian sheet with covariance

$$E\{W(x,y)W(x',y')\} = (x \wedge x')(y \wedge y').$$

Then the Gaussian process

$$Y(t,s) = (1 - F(s))\int_{u=0}^{s}\int_{v=0}^{t-u}\{H(t-u)(1-F(u))\}^{-1} \times (1 - G(u-))^{-1/2}\, dW(H(v), F(u)), \quad (t,s) \in \Delta_a, \tag{4.27}$$

has covariance kernel Γ_1 given by (4.22). Hence an argument similar to that used in proving the representation (1.30) can be used to show that

$$K_1^a = \Big\{\psi\colon \psi(t,s) = (1-F(s))\int_{u=0}^{s}\int_{v=0}^{t-u}\{H(t-u)(1-F(u))\}^{-1} \times (1-G(u-))^{-1/2}\phi(H(v),F(u))\, dH(v)\, dF(u),$$
$$(t,s) \in \Delta_a \text{ for some } \phi \in L_2([0,1] \times [0, F(\tau)])$$
$$\text{with} \iint_{[0,1]\times[0,F(\tau)]} \phi^2(x,y)\, dx\, dy \leq 1\Big\},$$

$$K_2^a = \Big\{\psi\colon \psi(r,t,s) = (1-F(s))\int_{\rho=0}^{r}\int_{u=0}^{s}\int_{v=0}^{t-u}\{H(t-u)(1-F(u))\}^{-1} \times (1-G(u-))^{-1/2}\phi(\rho, H(v), F(u))\, d\rho\, dH(v)\, dF(u),$$
$$0 \leq r \leq 1, (t,s) \in \Delta_a \text{ for some } \phi \in L_2([0,1]^2 \times [0, F(\tau)])$$
$$\text{with} \iiint_{[0,1]^2\times[0,F(\tau)]} \phi^2(\rho,x,y)\, d\rho\, dx\, dy \leq 1\Big\}.$$

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